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ABSTRACT

This yearbook is organized into three main sections: arithmetic, junior high school mathematics, and high school mathematics. Chapter 1 identifies and discusses specific questions in arithmetic instruction and learning. The kinds of research needed and suggestions about students needing remedial work are also discussed. Chapter 2 deals with reading difficulties in arithmetic, textbooks, different treatments of the number system, drill, and grade placement. Chapter 3 discusses 12 areas where arithmetic learning and teaching may be most difficult. The opening chapter in the junior high section discusses some problems concerning the general mathematics program in grades 7 and 8 and the restructuring of the ninth grade algebra course. Chapter 5 considers the time allotment and sequencing of topics for arithmetic, algebra, and geometry. Chapter 6 gives particular emphasis to the idea of social utility. Chapter 7 provides a discussion of objectives and a listing of topics and abilities to be cultivated. In the senior high section, chapter 8 concerns the aims of mathematics instruction and appropriate content. Chapter 9 continues this, and includes a section on possible curricular changes. The final chapter gives specific recommendations on content and instruction. (LS)

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE SECOND YEARBOOK

CURRICULUM PROBLEMS IN TEACHING MATHEMATICS

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INTRODUCTION

At the annual meeting of the National Council of Teachers of Mathematics in February, 1926, it was decided to publish a second yearbook. The topic, "Curriculum Problems in Teaching Mathematics," was chosen as the central theme because of the present interest in curriculum revision. Since it should be understood that such revision ought to be a continuous process, the discussions herein presented are not final. However, they furnish a basis that will help us to find better ways of determining how the proper content should be selected, arranged, and presented to the pupils. It must be remembered that unless the training of teachers in service keeps pace with the improvement in subject matter little good can be accomplished.

The Committee tried to obtain contributors holding many different points of view and representing as widely separated sections of the country as possible. The result can be labelled neither liberal nor conservative. Thus, in Part I, which is devoted to arithmetic, we have reflected the point of view of the trained supervisor, the psychologist, the classroom teacher, and the college professor. In Part II, on the mathematics of the junior high school, we find represented the attitude of the classroom teacher, the supervisor, the public school administrator, and the college professor. Part III, which deals with senior high school mathematics, presents the problems as seen by the college professor, the private school administrator and teacher, and a committee of classroom teachers studying the needs of the public school curriculum.

No attempt has been made to reconcile the various opinions. In fact, it was thought better not to do so. A thorough discussion of all the worthwhile suggestions is more desirable. This can be done in various meetings throughout the country and through the columns of *The Mathematics Teacher*.

The Committee desires to take this opportunity to thank those who have helped to make this yearbook. The credit for any success it may have should go to those who have given so unselfishly and unsparingly of their time and energy in its preparation.

With the hope that the reader will receive help in solving the many problems that arise in teaching mathematics, this Second Yearbook is respectfully submitted.

Second Yearbook Committee
W. D. Reeve, *Chairman.*
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C. B. Marquand.

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PART I

ARITHMETIC

SOME ASPECTS OF ELEMENTARY ARITHMETIC

By F. B. KNIGHT

College of Education, University of Iowa

Introduction. Discussions of presumably important aspects of elementary arithmetic which appear on pages 4 to 72 of this book have been assembled by F. B. Knight, Professor of Educational Psychology at the State University of Iowa. These reports are samples of the aspects of the total field of arithmetic with which workers in that field are at the present time largely concerning themselves. The following discussions do not claim to be an organized attack on the whole field of arithmetic or a final statement of sufficient truth about any single aspect. A perusal of these discussions will, however, suggest to those who are primarily interested in high school mathematics some phases of the general scope of investigation and thought in the elementary school section of mathematics. These discussions may also suggest types of studies and methods of study applicable to certain phases of high school mathematics.

Discussion Topics. The following topics will be discussed in subsequent pages:

1. What differences of opinion and practices relative to methods are at present claiming attention?

   Sample: Comments on thirteen differences of varying importance. (Knight)

2. What are the main agreements and disagreements in the nature of representative standard tests in arithmetic?

   Sample: A digest of the specifications of some standard tests. (Ruch)

3. How do representative textbooks of arithmetic agree and differ in their fundamental work of instruction as such?

   Sample: The differences in text material in addition of whole numbers. (Wells and Olander)

4. What types of analysis and investigation may be used to help settle questions relative to the better of two possible methods?

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7. What types of investigations will yield useful information about the teaching nature of a text?

Sample: A report of how two texts distributed their pupil time on the several teaching aspects when these texts were taught as printed for one year. (Thies)

8. Is it possible to approach an objective method in appraising the relative merits of texts as teaching instruments?

Sample: A summary of a procedure for rating the drill aspects of texts. (Lutes and Samuelson)

9. What can be done for pupils who need remedial work?

Sample: Possibilities in remedial work in problem solving. (Greene)

It goes without saying that the above discussions do not by any means exhaust the possibilities of or the need for scientific work in elementary arithmetic. They are presented as typical examples of a much broader field.

DISCUSSION ONE

What Differences of Opinion and Practices Relative to Methods Are at Present Claiming Attention?

Professor Thorndike says that there are three disputes in arithmetic which occasion more heat and hard feelings than perhaps any dozen other disputes.[1] It happens that at least two of these disputes are about aspects of teaching problems which can hardly be estimated as vital problems. And on no one of these disputes are

[1] E. L. Thorndike, *New Methods in Arithmetic*, pp. 208 ff. Rand McNally and Co.

there at the present time data of sufficient merit to close debates about them.

When, however, we extend the difference of opinion from the field of discussion to the area of actual classroom procedures, we find many differences of opinion and practice which are so confusing that a short discussion of some of them is in point.

I. Subtraction

How Shall Subtraction be Taught? This is one of the disputes mentioned by Professor Thorndike. Presumably equally competent people in about equal numbers advocate each method of teaching subtraction. At the present time conclusive evidence to support one method as against another is not available.[2]

Space is lacking here for an adequate discussion of the various problems at issue in the debate on how subtraction should be taught.[3] Three comments only are offered as follows:

1. There is at the present time an unfortunate confusion of terms, which involves two issues in the teaching of subtraction. These two issues are distinct and should not be confused.

Issue One. How shall the child think the subtraction, for example, in taking 4 from 9? There are two possibilities.

First, if the child essentially thinks 9 minus 4 leaves 5, he uses the *subtractive method.* There are many ways in which this essentially subtractive process may be stated.

Second, if the child essentially thinks 4 and 5 make 9, he uses the *additive method.* There are many ways in which this essentially additive process may be stated.

Much can be said on both sides of this issue: subtractive versus additive subtraction. Scientific data do not at present clearly support either side. Excellent results are obtained by either method,

[2] For a competent discussion of investigations on method in subtraction to date, see G. M. Ruch, F. B. Knight, O. S. Lutes, "On the Relative Merits of Subtraction Methods: Another View," *Journal of Educational Research,* 11: 154, 155, February, 1925. In this connection the reader should distinguish between evidence of an experimental nature, argument, and illustration usually contributed by writers who possess a vested interest in a certain method.

[3] For a systematic treatment of the criteria by which decisions relative to choices of method can well be made and an application of these criteria to the possible methods of teaching subtraction, see F. B. Knight, G. M. Ruch, and O. S. Lutes, "How Shall Subtraction Be Taught?" *Journal of Educational Research,* 11: 157-168, March, 1925.

or in spite of either method, as the case may be. Again, faulty work results from either, or in spite of either, as the case may be.

Issue Two. How shall the child handle the difficulty represented by the black faced type in the example: Subtract $\begin{array}{r}3439\\ \underline{\mathbf{6573}}\end{array}$ Here again there are two possibilities.

First, the "borrowing" method. In the example above the use of the "borrowing" method would involve changing the form of the 73 in the minuend to 6 tens and 13 ones.

Second, the "Austrian" or "equal additions" method. In this case 10 ones are added to the 3 ones in the minuend and 1 ten is added to the 3 tens in the subtrahend.

As in the case of Issue One, reliable scientific data are not available to support conclusively either the "borrowing" or the "Austrian" method.

Combining the possibilities in Issues One and Two, it is quite possible to teach each of the following combinations, and each one of these combinations no doubt is actually taught:

a. Subtractive-borrowing;
b. Subtractive-Austrian (equal-additions);
c. Additive-borrowing;
d. Additive-Austrian.

In current discussions and literature there is frequently a use of the word "additive" to mean "Austrian," a use of the word "subtractive" to mean "borrowing," and so on. Misunderstanding in the disputes relative to the teaching of subtraction is greatly increased by the careless use of terms.

2. A second comment on the teaching of subtraction which seems pertinent is the suggestion that the secrets of the successful teaching of arithmetic lie not in the matter of which method is to be used but in other aspects of teaching. In all probability if one method possessed significant advantage over another, such superiority would have been revealed by investigations carried out for that purpose. Essential equality of method is the main result of such experimentation.

Energy now spent in argument about which method to use would be better directed upon research and experimentation calculated to increase the efficiency of the teaching of subtraction by means of any substantial method. A critical examination of the actual procedures of teaching subtraction in texts and courses of study reveals wide differences in the amount and quality of the instructional material used. The problem of how best to teach any good method is the crucial point, in contrast to the problem of which particular method to use.

3. A third comment on the teaching of subtraction may justify itself by stressing the importance of insuring a wide experience with the three fundamental uses or ideas of subtraction. It is an error to present subtraction as if it were the technique to use in working solely with the "how much is left" idea or solely with the "how much must be added" idea. Subtraction thinking and computation is needed in three situations; namely, (1) how much is left, (2) how much more is needed, and (3) what is the difference of two quantities. Specific instruction in all three of these ideas is required for real insight into the subtraction process. Overstressing of one of the above ideas and neglect of the others is all too common in present-day teaching of subtraction.

In this connection a wider experience with the vocabulary of subtraction would prove beneficial. Many pupils go through the elementary school meeting subtraction work expressed in one or two set ways. In everyday life the subtraction idea is expressed in a legion of ways. Similar variety of expression should be included in the child's school experience with subtraction.

II. Division of Decimal Numbers

Methods and Rules. It is comparatively easy to start an argument about the methods and rules to use in taking care of the decimal point in the quotient of division examples involving decimal numbers and decimal fractions. Unfortunately, objective inquiry into the relative merits of various methods to match the voluminous quantity of speculation on this point does not exist. Comments here will be limited to a suggestion relative to the effective presentation of one method.

Let us assume that the child is to be taught to take care of the decimal point in the quotient by a procedure based on this rule: "Multiply the divisor by that multiple of 10 which will make it a whole number. Multiply the dividend by the same multiple of 10. Place the decimal point in the quotient directly above the decimal point in the dividend resulting from this procedure."

In actual computation the child does not really perform a written multiplication of divisor and dividend by 10 or a multiple thereof. He counts the number of decimal places in the divisor and then counts off a similar number of places in the dividend to the right of the decimal point, and places the decimal point in the quotient above the place where such counting leads him. In the example: $.234\overline{)50.7684}$ his counting would bring him to a place between the 8 and the 4 in the dividend. He would place the decimal point in the quotient directly above that position or place.

Improvement of Procedure. This procedure causes many children an excess of difficulty. But much of that difficulty can be relieved by improved techniques of drill rather than by a substitution of other rules.

This "counting off" of a varying number of places in the dividend involves an entirely new set of habits relative to reading and working with numbers. Until division of decimals is met the child has never counted off places in a number, he has never "looked at numbers that way." A new set of eye movements must be built up. It is wise in initial practice to let the hand help the eye as suggested in the following examples:

$$2.47\overline{)3.1689} \qquad .390\overline{)974.800}$$

Although the arrows as printed represent no mathematical procedure, they do represent a pedagogical device which relieves the child of difficulty often leading to error and discouragement. It has been the observation of the writer that children drop the habit of actually writing the arrows as soon as their new eye-movement habits are strong enough to operate smoothly without the aid of the hand. The dropping out of the arrows can be thought of as a functioning of the "law of laziness." Pupils as well as adults save themselves procedures which do not help them.

The mathematician may object to the procedure illustrated above on the score that the arrows imply at least two decimal points in a number, which is of course an absurdity. It should be borne in mind, however, that the child has to think this procedure anyway. The writing of the arrows introduces no new thinking; it only makes the thinking more sure and accurate while new eye-movement habits are being built up.

Increased Quantity of Drill. Increasing the effectiveness of method is a matter of sheer quantity of experience with the "shifting of the decimal points." The usual type of drill work in division of decimals involves the entire working of each drill example. It is fair to assume that the thing to stress in division of decimals is the matter of the decimal points, since the computation of the numbers themselves presumably has been previously mastered. With a limited amount of time to spend on drill involving decimal points it is questionable if all the time should be spent on the type of drill which gives most of the time to estimating quotient figures, and to the multiplication and subtraction involved in division. When this is the only type of drill provided, the child never gets very much experience with the handling of the decimal points themselves. A dozen examples, that is a dozen experiences with decimal points, would be an outside limit for one drill period of considerable length. When, however, the drill material is built to give practice on the handling of decimal points only, 30 to 40 practices on decimal points (the new skill to be mastered) is possible in a five- to ten-minute drill period. The following sample of drill work will illustrate the phrase, "increasing the effectiveness of a method by using better types of drill in connection with that method."

Placing Decimal Points in Quotients.—The figures in the quotients below are correct, but the decimal points and some zeros are left out. Study each example and then place the decimal points in the quotients, adding zeros when you need to.[4]

	(A)	*(B)*	*(C)*	*(D)*
1.	$\begin{array}{r} 724 \\ 236\overline{)17086.4} \end{array}$	$\begin{array}{r} 724 \\ 23.6\overline{)170864} \end{array}$	$\begin{array}{r} 724 \\ 23.6\overline{)170.864} \end{array}$	$\begin{array}{r} 724 \\ 23.6\overline{)17086.4} \end{array}$
2.	$\begin{array}{r} 724 \\ 236\overline{)1708.64} \end{array}$	$\begin{array}{r} 724 \\ 2.36\overline{)170864} \end{array}$	$\begin{array}{r} 724 \\ 2.36\overline{)1708.64} \end{array}$	$\begin{array}{r} 724 \\ 23.6\overline{)1708.64} \end{array}$
3.	$\begin{array}{r} 724 \\ 2.36\overline{)170.864} \end{array}$	$\begin{array}{r} 724 \\ 23.6\overline{)17.0864} \end{array}$	$\begin{array}{r} 724 \\ 2.36\overline{)17.0864} \end{array}$	$\begin{array}{r} 724 \\ 236\overline{)170.864} \end{array}$

The above two comments do not exhaust the possibilities of improving the teaching of division of decimals. They are but samples of several possibilities which seem to contain merit.

[4] Section of a drill unit from F. B. Knight, G. M. Ruch, J. W. Studebaker, *Arithmetic Work-Book, Grade 7*, p. 52. Scott, Foresman and Co., 1926.

III. Should a Pupil Add Up or Down a Column?

Importance of the Question. For some reason which utterly baffles the writer considerable importance is attached to the answer to this question. A worker in the pedagogy and psychology of arithmetic is asked this question about as often as any other, and his answer is waited for with interest comparable to the interest evinced by a sophomoric collegian when he asks a new acquaintance to which fraternity he belongs.

Of course, if it is really important that an answer to an addition example be correct, it is well to add *both* up and down the columns of the example. Whether one first adds up and then adds down to check the answer or whether one reverses the order cannot very much matter. There can be no magic virtue in the sequence here.

There has been considerable furore over the possibility that more of the eye-movement habits used in reading context are used in adding down a column than in adding up a column. No experimental data now available support this contention. If transfer from reading context to adding exists at all, it likely exists in almost negligible amounts. It should also be held in mind that transfer may be negative as well as positive, harmful as well as helpful. If the reader will actually trace his eye movements in adding down a column, he will probably conclude that for every instance in which eye-movement habits of reading are used in adding down a column of an example, other eye-movement habits of reading are contradicted; for every habit that is facilitated another is interfered with. For every facilitation factor between reading and adding there is certainly a companion "interference factor." When two skills are somewhat alike and somewhat different, the one is just as apt to hurt the other as to help it.

No one of course flatters himself that he has contributed to the improvement of the teaching of arithmetic in a vital way by insisting that pupils add up a column instead of down it, or the reverse. If one procedure were significantly better than the other, that fact would have been known and used these many years. Those who have the skill and energy for useful research, however much they may argue in spare moments about the direction of adding, also have enough judgment actually to spend their time for research with problems of teaching which have better chances of yielding more useful findings. Adding up versus adding down is a trivial dispute and should be treated as such.

IV. Crutches

Use. The use of crutches is another discordant element which infests practice and discussion. The writer has been unable to find a straightforward listing of the specifications for devices which make them a "crutch" in contrast with a defensible device in computation. An adding machine is certainly a crutch in one sense, and so even is multiplication itself, when thought of as an easy way to add or as the "lazy" (and prudent) man's way of adding certain types of examples. To an experienced mathematician much if not all of the written work in elementary school arithmetic is a crutch, since he can do such work in his head. And to the prodigy for one to write any partial products for the example 4896×5086 would be as much of a crutch as the writing of the partial remainders and dividends for the example $45\overline{)3456}$ would be for a fifth grade child.

In other words, when is a crutch a crutch? There may very well be certain devices which are so harmful for all workers of all degrees of ability and irrespective of the stage of learning involved that they should be intolerable under all circumstances and dubbed crutches in the unsavory sense of the word. There have been many careless statements about the relation between crutches and errors, between crutches and slowness of work, and between crutches and the untidy appearance of papers. But the writer does not know whether these relations exist or not. If they do exist, their existence should be adequately demonstrated. If some but not others exist the harmful ones should be determined. A consideration of two famous crutches may prove of interest.

1. Some claim that Sample B below is an outrageous crutch which should never be tolerated in print. It actually tells the child what to do. It does not give the child a chance to use any critical judgment of his own. It stultifies him. (Every criticism immediately above is a fair paraphrasing of the opinion of a well-known leader in elementary arithmetic.)

$$\begin{array}{lcccc} & \text{A} & & \text{B} \\ \text{Subtract:} & 3456 & vs. & 3456 \\ & \underline{1208} & & \underline{-1208} \end{array}$$

The writer must confess to the obtuseness which makes it impossible for him to see just why the minus sign (which the child must know means subtraction in order to use it) is any more of a crutch than the word "subtract" printed in plain English. Why is a sign more of a crutch than a word? [5]

2. Consider the writing of *partial* sums in an addition example. There is not, so far as the writer knows, any objection to writing *partial* products in multiplication examples, as shown below.

A		B	
23			
437	The partial sums appear. This is a crutch.	437	No written record of partial sums is made.
28		28	
96		96	
728		728	
72		72	
1361		1361	

Example A above shows the use of a crutch. The objection to this crutch is that should the child rework the example he will or may add the partial sum (*23*) twice, once as he has written it on his first trial and again as he reworks the example. Whether a pupil so stupid that he does as suggested is still bright enough to get correct answers without utilizing this crutch is not known.

This example is not presented as a defense for the use of the crutch but to create a situation in which it is convenient to ask several questions about crutches. These questions are:

1. Is it possible that certain practices are crutches, lazy and indefensible practices, for pupils of certain mental levels but useful and defensible practices for pupils of more limited ability, especially in the field of attention span and immediate memory?

2. Is it possible that the absence of a certain device may lay no strain on one type of pupil but may at the same time lay an undue strain upon another? The writer has no immediate need for the health "crutch," cod liver oil, but as a matter of fact his next-door neighbor has.

3. Is it possible that a device which is not needed for permanent use at the end of a learning enterprise may be very useful at the be-

[5] The objection to the use of the minus sign on the score of confusing the mind of the child about positive and negative numbers in algebra is of course another matter. It is possible that the subtraction sign as used in third grade arithmetic is a distant relative to confusion in ninth grade algebra. But this possibility is not the point at issue here.

ginning of that enterprise? The direct retort that we should learn things at the beginning as we are to use them at the end is hardly in point here. Helpful devices at the beginning will drop out when no longer needed, under the "law of laziness." For example, the child may learn his number combinations with individual cards as crutches. He will discard them when he no longer needs them.

4. Is it possible that the objection to crutches on the score that they are time consuming is a bit overstated? It may well be that the energy saved more than offsets the minute or two a week taken to write down a number (crutch) instead of holding it in mind.

5. Would it not be advisable to match the heated arguments over crutches and the cocksure attitude about them with a little research to determine whether our attitude toward crutches one way or the other is supported by the facts in the matter?

Teachers of high school mathematics may with some reason feel that discussions in elementary school arithmetic center around rather trifling topics. The situation may appear to be a case of exaggerating the irrelevancies of teaching to the neglect of the vital and dynamic essentials of method and content. The writer must confess to the presence of some ground for such an estimate of the status of elementary school mathematics. There is more than one methods book in which page after page of elaborations of the obvious may be found and no pages dealing directly with the heart and soul of the teaching of arithmetic. There are at least a dozen series of textbooks which vary one from the other in no significant way except that different authors have composed them and different publishing houses are responsible for their prosperity. It must not be thought, however, that arithmetic is entirely in the doldrums or that there has been no significant work in arithmetic for many years.

Professor Thorndike published his *Psychology of Arithmetic* in 1922.[6] Since then advances in efficiency in the teaching of arithmetic have been gaining steadily in merit and influence. It seems as if the dawn of a new day in the teaching of arithmetic was well started. The following section of Discussion One: What Differences of Opinion and Practices Relative to Methods Are at Present Claiming Attention? will impress the reader as indicative of real progress in the elementary school field.

[6] Quotations from this important work are infrequent in this report because it is assumed that those who read these pages are already familiar with Professor Thorndike's basic contributions.

V. How Should a Pupil's Time Be Distributed in Arithmetic?

Fundamental Questions. This question seems to strike at a vital part of the total problem of method. Its significance will be clear from the perusal of the following questions:

1. What per cent of the total time should be spent on the sheer learning of computational skills? That is, of every one hundred hours spent on arithmetic should ten, twenty, thirty, fifty or more hours be spent on sheer learning how to manipulate the several arithmetical processes?
2. Does the per cent of total time properly spent on gaining skill in computation vary from grade to grade? Would 60 per cent of the total time, for example, be correct for the third grade and but 20 per cent for the eighth grade?
3. What per cent of time should be spent upon enterprises calculated to maintain skills which have been previously learned?
4. What per cent of time should be spent upon the application of skill in process manipulation to problems and problem situations?

Of course the answers to such questions as these are not at present available. But workers in elementary school arithmetic are beginning to discuss these questions and others related to them.

VI. What Does a Year's Work in Arithmetic Mean?

Need for More Exact Knowledge. The need for more exact knowledge relative to the minutiae of a child's experiences in arithmetic year by year is becoming recognized. A demand for answers to questions like the following is beginning to be felt.

1. If Johnny does a good year's work in fifth grade arithmetic, how many examples in the addition of whole numbers will he try and what kind of examples will these be? Of the number attempted how many will he solve correctly on his first trial? Exactly what does he do about those on which he failed at his first trial? How many times does Johnny actually meet the combination $9 + 7$? How many times should he meet this combination and the many others similar to it?
2. If Mary passes with credit her sixth grade arithmetic, what does this mean in number and kinds of verbal problems solved? What should successful sixth grade work mean so far as problem solving is concerned? How much direct and indirect aid did and

should Mary receive from her teacher on the problems she attempted?

In general, the advantage of illuminating case studies of the details of a child's experience in arithmetic is being recognized. Even a hastly reading of actual case studies of the minutiae will impress the reader with two facts at least; namely, (*a*) the picture a teacher has in her mind about what a child actually did and did not do probably differs significantly from the facts; (*b*) proper organization of the facts gained from many case studies can develop a sagacious insight into the work and minds of children which will form a sound basis of experimental evidence for attempts to provide fundamentally better learning situations in the teaching of arithmetic.[7]

VII. What Is the Learning Nature of a Skill?

Useful Variance. There may be a useful variance in the nature of a skill from the standpoint of pure mathematics and from the standpoint of its presentation to a young learner. For example, a mathematical analysis of the addition process would exclude certain aspects which a learning analysis of the same process would necessarily include, because the mathematical aspects involve common difficulties for youngsters trying to master the skill. Until recently the aspects or items of the different processes in arithmetic which were presented to the child were fundamentally the items as listed from the standpoint of the mathematics involved. And the order of presenting such items was essentially a logical order.

Guarded observation of pupils' learning and other types of investigation are beginning to show that mathematical analysis and logical order of presentation of skills are not always the best ones for the young learner. Consequently, we are asking and seeking to find the answers to such questions as the following: What are the specific difficulties which we should teach in presenting each process? We are finding that almost every process in arithmetic is far more complicated from the standpoint of its learning than has been assumed. For example, instead of specifically teaching two or three things about addition of whole numbers, we are teaching many

[7] For an example of case studies of children's arithmetic, see W. W. Cook, *The History of Learning in Arithmetic of Four Fifth Grade Pupils*, 1926. An unpublished study obtainable from the Library of the College of Education, State University of Iowa.

more. In contrast to a presentation of long division on the theory that it is a complex of four or five learning units, we are finding that for the learner it is a complex of many more "unit-skills." Of course the above does not mean that individual teachers may or may not have pointed out learning units or difficulty units in their teaching of the several processes. We mean here that the teaching of arithmetic as a science reported in methods books, courses of study, and textbooks formerly failed to recognize the difference between mathematical analysis and learning analysis of the processes taught.[8]

VIII. WHAT IS AN ADEQUATE THEORY OF LEARNING?

Truth or Falsity of the Theory. Let us assume for the moment that the success or failure of the details of actual instruction are related to the truth or falsity of the theory of learning which is the foundation of the specific practices. Every course of study, every textbook, and every classroom technique is based on some theory or theories of learning. Of course a writer or a teacher may be quite unconscious of the fact that he is using some theory of learning. It is clear to the careful observer of the teaching of arithmetic that this science would have been saved from many vicious as well as futile methods if the inventors of these methods before writing them had taken time to ask themselves the next two questions: (*a*) Upon what theory of learning is this method based? (*b*) Is the theory upon which this method rests *as a mere matter of fact* both true and potent?

There are at present evidences of several differences of opinion on

[8] Space here permits the raising of this discussion only. For examples of learning analysis, see Knight, Luse, and Ruch, *Problems in the Teaching of Arithmetic*, pp. 13, 19, 27, 37, 47. (Iowa Supply Co., Iowa City, Iowa.) For examples of these "differences of practice" in courses of study, compare almost any course of study published before 1918 with the Denver Course of Study. (Denver is used only as an example of many modern courses.) For studying these "differences" in terms of textbook material, contrast the treatment of the four processes in any text published before 1923 (or based on ideas current before 1923 if published since that date) with Schorling and Clark, *Junior High School Mathematics* (World Book Company), and *The Standard Service Arithmetics* (Scott, Foresman and Co.). The writer does not mean to imply a judgment relative to the merits of various textbooks here. The above references will illustrate the differences of practice on the topic of what items of a skill should be specifically presented to pupils when they are first learning a new skill.

the desirability of having a well-organized and defensible theory of learning as a basis for the teaching of arithmetic. Two of these differences may be noted: (*a*) Many authors of material and the users of many methods of classroom teaching would be somewhat confused if they were asked to write down an exact statement of the theory of learning used in a particular section of their work. But such a practice would prove exceedingly illuminating to them. (*b*) There seems to be a difference of opinion among authors as to whether a sound theory of learning should be stated in a preface and then used in the actual material of the practice given in the body of the text. This state of affairs is evident in methods books versus classroom techniques; in introductions to courses of study versus the nature of the material in the book itself.

One of the most definite differences in theory itself may be noted. Relative to the learning of a new process actual practices may be divided under two main theories. In question form these are: (*a*) Shall the explanation of a new process be so organized that the teacher shall carry the burden of instruction? This is usually accomplished by making the reading difficulty of explanations in textbooks so great that the child is not able to figure out what is meant. He has to have the teacher explain the matter to him. This theory we may call the "passive attitude theory" since it forces the child to depend, not on his own powers of study, but on the teacher. (*b*) Shall the explanation of a new process be so organized that the pupil shall carry the burden of his own first learning? This is usually accomplished by providing him with expanded explanatory material often in the form of focalized reading lessons of such a reading difficulty that he can learn for himself. This may be classed the "active attitude" theory.

The writer wishes to call attention here not to the superiority of one method over the other but to these facts: (*a*) that our pupils are learning arithmetic by practices based primarily on one method or the other; (*b*) that classroom methods and texts are consciously or unconsciously based on one method or the other; (*c*) that those responsible for the first learning should know which theory they prefer to use and which method they are actually using.

The "passive attitude" theory versus the "active attitude" theory is but one of many differences in the field of the first learning of arithmetical skills and applications.

IX. How Shall Skills Be Maintained?

Difference of Opinion. It is obvious that there is much difference of opinion concerning the proper answer to this question. Most of the instructional material in arithmetic is organized as though the proper answer to the question above were this: Skills are best maintained by providing large amounts of practice at the beginning and end of a year's or a semester's work. This is accomplished by providing large amounts of drill on those pages of a textbook which pupils work upon at the beginning and end of a year or a semester. The answer is somewhat compromised by notes telling the teacher in one way or the other (or assuming that she will tell herself) that such material may be used at other times.

Although most textbooks are constructed according to the answer given above, it is very often repudiated by experienced teachers. This is evidenced by the great popularity of supplementary drill work used in connection with the regular textbook work on a daily or weekly schedule.[9]

The other main answer to the question about maintaining skills is that short mixed drills of an inventory nature should be scattered through the pages of the text as an integral part of text-material. A sample of such a drill is inserted here.

Self-Testing Drill No. 14[10]

Directions. Time allowed: 15 minutes. Follow the directions given for Drills 1 to 5. Try to beat all your past records.

1. $\begin{array}{r} 145 \\ \times 4 \\ \hline \end{array}$	2. $\begin{array}{r} 336 \\ 68 \\ 327 \\ 346 \\ \hline \end{array}$	3. $\begin{array}{r} 29 \\ 48 \\ 507 \\ 407 \\ \hline \end{array}$	4. $4\overline{)164}$	5. $\begin{array}{r} 922 \\ -533 \\ \hline \end{array}$
6. $5\overline{)400}$			7. $\begin{array}{r} 618 \\ -260 \\ \hline \end{array}$	8. $\begin{array}{r} 720 \\ -322 \\ \hline \end{array}$

9. $5 \times 32 = ?$ 10. Divide 276 by 3. 11. $5\overline{)300}$

[9] Examples of such supplementary drill work are: *The Studebaker Economy Practice Exercises*, Scott, Foresman and Co.; *The Courtis Drill Pads*, World Book Co.; *The Lennes Drill Sheets*, Laidlaw Bros.; *The Arithmetic Work-Books*, Scott, Foresman and Co.

[10] *Standard Service Arithmetics, Book One*, p. 284. Scott, Foresman and Co.

12. The committee in charge of a third-grade picnic bought ten pints of ice cream in pint bricks. How many quarts were there in the ten pints?

13. The druggist sold the ice-cream bricks to the pupils at his regular price per quart, which was 50c. How much did the ice cream cost?

14. The committee also bought two boxes of stick candy at 35c a box. How much did the candy cost?

15. There were 29 pupils and the teacher at the picnic. Each one gave 20c as his share of the expenses. The amount given was $6.00. The picnic cost $5.6?. How much was left to put in the class treasury?

STANDARDS

Number correct......	0	1–3	4	5	6	7	8	9	10	11–14	15
Scores	0	1	2	3	4	5	6	7	8	9	10

It is important in this connection to point out that such a maintenance program will merit drill material built with care according to accurate specifications.[11]

X. What About the Difficulty of Verbal Problems?

Present Practice. The bulk of the practice relative to problem material at the present time is to present verbal problems the difficulty of which has never been determined. In other words, the class uses problems somewhat analogous to the purchase of a "pig in a poke." It is possible that those who make the problems can guess the levels of difficulty of problem material best suited to the grades for which the problem material is made. But that the ability to make successful guesses is somewhat overrated is probable in view of the fact that with one exception no text has ever supplied scientifically obtained standards for work of problem material grade by grade in any considerable amounts.

One textbook does supply a series of problem scales. The problems are of known pupil-difficulty, the difficulty being derived from adequate experimentation. Here is a clean-cut difference of opinion; namely, the use of problems the difficulty of which is not known versus the use of problems the difficulty of which has been determined.

[11] For an outline of proper specifications see the *Third Year Book, Department of Superintendence*, pp. 62 ff. Washington, D. C.

Most problems now supplied to children are too difficult for them. For example, the counterpart of problems known to be useful in determining the intellectual difference of twelve- and thirteen-year-old pupils can frequently be found in material supplied to ten- and eleven-year-old pupils. Our present practice relative to problem material is based on a gross over-estimation of the child's ability to reason. This practice doubtless is largely responsible for the frequent over-helpfulness of teachers on problem material, which is a crutch few would defend. It is also probably responsible for the nervous and often ill-formed attempts to use devices to teach children how to solve problems. It is moreover probably responsible in part for the criticism of the public relative to problem solving in the schools. To the writer the situation seems to be something like this. We lead the public to think that a child can solve problems of a given nature since we give him these problems to solve. But either the child is too immature to solve these problems or the available methods for teaching problem solving are so inferior that the child cannot work them. The public notes the persistent gap between what we ask the child to do and what he actually can do, and blames the school.

At the present time we should find out what types of problems children can do and what are effective ways of teaching problem solving. It is quite possible that ability to solve verbal problems is far more a gift of nature than we at present assume.

XI. What Is the Status of Testing?

Difference of Opinion and Practice. The difference of opinion and practice here is on the point whether the gross unanalyzable score of a survey test yields enough information to pay for the trouble of testing. The possibility of substituting or evolving genuinely diagnostic tests in contrast to survey tests is also a subject of discussion and experimentation. Competent diagnostic tests are now available and are being used. Whether these tests sufficiently supplement the teacher's judgment is a mooted problem not yet settled in the minds of many teachers and supervisors. The writer, however, would be willing to represent the affirmative on both of the following assertions:

1. The usefulness of gross survey testing is far less than previously assumed.

2. A prudent use of competent diagnostic tests should be an integral part of arithmetic teaching and supervision.

XII. What About the Arithmetic Curriculum?

Social Utility Theory. There are several differences which should be noted here. There is first of all the matter of the theory upon which or by which items shall be admitted or omitted from an enlightened curriculum.

The social utility theory as a theory of curriculum building is quite properly the dominating one. But the exact nature of a defensible theory of social usefulness, to say nothing of its application in particular situations, is not so simple a matter as it seems at first glance. In the writer's opinion there has been some rather naïve and unsophisticated writing about the social utility theory where the dignity and cleverness of the phraseology used has badly outrun the worth of the ideas carried by the phraseology. We are agreed, however, in principle that usefulness of the material learned should largely determine what material is to be learned. Surely the fact that a topic is known, that it can be taught, and that it always has been taught would not, in any combination, form a defensible theory of curriculum building.

Just what is useful is not always clear. For illustration, a favorite diversion is to take a fling at the second case of percentage. This is done in spite of the frequency with which the second case of percentage must be used in coming to exact decisions relative to the price of baby bonds from advertisements about them in the newspapers. And baby bonds come with increasing frequency within the scope of the average man's interests. It seems to the writer that the idea used in the second case of percentage would be extremely useful if any one knew how to use that idea. We simply do not know to how many social uses the second case of percentage can be put and will be put within our time. No one has ever tried to make it useful. There are many topics in arithmetic judged socially useless which find themselves in the same situation which many inventions were in before some one had wit enough to find commercial uses for them. There are many topics in arithmetic which have appeared in courses of study and in textbooks but have never gotten over into the nervous systems of pupils so that the pupils could use them. Any poorly mastered skill is of doubtful usefulness. With the advance of skill in teaching arithmetic many skills which now seem socially hopeless may prove of enough use to pay for themselves because pupils can then use them; now they cannot through lack of mastery of them. It may be a bit hasty to ruthlessly reduce the arithmetic curriculum. It was greatly reduced but a few years ago. Perhaps we are not so clever in finding and increasing the uses of arithmetic as we think. Better arithmetic instead of less arithmetic contains possibilities.

Grade Placement. Another matter relative to curriculum is grade placement of a given topic. This matter deserves attention. *Ex cathedra* decisions are not the only ways in which problems of grade placement may be solved.

One caution relative to grade placement may well be made. Take as an illustration a recent *ex cathedra* decision that long division should be transferred from grade four to grade five, presumably because of its difficulty. Teaching of long division based on an inadequate analysis of the learning problems involved may result in a treatment too hard for the fourth grade or even the fifth. But a teaching of long division based on an adequate analysis of its difficulties and buttressed with skillfully organized drill is within the ability of fourth grade children, and provides an almost exciting experience for them. Should we settle the problem of long division, then, by putting it in the fifth grade or by teaching it better in the fourth?

Level of Difficulty. A curriculum must or should decide not only whether or not a skill should be taught and where, but also to what level of difficulty that skill should be brought for a given grade. For example, which of the following examples should be a model for long division in the fourth grade?

$$58\overline{)4089} \qquad \text{or} \qquad 5087\overline{)13099463}$$

In making decisions relative to the level of difficulty of each topic the social utility theory will be of yeoman's service.

XIII. Group vs. Individual Instruction

A Widely Discussed Topic. This topic is widely discussed in arithmetic as well as in the other school subjects. There is of course much to be said on both sides. Experimentation with individual methods of instruction is increasing and a growing support for it seems to be a fact. At least this issue can neither be laughed out of court nor dismissed lightly. The writer shares with others insufficient experience with individual instruction over long periods of time with the same pupils involved to warrant any general conclusions. Several comments on this issue are current and may be recorded here.

1. Exceptionally competent materials are required for the working of a system of individual instruction. There is considerable question concerning the satisfactory nature of most materials now offered for this purpose.

2. Any form of individual instruction that is a return to the ungraded school under a different name may gain temporary notoriety but will have a poor chance of persisting.

3. Adequate methods of sectioning classes on a basis of ability and skill will tend to lessen the weaknesses of group instruction and will dull the demand for individual instruction.

4. A mixture of group and individual work may be at least a substantial improvement. Of course teachers have always given individual instruction but average technique here is susceptible to real improvement. A combination of the two basic methods, namely, keeping the class together on enterprises of first learning and conducting remedial and much of the drill work on an individual basis, may commend itself to many.

5. The social implications of over-individualized work will evoke comment from several sources.

6. At all events individual instruction as an idea has gained a position of weight and respectability. The practical details of actualizing this idea in typical classrooms are yet to be perfected. Workers of varying degrees of industry and competence are engaged on these details at the present time.

Conclusion. The foregoing paragraphs seem to be a fair sampling of the type of problems which seem of importance and interest in the elementary school field. They, of course, do not exhaust the topics now much discussed and another writer might very well have presented quite a different list.

DISCUSSION TWO

What Are the Main Agreements and Disagreements in the Nature of Representative Standard Tests?

The following treatment of standard tests in arithmetic is taken from the work of Professor G. M. Ruch, University of California.

I. INTRODUCTION

Standard Tests. Standard tests in arithmetic may be classified roughly in two categories:

1. *Survey tests*, i.e., measures of general accomplishment in arithmetic *in toto*, or of broad groups of processes or functions. These tests should not, and usually do not, make claims to diagnostic powers. Since the present report does not deal with tests of this type, they may be dismissed without discussion.[1]

2. *Diagnostic tests*, i.e., tests which yield separate scores on isolated unit-skills, operations, or processes. Survey scores may or may not be provided incidentally as well. The better-known tests of this type include:

   a. Cleveland Survey Arithmetic Tests.
   (Contrary to the title, these are also diagnostic tests.)
   b. Compass Diagnostic Tests in Arithmetic.
   c. Courtis Research Tests, Series B. (Not diagnostic except in a very general way.)
   d. Monroe Diagnostic Tests in Arithmetic.
   e. Spencer Diagnostic Arithmetic Tests.
   f. Stevenson Problem Analysis Test.
   g. Wisconsin Inventory Tests.
   h. Woody Arithmetic Scales. (Not diagnostic except in a very general way.)

[1] The better known survey tests in arithmetic computation include the following: Woody-McCall Mixed Fundamentals Test, Stanford Arithmetic Computation Test, Monroe General Survey Tests in Arithmetic, Courtis Standard Supervisory Tests in Arithmetic, and the Lippincott-Chapman Arithmetic Fundamentals Test. As problem scales or reasoning tests the following are well known: Buckingham Scale for Problems in Arithmetic, Stone Reasoning Test, Stanford Reasoning Test, Monroe Standardized Reasoning Tests in Arithmetic, and the Otis Reasoning Test.

It is not always possible to tell from the title or the advertisements of a test whether or not it is diagnostic. This is probably due to a variety of causes, such as, the loose connotation of the term diagnostic, over-zealousness on the part of publishers, and genuine differences in opinions as to the characteristics of diagnostic tests. The entire situation will be cleared up only through a careful discussion of these misunderstandings among teachers of mathematics. On the other hand, the Cleveland Survey Tests must be classified as diagnostic in spite of their name, which had its origin in the fact that they were constructed for use in a survey of the Cleveland schools. The writer is somewhat disturbed lest he should do certain tests like those of Woody and Courtis an injustice by including them in the above list of diagnostic tests since these authors make no such claims for their tests. The tests, however, possess some general diagnostic value as far as differentiating among the four operations with integers. The table to be presented later, it should be stated, includes several types of tests, as follows: (*a*) those properly called diagnostic, (*b*) those named diagnostic but not necessarily genuinely such (in the light of the following list of requirements), and (*c*) those possessing some general diagnostic features without necessarily making claim to such.

II. The Requirements to Be Met by a Diagnostic Test

Criteria. Before proceeding with the discussion of the various diagnostic tests, it will be necessary to present the criteria by which the claims to genuine diagnosis can be evaluated. Great confusion exists in the minds of both schoolmen and test workers as to the differentiation between survey and diagnostic tests. The essential requirements of a truly diagnostic test are:

1. *Validity.* Within the function or process to be measured, the test items must sample widely the important skills and processes which have marked social utility, which are common to standard courses of study in arithmetic, and which possess marked learning difficulties for pupils. Each test, or section of a test, for which a score for an *individual pupil* is obtained and used in thinking about the ability of that pupil, should contain all of the known difficulties in learning which fall within that category. All test items should be submitted to experimental tryout in order to insure the proper degree of difficulty.

2. *Reliability.* Each separate score yielded by the test (whether for the entire test or any section), *if it is to be used alone as a measure of an individual pupil,* must represent a stable sampling, i.e., if successive samplings or repetitions of equal numbers of test items were administered to the same pupils, the scores on each sampling (often called "forms" of the tests) must not show very marked numerical fluctuations from sample to sample. Reliability is thus seen to be, to a marked degree, a product of the extensity of the sampling, i.e., the number of test items or examples entering into a pupil's score. In general, narrow functions or unit-skills require as long, or nearly as long, a sampling as do broad functions.

Brevity has been the greatest single sin of all educational testing in the past. The typical arithmetic test in the past has rested its diagnosis too often on 5 to 10 examples or 30 to 180 seconds of testing. There never has been, and there probably never will be, any experimental justification for taking seriously scores based upon such samplings *as far as individual pupils are concerned,* granting, of course, the adequacy of such tests for surveying whole school systems. Teachers, however, teach individual pupils, not school systems, and testing should be as individual as teaching. No teacher would risk her opinion about a child's knowledge of the multiplication table upon his success or failure on 5 of 100 combinations chosen at random, yet this is relatively a larger sampling in per cents than standard tests sometimes provide for very much more complex processes. For narrow functions approaching unit-skills, perhaps 20 to 25 or more examples and 3 to 5 minutes or more of time represent the sampling permissible for reliable results.

3. *Time Requirements.* This consideration grows directly out of the preceding one. The general significance of the time factor in testing has been set forth. The mere time required to give a test is alone valueless in determining the worth of the test. The criterion is *validity and reliability per unit of time employed.* A diagnostic test necessarily implies analysis, i.e., the breaking up into separate unit-skills which are measured and then are integrated as a total process or skill.

It follows, therefore, that when a total skill like long division is analyzed into, say, six unit-skills, it will require approximately six times as great a working time as it would if a mere survey of general accomplishment in long division were the objective. This issue must be faced by test makers. They will have to convince the superintendent that a test which pictures the *specific* weaknesses of an individual pupil is worth the additional time and money cost. It has been said that it takes God 200 years to grow an oak tree, but that He can raise a squash in about 90 days.

4. *Degree of Analysis.* Theoretically, no limits can be set to the fineness to which the analysis embodied in a diagnostis test should be carried. Practical considerations suggest that it is sufficient to break total processes into unit-skills or small units which are parallel to actual teaching situations, e.g., finding common denominators, estimating quotient figures, pointing off in decimal numbers, carrying in column addition, cancellation, computation of interest by tables, and the like. The criterion here can be stated thus: Is the diagnosis sufficiently exact to yield specific direction as to the kind of remedial work needed by each pupil? The trouble with much of our educational testing to-day is that it does little more than satisfy an idle curiosity about our pupils. In the main this is a reflection upon the type of tests available rather than upon the schools.

5. *Norms or Standards.* The essential conditions here are those common to all educational measurement; the representative adequacy of the sampling entering into the norm is far more important than the mere numbers of cases employed.

6. *Administration of the Test.* Clarity of directions, use of sample, adequate manuals of instruction, and general attractiveness of the printed page are the main desiderata here as in all measurement.

7. *Ease of Interpretation.* The diagnosis should be *direct,* i.e., should not require the tabulation of specific errors. Each separate part of the test which yields an individual score should represent a unit-skill (or some narrow function), and should be paralleled by a table of norms for purposes of interpretation. Tests which require tabulations of errors in order to permit of diagnosis should not be considered as genuinely diagnostic tests since any test, thus treated, becomes "diagnostic."

8. *Relation of Testing to Teaching.* Survey tests are usually given at the end or beginning of semesters. Diagnostic tests, on the other hand, should be given when, in the judgment of the teacher or supervisor, the teaching of a process has been completed. Only in this way can the teaching, testing, and remedial programs be integrated. There is no very good reason why long division should be

tested in June if it has been completed in November. Another important angle of this question is the choice of particular tests to be given. If a policy of "testing-after-teaching" is followed, the choice of tests is settled by the course of study, and one inexcusable practice in educational measurement will be avoided, viz., the wholesale administration of tests of the four operations with whole numbers in Grades V to VIII, long after the direct teaching of these operations has been finished. Instead, the testing program of Grades V to VIII will substitute measurements in fractions, decimals, denominate numbers, mensuration, percentage, interest, and the like; in a word, the text program will follow the curriculum.

9. *Provision for Remedial Programs.* This is the most fundamental educational objective of any test or scale. At the same time, the greatest weakness in the test movement to-day is the lack of provision for follow-up work on the weaknesses revealed by the testing. A genuinely diagnostic test must carry the analysis of processes to units detailed enough so that the average teacher can locate in a standard textbook teaching units which parallel the unit scores of the test.[2]

The failure of test makers to provide remedial programs based upon the results of their tests is alone responsible for most of the dissatisfaction relative to the use of standard educational tests.

III. Current Diagnostic Tests

Analysis of Existing Diagnostic Tests. The following table presents an analytical summary of the nature, scope, and content of the more important diagnostic tests available at the present time.

Two figures appear in each cell of the table. The lower and larger figure shows the extent of the sampling, i.e., the number of examples or test items included. The smaller figure in the upper right-hand corner of each cell shows the number of sub-processes or unit-skills into which the general processes (listed at the left) are broken for diagnostic purposes. The letter *M* means that the examples occur in mixed fashion and require laborious tabulations to secure diagnosis.

Attention is especially directed to the summary at the bottom of

[2] The only diagnostic tests in arithmetic which seem to have made definite provision for correlated remedial exercises are the Compass Diagnostic Tests in Arithmetic, Scott, Foresman and Co., 1925.

AN ANALYSIS OF DIAGNOSTIC ARITHMETIC TESTS

Processes	Cleveland-Survey Arithmetic Tests	Compass Diagnostic Tests in Arithmetic	Courtis Research Tests, Series B	Monroe Diagnostic Tests in Arithmetic	Spencer Diagnostic Tests in Arithmetic Test 1. Grades 3-4	Spencer Diagnostic Tests in Arithmetic Test 2. Grades 5-6	Spencer Diagnostic Tests in Arithmetic Test 3. Grades 7-8	Stevenson Problem Analysis Test	Wisconsin Inventory Tests (Osburn)	Woody Arithmetic Scales
Addition of Whole Numbers	41^{4}	168^{5}	24^{1}	53^{3}	47^{5}	37^{6}			411^{3}	38^{M}
Subtraction of Whole Numbers	15^{3}	106^{4}	24^{1}	48^{3}	47^{7}	32^{5}			100^{1}	35^{M}
Multiplication of Whole Numbers	20^{3}	191^{6}	24^{1}	61^{3}	65^{11}	34^{5}			100^{1}	38^{M}
Division of Whole Numbers	19^{4}	257^{7}	24^{1}	51^{3}	60^{13}	27^{7}			126^{1}	36^{M}
Addition of Fractions and Mixed Numbers	3^{M}	145^{5}		30^{3}		34^{5}				
Subtraction of Fractions and Mixed Numbers	3^{M}	105^{6}		15^{1}		23^{5}				
Multiplication of Fractions and Mixed Numbers	1^{M}	89^{5}		15^{1}		20^{6}				
Division of Fractions and Mixed Numbers	1^{M}	114^{6}		15^{2}		15^{4}				
Addition, Subtraction, and Multiplication of Decimals		136^{5}		44^{3}		16^{5}				
Division of Decimals		73^{5}		68^{3}		19^{1}				
Addition and Subtraction of Denominate Numbers		108^{5}				1^{1}				
Multiplication and Division of Denominate Numbers		69^{5}				2^{1}				
Mensuration		114^{5}					7^{3}			
Percentage		113^{5}					12^{1}			
Interest, Business Forms, etc.		104^{4}								
Vocabulary, Rules, Definitions, etc.		145^{2}								
Ratio and Proportion							3^{1}			
Problem Scales		15^{1}								
Problem Analysis		75^{5}					18^{3}	24^{1}		
Summary										
Diagnosis within Total Process or Function	D	D	D	D	D	D	T	T	D	T
Total Number of Processes or Functions Covered	15	85	4	21	36	48	7	1	8	4
Total Number of Examples or Test Items	105	2127	96	400	219	260	40	24	737	147
Average Sampling of each Function Covered	7.0	25.0	24.0	19.0	6.1	5.4	5.7	24.0	92.1	36.7
Publisher	P	S	C	P	B	B	B	P	P	T

the table. The letters, *D* and *T*, respectively, refer to whether the diagnosis is "direct" or requires "tabulation" of errors. The letters in the last row are abbreviations of the following: *B* Bureau of Reference and Research, University of Oregon, Eugene, Oregon. *C* Courtis Standard Tests, 1807 E. Grand Blvd., Detroit, Michigan. *P* Public School Publishing Co., Bloomington, Illinois. *S* Scott, Foresman and Co., Chicago, Illinois. *T* Bureau of Publications, Teachers College, New York City.

Condensed Descriptions of Diagnostic Tests. In suggesting the grade ranges printed below, it should be kept in mind that the figures given refer to the *basic* grades, i.e., those in which the functions covered are normally taught, not necessarily the grade range through which norms are provided. The comments on reliability are based in part upon experimentation and in part upon observation of the behavior of similar materials when treated statistically.

Cleveland-Survey Arithmetic Tests

Grade range: III to VI (?).

Scope: Whole numbers fairly well covered; fractions inadequately. No provision for basal arithmetic work in Grades V to VIII. Consists of one eight-page, 6 x 9, booklet. Functions covered: 15.

Reliability: Diagnosis properly confined to class averages; not sufficiently reliable for certain individual diagnosis (average sampling about 7 items; time limits 30 to 180 seconds).

Diagnosis: No tabulations of individual errors needed.

Compass Diagnostic Tests in Arithmetic

Grade range: III to VIII (each of the 20 tests has a recommended grade range).

Scope: Whole numbers, fractions and mixed numbers, decimals, denominate numbers, mensuration, percentage, interest and business forms, vocabulary, rules, and definitions, problem scales, and problem analysis. Twenty tests of 1 to 12 pages each, 8½ x 11. Functions covered: 85.

Reliability: Sufficiently reliable for either individual or class diagnosis. Average sampling about 25 items; time limits, 2 to 20 minutes.

Diagnosis: No tabulations of individual errors needed; tests are planned to "gear into" a correlated remedial program to correct weaknesses in any of the 85 unit-skills or functions covered.

Courtis Standard Research Tests, Series B

Grade range: III to IV.
Scope: Whole numbers only. No provision for basal work above fourth grade. One four-page, 6 x 9, booklet.
Reliability: Sufficiently accurate for individual or class diagnosis; diagnosis confined to four operations with integers. Average sampling, 24 items; time limits, 4-8 minutes.
Diagnosis: No tabulations of individual errors needed. Rate and accuracy scored separately. Does not attempt to reveal weaknesses hidden within the 4 coarse functions covered.

Monroe Diagnostic Tests in Arithmetic

Grade range: IV to VI.
Scope: Whole numbers, fractions, and decimals. (No tests on basic number combinations.) Four four-page, 6 x 9, booklets. Functions covered: 21.
Reliability: Fairly accurate for either individual or class diagnosis; average sampling, 19 items; time limits, 30 seconds to 4 minutes.
Diagnosis: No tabulations of individual errors needed.

Spencer Diagnostic Arithmetic Tests

Grade range: III to VIII (three tests, each with its recommended range).
Scope: Whole numbers, fractions, decimals, denominate numbers, mensuration, percentage, ratio and proportion, and problem analysis. Three four-page, 8½ x 11, booklets.
Reliability: Except for certain sections, sampling probably too short for accurate individual diagnosis. Will serve very well for class diagnosis. Average sampling, 6 items. Time limits, unknown, apparently not timed.
Diagnosis: Partially requiring tabulations and partially direct. There appear to be no published norms.

Stevenson Problem Analysis Test

Grade range: IV to IX.
Scope: Problem analysis only. Two four-page, 6 x 9, booklets. Functions covered: 1.
Reliability: General score reliable for either individual or class measurement. Sampling, 24 items.
Diagnosis: Possible only through tabulation of errors; four sub-scores thus possible. Such diagnostic scores should be confined to class diagnosis (sampling, 6 items).

Wisconsin Inventory Tests

Grade range: II to IV.
Scope: Operations with whole numbers. Eight four-page, 6 x 9, booklets. Functions covered: 8.
Reliability: High, accurate for individual diagnosis. Certain tests involve no errors of sampling since all possible combinations are covered. Average sampling: about 90 items.
Diagnosis: No tabulations or norms are needed since pupil mastery should be approximately perfect on tests involving only the basic operations.

Woody Arithmetic Scales

Grade range: III to VIII.
Scope: Mixed fundamentals of the four operations with integers, fractions, decimals, and denominate numbers. Functions covered: 4.
Reliability: Accurate for individual or class measurement. Average sampling: 37 items.
Diagnosis: Confined to four operations (addition, subtraction, multiplication, and division). Further diagnosis requires tabulation of errors. Not designated as a diagnostic test by the author.

DISCUSSION THREE

How Do Representative Textbooks of Arithmetic Agree and Differ in Their Fundamental Work of Instruction as Such?

This report will be suggestive to those responsible for the selection of textbooks and those interested in the problem of what texts should do relative to instruction. It is taken from the work of Frank L. Wells and E. A. Olander, College of Education, University of Iowa.

A critical study of the instructional features of textbooks reveals almost astounding differences. Where one text will teach and illustrate and on subsequent occasions recall a given computational "hard spot," another text will print nothing whatever about it. When this "hard spot" is met in drill or problem work by pupils taught by the two textbooks their reactions to it will presumably vary.

Let us assume for the moment that a given topic is made up of items or skills A, B, C, D, E, and F. One text will explain A, B, and C, but not D, E, and F. Another will teach A, D, E, and F, but not B and C. And where two texts both print something about

A, one may devote a single line to it while another devotes half a page. At the present time textbooks agree on what to teach with considerable unanimity of practice, but their disagreements on how to teach topics effectively vary from scant adult explanations to expanded explanations of a self-instructional nature. And most texts seem to be at war with themselves, teaching one topic fully and another presumably of equal importance hurriedly and carelessly.

The following report reduces to a quantitative basis some of these differences relative to the teaching functions of texts.[1]

Introduction. In the elementary education process of to-day, the textbook is a very necessary instrument—a tool which is a highly complicated mechanism in the educational production plant. The quality of output achieved through this textbook machinery depends, of course, in large measure upon the ability of the individual classroom teacher to manipulate it. Obviously, the better the tool, the better it can be utilized by the skilled workman. Hence, there is need for objective means of appraising the adequacy of textbook instruction. The capable administrator or supervisor in elementary education wants to know what text will be the best instrument to put into the hands of the teachers working under his direction. Educational research in its analysis of the arithmetic teaching process has reached a point where it now seems feasible to evolve objective techniques for evaluating the comparative merits of various series of texts. Upon these hypotheses, the study briefly suggested here has been initiated and is being worked out.

This discussion is a section from a more comprehensive study which will analyze the entire instructional content of seven sets of elementary arithmetic textbooks now in current use. The entire study upon its completion will appear in the University of Iowa Monographs in Education series, published by the College of Education of the University.

The Problem. The study is an attempt to set up objective techniques for analysis of the instructional adequacy of textbooks of elementary arithmetic. The problem presents two phases: (1) *what* is taught, or the extent to which the unit-skills underlying the fundamental processes of elementary arithmetic are identified in the actual instructional content of arithmetic texts; (2) *how well*

[1] Teaching functions are here contrasted with drill and problem solving functions.

the subject matter is taught, or the appraisal of the adequacy of the instructional content whereby such texts bring these unit-skills together in learning situations.

The present discussion will not attempt to touch upon the second phase of the problem, but will be confined to a presentation of what constitutes the instructional content of seven current arithmetic texts relative to the addition of whole numbers.

Instruction, for the purposes of this discussion, may be defined as that textual material which is designed to aid the pupil in identifying and mastering the skills of arithmetic. It is exclusive of the problems and examples which the pupil is called upon to do without immediate aid from the book. Instructional material endeavors to present before the child upon the printed page the total array of specific facts (or sufficient of them consistent with his abilities or needs up to that point for a child to grasp the sequence) in the order in which these facts are utilized in the manipulating of the actual arithmetic skill which is being taught. Presumably the child can follow through the steps of the process and thus learn to manipulate the skill for himself.

Procedure. Obviously, the handling of the instructional material in a uniform manner for all texts necessitated some scheme of classification and organization of the data. The most feasible plan seemed to be that of reorganizing all those portions of textbook content pertaining to the problem upon the basis of the fundamental "blocks of instruction" about which arithmetic texts are built. The block chosen here to illustrate the procedure is the addition of whole numbers.

The instructional material dealing with the addition of whole numbers was cut out of the various texts, designated as A, B, C, D, E, F, and G, and assembled in charts. All the material from Text A was mounted together in one chart, and the corresponding material from each of the other texts was treated in similar fashion. These charts gave a gross measure of the instruction and also facilitated further study by assembling all the data for this particular unit in a very convenient form so that it could be seen and studied as a whole.[2] These charts show startling differences in textbook instructional content. The wide range of sheer area of printed pages

[2] In discriminating between instructional versus non-instructional material, certain decisions were made arbitrarily. The definition of instruction, as already stated in the discussion, was the criterion for such arbitraiy decisions.

devoted to an initial and highly important topic of elementary arithmetic is astounding.

Area Analysis. Table I presents an area analysis of instructional content in terms of (1) total area, (2) amount devoted to pictorial matter, (3) number of words, and (4) number of figures and symbols.

TABLE I

AREA ANALYSIS OF INSTRUCTIONAL CONTENT DEALING WITH ADDITION OF WHOLE NUMBERS

For Seven Current Arithmetic Texts

TEXT	NO. OF SQ. IN. OF TOTAL AREA	NO. OF SQ. IN. OF PICTORIAL MATTER	TOTAL NO. OF WORDS	TOTAL NO. OF FIGURES * AND SYMBOLS
A.......	91	2	570	267
B.......	92	28	370	157
C.......	82	12	266	427
D.......	78	19	373	256
E.......	128	40	401	311
F.......	133	22	810	656
G.......	527	63	3389	894

* Two-, three-, and four-digit numbers are counted as one figure.

Comparisons between the texts which are at the two extremes in Table I emphasize the divergence in the treatment of addition. Text G employs nearly seven times as many square inches for the topic as Text D. Text A devotes approximately 2 per cent of its total space to pictorial matter where Text E uses 31 per cent for this purpose. Text C finds 266 printed words ample for the explanation of addition, a condensed, highly compact handling of subject matter, while the printed words of Text G mount up to 3,389, a widely expanded type of explanatory discussion. Likewise, Text G utilizes 894 figures and symbols in illustrative problems and examples as contrasted with 157 for Text B.

It is well to bear in mind in reading Table I that in general when subject matter is condensed reading difficulty is increased.

Table II shows how each text apportions its total area (as measured in terms of square inches) among the constituent elements which enter into the addition process.

Table II discloses the fact that the texts are virtually unanimously agreed that some instruction should be devoted to each of the usual

TABLE II

DISTRIBUTION OF AREA ANALYSIS

For Seven Current Arithmetic Texts

Text	100 Basic Addition Facts		Higher Decade Addition		Column		Vocabulary		Total	
	Sq. In.	%	Sq. In.	%	Sq. In.	%	Sq. In.	%	Sq. In.	%
A	33	36	1	1	51	56	6	7	91	100
B	52	56	7	8	26	28	7	8	92	100
C	34	41	18	22	23	28	7	9	82	100
D	40	51	0	0	37	47	1	1	78	99
E	59	46	8	6	58	45	3	2	128	99
F	29	22	8	6	95	71	1	1	133	100
G	102	19	71	14	300	57	54	10	527	100

four main features of the addition process; namely, (1) the 100 basic addition facts, (2) higher decade addition, (3) column addition, (4) vocabulary. The sole exception occurs in the case of Text D, which contains no instruction upon higher decade addition. The reader can see for himself that the seven texts show marked disagreement concerning the relative instructional emphasis, as measured in terms of area, which should be placed upon each of the four elements involved in adding whole numbers.

Analysis of Unit-Skills. In the light of this disagreement, it is important to note what items of the four elements listed in Table II are taken into consideration in the various texts. In other words, do the texts disagree upon what specific unit-skills are to be presented in teaching addition? The data in Tables III to VII inclusive answer this question in the affirmative. These tables give a detailed analysis of the unit-skills into which each of the four constituent elements resolve themselves, in so far as these unit-skills are taught by the various texts.

Conclusion. The aim of this discussion has been to present objectively the facts as to what is taught in the addition of whole numbers in the seven arithmetic texts examined. With these facts before us one thing is very evident; namely, that whether we take the analysis made on the basis of area or that made on the basis of skills taught, there is wide variation among the texts.

A further fact to be noted also is the correspondence between the gross amounts of instruction in the various texts and the number of unit-skills and abilities taught. Those texts which have the

[*Text continued on page 41*]

TABLE III

THE 100 BASIC ADDITION FACTS

Unit-Skills in Seven Current Arithmetic Texts

UNIT-SKILLS	TEXTS						
	A	B	C	D	E	F	G
Identification of the 100 facts (number of combinations taught)	12	24	76	38	46	73	83
The correct form of writing addends Form A 6 + 3					T*		T
Form B $\begin{array}{r}3\\ \underline{6}\end{array}$			T				T
Correct placement of answer Form B $\begin{array}{r}3\\ \underline{6}\\ 9\end{array}$				T			

	% Taught	% Untaught		% Taught	% Untaught
Text A	12	88	Text E	46	54
Text B	24	76	Text F	73	27
Text C	76	24	Text G	83	17
Text D	38	62			

* In Tables III to VII, T indicates that the specified unit-skill is taught. This study does not commit itself on how well it is taught. A blank indicates that it is not taught.

TABLE IV

Amount and Distribution of Instruction on the 100 Basic Addition Facts

In Seven Current Arithmetic Texts

Table IV is interpreted as follows:
The fact 0 + 0 is taught in Text A only.
The fact 0 + 7 is taught in Text C only.
The fact 4 + 2 is taught in Texts A, B, C, D, E, F, G.
The fact 7 + 9 is taught in Texts C, F, G.

	0	1	2	3	4	5	6	7	8	9
0	A	D	DE				C	C		G
1	D	ABC DE G	C DEF G	AC DF G	C F G	C F G	C F G	C F G	C F G	C F G
2	B D	BC DE G	ABC DEF G	C DEF G	ABC DEF G	BC DF G	BC DF G	BC F G	BC F G	C F G
3	D	C DE G	BC DEF G	BC DEF G	C DEF G	C DF G	C DF G	C F G	C F G	C F G
4		BC DE G	ABC DEF G	BC DEF G	ABC DEF G	C DF G	BC DF G	C F G	C F G	C F G
5	D G	BC DEF G	AC DEF G	ABC DF G	BC DEF G	BC DEF G	C DF G	C F G	C F G	C F G
6	C	C E G	AC DEF G	ABC DEF G	C DEF G	C DEF G	BC DEF G	C EF G	C F G	C F G
7	C	C G	C EF G	BC EF G	C EF G	C EF G	F G	C EF G	C F G	C F G
8		C G	AC EF G	C EF G	C EF G	EF G	EF G	EF G	C EF G	C EF G
9		C G	BC EF G	C EF G	EF G	EF G	EF G	EF G	EF G	C EF G

TABLE V

Higher Decade Addition

Unit-Skills in Seven Current Arithmetic Texts

Unit-Skills	Texts						
	A	B	C	D	E	F	G
Identification of the facts (number of combinations taught)	2	2	5	0	5	3	11
The correct form of writing addends Form B 2 19			T				T
Adding digits in units column first		T	T				T
Carrying one from units column		T	T				
Correct placement of answer Form B 2 19 21			T				

TABLE VI

Addition Vocabulary

Unit-Skills in Seven Current Arithmetic Texts

Unit-Skills	Texts						
	A	B	C	D	E	F	G
add	T				T		T
plus	T		T				T
+	T	T	T	T			T
=		T	T	T			T
sum	T	T			T	T	T
addends		T				T	
addition			T				
check					T		T
column							T
carry							T

TABLE VII

Column Addition *

Unit-Skills in Seven Current Arithmetic Texts

Unit-Skills	Texts A	B	C	D	E	F	G
One-column							
Digits in units column only in answer..	T	T			T		T
Digits in units and tens column in answer						T	T
Two-column							
No carrying	T		T	T		T	T
Carrying one from units column		T	T	T	T	T	T
Carrying two from units column		T					T
Carrying three from units column	T						T
Zero in units column of answer	T						T
Zero in tens column of answer							T
Zero in units and tens column of answer							T
Digit in hundreds column in answer, no carrying from units column ...		T					T
Digit in hundreds column in answer, with carrying from units column ..				T	T		T
Broken or unequal columns							T
Three-column							
Carrying from units column only						T	T
Carrying from tens column only				T		T	T
Double carrying			T		T	T	T
Digit in thousands column in answer..					T	T	
Broken or unequal columns				T			T
Four-column addition (broken column)			T				
Checking	T				T	T	T

* Addition examples of two numbers in which one of the numbers is 0 or less are arbitrarily excluded from consideration here, since such examples have already been considered under the treatment of the 100 basic facts or higher decade addition.

larger instructional areas tend to teach a greater number of the unit-skills of arithmetic.

Of what significance, then, is variability in number of unit-skills taught? If we agree with Professor Thorndike that education is the formation of specific bonds, we have a basis for evaluating the findings; the necessity for the identification in the instruction of the skills to be acquired is thus established. Other things being equal, that textbook is best which most completely identifies the skills to be acquired. Both teacher and pupil profit most from such a text in that the objectives are made more evident.

The reader is cautioned against thinking that the above study is a complete analysis of the instruction on addition of whole numbers in the texts examined. It is complete as far as it goes. It is not complete on several points, among which are: (1) how *well* the unit-skills are identified, i.e., quality in contrast to quantity; (2) how effectively drill organizations of proper kinds are woven into the instructional material; (3) how differences in total adequacy may be numerically expressed for purposes of scoring and of text selection.

DISCUSSION FOUR

What Types of Analysis and Investigation May Be Used to Help Settle Questions Relative to the Better of Two Possible Methods?

I. Data Pertinent to Decisions

Long Division. As far as the writer knows no thorough analysis of the quantity of computations possible in long division, even when long division is restricted to examples using two-digit divisors, had appeared until a report on this topic was included in the *Fourth Year Book of the Department of Superintendence* under the title "Some of the Psychology of Long Division." The following discussion of certain data pertinent to decisions about rules of and procedures for the finding of quotient figures illustrates the kind of information which is useful in critical thinking relative to such decisions. This discussion is the work of H. H. Jeep, Professor of Education, State Teachers College, Mayville, North Dakota.

Quotient Difficulty. Of all the difficulties involved in long division, probably the hardest one that the child is asked to cope with is the quotient difficulty. This difficulty is real and perhaps will always stand as the major difficulty in long division. But it is very often greatly magnified because of the unfamiliarity on the part of the teacher or textbook author with its nature and the skills involved. There is no agreement as to the best method of estimating the quotient. In large measure this is due to the lack of certain data, which lack has up to this time made it practically impossible to arrive at any satisfactory conclusions as to what is the best method. Perhaps the only way to determine definitely the best method for the child to use in estimating the quotient figure is by actual experimentation on a large number of children. However, it is evident that such a research can be carried on satisfactorily only after the skills involved are understood at least in part. It is the purpose of this paper to present something of the nature and extent of the quotient difficulty.

With modifications of varying importance, two methods of estimating the quotient in long division are in use at the present time. The method generally taught uses the first digit of the divisor as the trial divisor into the first one or two digits of the dividend. Thus:

$\begin{array}{r} 2 \\ 46\overline{)95} \end{array}$	4 is the trial divisor. When divided into 9, it gives 2 as the quotient.
$\begin{array}{r} 4 \\ 46\overline{)185} \end{array}$	4 is the trial divisor. When divided into 18, it gives 4 as the quotient.

The second method, not so commonly taught, is that of increasing the first digit of the divisor by unity and then using the digit thus secured as the trial divisor into the first one or two digits of the dividend. Thus:

$\begin{array}{r} 2 \\ 37\overline{)93} \end{array}$	3 plus 1 equals 4 as the trial divisor, which divided into 9 gives 2 as the quotient.
$\begin{array}{r} 4 \\ 37\overline{)163} \end{array}$	3 plus 1 equals 4 as the trial divisor, which divided into 16 gives 4 as the quotient.

The first of the above methods will be spoken of in this discussion as the apparent-quotient rule and the second as the increase-by-one rule. The apparent-quotient rule is very often taught to the exclusion of the increase-by-one rule. On the other hand, when the increase-by-one rule is taught, it is only for those divisors in which the second digit is large. Generally it is taught for only those divisors in which the second digit is 7, 8, or 9, but it is sometimes used for those divisors in which the second digit is 6. Thus where the increase-by-one rule is taught, the apparent-quotient rule must also be taught. There are then, roughly speaking, two procedures used in teaching the estimation of the quotient in long division; i.e., (1) teaching of the apparent-quotient rule for all divisors, or (2) teaching of the increase-by-one rule for those divisors in which the second digit is large. There seems to be no question as to what rule should be used for those divisors in which the second digit is small. The variance of opinion occurs only with those divisors in which the second digit is large.

The following are some of the questions that should be answered before deciding which of the above two procedures to follow:

a. What percentage of times does the increase-by-one rule yield the true quotient for those divisors in which the second digit is large?
b. What percentage of times does the apparent-quotient rule yield the true quotient for the same divisors?
c. What is the relative complexity of the two rules?
d. What other factors are there that might have a bearing on the decision?

In this study, only double-digit[1] divisor, single-digit[2] quotient long division examples are considered.[3] Of the total number of such examples, we find that there are 21,465 examples with divisors in which the second digit is less than 6 and 18,630 examples with divisors in which the second digit is 6 or larger. It is with the latter number that Table I deals. This table shows the distribution of the 18,630 examples according to the quotient difficulties. Table II is a summary of Table I.

[1] So far as the quotient difficulties are concerned, the difficulties are practically the same for two-digit divisor examples and those examples that have three or more digit divisors.

[2] Examples with more than one digit in the quotient would be considered as so many examples, each with one digit in the quotient. For instance, the first example at the right would be considered in this discussion as two examples like the second and third at the right.

$$\begin{array}{r} 23 \\ 24\overline{)552} \\ \underline{48} \\ 72 \\ \underline{72} \end{array} \qquad \begin{array}{r} 2 \\ 24\overline{)55} \end{array} \qquad \begin{array}{r} 3 \\ 24\overline{)72} \end{array}$$

[3] Those examples in which the second digit of the divisor is zero are not included because such examples may be thought of as short division examples so far as the scope of this discussion is concerned.

TABLE I: DISTRIBUTION OF 18,630 EXAMPLES ACCORDING TO THE QUOTIENT DIFFICULTIES

Group	Quotient by: Apparent-Quotient Rule Gives	Quotient by: Increase-by-One Rule Gives	Number of Examples	%	Examples	Illustration: Apparent-Quotient Rule	Illustration: Increase-by-One Rule
Group I *	True quot.	True quot.	4800	25.76	3 48)154 144 10	Trial div. = 4 Est. quot. = 3 True quot. = 3	Trial div. = 5 Est. quot. = 3 True quot. = 3
Group II *	True quot. + 1	True quot.	7241	38.86	2 88)243 176 67	Trial div. = 8 Est. quot. = 3 True quot. = 2	Trial div. = 9 Est. quot. = 2 True quot. = 2
	True quot. + 2	True quot.	716	3.84	5 59)351 295 56	Trial div. = 5 Est. quot. = 7 True quot. = 5	Trial div. = 6 Est. quot. = 5 True quot. = 5
	True quot. + 3	True quot.	142	0.70	6 27)188 162 26	Trial div. = 2 Est. quot. = 9 True quot. = 6	Trial div. = 3 Est. quot. = 6 True quot. = 6
	True quot. + 4	True quot.	33	0.17	4 19)89 76 13	Trial div. = 1 Est. quot. = 8 True quot. = 4	Trial div. = 2 Est. quot. = 4 True quot. = 4
	True quot. + 5	True quot.	5	0.02	4 19)94 76 18	Trial div. = 1 Est. quot. = 9 True quot. = 4	Trial div. = 2 Est. quot. = 4 True quot. = 4

Group III *	True quot.	True quot. — 1	1799	9.65	6 67)419 402 17	Trial div. = 6 Est. quot. = 6 True quot. = 6	Trial div. = 7 Est. quot. = 5 True quot. = 6
Group IV *	True quot. + 1	True quot. — 1	855	4.58	8 48)399 384 15	Trial div. = 4 Est. quot. = 9 True quot. = 8	Trial div. = 5 Est. quot. = 7 True quot. = 8
	True quot. + 2	True quot. — 1	186	0.99	7 39)279 273 6	Trial div. = 3 Est. quot. = 9 True quot. = 7	Trial div. = 4 Est. quot. = 6 True quot. = 7
	True quot. + 3	True quot. — 1	47	0.25	4 19)79 76 3	Trial div. = 1 Est. quot. = 7 True quot. = 4	Trial div. = 2 Est. quot. = 3 True quot. = 4
	True quot. + 4	True quot. — 1	31	0.16	5 17)99 85 14	Trial div. = 1 Est. quot. = 9 True quot. = 5	Trial div. = 2 Est. quot. = 4 True quot. = 5
	True quot. + 3	True quot. — 2	4	0.02	6 16)99 96 3	Trial div. = 1 Est. quot. = 9 True quot. = 6	Trial div. = 2 Est. quot. = 4 True quot. = 6
Group V *	True quot.	More than true quot.	90	0.48	10 57)587 57 17	Trial div. = 5 Est. quot. = 1 True quot. = 1	Trial div. = 6 Est. quot. = 9 True quot. = 1

TABLE I—(Continued)

Group	Quotient by: Apparent-Quotient Rule Gives	Quotient by: Increase-by-One Rule Gives	Number of Examples	%	Examples	Illustration: Apparent-Quotient Rule	Illustration: Increase-by-One Rule
Group VI *	10 or more (9 = true quot.)	True quot.	1300	6.97	$\begin{array}{r} 9 \\ 78\overline{)779} \\ \underline{702} \\ 77 \end{array}$	Trial div. = 7 Est. quot. = 10 True quot. = 9	Trial div. = 8 Est. quot. = 9 True quot. = 9
	10 or more (9 = true quot.)	Less than true quot.	600	3.22	$\begin{array}{r} 9 \\ 69\overline{)629} \\ \underline{621} \\ 8 \end{array}$	Trial div. = 6 Est. quot. = 10 True quot. = 9	Trial div. = 7 Est. quot. = 8 True quot. = 9
	10 or more (9 = true quot.)	True quot.	450	2.41	$\begin{array}{r} 8 \\ 38\overline{)341} \\ \underline{304} \\ 37 \end{array}$	Trial div. = 3 Est. quot. = 11 True quot. = 8	Trial div. = 4 Est. quot. = 8 True quot. = 8
	10 or more (true quot. is less than 9)	Less than true quot.	331	1.77	$\begin{array}{r} 8 \\ 29\overline{)239} \\ \underline{232} \\ 7 \end{array}$	Trial div. = 2 Est. quot. = 11 True quot. = 8	Trial div. = 3 Est. quot. = 7 True quot. = 8

Group I are examples with no quotient difficulty. Group II are examples in which the apparent-quotient rule yields too large a quotient and the increase-by-one rule yields the true quotient. Group III are examples in which the apparent-quotient rule yields the true quotient and the increase-by-one rule yields too small a quotient. Group IV are examples in which the apparent-quotient rule yields too large a quotient and the increase-by-one rule yields too small a quotient.

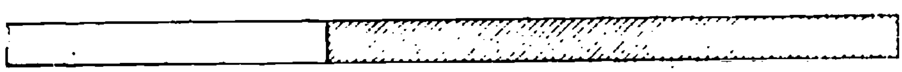
Group V are examples in which the apparent-quotient rule yields the true quotient and the increase-by-one rule yields a quotient which is larger than the true quotient. When the increase-by-one rule is taught, there is a quotient difficulty which is peculiar to just 90 examples. These are the examples in which the first digit of the divisor is the same as the first digit of the dividend, and the second digit of the divisor is less than the second digit of the dividend. For these 90 examples, the increase-by-one rule yields too large a digit while in all other cases in which the increase-by-one rule fails to yield the true quotient, it yields too small a digit. It is not possible to overcome this difficulty by considering the second digit of the divisor, as in the next group of examples. This means that when the increase-by-one rule is taught, the child must be taught sometimes to increase the estimated digit and at other times to decrease the estimated digit.

Group VI are examples in which the apparent-quotient rule yields a quotient which is larger than 9 and the increase-by-one rule may or may not yield the true quotient. When the child is taught the apparent-quotient rule, the quotient difficulties for such examples should be very carefully considered. Of course, this quotient difficulty is not peculiar to those examples in which the second digit of the divisor is 6, 7, 8, or 9; it occurs for any divisor when the first digit of the divisor is the same as the first digit of the dividend and the second digit of the divisor is larger than the second digit of the dividend. In such examples the child should be taught to con-

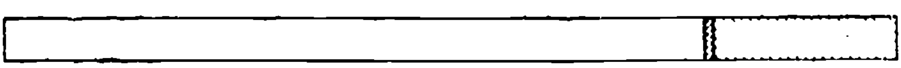
TABLE II

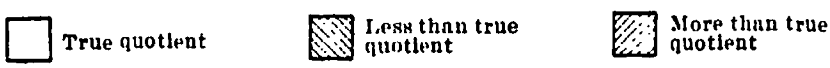
Rule Gives	Apparent-Quotient Rule				Increase-by-One Rule			
	No. of Ex.	%	No. of Ex.	%	No. of Ex.	%	No. of Ex.	%
True quotient.......			6689	35.90			14687	78.83
More than true quot..			11941	64.07			90	.48
by 1	8096	43.45						
by 2	902	4.84						
by 3	193	1.03						
by 4	64	.34						
by 5	5	.02						
	9260	49.68						
by more than 9	2681	14.39						
Less than true quot...							3853	20.68
by 1					3793	20.35		
by 2					60	.32		
			18630	99.97			18630	99.99

The Apparent-Quotient Rule



The Increase-by-One Rule





II. Conclusions

1. Neither the apparent-quotient rule nor the increase-by-one rule will yield the true quotient in every instance.
2. When the apparent-quotient rule is used:
   a. The estimated quotient is the true quotient in 6,689 examples.
   b. The estimated quotient is too large in 9,260 examples.
   c. The correction in those cases in which the rule fails to yield the true quotient ranges from one to five. This is important because the difficulty of obtaining the correct quotient increases in the same ratio as the size of the correction. For instance, in the example $28\overline{)166}$ when the apparent-quotient

rule is used the child estimates 8 as the quotient. This is too large; so he tries 7, then 6, and then 5. Thus we see that the apparent-quotient rule not only fails to yield the true quotient in more instances than does the increase-by-one rule, but the degree of difficulty is also greater.

d. The estimated quotient is more than 10 in 2,681 examples.

e. If the rule fails to yield the true quotient, the correction is always a decrease of the estimated quotient.

f. If this rule is used exclusively, only one method of estimating the quotient need be taught.

3. When the increase-by-one rule is used:

a. The estimated quotient is the true quotient in 14,687 examples.

b. The estimated quotient is too small in 3,853 examples.

c. The correction in those cases in which the rule fails to yield the true quotient is never more than two. Thus the increase-by-one rule fails to yield the true quotient less often than does the apparent-quotient rule and it also has a smaller degree of quotient difficulty than does the apparent-quotient rule. (See 2*c*, above.)

d. The estimated quotient is larger in 90 examples.

e. If the rule fails to yield the true quotient, the correction may be either an increase or a decrease, and thus there is the danger of confusing the child.

f. It would undoubtedly mean that both rules would be taught, and here again there would be danger of confusing the child with two methods for estimating the quotient.

DISCUSSION FIVE

What Types of Investigation Will Contribute Evidence Relative to Effective Methods for Improving Skills?

Introduction. The following discussion pertains to one of the main problems in arithmetic teaching, as attacked from the standpoint of effective classroom methods. A teacher may well be impressed with the fact that we know far more about and have much greater control over the learning of sheer computation than we have over the more complex and probably more important division of arithmetic teaching called problem solving. The study here reported was made by O. S. Lutes, Professor of Education at the University of Maine, Orono, Maine. It is an analytical account of what occurred when different methods for improving problem-solving skills were employed.[1]

Skill of Problem Solving. The purpose of this study is the discovery of effective classroom techniques for teaching the skill of problem solving in elementary school arithmetic.

As a basis for the study 1,025 errors actually made by children in attempting to solve problems were carefully analyzed and found to fall under three mains heads: (1) ignorance of principle, or wrong operation, (2) comprehension difficulties, (3) computation errors.

The Problem. The problem then became one of devising methods of drill or instruction which would attack directly the three major types of errors found in the analysis. The first main type of errors, which seemed to be due chiefly to comprehension difficulty, was met by drills in choosing solutions to problems.

The *operation* errors were provided for by devising a drill which afforded practice in choosing the correct operations in problems without doing the computation. This type of drill was used as the special technique for one experimental group. *Comprehension* difficulties were met by drills containing several solutions to problems, one of which was usually right and the others wrong. The first and second doubtless overlap greatly and both may be reduced perhaps to comprehension difficulties. Then the first two types of techniques used in this experiment may be considered simply as two different ways of improving the comprehension of written problems.

For errors in *computation,* drills were prepared in computation only, and these drills became the techniques for another experimental group.

The experiment was conducted in the 6B class of each of twelve elementary schools in Des Moines, Iowa, and lasted twelve weeks from March to May, inclusive, 1925. These schools were scattered well over the city in such a way as to include pupil groups which were representative of widely diverse elements of the population, having a wide range of native intelligence, social status, and teachers of representative personality.

[1] For the complete published account of the study see: O. S. Lutes, *An Evaluation of Three Techniques for Improving Ability to Solve Arithmetic Problems* (A Study in the Psychology of Problem Solving), University of Iowa Monographs in Education, No. 6, College of Education, University of Iowa.

For convenience the weekly cycle of work was started on Thursday, and continued as follows:

Thursday—Use of the special problem-solving technique assigned to each of all experimental groups.

Friday—Problem test taken by all experimental groups.

Monday—Use of the special remedial drills based on the results of the previous Friday tests.

Tuesday—New skills taught as usual, entirely regardless of the experiment.

Wednesday—New skills taught as usual.

The Technique Used. In order to explain the above procedure more in detail it will be necessary to give the different techniques used with each experimental group. The groups were designated as follows:

1. *Computation Method.* This group was drilled on Thursday in computation only. The attempt was made to include in the drills all the number combinations and computational skills which would occur in the problem test to be given the following day. The drills were so constructed that the particular combinations and computations which would occur in the problem test were hidden among others.

2. *Choosing Operations Procedure.* For this group the Thursday drills contained chiefly two aspects: first, three or four questions followed each problem, the purpose of which was to call the attention of the pupil to crucial points in the problem to guide his thinking about it; then a key was printed at the top of each drill in which a number was given to each of the more common operations, such as addition, subtraction, reducing to lowest terms, and the pupil was asked to indicate on a line provided after each problem which of these operations was required in each case.

3. *Choosing Solutions Method.* The Thursday drills for this group contained a list of problems with different solutions (usually one to three) presented for each, one of which was usually correct and the others wrong. The pupil was asked to choose the right solution and indicate his choice by inserting a cross in the square opposite the problem.

4. *The Control Group.* The schools of this group were instructed to continue teaching arithmetic as usual by following their course of study and text.

On Friday all three experimental schools were given the same problem test. These tests were collected and corrected, being returned to the schools by Monday, together with remedial drills based upon the errors made in the test. The arithmetic period on Monday was spent explaining the errors made and doing the remedial work. In the remedial work each group attempted to stress only the errors which occurred in the particular aspects practiced in the Thursday drills. On Tuesdays and Wednesdays all the experimental groups were taught from the regular course of study as though no experiment was being conducted.

In order to equalize the four groups of schools used in the experiment a measure both of arithmetical attainment and of general intelligence was secured. For the former the Stanford Achievement Test, Parts 4 and 5, was used. For the latter, Scale A, Form 1 of the National Intelligence Test was used. The various factors entering into the experiment were equalized between groups wherever possible.

Conclusions. Dr. Lutes summarized the conclusions of his study as follows:

"1. All pupils of normal intelligence can profit from instruction in problem solving. Those of higher intelligence profit most, but all within the normal mental age range make gains sufficient to justify spending time on such instruction.

"2. Of the techniques used in this experiment, that which had for its main purpose the emphasis on computational skill produced the greatest gains in both computation and reasoning.

"3. All the groups in this experiment made gains in problem solving as measured by the Stanford Achievement Test which are statistically significant. This includes the Control Group as well as the three experimental groups.

"4. Motivation is an important factor in securing improvement in problem solving ability. This conclusion is supported by the fact that the Control Group made the second largest gains of the four groups, and that all groups made greater than normal or expected gains.

"5. Improvement in computational accuracy does increase ability to solve verbal problems; whether it increases ability to do arithmetical reasoning cannot be stated from the results of this study. It does improve ability to earn scores on tests of ability to solve verbal problems, which is the school's definition of arithmetical reasoning."

DISCUSSION SIX

What Types of Research Will Contribute Data to Support Useful Specifications for the Building of Drill Units?

Distributed vs. Non-distributed Drill. The problem here is vital to drill construction. If quantity of drill is the main factor it is not necessary to build drill material carefully so that every number combination will appear a calculated number of times. But if quality shares importance with quantity, drill should be so built that each number combination appears with a calculated frequency. The issue here may be expressed as "distributed vs. non-distributed" drill.

Dr. Luse studied the effects of these two types of drill on 600 fifth grade pupils equally divided into two groups on a basis of general arithmetical ability.[1]

These two groups were given fifty consecutive drill periods of fifteen minutes each. One group used drill material carefully constructed as to the distribution of practice in addition, subtraction, multiplication, and division of whole numbers.[2] The other group used material slightly in excess as to sheer amount but so built that certain combinations were slighted. All other conditions were held constant. The appearance of the drill sheets was such that even experienced teachers and superintendents when asked to select the body of drill which they, upon a half hour's study, judged to be the better chose the poorly distributed drill as often as the other. In constructing the drill all possible practice was studied and controlled. For example, the addition practice concealed in checking addition examples, the kind used in carrying in multiplication, in adding the partial products and in checking multiplication, all addition practice in carrying in multiplication used in long division and in checking long division, were controlled.

The Issue. Dr. Luse had a clean-cut issue. Here are 600 pupils who since the second grade have had approximately equal experi-

[1] Space forbids a complete description of the administration of Dr. Luse's investigation. For a complete report of this fundamental research, see Eva M. Luse, *Transfer within Narrow Mental Functions* (A Study of the Effects of Distributed vs. Non-distributed Drill in Arithmetic), University of Iowa Monographs in Education, No. 5, College of Education, University of Iowa.

[2] This material is now available for school use. *The Arithmetic Work-Books*, Scott, Foresman and Company, Chicago, Ill.

ence with the four fundamental processes involving whole numbers. They are now divided into two equal groups and for 12½ hours the experience of one group varies from that of the other in the matter of frequency with which the number combinations are present in all drill work.

Addition. A complete inventory of all addition used by both groups appears in the next tables.

Table A shows the practice given to the group which used well distributed drill. Table B shows the practice in the material which was rather poorly distributed.

These tables are read as follows: In Table A zero was added to zero 33 times, zero was added to one 102 times, zero was added to two 70 times, and so on.

Other Operations. The experiences with subtraction, multiplication, and division were similar to the addition experience.

After the fifty consecutive drill periods the two groups were given a series of tests. One series of tests was given a few days after the drill periods and another three months later. Each series was so built that one part contained combinations upon which both groups had received much practice, another part contained many combinations that one group had practiced liberally but the other not so much, and a third part contained combinations upon which one of the groups had received relatively little practice. This made it possible not only to look for gross mean differences but to study the flow of the size of difference as examples based on disparity of practice relatively increased.

Further, each group was divided into fifths on the basis of performance on tests given at the beginning of the investigation. This made it possible to inquire into the relative effects of distributed versus non-distributed drill on different levels of general arithmetical competence.

Conclusions. The more pertinent of Dr. Luse's findings are quoted from her monograph:

"1. Both groups made a decided gain from the fifty periods of drill. The gain for the distributed drill was from 19.6% to 53.7% in Attempts and 31.1% to 84.8% in Rights. The gain from the haphazard type of drills was from 11.2% to 39.8% in Attempts and from 13.3% to 60.8% in Rights.

"2. The distributed drill gave an excess over the non-distributed

TABLE A

	0	1	2	3	4	5	6	7	8	9	Totals
0	33	102	70	84	89	79	43	46	43	36	625
1	47	78	101	92	91	86	80	72	80	76	803
2	26	86	85	85	78	72	77	90	68	47	714
3	35	62	67	34	58	60	50	49	54	46	515
4	31	83	55	89	68	73	60	34	52	35	580
5	34	57	73	60	70	51	53	42	41	27	508
6	36	74	73	57	79	45	60	55	48	41	568
7	23	62	39	34	50	56	46	43	44	35	432
8	41	81	56	63	58	60	57	51	48	53	568
9	35	39	58	36	41	52	36	40	56	42	435
Totals	341	724	677	634	682	634	562	522	534	438	5748

TABLE B

	0	1	2	3	4	5	6	7	8	9	Totals
0	131	242	211	97	87	64	74	47	65	51	1069
1	112	245	217	187	154	92	108	75	60	49	1299
2	76	228	179	122	67	59	42	31	18	14	828
3	58	155	115	96	72	25	30	15	16	21	603
4	62	143	158	104	72	49	34	22	19	10	673
5	67	191	168	117	75	68	34	21	3	17	761
6	69	208	112	90	57	30	32	17	13	12	640
7	67	124	117	66	42	21	20	18	12	9	496
8	60	173	137	50	51	17	20	23	16	4	551
9	60	106	63	51	31	14	17	9	8	11	370
Totals	762	1807	1477	980	708	439	411	278	230	198	7290

drill in examples solved correctly of 17.7% in Addition, 18.8% in Subtraction, 35% in Multiplication, and 23.9% in Division.

"3. The same relative differences in gain from the distributed and non-distributed drill held for the different levels of ability as for the whole group.

"4. The residuum after the summer vacation in actual number of examples was greater for the distributed drill group.

"5. In the comparisons made there were 126 opportunities for either form of drill to yield results higher, equal or lower than the other form of drill. The distributed drill excelled in 120 and the non-distributed in 6 of these opportunities. Of the 120 in which the distributed drill excelled, 75 showed statistically significant differences (differences of the means varying from 3 to 12.9 times the P.E. of the difference). Of the 6 cases in which the non-distributed drill excelled no one of the differences was statistically significant."

Most of us will agree that Dr. Luse's research closes the issue of "distributed vs. non-distributed" drill, at least until a much better study discredits her data. Such evidence is a bit unlikely to be forthcoming. Pending experimental data, distributed practice among the many items which go to make up total skills in algebra and geometry could well be assumed to be preferable also.

DISCUSSION SEVEN

What Types of Investigations Will Yield Useful Information about the Teaching Nature of a Text?

Conventional Methods. For some time it has been the custom to appraise the amount of drill or the amount of instruction or the amount of problem material in a textbook by the number of pages given to a topic. We have thought of 20 pages of drill as equal to 20 pages of drill. Measurement by such gross methods often hides more facts than it reveals. Or, we have said that since one text has 700 verbal problems and since a second text has 500 problems therefore the first text has the better problem provision. This in all sobriety is sheer guessing. A text with 700 problems of which 250 can be worked by the pupils is certainly not even equal in problem provision to one having but 500 problems, 415 of which can be worked by average pupils.

Method Used Here. There are several attacks on the general problem of the determination of the actual nature of a text. The investigation reported in the following paragraphs approaches the problem from the viewpoint of time requirements.[1]

This study of time analysis was conducted for the purpose of determining how much time is actually spent on the various processes in arithmetic in the fifth and sixth grades during a whole year when a textbook is taught as printed. The original data were secured by Mr. Theis with the assistance of his local fifth and sixth grade teachers and a departmental teacher of arithmetic in a neighboring school. The textbooks which were used are called Text A and Text B in the tables and in the discussion.

The textbooks were arranged into divisions or articles of work and blanks with the number of the articles in the textbook were supplied to each teacher. The instructions to the teachers were to put down the amount of time that was spent in study and the amount of time that was spent in recitation on each article, each day of the school year. These reports on time spent on each unit of work were then reorganized. The summaries appear in the tables which report the facts relative to how time given to each topic in the texts studied was actually spent in typical classroom situations. The numbers in roman type are for Text A in all cases. They are the lower of each pair. The numbers in italics are for Text B in all cases. They are the upper of each pair.

Sample Tables. Lack of space makes it necessary to reduce the tables to a minimum but from the following samples it is possible to see their character. The topics are 60 in number, each one being shown in detail in tables following the summary tables.

SUMMARY OF THE TOTAL TIME SPENT BY THE FIFTH GRADE

FIFTH GRADE ARITHMETIC TOPICS	TOTAL TIME TEXT-BOOK A	TOTAL TIME TEXT-BOOK B	A % OF TOTAL TIME	B % OF TOTAL TIME
1. Reading and writing whole numbers.	4	131	.001	2.14
2. Addition of whole numbers	363	131	4.81	2.14
3. Subtraction of whole numbers	472	91	6.25	1.49
4. Multiplication of whole numbers ...	253	390	3.35	6.36
5. Division of whole numbers	463	666	5.42	10.89
6. Supplementary to long division	107		1.41	0.00
7. Supplementary to whole numbers ...	39		.52	0.00

[1] For the complete published account of the study see: L. J. Thies, *The Time Factor in Arithmetic Texts,* University of Iowa Monographs in Education. No. 2, College of Education, University of Iowa.

Summary of the Total Time Spent by the Sixth Grade

Sixth Grade Arithmetic Topics	Total Time Text-book A	Total Time Text-book B	A % of Total Time	B % of Total Time
46. Introduction to decimals	52	286	.75	4.37
47. Addition of decimals	48	179	.69	2.74
48. Subtraction of decimals	54	175	.78	2.66
49. Multiplication of decimals	323	676	4.68	10.35
50. Division of decimals	766	554	11.09	8.48
51. Introduction to denominate numbers	51	92	.74	1.40
52. Addition of denominate numbers ...	23	4	.33	0.00
53. Subtraction of denominate numbers.	29		.42	0.00
54. Multiplication of denominate numbers	18		.26	0.00
55. Division of denominate numbers ...	14		.20	0.00

Multiplication of Decimals

		No. Art.	Oral	Written	Total	%
Instruction		*7* 5			*65* 105	*10* 32
Drill	Teaching process	*9* 7	*110* 14	*271* 67	*381* 82	25
	Maintaining skills	*9* 32	*35* 4	*148* 78	*283* 82	25
Verbal Problems	Teaching process	*6* 3	*4* 1	*33* 47	*37* 48	14
	Maintaining skills	*3* 2		*10* 6	*10* 6	2
Totals		*34* 49	*149* 19	*462* 198	*776* 323	*10* 98

Division of Decimals

		No. Art.	Oral	Written	Total	%
Instruction		*4* 9			*25* 265	*4* 34
Drill	Teaching process	*8* 8	*80* 65	*304* 209	*384* 274	*70* 36
	Maintaining skills	*5* 29	30	*62* 111	*62* 141	*11* 18
Verbal Problems	Teaching process	*4* 3	*10*	*11* 75	*21* 75	*3* 10
	Maintaining skills	*2* 3	*7* 3	*55* 8	*62* 11	*11* 1
Totals		*23* 52	*97* 98	*432* 403	*554* 766	*99* 99

Findings of the Investigations. A study of the tables reveals definite differences in the structure of Texts A and B, such as:

1. A radical difference in the theories of learning upon which these texts are consciously or unconsciously based. A good illustration of this is the relative emphasis placed on instruction as an integral part of a text. Text A is built on the theory that sheer instruction is of first importance; Text B is built on the theory that instruction as such need take but relatively little time in contrast with drill and problem work.

2. A difference in the division of time between drill work and problem work. It is evident that both texts spend practically the same amount of time upon drill and problem work together. But they spend this time quite differently. Text A tends to invest time in problem work more generously than does Text B. This comparison is upon a time basis alone and does not warrant a conclusion of comparative merit in the case of either text, inasmuch as other factors than time are of course pertinent. It is useful to know, however, that one text does stress problem work in a way that the other does not.

3. A difference in the division of time between oral and written work. Text A spends 729 more minutes upon oral work than does Text B. Allowing 60 weeks for grades five and six, it is evident that Text A provides somewhat over 10 minutes a week more oral work than does Text B.

4. A difference in what the texts do toward maintaining skills after they have been built. Text A spends roughly twice as much time in maintaining skills as does Text B. Here again it should be pointed out that assertions relative to superiority of one text over the other are not made. The point is that the two texts are built on genuinely different theories of textbook construction.

5. A difference closely allied to the one just commented upon has to do with the number of different sections or articles in the text in which any given topic appears. The number of articles given to a topic is at least a rough index of persistency of treatment. A topic on which few articles are reported is a topic which tends to be taught at a given time and then rather neglected. A topic on which many articles are reported tends to be a topic which is used again and again even after initial instruction has been completed. In treating 58 topics Text A provides a use of each of these 58 topics in 30 articles on the average. In treating the same topic

Text B provides appearance of each in 12 articles on the average. That is, Text A endeavors to weave the topics into a general power over arithmetic; Text B tends to teach in "chunks."

The agreements and disagreements relative to the teaching of fractions are interesting. Addition and subtraction require the most instructional time in Text A; in Text B multiplication requires by far the most, with division clearly ranking second.

Text A needs more time to *teach* division of decimals than it needs to teach multiplication of decimals, but Text B just reverses the matter.

Text A needs more drill to *maintain* division of decimals than it does to maintain multiplication of decimals, but Text B just reverses the matter.

Text A needs more time to use for *instructional drill* in division than in multiplication, but Text B does not.

While it is possible that both texts may be quite wrong on time provisions for multiplication and division of decimals, there is little likelihood that they are both *right.*

It is hoped that a perusal of the time factors in texts as given in this study will lead the reader to a serious consideration of the desirability of knowledge of this kind for use in the prudent administration of texts in the classroom and the use of this type of facts in making judgments relative to selection of textbooks for class use.

DISCUSSION EIGHT

Is It Possible to Approach an Objective Method in Appraising the Relative Merits of Texts as Teaching Instruments?

Rating Drill Provisions. Competent research has demonstrated that drill material constructed according to careful specifications yields a high percentage of efficient returns in the teaching of arithmetic. The study here reported has analyzed the minutiæ of drill embodied in six arithmetic texts having a wide current usage, with a view toward establishing a technique whereby the drill provisions of any arithmetic text can be objectively rated. The study is the work of O. S. Lutes, Professor of Education at the University of Maine, Orono, Maine, and Agnes Samuelson, State Superintendent of Public Instruction of Iowa.[1]

[1] Those interested in reading the complete study are referred to O. S. Lutes and Agnes Samuelson, *A Method of Rating the Drill Provisions in Arithmetic*

The authors set forth the main features of their monograph thus in Chapter I of the study:

1. This is an objective system for rating drill aspects of texts.
2. Ratings are based on an analysis of the drill in texts.
3. This system is limited to:
   a. Primary combinations of whole numbers (including decimals);
   b. The four fundamental operations;
   c. Drill exercises (no verbal problems, denominate numbers, etc.).
4. Main steps in rating are:
   a. First credit each operation (as addition, etc.) with 1,000 points for a three-book series, as follows:
   Books One (Grades 3 and 4) .... 500.
   Book Two (Grades 5 and 6) .... 300.
   Book Three (Grades 7 and 8) .... 200.
   b. Deduct penalties from these initial credits according to the scoring system described later in the study.
   c. Add credits for specific observance of certain aspects of good drill. See *e, f, g, h* in 5 below. (Total maximum score for a series, about 5,000.)
5. The eight aspects of drill rated are:
   a. Distribution of practice on the primary combination ,
   b. Bulk of practice on the primary combinations;
   c. Relative practice on hard and easy combinations;
   d. Amount of carrying practice (in addition, multiplication and division; practice on unseen combinations in subtraction);
   e. The placement of drill units in the text;
   f. Arrangement of examples within drill units in order of difficulty;
   g. Use of standards with drill work;
   h. Use of mixed *versus* isolated drills.

The study reports tables showing the distribution of drill practice by 15 page units in Book I, 30 page units in Book II, and 45 page units in Book III for each of six arithmetic textbook series. As the authors point out, "This is important for we must

Text Books, University of Iowa Monographs in Education, No. 3, College of Education, University of Iowa.

know how practice is spread through the book. One text which allows addition to lie idle for short periods only is certainly better on this score than another text which may give the same amount of drill but bunches it at infrequent intervals." Summary tables for Books I, II and III of each text series are also given.

All material in the texts which affords appreciable amounts of drill, as such, in the four fundamental operations is analyzed. All practice in verbal problems is omitted on the grounds of the scanty drill practice afforded.

Addition Drill. The addition drill material is analyzed according to the distribution of practice on the 100 basic addition facts. Table I is a sample of the analysis tables for this process contained in the study. In order to understand the table it should be read as follows:

0 is added to 0 three times, 0 is added to 1 two times, 0 is added to 2 two times, and so on. The practice in carrying is indicated at the bottom of the table: 1 is carried 6 times; 2 is carried 3 times.

In the addition and multiplication tables, a heavy line marks off the lower right quarter of the cells or combinations. These are called the "fourth quadrant" combinations. There is evidence to show that they are the most difficult combinations; hence the scoring system penalizes drill which fails to give them at least an equal amount of practice.

Subtraction Drill. Table II illustrates the process of analyzing the subtraction drill. In order to get the 100 basic combinations the minuends must run to 18 inclusive. The diagonal line which occurs in each cell or combination space containing any practice separates what we call "seen" from "unseen" combinations. If the digit in the minuend is changed by borrowing, then the pupil must subtract the subtrahend from an "unseen" number. Thus in the example 175 minus 96, 1 is borrowed from the 7 so that 9 is taken from 16, an "unseen" number. This assumes the use of the subtractive or the decomposition method of subtraction. But while the particular combinations may be changed by other methods, we still have the same proportion of "unseen" numbers to handle whether in the subtrahend or in the minuend.

Thus to read the table, 2 is taken from "seen" 4 once, and from "unseen" 4 three times, or four times in all.

The heavy vertical line between the minuends 9 and 10 marks off the 45 combinations to the right of this line which involve upper

TABLE I

NUMBERS ADDED TO

NUMBERS ADDED	0	1	2	3	4	5	6	7	8	9
0	3	2	2	1	2	4	1	1		
1		1	2	2	1	1			1	1
2		2	2	2	1				1	
3		1	1	2	2			1	2	1
4		1		1			1			1
5		4	3			2				
6		2	2			1		1		
7		3	1	1						
8					2			1		
9					2					

NUMBERS CARRIED

1	2	3	4	5	6	7	8	9
6	3							

TABLE II

Minuend

Subtrahend	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	2/				1/			1/											
1		1/3	2/1	/1	1/4	1/				/1	2/								
2			2/	2/2	1/3	2/	2/	2/	1/	1/2	5/								
3					2/2	3/		1/	/1	/2	2/								
4					/2		/1	1/	1/	1/1	2/			2/					
5						2/	1/1	1/		/1	1/1								
6							1/	/1	1/1	/1	2/					3/			
7									1/	1/2	3/	1/				2/			
8										/1	3/		1/			2/	3/		
9										/1	3/		2/	1/		1/	1/1		1/

decade minuends. The proportion of practice found in these combinations is considered in the rating system.

Multiplication Drill. In the analysis of multiplication drill, the table arrangement is identical with that illustrated in Table I, except that "Multiplier" is substituted for "Numbers added" and "Multiplicand" for "Numbers added to."

TABLE III

	Initial Credit	Score A	Score B	Score C	Score D	Score E	Score F
Addition							
Distribution	150	43	69	102	76	2	56
Bulk	75	5	26	33	54	75	30
4th Quadrant	45	35	15	15	45	45	15
Carrying	30	–20	–5	10	5	–10	5
Totals	300	63	105	160	180	112	106
Subtraction							
Distribution	150	–45	–21	13	–8	10	23
Bulk	75	–128	–128	–114	–114	–119	–96
Harder 45	45	15	45	45	45	45	27
Unseen Minuends	30	30	30	30	30	30	30
Totals	300	–128	–74	–26	–47	–34	–16
Multiplication							
Distribution	150	94	105	94	105	93	62
Bulk	75	3	40	44	27	70	37
4th Quadrant	45	–35	5	25	–5	45	45
Carrying	30	10	30	30	30	30	30
Totals	300	72	180	193	157	238	174
Division							
Distribution	150	38	43	69	72	45	54
Bulk	75	5	5	19	9	68	75
4th Quadrant	45	–5	–5	5	–35	45	45
Carrying	30	–10	–10	10	–10	30	30
Totals	300	28	33	103	36	188	204
Grand Totals	1200	35	244	430	326	504	468

Division Drill. The analysis of division drill is that of the multiplication drill in division; hence the organization of these data is identical with that for multiplication. Thus in the example $\begin{array}{r}213\\3\overline{)639}\end{array}$ one practice each on 2×3, 1×3, 3×3 would be reported. The other aspects of division drill were not reported for two chief reasons; namely, (1) the unpractical amount of time required for such analysis; (2) the probability that textbook writers who take time properly to distribute the multiplication practice involved in division are likely sensitive to the other characteristics of good drill.

Ratings. The scores for Book II in each of the six text series analyzed will serve as concrete illustrations of the ratings. These ratings are presented in Table III.

Lack of space prevents a statement here of the reasoning upon which the authors base the organization of their rating system, the main features of which were outlined in one of the opening paragraphs. Obviously, other individuals employing a similar technique could evolve a rating plan with a markedly different emphasis and weighting of the constituent factors. This may be a profitable procedure for others to follow. But the utilization of scientific, detailed analysis of drill material in the appraisal of the drill aspects of a teaching instrument is essentially sound procedure. It reveals many hitherto unsuspected strengths and weaknesses, and enables the administrator or supervisor to choose with more discerning discrimination an adequate arithmetic text to place in the hands of the classroom teachers.

DISCUSSION NINE

What Can Be Done for Pupils Who Need Remedial Work?

Remedial Drill on Arithmetic Problem Solving. The following treatment of remedial drill on arithmetic problem solving is the work of Professor H. A. Greene, University of Iowa.

Two large and specific tasks confront teachers of elementary school mathematics. These are (1) the development of efficient pupil skill in the use of the fundamental processes, and (2) the development of the knowledge of *when* to use them. The first deals with the mastery of tool skills, the *how;* the second deals with the knowledge of when to apply the specific skill. The proper use of correctly designed practice and remedial materials will rapidly produce control of these tool skills. The second is a much more difficult problem, however, involving as it does the development of technical judgment in the matter of *when* and in *what order* to use these basic skills.

The matter of the development of this judgment on the part of the pupil is probably the greatest single difficulty in the way of efficiency in problem solving. The results of measurement and the analysis of pupils' work both show clearly that many children have an adequate knowledge of the fundamental combinations as such yet cannot comprehend the problem setting when the same numerical situations are presented in verbal form. A little observation of pupils at work on verbal problems leads inevitably to the conclusion that *trial and error* is too often the method used. Since trial and error as a method is used only when no other satisfactory solution is available, it seems quite clear that for the majority of pupils something is lacking in the instruction in this particular field of arithmetic.

It has been frequently stated that *a problem in arithmetic is a statement of an incomplete numerical situation which the child in his solution is called upon to complete.* This definition sets up the elements of the situation which according to Dewey involves "the act of complete thought." Closely paralleling the steps in Dewey's analysis of the thinking process five fundamental steps in problem solving are noted, each affording a clue to a number of possible remedial procedures.

1. The first step in the solution of verbal problems involves a complete *comprehension* of the items, elements, and processes which are stated or implied in it. Such factors as rate of reading, vocabulary difficulties, ability to read numerals, the organization of the problem, as well as its complexity in terms of the number and order of the fundamental processes used, are here involved.

2. The second step is the *analysis* and *organization* of the elements of the problem. In this process the unnecessary facts or implications are discarded and only those which have definite bearing on the solution of the problem are retained.

3. The third step is almost a part of the second since the act of *recognition* of the process or processes involved in the problem is itself a part of the analysis.

4. The fourth step is the *solution* in which the pupil applies to the specific situation his knowledge of the tools required.

5. The fifth step is *verification*. This may be merely a matter

of estimating the probable answer or it may involve a complete re-checking of the computations and processes.

An examination of each of these basic steps in verbal problem solving as presented in the foregoing discussion and in the accompanying chart reveals some very real opportunities for the development and application of remedial materials. The chart shows the

CHART SHOWING ANALYSIS OF STEPS IN THE PROCESS OF PROBLEM SOLVING

Steps in Problem Solving	Factors Underlying Problem Solving	Types of Drill Provided
Comprehension	Vocabulary Ability to read numerals Ability to read rapidly Ability to comprehend *a.* Follow directions *b.* Make generalizations *c.* Select potent elements *d.* Discard irrelevant elements *e.* Determine problem setting as a unit *f.* Determine the outcome of the problem *g.* Grasp significance of problem cues	Vocabulary drill Comprehension drills *a.* Directions exercise *b.* Completion exercise *c.* Multiple choice exercises
Analysis and Organization	Selection of potent factors Selection of processes involved Determining what the problem calls for Determining what is given in the problem Determining process relationships	What is called for Process analysis What is given Problem relationships
Recognition	Choice of procedure Determining problem conditions Determining purpose of the problem Determining relevant elements in problem	Process analysis What is given What is called for
Solution	Selection of processes Organization of processes in order Knowledge of combinations Problem relationships	Process analysis Problem relationships Problem scales
Verification	Probable form of answer Probable magnitude of answer	Probable answer

steps in the process of problem solving further analyzed into the more elementary factors underlying each. It also shows suggested types of drill material for providing much needed additional practice on these underlying factors.

Remedial Materials. Samples of types of remedial materials designed to give the pupil practice in reading verbal problems and training in the application of the technical judgment which he is constantly called upon to apply in solving such problems are given below.

SAMPLE 1

EXERCISES STRESSING VOCABULARY

A[1]

DIRECTIONS: The words below are often used in your arithmetic work. Following each word you will find four statements, ONE of which tells correctly the meaning of the word when used in arithmetic work. Put a circle around the number of the exercise which is correct for each word.

1. *Acre*
   1. contains 160 sq. rds.
   2. is 16½ rods of land
   3. a square piece of land
   4. area of a small lot

2. *Area*
   1. the length and width of a surface
   2. amount of land in an acre
   3. surface having length and width
   4. distance around the base of a figure

B[2]

It takes 400 sq. ft. of carpet to cover a certain school room floor.

Underline the word, given below, that most nearly explains the meaning of the 400 sq. ft. in the above statement:

LENGTH AREA SIZE WIDTH

[1] These exercises are taken from a preliminary form of a vocabulary test now being standardized.

[2] These exercises and the rest of those presented under "B" are taken from "Practice Exercises for Comprehension and Speed in the Solution of Verbal Problems in Arithmetic" prepared and published by Walter Shriner, and printed by The Craft Typeshop, Ann Arbor, Michigan.

Sample 2

Exercises Stressing Problem Comprehension

A[*]

Comprehension—One Step Problems

Directions: Read each problem carefully. Then read each statement (a-e) below the problem and write in the box the letter which stands for the ONE statement which is true as you understand the problem. Read the problem as often as you need to.

1. We had a total of 39 words in spelling this week. I spelled only 25 of them correctly. How many did I misspell?	2. To what number must 33 be added to get 67 for an answer?
(a) I missed more words than I spelled correctly. (b) I spelled 39 words correctly. (c) I misspelled 25 words. (d) I spelled correctly more than one-half of them. (e) We had only 25 words in spelling this week.	(a) 33 is to be added to 67 ....(a) (b) The number is the difference of 33 and 67 ...(b) (c) The number is larger than 67 ....(c) (d) 67 is to be multiplied by 33 ....(d) (e) The number is less than 33 ....(e)

B

We have three new words added to our spelling list each day. How many new words have we in 5 days?

If you are asked to find the number of new words that we have on the fifth day, underline the word YES, given below; if not, underline the word DAY:

FIVE DAY NO YES

[*] These exercises and the rest of those presented under "A" are taken from a set of remedial exercise cards in problem solving published by Scott, Foresman & Company, Chicago. These exercises are prepared in permanent card form in such a manner as to permit their repeated use. The cards are arranged in series providing drill on each of the basic steps in problems solving on exercises varying in complexity from one step problems to those involving three steps. The six such series of cards furnishing practice on at least 12 exercises on each skill are so arranged as to give perfectly balanced drill on each of processes and combinations of processes. One set of directions at the top of the card is adequate for all exercises on the card.

Sample 3

Exercises Stressing What Is Given in the Problem

A

What Is Given—One-Step Problems

Directions: Read each problem carefully. Then read each statement (a-e) below the problem and write in the box every letter which stands for a fact given in the problem. Several of the statements may be true. Do not miss any. Refer to the problem as often as you need to.

1. A lady took a train to the city, a distance of 19 miles. There she took another train on which she rode 172 miles. How far did she ride on the two trains?

(a) Speed of the train to the city.
(b) Distance the lady rode on the first train.
(c) Time she waited for the train to come.
(d) Length of time lady was on trains.
(e) Distance the lady rode on second train.

2. A factory employing 285 workmen found it necessary to discharge 90 of them. How many workmen were retained?

(a) The amount paid the workmen ....(a)
(b) The product manufactured ....(b)
(c) Number employed before discharging 90 men..(c)
(d) Number employed after discharging 90 men..(d)
(e) Number of men discharged ....(e)

B

A boy in the city rode his new bicycle 5 times around the Public Square. How many blocks did he ride?

Underline the number given below that tells how many blocks the boy rode:

3 4 10 20

Sample 4

Exercises Stressing What Is Called For in the Problem

A

What Is Called For—One-Step Problems

Directions: Read each problem carefully. Then read each statement (a-e) below the problem and write in the box the letter which stands for the ONE statement which tells what is called for by the problem.

1. A train leaves the station at 2:30 in the afternoon and reaches the city 3 hours and 20 minutes later. What time does the train reach the city?

(a) The time the train leaves the station.
(b) The speed of the train.
(c) The time the train arrives in the city.
(d) The time required for the train to run to the city.
(e) The distance the train runs.

2. This morning the temperature was 41 degrees. At noon it was 68 degrees. What was the difference in the morning and noon temperatures?

(a) The temperature yesterday morning ....(a)
(b) The temperature at noon today ....(b)
(c) The temperature this morning ....(c)
(d) The difference in temperature in morning and at noon ....(d)
(e) The difference in the temperature at noon and at evening ....(e)

B

A man and two boys together caught 320 fish. If the man is to have 2 times as many fish as each of the boys, how many does each get?

Underline the number given below that tells how many things you are asked to find in the problem:

1 2 3 4

Sample 5

Exercises Stressing the Estimation of Answers

A

Probable Answer—One-Step Problems

Directions: Read each problem carefully. There are five suggested answers (a-e) below each problem. One of them is the most probable answer of the five. In the box write the letter which stands for the most probable answer. Read the problem as often as you need to.

1. How many hours will it take a train which averages 39 miles per hour to run 364 miles?

(a) 10 miles.
(b) About 9 hours.
(c) 92 hours.
(d) About ⅔ of a mile in 1 minute.
(e) About 364 minutes.

2. A girl sold tickets for a moving picture theater On a night when the admission was 44¢, she sold 529 tickets. How much money did she take in that night?

(a) About $539 ....(a)
(b) About $44.00 ....(b)
(c) $681.00 ....(c)
(d) About $235.00 ......(d)
(e) Exactly $118.53 .....(e)

B

There are 34 children belonging to our class, but only 23 of them were present today. How many children were absent?

If the answer should be greater than 24, underline the word MORE, given below; if not, underline the word LESS:

ABSENT MORE PRESENT LESS

SAMPLE 6

EXERCISES STRESSING CHOICE OF PROCEDURE

A

Process Analysis—One-Step Problems

KEY
A–Addition
S–Subtraction
M–Multiplication
D–Division

DIRECTIONS: Read each problem carefully, but do not try to get the answer. Write the proper letter—A, S, M, or D—to show which one of the four processes you would use if you were to work the problem. (Addition, Subtraction, Multiplication, and Division are the four processes.) Read the problem as often as you need to.

S	*Sample:* We have ten boys in our room at school. Four were absent today. How many were present?

	1. A newsboy sold 24 papers in the forenoon and 37 papers in the afternoon and evening. How many papers in all did he sell that day?	2. From a class of 24 pupils 7 were absent. How many were present?	

B

If you knew the amount paid for 2 dozen apples, how would you find the cost of one dozen apples?

Underline the word given below that tells the right thing to do:

ADD SUBTRACT DIVIDE MULTIPLY

SAMPLE 7

EXERCISES STRESSING RELATIONSHIPS IN PROBLEMS

B

John walked 10 miles at the rate of 4 miles an hour. How many hours did John walk?

Underline the relationship, given below, that is the correct one to use:

(a) TIME = DISTANCE + RATE
(b) TIME = DISTANCE — RATE
(c) TIME = DISTANCE × RATE
(d) TIME = DISTANCE ÷ RATE
(e) TIME = RATE ÷ DISTANCE

The possibilities of the development and application of remedial instruction units such as the above are almost entirely untouched. More and more instruction in arithmetic as in other fields will become individualized as adequate materials of instruction are prepared. Certainly the most efficient method of maintaining desired levels of permanent mastery of all types of arithmetical skills is found in the careful diagnosis of individual weaknesses of pupils and the application of specific corrective material built upon carefully written specifications.

GENERAL CONCLUSION

The report on elementary school arithmetic as printed on pages 3 to 72 has attempted to do two things: (*a*) to suggest the type of problems with which workers in this field concern themselves; (*b*) to present typical examples of objective and experimental work on problems of apparent importance. Writings and researches which have been available in printed form for some time have been slighted in this report only because it is assumed that the reader is already familiar with them. For example, no mention is made of Buswell's *Diagnostic Studies in Arithmetic* because those interested in arithmetic will have mastered that most excellent contribution long before these pages are read.[1] Similarly, other important contributions of which a complete report would require much space unfortunately are omitted.

[1] Buswell, G. T.: *Diagnostic Studies in Arithmetic.* University of Chicago Press.

CURRICULUM PROBLEMS IN ARITHMETIC

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Introduction. For some reason students of education have not given to the subject of arithmetic that detailed kind of scientific analysis which they have given to the subject of reading. From 1901 to 1924 there were published 421 studies relating to reading and only 277 studies relating to arithmetic. During the four-year period from 1921 to 1924 there were 201 studies published relating to reading, while there were only 66 studies relating to arithmetic. The reason for this lack of investigation of the problems of arithmetic is somewhat difficult to explain. The techniques of investigation for the subject of arithmetic are no more difficult than those for the subject of reading. Apparently there has been a greater degree of satisfaction with the results obtained in arithmetic and a smaller appreciation of the fact that there are problems in this field which need investigation. There is some evidence that, at the present time, arithmetic is receiving an increasing amount of attention. The number of published studies (unpublished theses not included) since 1924 has been somewhat greater in the field of arithmetic than in the field of reading. Evidently educators are becoming more acutely aware of the problems which need solution and are beginning to recognize in the subject of arithmetic a productive field for research. It is the purpose of this article to present certain problems which need further investigation.

Consideration will be given here to five general problems. These are (1) the difficulties in reading encountered in arithmetic; (2) the teaching contribution of arithmetic textbooks; (3) the treatment given to the number system as such; (4) the preparation of drill exercises keyed to specific needs as revealed by diagnoses of pupils' work; and (5) the grade location of the different arithmetical processes.

I. The Reading Difficulties Encountered in Arithmetic

Difficulties Due to Vocabulary. The vocabulary studies of Thorndike, Gates, and others have enabled the authors of textbooks in reading to select the words to be used on the basis of their familiarity and common use. Particularly in the preparation of primers and first readers, great care has been exercised in the gradual introduction of new words and in the provision of sufficient repetition of these words to insure familiarity. In the preparation of readers for the grades above the first, there has also been a careful check of vocabulary to make sure that the words used are within the normal experiences of the children. While analyses of vocabularies of arithmetic books indicate that in certain cases, for example the Thorndike arithmetics, car. ul attention has been given to vocabulary, in most cases there is no evidence of any regard for this matter. Since arithmetic textbooks must be read, it is obviously important that the vocabulary be controlled with this fact in mind.

An excellent example of the results exhibited by an analysis of arithmetic textbooks has been furnished in a Master's thesis which has just been completed by Miss Ava Hunt.[1] In this study Miss Hunt made an analysis of the vocabularies of six third grade arithmetic textbooks. She then compared the results with the vocabulary of ten third grade readers and also with the Thorndike word list. The following quotation gives a summary of the facts discovered:

1. The total vocabulary of six third grade arithmetic textbooks examined is 2,993 different words.
2. The average vocabulary of the six textbooks is 1,262 different words.
3. The common vocabulary of the six arithmetics is small, there being only 350 words, or 11.7 per cent of the total vocabulary, that occur in all six textbooks.
4. There are 1,345 words, or 44.9 per cent of the total vocabulary, that are used in only one textbook.
5. The percentage of words which are used in only one textbook ranges in the six textbooks from 11.2 per cent to 21.03 per cent.
6. An average of 32.3 per cent of the words occur only once in the textbooks in which they are used.
7. Of the 1,345 words used in only one textbook, 1,214 are used less than four times in the book in which they occur.
8. Of the total vocabulary of 2,993 words, 328 rank in the first 1,000 for importance in the *Teacher's Word Book*.
9. There are 2,467 words, or 82.4 per cent of the vocabulary, that rank

[1] Ava Farwell Hunt: "A Comparison of the Vocabularies of Third-Grade Textbooks in Arithmetic and in Reading." Unpublished Master's thesis, Department of Education, University of Chicago, 1926.

in the first 5,000 for importance; 284, or 9.5 per cent, that rank in the second 5,000; and 242, or 8.1 per cent, that are not found in the list.

10. The technical vocabulary of the six arithmetic textbooks consists of 306 words, or 10.2 per cent of the total arithmetic vocabulary.

11. Only 34 of the words of the technical vocabulary are used in all of the textbooks, and 75 are used in only one book.

12. Of the 306 words of the technical vocabulary of the arithmetics, 261 rank in the first 5,000 for importance in the *Teacher's Word Book.*

13. There are 514 words, or 17.2 per cent of the arithmetic vocabulary, that are found in the common vocabulary of ten third grade readers.

14. There are 980 words, or 32.7 per cent of the arithmetic vocabulary, which do not occur in any of the ten readers.

15. There are 1,043 words, or 34.9 per cent of the arithmetic vocabulary, which are used in half or less than half of the readers.

16. There are 616 words, or 20.6 per cent of the arithmetic vocabulary, which occur in only one of the arithmetic textbooks and are not found in any of the readers.

17. Of the 616 words occurring in only one arithmetic and in none of the readers, 292 rank in the first 5,000 for importance in the *Teacher's Word Book*, 139 rank in the second 5,000, and 185 are not in the list.

18. Of the 616 words mentioned above, 303 occur only once in the textbook, and only 57 occur more than three times.

19. Of the 306 technical words, 164 do not occur in any of the readers, and only 12 occur in all ten of the readers (Chapter V).

The facts presented in the preceding summary indicate that an arithmetic textbook would present considerable difficulty to children if the only requirement were that it be read and if the sole criterion of difficulty were the number of new words encountered. A third of the words of the six textbooks occur only once, no provision being made for sufficient repetition to make them familiar to the child. A third of the words in the arithmetics do not occur in any of the ten readers. The teacher of arithmetic, therefore, becomes responsible for teaching these new words in the arithmetic class. Even in respect to the technical vocabulary used by textbooks in arithmetic, there has been little common practice, only 34 of the 306 technical words being common to all of the six books.

The situation disclosed in the preceding summary is still more striking when one observes the variation in practice for single sets of textbooks. Authors have simply drawn upon their own private vocabulary without regard for the difficulties which the limitations of their vocabulary may cause children. One of the first problems which must be solved before the reading difficulties encountered in arithmetic are removed is a standardization of the vocabularies used in the different books and a correlation, grade by grade, between the vocabularies used in arithmetics and the vocabularies encountered in general reading in the elementary school.

Difficulties Due to Unfamiliarity with the Field. It has been frequently pointed out that the problems taken from arithmetic textbooks are not common to the experiences of children, and that the very terms employed in the statement of these problems are often meaningless. A typical illustration recently came to the writer's attention. A pupil was trying to solve the problem, "Find the perimeter of a room 10 ft. $\times$ 12 ft." The word "perimeter" was new to her, and she had no idea of its meaning. Consequently she was entirely unable to proceed with the problem, and she failed on it completely. Her difficulty, however, was not a difficulty in arithmetic but a difficulty in reading. Many problems of this sort occur in the formal exercises in arithmetic textbooks. However, this type of difficulty generally goes much beyond unfamiliarity with words. Children who live in cities have vague ideas in regard to the social significance of the problems which relate to farm life and farm products. Likewise, rural children are frequently mystified by the situations encountered in problems based upon city life. A problem is not genuine unless a child senses the situation upon which it is based. Doubt is frequently expressed as to whether any printed set of problems can ever be satisfactory for children in different types of environment. It is quite certain that if problems are to be genuine more space must be given to developing the situation upon which they are based. This has produced a commendable practice, followed by some authors, of grouping problems around a single social situation, where the situation is explained in sufficient detail that the meaning of the specific problem becomes entirely clear. For example, instead of devoting four pages to a miscellaneous list of disconnected problems, three of the pages may be devoted to a general narrative presentation of a situation which is interesting as a story. Following this interesting introduction, a whole series of problems will be presented which grow directly out of the situation described. In this case the child is made familiar with the field to be covered by the problem before the specific exercises are presented, and his reactions become correspondingly more genuine.

The Technique of Reading Problems. In 1922 Professor Terry carried on a laboratory study [1] to determine how children read problems. One very significant fact pointed out in the investigation is that children ordinarily require two readings for a satisfactory understanding of a problem: the first reading to get the general situation described and the second reading to get the exact numerical terms from which the answer is to be secured. When children begin to work immediately following the first reading, they frequently make one of two possible errors: either they fail to read accurately the numerical terms in the problems or they fail to recognize some important condition necessary for its solution. Investigators in reading have repeatedly emphasized the fact that different reading habits are required for different types and purposes of reading. Reading habits which are entirely satisfactory for dealing with fiction cannot be carried over with equal success to the reading of problems. Furthermore, these investigators have pointed out that the study type of reading is produced only by specific instruction with that end in view. This kind of instruction ordinarily is not supplied in the regular reading period. Therefore, one of the problems which remain for the teachers of arithmetic is the development of materials and methods for teaching children to read arithmetic problems precisely and accurately.

After a study of difficulties in reading arithmetic problems, Professor Osburn [2] lists the nine causes of misunderstanding:

1. Lack of vocabulary.
2. Failure to read or see all the elements in the problem.
3. Failure to resist the disturbance caused by preconceived ideas.
4. Inability to read between the lines.
5. Failure to understand fundamental relations, particularly those of the inverse type.
6. Failure to make a quick change of mental set.
7. Failure to generalize or transfer meanings.
8. Failure to interpret cues correctly.
9. Responding to irrelevant elements.

[1] Paul Washington Terry: *How Numerals Are Read.* Supplementary Educational Monographs, No. 18. Department of Education, University of Chicago, 1922. Pp. xiv + 110.

[2] W. J. Osburn: *Reading Difficulties in Arithmetic.* State Department of Public Instruction, Madison, Wisconsin, 1925. Pp. 8.

One of the major curriculum problems in arithmetic is the formulation of arithmetic materials which are readable, which are expressed in a vocabulary common to the experience of the child for whom they are made, and which are based upon social situations to which the child can react intelligently.

II. The Teaching Contributions of Textbooks

Two Kinds of Material. Textbooks in arithmetic contain two kinds of material. One of these may be designated as content material which gives information about the subject itself and which explains the methods by which the arithmetic processes are to be carried on. The other kind may be designated as formal material consisting chiefly of practice exercises to illustrate and establish the various processes in arithmetic. If the content of a textbook in arithmetic were to be separated, the explanatory material being placed in one book and the practice material in another a striking fact would be apparent. The book of explanatory material would be very much smaller than the book of practice exercises. Textbooks in arithmetic have, for the most part, been exercise books which would correspond to the laboratory manuals in science. The schools have depended very much more upon the teacher than upon the textbook to supply the necessary information to the pupils.

Explanatory Material. The effects of supplying such a small amount of explanatory material are apparent in both the upper and the lower grades. In the upper grades it results in a superficial presentation of such topics as "interest," "taxes," "bonds," "insurance," and the like, which gives the pupil a very meager understanding of the social character of these processes. One frequently gets the impression that these various processes are inserted simply to provide fields for more problems rather than presented as important phases of social activity which involve quantitative relationships. Many of the errors in the problems relating to these fields result not from an inadequate understanding of numerical relationships but rather from an inadequate understanding of the character of the social process.

A typical problem in a textbook reads, "If a broker charges $2.00 for buying a $1000 bond, what is his rate of commission based on the par value?" Before a pupil can deal intelligently with such a problem he should understand the nature of the broker's operations, what bonds are and the part they play in social life, some idea of normal rates of commission, an understanding of the term "par value" and all that it signifies. To be sure, the pupil may be able to get the answer without understanding these various things, but one may well raise the question as to whether his understanding of the situation will be increased by getting such an answer. If a pupil is to work intelligently with such problems as this he must have a fund of information which ordinarily has neither been supplied in arithmetic textbooks nor been given by other subjects in the elementary school curriculum at the corresponding grade level. When arithmetics are so filled with formal practice exercises, it is perfectly obvious that there is no room for more adequate explanations. The question remains, therefore, as to how an adjustment is to be made between problem exercises and adequate explanatory material.

Adjustment in Lower Grade Levels. At the lower grade levels the adjustment between explanatory and drill materials presents itself in somewhat different form. Here the child is chiefly concerned with the fundamental processes of arithmetic, and here, more than for the upper grades, the textbooks are filled with practice exercises. They are virtually drill manuals. There is little explanation to tell the child just how to proceed even in such simple matters as addition and subtraction. Consequently, the child adopts a variety of practices of his own. A careful survey of his methods of work reveals a diverse array of habits of procedure, some of them good but many of them wasteful. The necessary teaching has neither been supplied by the textbook, nor, in many cases, has it been supplied by the teacher. There is evidence that recently some authors have become aware of the acuteness of this problem, and they have used considerably more space in explaining to the children just how they should proceed in working their examples. In these cases the textbook presents detailed directions as to methods of work before any formal exercises for practice are given. They attempt to assure correct practice before introducing drill.

An Urgent Need. The very urgent need for curriculum material which explains to the child the proper method of procedure is indicated in a survey of habits of work of some four hundred different pupils. In this diagnostic survey the experimenter observed care-

[*Text continued on page 84*]

TABLE I

FREQUENCY OF HABITS IN ADDITION (ALL CASES)

	Habit	Grade				Total
		III	IV	V	VI	
a1	Errors in combinations	81	103	78	58	320
a2	Counting	61	83	54	17	215
a3	Added carried number last	39	45	45	26	155
a4	Forgot to add carried number	37	38	34	17	126
a5	Retraced work after partly done	26	34	39	22	121
a6	Added carried number irregularly	26	30	28	18	102
a7	Wrote number to be carried	34	25	18	12	89
a8	Carried wrong number	28	19	26	14	87
a9	Irregular procedure in column	16	29	23	18	86
a10	Grouped two or more numbers	25	22	21	16	84
a11	Split numbers	12	29	25	14	80
a12	Used wrong fundamental operation	23	25	20	11	79
a13	Lost place in column	17	17	17	14	65
a14	Depended on visualization	24	8	27	2	61
a15	Disregarded column position	34	11	9	1	55
a16	Omitted one or more digits	13	21	13	5	52
a17	Errors in reading numbers	14	10	21	7	52
a18	Dropped back one or more tens	13	12	17	5	47
a19	Derived unknown combination from familiar one	13	7	11	11	42
a20	Disregarded one column	15	11	8	2	36
a21	Error in writing answer	12	3	14	5	34
a22	Skipped one or more decades	11	7	9	5	32
a23	Carried when there was nothing to carry	6	9	9	5	29
a24	Used scratch paper	7	5	9	0	21
a25	Added in pairs, giving last sum as answer	6	6	9	2	20
a26	Added same digit in two columns	10	6	1	1	18
a27	Wrote carried number in answer	10	2	2	1	15
a28	Added same number twice	4	1	3	3	11
a29	Began with left column	1	1	1	0	3
a30	Confused columns	1	0	0	0	1
a31	Added carried number twice	0	1	0	0	1
a32	Subtracted carried number	0	0	0	1	1
a33	Added imaginary column	0	0	1	0	1
	Total Number of Subjects	96	124	116	78	414

TABLE II

FREQUENCY OF HABITS IN SUBTRACTION (ALL CASES)

	Habit	Grade III	Grade IV	Grade V	Grade VI	Total
s1	Errors in combinations	62	75	69	40	246
s2	Did not allow for having borrowed	19	50	57	36	162
s3	Counting	43	44	39	10	136
s4	Errors due to zero in minuend	25	39	26	15	105
s5	Said example backward	21	38	29	12	100
s6	Subtracted minuend from subtrahend	47	33	12	4	96
s7	Failed to borrow, gave zero as answer	21	20	14	4	59
s8	Added instead of subtracting	18	9	19	1	47
s9	Error in reading	14	5	13	10	42
s10	Used same digit in two columns	18	15	3	4	40
s11	Derived unknown from known combination	12	9	13	3	37
s12	Omitted a column	9	13	8	5	35
s13	Deducted from minuend when borrowing was not necessary	2	8	10	5	25
s14	Split numbers	7	5	10	2	24
s15	Used trial-and-error addition	6	7	7	4	24
s16	Ignored a digit	12	6	2	3	23
s17	Deducted 2 from minuend after borrowing	1	5	8	6	20
s18	Error due to minuend and subtrahend digits being same	1	5	10	3	19
s19	Used minuend or subtrahend as remainder	10	6	2	0	18
s20	Reversed digits in remainder	4	7	2	4	17
s21	Confused process with division or multiplication	5	6	3	2	16
s22	Skipped one or more decades	3	4	7	0	14
s23	Increased minuend digit after borrowing	2	2	6	2	12
s24	Based subtraction on multiplication combination	1	2	3	0	6
s25	Error in writing answer	2	1	0	1	4
s26	Began at left column	2	0	1	0	3
s27	Deducted all borrowed numbers from left-hand digit	1	0	1	0	2
	Total Number of Subjects	84	109	109	70	372

TABLE III

Frequency of Habits in Multiplication (All Cases)

	Habit	Grade III	Grade IV	Grade V	Grade VI	Total
m1	Errors in multiplication combinations	36	59	60	41	196
m2	Error in adding the carried number..	6	40	58	45	149
m3	Wrote rows of zeros	2	33	40	34	109
m4	Errors in addition	5	31	41	21	98
m5	Carried a wrong number	5	28	40	22	95
m6	Used multiplicand as multiplier	18	33	23	15	89
m7	Forgot to carry	10	30	27	22	89
m8	Error in single zero combinations, zero as multiplier	11	20	23	27	81
m9	Errors due to zero in multiplier	5	26	30	17	78
m10	Used wrong process	18	22	16	10	66
m11	Counted to carry	4	20	28	9	61
m12	Omitted digit in multiplier	1	15	20	16	52
m13	Wrote carried number	8	16	14	9	47
m14	Omitted digit in multiplicand	2	17	12	12	43
m15	Errors due to zero in multiplicand....	4	14	15	9	42
m16	Counted to get multiplication combinations	15	11	9	5	40
m17	Error in position of partial products..	0	15	15	9	39
m18	Error in single zero combinations, zero as multiplicand	7	13	11	8	39
m19	Confused products when multiplier had two or more digits	1	13	9	9	32
m20	Repeated part of table	3	11	11	6	31
m21	Multiplied by adding	6	11	8	4	29
m22	Did not multiply a digit in multiplicand	5	9	7	7	28
m23	Derived unknown combination from another	3	11	6	6	26
m24	Errors in reading	6	5	11	3	25
m25	Omitted digit in product	0	5	7	5	17
m26	Errors in writing product	2	4	8	2	16
m27	Error in carrying into zero	1	6	7	1	15
m28	Illegible figures	0	3	5	7	15
m29	Forgot to add partial products	0	3	7	2	12
m30	Split multiplier	0	1	6	4	11
m31	Wrote wrong digit of product	0	3	4	2	9
m32	Multiplied by same digit twice	1	1	3	2	7
m33	Reversed digits in product	1	1	2	2	6
m34	Wrote tables	0	0	4	1	5
m35	Used multiplicand or multiplier as product	1	1	1	1	4
m36	Multiplied carried number	2	1	0	1	4
m37	Used digit in product twice	0	1	2	0	3
m38	Added carried number twice	0	1	1	0	2
m39	Carried when there was nothing to carry	0	0	1	0	1
m40	Began at left side	1	0	0	0	1
m41	Multiplied partial products	0	1	0	0	1
	Total Number of Subjects	47	98	102	82	329

TABLE IV

FREQUENCY OF HABITS IN DIVISION (ALL CASES)

	HABIT	GRADE III	GRADE IV	GRADE V	GRADE VI	TOTAL
d1	Errors in division combinations	35	55	59	42	191
d2	Errors in subtraction	4	25	47	37	113
d3	Errors in multiplication	1	20	48	36	105
d4	Used remainder larger than divisor	1	17	39	29	86
d5	Found quotient by trial multiplication	1	8	49	24	82
d6	Neglected to use remainder within example	5	27	25	13	70
d7	Omitted zero resulting from another digit	0	20	22	24	66
d8	Used wrong operation	17	17	24	6	64
d9	Omitted digit in dividend	4	15	27	18	64
d10	Counted to get quotient	5	25	24	4	58
d11	Repeated part of multiplication table	4	15	27	9	55
d12	Used short-division form for long division	0	16	24	10	50
d13	Wrote remainders within example	8	11	17	13	49
d14	Omitted final remainder	4	16	18	11	49
d15	Omitted zero resulting from zero in dividend	3	12	19	12	46
d16	Used long division form for short division	0	4	27	13	44
d17	Counted in subtracting	3	15	18	6	42
d18	Used too large a product	0	7	21	12	40
d19	Said example backward	9	11	8	7	35
d20	Used remainder without new dividend figure	0	6	14	9	29
d21	Derived unknown combination from known one	1	6	11	8	26
d22	Grouped too many digits in dividend	1	4	12	5	22
d23	Had right answer but used wrong one	0	7	10	4	21
d24	Error in reading	3	2	10	2	17
d25	Used dividend or divisor as quotient	2	4	6	4	16
d26	Reversed dividend and divisor	8	3	2	2	15
d27	Found quotient by adding	1	3	6	4	14
d28	Used digits of divisor separately	0	1	8	1	10
d29	Wrote all remainders at end of example	1	0	6	2	9
d30	Misinterpreted table	0	2	5	2	9
d31	Used digit in dividend twice	0	2	5	2	9
d32	Used second digit of divisor to find quotient	0	1	5	1	7
d33	Began dividing at units' digit of dividend	1	1	4	1	7
d34	Split dividend	0	0	5	1	6
d35	Used endings to find quotient (long division)	0	2	3	1	6
d36	Added remainder to quotient	0	2	1	0	3
d37	Added zeros to dividend when quotient was not a whole number	0	0	1	2	3
d38	Added remainder to next digit of dividend	0	0	0	1	1
d39	Wrote rows of zeros	0	0	1	0	1
d40	Illegible figures	0	0	0	1	1
d41	Dropped zero from divisor and not from dividend	0	0	0	1	1
	Total Number of Subjects	44	77	103	76	300

fully the work of the pupils and by questions and suggestions ascertained the character of their mental processes. After the work of the pupils had been observed the various habits exhibited were tabulated. The pupils studied were selected from the third, fourth, fifth, and sixth grades. The results of this diagnostic survey are exhibited in Tables I, II, III, and IV.[1] In these tables the frequency of the habits is listed separately for each grade, and the total frequency for all grades is also given. The habits are listed according to the frequency for all grades.

Lack of Uniformity. The fact that pupils display such a variety of habits of work indicates that their methods of instruction have been far from uniform. Many of the habits listed are wasteful and time-consuming, and, in many cases, they lead directly to errors. Some of the methods of work were devised by the pupils themselves, others were learned at home or from their friends, still others were taught in the school. These many ways of adding, subtracting, multiplying, and dividing are, in a large measure, directly due to a lack of detailed and specific instruction in the textbook regarding procedure. One of the most common methods of adding and multiplying is to count on the fingers rather than to perform the proper number operations. In the textbooks can be found little instruction which cautions children against this practice and explains that it is a wasteful form of procedure. Children add the carried number last because they have never been told to add it first, and in attempting to add it last they frequently forget it before the end of the column of digits. Children skip about when adding a column of numbers rather than proceed regularly up or down the column because they have not been instructed that regular procedure is the better practice. The process of carrying produces no end of difficulty, and yet the explanation as to why carrying is necessary and as to just what it means is ordinarily not given. Children get the quotient in division by a pure process of trial multiplication because the book has not pointed out a more logical method of getting the trial quotient.

A Book as a Teaching Device. If a book is to be a teaching device rather than a simple manual of exercises, adequate attention must be given to the methods which pupils employ in their work.

[1] G. T. Buswell, with the co-operation of Lenore John: *Diagnostic Studies in Arithmetic*. Supplementary Educational Monographs, No. 30. Department of Education, University of Chicago, 1926. Pp. xiv + 212.

This means that textbooks must be written for children and must be written in such a way that the necessary forms of procedure can be understood after reading the explanation. Furthermore, the processes need to be dealt with step by step so that the pupil may secure a logical set of instructions from the beginning stages of any of the fundamental operations up to the final stages. This is not an impossible undertaking. There is a considerable body of facts now available upon which such a book could be built. The teaching contribution of such a book would be very much greater than that of the traditional book which devotes the major part of its space to practice exercises. Obviously, practice exercises are necessary, but they might be supplied to a greater extent in supplementary manuals, thus leaving more space in the textbooks for the fundamental instruction and the basic information relating to the number system in the various fields in which it is to be applied.

Problem-Solving. The mental processes employed in problem-solving have not been subjected to the same degree of analysis as the processes employed in working with the four fundamental operations. Ultimately, such detailed diagnoses will be made. It has been assumed quite generally that children solve problems by the logical methods which are supposed to be characteristic of adult thinking. This logical procedure is described ordinarily in such steps as the following: (*a*) defining the problem; (*b*) recalling related facts which bear upon it; (*c*) setting up and evaluating hypotheses as to its solution; (*d*) selection of one hypothesis; and (*e*) the final verification of the hypothesis and the solution of the problem. If one will observe carefully the mental processes employed by children in the actual solution of problems, he will find that there are many deviations from this so-called logical method. Children make frequent short cuts in their thinking. They frequently act upon the first hypothesis which comes to their minds rather than make a careful evaluation of several hypotheses. More commonly still, they take their cue from certain words in the problem or from certain forms of expression and proceed with very little logical thinking at all. While the logical steps by which the process of reasoning has been described may represent the way one should think, they certainly do not represent the way in which many children do think. An investigation which is badly needed is a detailed individual diagnosis of the actual thinking carried on by children in solving the ordinary problems presented in arithmetic. Until such a diagnostic survey of children's reasoning is available, it will be difficult to supply suitable instructional material for problem-solving.

An Analysis of Written Work. At present the most significant facts available showing the way children solve problems have been obtained by an analysis of the written work of children. Any analysis of written work has an outstanding handicap, namely, the impossibility of probing back into the actual mental processes carried on by the pupil. One may infer some of these mental processes from the types of errors which are exhibited in the written work, but the method produces meager information at the best. A good illustration of the facts obtainable from such a method may be taken from a study reported by Osburn in which he made an analysis of difficulties in reasoning based on the Buckingham Scale for Problems in Arithmetic.[1] Thirty thousand errors made by six thousand children in eighteen counties and one large city were analyzed. The classification of the errors discovered is shown in the following table:

TABLE V

TYPES OF ERROR AND THEIR APPROXIMATE FREQUENCY

	Per Cent
1. Total failure to comprehend the problem	30
2. Procedure partly correct but with the omission of one or two essential elements	20
3. Ignorance of fundamental quantitative relations	10
Total	60
4. Errors in fundamentals	20
5. Miscellaneous errors	2
6. Errors whose cause could not be discovered	18
Total	100

Osburn records as the greatest cause of error in problem-solving, "total failure to comprehend the problem." which he notes in 30 per cent of the cases. This is a significant finding for curriculum makers in arithmetic. When children of normal intelligence show

[1] W. J. Osburn: "Diagnostic and Remedial Treatment for Errors in Arithmetical Reasoning." State Department of Public Instruction, Madison, Wisconsin, 1922 (mimeographed). Pp. 12.

a total failure to comprehend a problem, one suspects that the problem is not well stated, that sufficient information as to its character is not provided, or that it is too difficult for children of the degree of maturity to whom it is presented. Certainly the school cannot expect proper reasoning responses when children are presented with situations which they totally fail to comprehend. Furthermore, the burden for failure to comprehend must be placed upon the problems rather than upon the children. A very productive undertaking for the curriculum maker would be an investigation of the actual mental reactions which children make to a complete set of problems used during a particular unit of the course in arithmetic. This type of investigation requires detailed and tedious procedure. The expenditure of a great deal of time and the coöperation of many individuals would be required to secure a sufficient body of information to give adequate direction to authors of textbooks.

The Remedy. Textbooks should tell children how to solve problems, not by giving them a formal statement summarizing the thinking experiences of adults but by pointing out to them the many ways in which problems may be solved by children, and critically evaluating these various ways. If children can be shown that certain methods of reasoning are open to erroneous results and if they can be shown why, there is reason to believe that they will tend to avoid these ways and search for more productive methods of reasoning. At the present time, children are left very much to their own devices in the solution of problems.

The Organization of Textbook Materials. The purpose of this discussion is to present curriculum problems. Problems are solved through experiments. The writer would like to propose an experiment relating to the preceding discussion of the teaching contribution of textbooks. Although he has no objective evidence to show that the results of the experiment to be proposed would be superior to the results obtained at the present time, he believes that a serious trial of the experiment would be justifiable.

The subject of arithmetic includes two specific bodies of materials. The first of these is the number system itself, involving the four fundamental operations, decimals, and fractions. The second is the application of number facts to the quantitative relationships involved in socially significant activities. The first has to do with what may be considered a closed number system complete in itself.

The second has such wide ramifications that the limits cannot be defined. The mastery of the first body of material is necessary for the applications involved in the second body of material. However, the successful application of number to the quantitative relationships involved in socially significant activities depends upon reasoning of a general character as much as upon familiarity with purely numerical relationships. The reasoning involved in arithmetic is very similar to the reasoning involved in any other situation. Training in this type of reasoning cannot be thought of as the sole obligation of arithmetic. Other subjects also teach children to reason.

The satisfactory attainment of the objectives of the teaching of arithmetic requires, first, an understanding of the numerical relationships to be applied and the situations in which they are to be applied, and, second, a sufficient amount of practice and drill to enable the pupil to make the applications quickly and accurately. The first of these is an explanatory process contributed to by both the textbook and the teacher. The second is a drill process in which the requirements are suitab'e practice exercises and a sufficient amount of work by the pupil.

Practice Materials. In the preparation of practice materials for the fundamental operations much progress has been made in the last few years. In the place of haphazard examples collected without any regard for the amount of practice given on the particular combinations, there are now available systematic and scientifically prepared practice exercises which cover the entire field of fundamental operations and which provide a known amount of practice for each combination. These scientifically prepared practice exercises have been shown to be vastly superior to the materials found in the ordinary textbook. Furthermore, they have been published in a form which is more conducive to economical practice than is the form of the examples found in textbooks. Practice exercises for reasoning problems have not been developed to the same extent as for the fundamental operations, but there is no reason to suppose that development will not follow in the course of a few years. The result of the special attention given to practice exercises has been the production of a body of material greatly superior to the material produced in the textbooks. This leaves the traditional textbook in a peculiar position. Its examples and problems are not as good as those provided independently. In order to provide space for the examples and problems which the textbook does give, it has been necessary to limit the space devoted to explanatory teaching, with the result that the actual instruction relating to procedure and to the processes to be covered has been left to the teacher. It is difficult to defend the traditional textbook on the basis of its teaching contribution and its explanatory material, since it contains little of these. Likewise, it is becoming difficult to defend the textbook on the basis of its examples and problems, since better practice materials have been developed separately.

A Desirable Experiment. In view of the facts just discussed, the experiment which the writer wishes to propose is the following. Let some qualified person prepare a textbook which is composed essentially of explanatory material, and then let him select or construct a manual of practice exercises to accompany the textbook. Let the space which has been previously used for examples and problems be given to detailed instructions to the pupils relating to their methods of procedure, to some genuine explanation of the various arithmetical operations, and to an ample presentation of the social situations in which arithmetic is to be applied. In the greater amount of space available in such a textbook, an ample presentation of the subject might be given. Certainly the teaching based upon such a textbook would be radically different from that which is now possible. Purely as an experiment, the writer would consider this proposal worthy of serious work on the part of some competent curriculum maker in the field of arithmetic. A logical method of procedure would be to develop certain units individually and try these before proceeding with an entire series of texts. For example, units on subtraction, fractions, percentage, and insurance should test the idea.

III. The Number System as Such

Neglect of the Number System. If one examines a textbook in arithmetic or visits classes in which this subject is taught he is impressed by the fact that, in most cases, a great amount of effort is expended in finding situations in which number may be applied. Many excellent and ingenious devices are used in order to stimulate the pupil's interest and to motivate his work. However, one notes that practically all of these devices are selected from outside of the number system itself. For example, it is commonly assumed that children will be interested in playing store, in making change, or in ascertaining the number of articles in different parts of the schoolroom. Ordinarily children do have such interests, although frequently in much less degree than the teacher supposes. Just why all children should be interested in grocery stores has never been pointed out. Certainly such an interest is not native to the child, and one may harbor the suspicion that such an interest is the result of the group coöperation in the activity rather than interest in the grocery store. However, the point which the writer wishes to note here is the almost complete neglect of the number system as a source of motivation. This neglect is particularly striking in view of the fact that during the period just prior to school entrance and during the first-grade period numbers *per se* are among the child's first interests.

While the development of the number system is one of the most difficult intellectual achievements of the human race, nevertheless some understanding of the nature of the system, once it is achieved, is not beyond the comprehension of third- and fourth-grade children. The point to be emphasized here is the fact that there is a number *system*, and that an understanding of the nature of this *system* will materially assist the child in applying it. When the child gets some notion of the recurrence of digits according to the decimal system, the number combinations begin to take on some logical character rather than being haphazard facts which are true merely because the book says they are true. The difficulty of some understanding of the number system has been overestimated. Children master other systems and are greatly aided thereby. For example, in any large city the systematic arrangement of house and street numbers makes possible the quick location of any particular residence which one may wish to find. To one who does not understand the nature of the system, house and street numbers appear to be assigned in a purely chaotic fashion. When once the system is understood, the whole matter of location assumes a logical aspect. After a comparatively small amount of experience in locating particular places, children master this system of numbering and are able to find their way about without difficulty. Once they master the system they are able to find new locations without help.

The teaching of the system of arabic numerals in arithmetic is not greatly different from the teaching of the street and house numbers in the city. When the child has mastered the basic combinations in the four fundamental processes and has been made to understand the nature of the system itself, he is able to make certain transfers and to "figure out" new combinations. The specific question which is involved in the construction of arithmetic materials is in regard to how large the sample of number facts to be taught must be before the child will be able to comprehend the nature of the system and discover new relationships in the system for himself. This presents an interesting experimental problem for the curriculum maker, namely, how far can the child be made to comprehend the systematic relationships of numbers and thereby avoid the notion that number relationships are purely chaotic and independent?

IV. Drill Exercises Keyed to Specific Needs

Individual Differences. Diagnoses of pupils' work in arithmetic have shown that the needs of instruction vary greatly from individual to individual. Even in such a simple matter as addition or subtraction, the particular habits which need correction show a surprising amount of variability. It is now possible for a teacher to make a fairly rapid diagnostic survey of a pupil's habits of work and to check the items where instruction is most needed. The curriculum maker would do the teacher a real service if he would provide for her a systematic set of drill exercises keyed to the needs disclosed in the diagnosis. For example, if the pupil has difficulty with particular number combinations, exercises affording special practice on these combinations are needed. Or, if pupils have some particular difficulty in carrying, exercises might be constructed which would give practice of this particular kind more than of other kinds. Some materials of this sort have already been constructed. The pupils' materials constructed by Superintendent Washburne at Winnetka should be mentioned in this respect. The need now is for the construction of a more elaborate and complete set of keyed drill exercises. These must be prepared in the light of facts revealed through careful diagnoses of pupils' work. When drill materials to accompany the various habits of work as revealed through diagnosis of pupils' mental processes are available, the teacher will be able to give a much more specific type of instruction than is now possible.

V. Grade Location of Arithmetic Processes

Effect of Curriculum Revision. While there has been a fair amount of agreement as to the location of the various arithmetical operations in the grades beginning with the third, this agreement is very likely to be disturbed by the modifications in the curriculum brought about through the junior high school movement. Interest in this problem is already apparent in the numerous attempts at curriculum revision which are taking place in many of our school systems. However, the problem is more acute at the primary-grade level, where, at the present time, there seems to be little but chaos in the teaching of arithmetic. Although the formal text-books in arithmetic ordinarily begin at the third grade, there is ample evidence to indicate that some instruction should be given much earlier. The six-year-old child has certain very definite number concepts and is able to perform certain simple arithmetic operations. It is highly important that the development of the child's arithmetical ability be not left to chance during the first two years of school. This does not necessarily imply that the text-books should be provided at the first grade level. It does mean, however, that definite curriculum materials should be supplied which are adjusted to the child's mental maturity rather than to his chronological age. The primary need is a definite survey of practices in the first and second grades. This survey should be followed by experimental evaluations of the various practices found and also by the trial of new practices which are not used at the present time. The problem of grade location of arithmetic material is particularly difficult, due to the fact that the answer cannot be given simply in terms of arithmetic itself. Correlations must be noted between arithmetic and other subjects, and these correlations are continually being disturbed by changes in the other subjects. Consequently, the curriculum maker cannot expect an immediate or final solution to the problem of grade placement. Furthermore, this problem is one which must be worked out in the light of pupil experiences rather than on the basis of logical organization of arithmetic itself.

Summary

A satisfactory solution of the five major problems discussed in this paper will depend entirely on the possibility of securing objective evidence bearing upon them. Curricula can no longer be built on the pronouncements of authorities. The major need is for experimentally determined facts. Only on the basis of such experimental evidence can the curriculum maker justify his recommendations. Scientific curriculum making cannot be carried out in a day. Answers to the five problems discussed in this paper will depend on how quickly and how thoroughly experimental investigations are carried on.

PROBLEMS OF A SUPERVISOR OF ARITHMETIC IN THE ELEMENTARY SCHOOLS

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Introduction. During the past few years much has been done to improve the teaching of arithmetic. Many of our recent textbooks have been prepared most scientifically, and this has relieved the teacher of much responsibility in the selection and organization of drill materials. In the recent studies of the psychology of arithmetic, principles have been formulated the application of which makes the teaching of arithmetic much easier and more certain in its results.

In spite of the valuable material that is available in the teaching of this subject, we frequently find ourselves confronted with difficulties that are not easy to overcome. I have selected a few of the things that seem to cause trouble, and will try to offer some suggestions that I have found helpful in my supervision of arithmetic in the elementary grades.

The topics which I will attempt to discuss, the lack of attention to which makes the teaching of arithmetic more difficult or more inefficient, are the following:

1. Failure to diagnose difficulties before attempting to apply the remedy.
2. Necessity of carefully grading the steps in teaching a new process.
3. Value of checking.
4. Difficulties presented by problems.
5. Use and misuse of concrete objects in the teaching of arithmetic.
6. When rationalization is an aid, and when a hindrance.
7. Failure to provide for the fixing of facts and processes.
8. Loss of time in teaching arithmetic resulting from the failure, on the part of the pupil, to feel the need of it.
9. Difficulties resulting from individual differences.
10. Failures on the part of the teacher to check the results of her teaching.

11. Content and arrangement of textbooks.
12. Arithmetical opportunities in pupil activities.

I. Failure to Diagnose Difficulties

Pupils' Errors in Addition. We often find in the work of a pupil that the answer to an example is wrong and frequently we require the child to re-work the example, thus giving him opportunity for practicing an error, before we make any attempt to diagnose the situation and find out what really caused the error in the answer. I have collected a number of errors found on pupils' papers and have tried to locate the possible causes of these errors. For instance, the problem of adding 29, 16, 8, and 36 by column addition may be solved incorrectly for the following reasons:

1. The pupil does not know the single combinations, such as adding 9 and 6, 2 and 1, or 3 and 3, as found in the example. If we have the pupil work this aloud, we can quickly find which combinations he does not know. These must be taken out of column addition and drilled upon as separate combinations until the pupil is sure of the correct response. They may then be put back into example form to determine whether the pupil can respond correctly to them in this slightly changed situation.

2. He may be unable to hold 15 in mind while adding 8.

3. He may be unable to add numbers of a higher order, like 15 and 8, and 23 and 6. If this is the case, rhythmic addition as in adding 5 and 8, 15 and 8, 25 and 8 may help. In this instance the 5 and 8 or 3 and 6 combinations should be taught as the key combinations and the relation shown between these and the numbers of higher order.

4. He may put down the wrong figure or both figures on the right hand side. This may be due to too much drill on single column addition, the sum of which exceeds 9, before introducing column addition with carrying. The pupil has formed the habit of putting down two figures when he adds, and this habit has to be broken. He may have had too much drill in carrying 1 before carrying 2 was introduced. This can be avoided by giving examples with 1, 2, and 3 to carry very soon after the idea of carrying is introduced. This is not difficult if the pupil has been taught his doubles before attempting column addition involving carrying. Such examples as 19, 29, and 24 or 28, 18, 36, and 9 will provide for this. If the pupil

has been taught to add from the bottom up the position of the doubles must be reversed.

5. He may carry the wrong figure or none at all. The suggestions given above may also help to avoid this.

6. He may begin to add on the left side instead of the right. The idea of following traffic rules, "Keep to the right," may help. This rule may be written on the board and kept before the class until the habit of beginning to add on the right is thoroughly established. Objective proof showing how ridiculous the answer is may help to correct the error indicated in reasons 4, 5, and 6.

7. The example may be copied incorrectly. This may be due to poor eyesight or to carelessness. If the former, arrangement must be made to seat the pupil so that he can see, until something can be done about his eyes. He may be given mimeographed or printed copies of the examples and thus be relieved of the strain of copying them. If carelessness is the cause, constant attention must be given to developing pride in correct work, and forming the habit of checking all work before handing it in may help.

Each of the situations described above is a separate habit and drill on the total situation will do little good until each specific habit has been correctly built up.

Errors in Short Division. The errors in the answer in short division of the following type

$$3\overline{)679}^{\,226}, \text{ 1 remainder}$$

may be due to any one of several causes:

1. The examples may be copied incorrectly. The pupil must be shown what caused the error and its effect.

2. The pupil may not know the facts of the third division table. Further work on short division is wasted until these facts are mastered. Let the pupil see why the example was missed, and then proceed to drill on the facts of the table until they are thoroughly mastered. From time to time during the drill the numbers may be put back into example form to test the pupil's mastery of them.

3. The pupil may be inaccurate in his subtraction, and for this reason secure the wrong remainder.

4. He may not know what to do with the remainder. Work in uneven division of single combinations (such as is illustrated by

the following sample) before further examples in short division are attempted will help to correct this.

$$\begin{array}{r}2\\3\overline{)7}\end{array}\text{, 1 remainder}$$

5. If this division sign (÷) is used instead of $\overline{)}$ the pupil may not know what to do. Each sign of division must be taught as a separate habit, just as each division fact is taught.[1]

Other Frequent Errors. Children frequently write their "teen" numbers backward. This may be due to lack of understanding of the meaning of number or to the habit of writing numbers as they hear them. In 16 they hear the 6 first and consequently are apt to write 61. This may be avoided by teaching all the "teen" numbers as separate bonds, emphasizing how these numbers are different from all other numbers.

We sometimes see 10016 for 116. Having the pupil write 100, then 16 under it, and adding will frequently help to correct this.

Little children very often make their numbers backward, writing Γ for 7, ∂ for 6, S for 2. This is due to an incorrect mental picture of the number. Make the number slowly and carefully for the pupil, call attention to how it looks, where to begin when writing, how it curves or bends; then erase it and have him try. This process has to be repeated again and again before satisfactory results are obtained.

There are possibly no combinations missed more frequently than the multiplication combinations with zero, as in $7 \times 0 = 7$. This may be due to the difficulty in understanding the meaning of zero, the unreality of the situation, and to the formation of the bond that 'multiply' means get a larger number. This difficulty may be avoided by not teaching the zero combinations until a need for them arises in long multiplication, by forming separate bonds for all zero combinations after all the other multiplication bonds have been formed. (Whenever a number is multiplied by zero, the result is always zero. Whenever zero is multiplied by any other number the result is always zero.)

[1] It should be understood that the placing of the quotient above the dividend in short division is entirely opposed to business usage. It has to be unlearned when the practical applications of the operation are made to the finding of discounts, averages, or products of a whole number multiplied by a fraction. Of course the argument advanced by certain psychologists is well known. *Editor.*

Necessity of Diagnosing Errors. The following errors recently collected from pupils' papers still further illustrate the necessity of diagnosing errors before attempting to apply the remedy. For example,

$$\frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}.$$

This pupil had not had enough experience with studying the correct form of division of fractions and proving that

$$\frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times \frac{3}{1} = \frac{9}{4} = 2\frac{1}{4}$$

is right because

$$\frac{\overset{3}{\cancel{9}}}{4} \times \frac{1}{\cancel{3}} = \frac{3}{4}.$$

The zeros give trouble wherever we find them. For instance, in the following examples such troubles occur.

$3\overline{)306}$ = 1 2	$3\overline{)360}$ = 12	$5\overline{)10}$ = 2	$32\overline{)25760}$ = 85
3246 × 302: 6492, 9738, 103872	2003 × 5 = 1015	1000 − 978 = 1082	240 − 136 = 116

Each zero difficulty is a separate bond and must be formed as such. We must form each of these habits well before taking up the next step, if we wish them to function correctly. "Form habits; do not expect them to come as miracles."

Much drill in column addition has possibly fixed the idea that all numbers on the right hand side must be placed directly under each other (as in the addition of the partial products in the first example at the right. This may also be the cause of the error in the following example: Add 4.6, 2.43, .1, and .246, which was worked as shown by the second example on the right.

142	4.6
× 35	2.4 3
710	.1
426	.24 6
1136	.53 6

Such errors as this may also be due to a failure to understand the meaning of decimals.

A child knows how to subtract in the first example below but fails to get the correct result in the other two.

$$\begin{array}{r} 8 \\ \underline{7} \\ 15 \end{array} \qquad \begin{array}{r} 15 \\ \underline{-8} \end{array} \qquad \begin{array}{r} 15 \\ \underline{-7} \end{array}$$

While the subtraction facts are based on the addition facts, they are quite different in a child's mind and must be taught as separate combinations. However, the relation between subtraction and addition should always be shown.

In the example at the right the pupil has added his numerators only and has written them over the larger denominator. Objective work that will make these fractions more real will help in this difficulty.

$$\begin{array}{r} 4\frac{1}{4} \\ \underline{2\frac{1}{2}} \\ 6\frac{2}{4} \end{array}$$

This error is possibly carried over from subtraction of whole numbers where it was necessary to "borrow" one 10 in order to make the numerator in the units column large enough to subtract the figure in the subtrahend from it. Again, work with objects will help to show that $\frac{4}{4} = 1$, $\frac{2}{2} = 1$, etc.

$$\begin{array}{r} 6\frac{1}{4} = 5\frac{11}{4} \\ \underline{1\frac{3}{4} = 1\frac{3}{4}} \\ 4\frac{8}{4} = 6 \end{array}$$

Sometimes we find curious things like the example at the right where it is almost impossible to tell why the child works in this way. Work with concrete material will help to prevent unusual errors.

$$\begin{array}{r} 16 \\ \underline{+7} \\ 14 \end{array}$$

II. Necessity of Carefully Grading the Steps in Teaching a New Process

Failures and Remedies. Teachers frequently look upon addition, multiplication, and division as single processes and fail to realize that an apparently slight change in the situation may cause the pupils much trouble. Because they look upon these processes as simple rather than very complex situations, they often fail to give careful attention to grading the steps in teaching a new process. If care is taken to develop only one step at a time and to present all steps in the easiest sequence, giving a sufficient amount of drill to each step, many of the errors which pupils make will be avoided. They will learn more quickly and with greater ease, and a higher degree of accuracy will characterize the work. No child should be called upon to solve a problem involving several difficulties until he has mastered exercises involving each difficulty separately. The following gradation in division may serve to illustrate what is meant by teaching one step at a time and in such sequence that the learning of one step will facilitate the learning of the next:

1. Develop the division idea through actually dividing materials among the children. At the same time the idea is being developed, the words *divide, dividing,* and *divided* should be added to the pupil's vocabulary.

2. Develop division facts one at a time, using objective material as long as it is necessary. Select the first facts from the 5th, 10th, or 2nd tables. This will depend upon which multiplication facts the pupils have learned prior to the introduction of the division idea. I have found it better to teach eight or ten multiplication facts, so that the multiplication idea may be thoroughly fixed before introducing the division idea. This method of procedure is based on the principle that a sufficient amount of practice should be given in the formation of a new bond to establish it firmly before the formation of a different bond is attempted. After both the multiplication and division ideas have been fixed, it is better to carry on the two processes together, as the learning of one facilitates the learning of the other.

3. Postpone zero combinations until the necessity for them arises in problems or examples.

4. Develop the meaning of the division symbols $\overline{)\ }$ and $\div$ in connection with the combinations worked out objectively. Divide ten sheets of paper among five pupils and illustrate the operation as follows:

$$5\overline{)10}^{\,2}, \qquad 10 \div 5 = 2.$$

5. Show the relation of division to multiplication by using multiplication as a proof of division:

$$\begin{array}{r}2\\5\overline{)10}\end{array} \text{ because } \begin{array}{r}2\\\times 5\\\hline 10\end{array}$$

6. Put individual facts into use as soon as possible after they are learned; for example, after the 2nd division table has been developed pupils enjoy working such examples as: $2\overline{)42}$ and $2\overline{)64}$. In this way they are getting practice on the division facts in dif-

ferent situations, and at the same time the foundation for short division as such is being laid.

7. Teach uneven division as follows: 5)6, 2)25.

8. Put uneven division into use through carrying in simple short division as follows: 5)65. Of course this step should be developed with the pupils before independent drill work is given.

9. Increase the size of the dividend, using carrying in one instance only as follows: 5)755.

10. Teach carrying and remainder in the same example as follows: 5)76.

11. Then teach carrying in two places, with a remainder: 2)315.

12. Give cases where the divisor is not contained in the first figure of the dividend as follows: 3)114.

13. As new division facts are developed, the difficulty of the divisors may be increased after the process is thoroughly understood.

14. Introduce zeros in the dividend in examples of the following type: 4)604, 5)560.

15. Introduce zeros in the quotient in examples of the following type: 2)840, 4)436.

16. Give examples including several of the difficulties discussed above: 6)1243.

17. As an introduction to long division, have an example in short division worked. After this has been worked, have the pupils analyze the process; and as the steps necessary in working the example are consciously brought to mind, write them on the board; *divide, multiply, subtract, carry*. Then work a problem in long division, having the pupils see that the process is exactly the same, the only difference being in the form of the work—in long division the numbers have to be written because they are too difficult to be carried in the mind. Instead of the last step, *carry*, we may substitute *bring down* the next figure.

18. Select as the first problems in long division examples with divisors of 21 or 31, then 22, 32, and 42 with three figures in the dividend; no remainder as follows: 21)441.

19. Using similar divisors, increase the size of the dividend in order to show the necessity of repeating the steps—*divide, multiply, subtract, examine remainder, bring down*—until all figures in the dividend have been used in order to secure the correct answer; such as the following: 31)6541.

20. Very soon after introducing long division give examples with remainders in order that the pupils may not form the habit of expecting every example to come out even in order to be right; such as the following: 21)205.

21. Give examples where the first three figures of the dividend have to be taken in order for the divisor to be contained in the dividend; such as the following: 21)1491.

22. Give examples where it is more difficult to tell how many times the divisor is contained in the first two figures of the dividend. Use 25 as the divisor in the first few examples as children are better acquainted with products of 25 and can more readily estimate the correct trial figure in the quotient.

23. Give similar examples, increasing the difficulty of the divisors and dividends.

24. Give examples calling for zero in the quotient; such as the following: 35)7185.

25. Constantly increase the size of the divisors and dividends providing for the recurrence of all the preceding difficulties.

III. Value of Checking

Inaccurate Work. A business firm will not accept inaccurate work from its employees. No employee who wishes to keep his position will submit to the head of his department figures that have not been checked. We often fail to realize that teaching pupils to check their work is a very vital part of the successful teaching of arithmetic, and as a result inaccuracies creep in until pupils form the habit of being satisfied with inaccurate work. Knowing how to check and forming the habit of doing it is invaluable to pupils. It inculcates a critical attitude toward one's work; it teaches the art of self-correction; it diminishes the number of absurd answers; it exalts accuracy in arithmetical computation; it doubles the amount of practice on the process or its reverse, if this system of checking is used; it develops confidence in one's ability to produce correct results, and creates a feeling of responsibility, thus providing for a greater amount of satisfaction which will spur the pupil to further work.

When Checking Should be Taught. Checking should be taught when each process is taught and not relegated to the sixth or seventh grades. The methods of checking will vary with the process being taught and the ability of the pupils. Following are a few suggestions for teaching pupils how to check:

1. One of the most helpful methods of checking, and one that can be used very easily in the arithmetical experiences of the pupil, is that of estimation. Have the pupil read the problem or example, estimate the answer, and give reasons for his estimate. Then he may work the example to see how nearly right he was, the teacher may give the correct answer, or he may look up the correct answer in the key.

2. Checking by the reverse process may be used as soon as the pupil is familiar with the two processes involved. This provides for drill in a real situation. The pupil really works two examples every time he works and checks one.

3. Pupils can be taught to check addition by adding the columns in reverse order as soon as all the combinations found in the examples have been taught. Checking in this way gives drill on an entirely different set of combinations. For example, in adding down the first column below the pupil needs to know the second and third combinations below and also the fourth. In adding up the column he receives drill on the fifth, sixth, and seventh.

$$\begin{array}{r} 3 \\ 4 \\ 2 \\ \underline{6} \end{array} \qquad \begin{array}{r} 3 \\ \underline{4} \end{array} \qquad \begin{array}{r} 7 \\ \underline{2} \end{array} \qquad \begin{array}{r} 9 \\ \underline{6} \end{array} \qquad \begin{array}{r} 6 \\ \underline{2} \end{array} \qquad \begin{array}{r} 8 \\ \underline{4} \end{array} \qquad \begin{array}{r} 12 \\ \underline{3} \end{array}$$

4. They may be taught to solve problems or examples by another method; for example,

$$8 \times 3\tfrac{1}{2} \qquad \begin{array}{r} 8 \\ \underline{\times 3\tfrac{1}{2}} \\ 24 \\ \underline{4} \\ 28 \end{array} \qquad \frac{\overset{4}{\cancel{8}}}{} \times \frac{7}{\cancel{2}} = 28 \qquad \begin{array}{r} 8 \\ \underline{3.5} \\ 40 \\ \underline{24} \\ 28.0 \end{array}$$

$$\begin{array}{r} 4362 \\ 9248 \\ 1439 \\ \underline{2713} \\ 17762 \end{array} \qquad \begin{array}{r} 22 \\ 14 \\ 16 \\ \underline{16} \\ 17762 \end{array}$$

5. They may interchange the multiplier and the multiplicand and see if they obtain the same product.

6. In the sixth and seventh grades a few of the short methods of checking may be given.

a. **Proof figure:**

The proof figure of any number is found by adding the digits continuously until the final sum is one figure.

Illustration of the use of proof figure in

1) Addition:

475	16	7	
822	12	3	
17		8	
1314	(9)	18	(9)

2) Subtraction:

942	15	(6)	
187	16	7	
755	17	+8	} = (6)

3) Multiplication:

449	17	8	
86	14	×5	
2694		40	(4)
3592			
38614	22	(4)	

4) Division:

6 385	16	7	6 × 7 = 42 (6)
24)9240	15	(6)	
72			
204			
192			
120			
120			

b. Casting out nines is substantially the same as the method of the proof figure.

IV. Difficulties Presented by Problems

Problems a Constant Source of Trouble. In addition to the difficulties that arise in connection with teaching new combinations and processes, we find that problems are a constant source of trouble for both teacher and pupil. In spite of their difficulty, it is necessary to give the pupils practice in problem solving; for it is through this type of work that the pupils have a chance to put into use the skills which they have mastered, to become acquainted with the common number situations of daily life, to practice right thinking in mathematical situations, and to become familiar with arithmetical language.

Some of the difficulties in problem work arise mainly because:

1. The language used in the problems is too difficult. It is beyond the reading standard for the grades in which it occurs.

2. The pupils have not had sufficient training in interpreting thought from silent reading.

3. The pupils lack understanding of the technical terms involved.

4. The situations described by the problems are not understood by the pupils because they are outside the range of their experiences. If this is true, we should provide opportunities for the pupils to become familiar with the situations either by means of excursions or through the use of projects worked out in the classroom. If situations are selected which are likely to occur often in reality—even if they are not actually present to the senses at the time—and if they are dealt with as life processes demand, the problems will be much more real to the pupils, and they will manifest a greater interest in them.

5. The fundamental combinations, facts, or processes called for in the solutions of the problems have not been habituated. If this is the trouble, give drill on number combinations similar to those called for in the problem, until they are mastered; then put them into problems again.

6. The pupils may be unable to see the relation between the steps called for in the solution of the problem. If you can use the same idea with smaller figures, the pupils can frequently solve the problem; then you can explain that the same processes are used only with larger numbers.

7. The pupils are so burdened with undue labeling and elaborate indication of steps that their minds are diverted from the real process of solution.

Remedies. If teachers will help the pupils form the following habits of attacking problems, many of the difficulties in problem solution will be eliminated:

1. Read the problem thoughtfully in order to grasp the situation.
2. Separate the problem into smaller units, if there are several steps in the problem, and decide what is given and what is to be found.
3. If necessary, state the problem orally or ask for help if the situation is not clear.
4. Plan the solution mentally before beginning to work.

5. Estimate the answer.
6. Solve the easiest and quickest way.
7. Prove the answer.

The formation of these habits of attack will prevent much of the careless, thoughtless work which so often characterizes the problem solving of pupils.

We have insisted upon an elaborate, formal, oral and written analysis of problems, which has often degenerated into mere words. If we encourage pupils to state the analysis in their own words and to make it as simple and brief as possible, we will stimulate thought instead of interfering with it.

Just as much care should be taken in teaching the language of arithmetic as in teaching arithmetical facts. If this is done, the technical terms will not present difficulties when they appear in problems.

Since many of the errors in problem solving are caused by the inability of the pupils to decide how to work, much practice should be given in reading problems and telling how to work them without actually doing the work. Problems without figures will often be the best to use in this type of work.

In order to make problem solution more real it is sometimes necessary to have the pupils use diagrams, money, or real objects in working out the problems.

Many of our newer textbooks are rich in interesting problem material. It may sometimes be well for the teacher to change the figures in the problems given in the textbooks in order to bring the prices up to date or to make them fit a local situation.

No matter how good the problem material found in the texts may be, in order to give arithmetic local color and vital meaning, the teacher may supplement these problems by making problems based on:

1. Children's experiences outside of school, plays and games, and projects being worked out in class.

2. Adult activities in which children are interested; for example, cost of clothing, keeping family accounts, upkeep of an automobile, taxes on home and property, land developments in and around their homes.

3. Advertisements of foods, clothing, special sales; accounts of election returns; disasters from fires or storms; and baseball scores.

4. Holidays and holiday experiences.

Little children are especially interested in making up number stories about the people or animals in their favorite fairy or folk tales.

Pupils often enjoy making problems themselves. This gives variety to the work and often furnishes an insight into the interests of children, even though the problems may not always appeal to an adult.

V. USE AND MISUSE OF CONCRETE MATERIAL

Nature of the Material. It may be that the arithmetic situations which we present to our pupils are so simple for us that we fail to realize the difficulties which the children are encountering in understanding them. It is often necessary for pupils to work with concrete material; for it is only through actual contact with things, events, qualities, and relations that words, numbers, signs, and symbols acquire meaning. Concrete material is a means of thinking, computing, and verifying results. It provides for learning through several senses, and adds interest to the learning process.

The newness of the experience or subject matter rather than the age of the pupils determines the necessity for objective work. Objective material should be used only until the pupils have grasped the meaning of the facts or ideas we are trying to present. It is just as bad to keep pupils in the objective stage of development too long as it is to neglect it entirely. Pupils become dependent upon objects, and if this is the case, their use will interfere with thinking instead of clarifying it.

The amount of concrete material necessary will depend upon what is to be made intelligible, the previous knowledge and experience of the pupils, and their general ability. Very large numbers need more concrete aid than is commonly given. This is due to the fact that the pupils have had little experience with these, and it is hard for them to visualize mile, acre, ton, and large amounts of money, as $1,000,000. We have often neglected giving concrete experiences with these large units because of this difficulty. Numbers very much smaller than 1 need to be objectified for the same reason as stated above. Majority and plurality should be clarified by actual votes in the civic clubs or class organizations of any kind. Pupil participation in school banking helps to make clear deposit slips, checks, interest, bank balance, tellers, and cashiers. This, however, should not be overdone.

Objective material that is most usable in the classroom should be selected. It should be varied as much as possible. Unless it is varied, the pupil is apt to associate what is being learned with one object only and to be unable to apply the knowledge to other things or situations. Real objects are more interesting and possess the additional advantage that pupils are acquiring ideas and facts in connection with the kind of material with which they will be used. The objects should not be so attractive that the attention of the pupil is drawn from the process he is learning to the material. It is better to have all the material used in a given recitation centered around one idea or situation; for example, things found in a store at Easter, school supplies, fair products, school exhibits, Indian relics, industrial arts materials, and the like.

VI. Rationalization

When an Aid and When a Hindrance. Not only do we need objective material to make situations clear, but sometimes the use of objects must be accompanied or followed by a certain amount of explanation. We have more frequently made the error of rationalizing too much than too little. Elaborate explanations frequently get in the way of learning instead of helping it. For example, a pupil can learn to divide with fractions much more quickly by doing it and proving that the method used gives the right answer than he can if an explanation is attempted. As a rule it is better to reserve explanations as to why a process must be right until the child can use the process accurately and knows that it is right. If it is necessary to formulate a rule, it should be built up as far as possible by the pupils and stated in their words.

Two Lists of Principles. Professor Klapper gives the following list of principles governing rationalization:

1. No process should be rationalized unless it is important.
2. When an arithmetical situation or type occurs in variable form, it is necessary to rationalize.
3. When there is doubt as to the children's comprehension of underlying reasons, it is wise to omit all rationalization.
4. When repeated use and increasing maturity will reveal the reason for

a process that cannot be explained early in the school course without the expenditure of much time and effort, it is advisable to postpone rationalization.

5. The same standards of rationalization must not be applied to all children regardless of native ability.

Professor Suzzallo gives the following principles for the use of rationalization:

1. Any fact or process which always recurs in an identical manner and occurs with sufficient frequency to be remembered ought not to be rationalized for the pupil but habituated.
2. If a process does recur in the same manner, but is so little used in after life that any formal method of solution would be forgotten, then the teacher should rationalize it; square root, for example.
3. If the process always does occur in the same manner, but with the frequency of its recurrence in doubt, the teacher should both "habituate and rationalize."
4. When a process or relation is likely to be expressed in a variable form, then the child must be taught to think through the relations involved and should not be permitted to treat it mechanically.

VII. Fixing Facts and Processes

Importance of a High Degree of Efficiency. If arithmetic is to function in our lives, it is necessary to reach a higher degree of efficiency in dealing with the fundamental processes than has formerly been attained. "If clerks get only six answers out of ten right, one would need to have at least four clerks make each computation and would even then have to check many of their discrepancies by the work of still other clerks, if he wanted his accounts to show less than one error per hundred accounting units." Ninety-nine per cent accuracy is more nearly what we should require than seventy-five per cent.

Unless these arithmetical bonds are formed more accurately and firmly, the pupil's time will be wasted in excessive checking to find his error; he will be constantly practicing errors, thus fixing incorrect rather than correct bonds; his attention will be divided in learning new facts and processes by the necessity of thinking out facts which are supposed to have been mastered; it will be harder for him to form new bonds because the bonds which should facilitate the learning of new bonds are not strong enough to do so.

How Bonds Are Established. Only through repeated practice accompanied by interest and resulting in satisfaction can we hope to fix these fundamental bonds so thoroughly that possible error will be reduced to a minimum. In order to accomplish a higher degree of efficiency it is necessary to have clearly in mind just what habits we wish to form. Surely it is not too much to expect every normal pupil to form the following habits thoroughly before he leaves elementary school: a correct response to the one hundred addition combinations and their reverse subtraction facts; every item of the multiplication and division tables; facts of weights and measures in common use; decimals as used in money and in measuring distances especially; the common per cent equivalents that are frequently used, and the ability to put these facts to use in situations similar to those which he meets every day in life. In addition to the above every pupil should form the habit of checking his work, and of arranging his work systematically, legibly, and neatly.

VIII. Importance of Realizing a Need of a Process in Arithmetic

Realization of Need. Time is wasted in drill exercises unless we know what the pupils need to drill upon. This can easily be determined through the use of standard tests and educational inventories, and by keeping a record of the facts missed from day to day. It is not only necessary for the teacher to know the needs of each individual pupil, but the pupil must know what he needs and why he needs it. Realization of a vital need is the first step in successful habit formation. For example, a pupil wants to be the storekeeper. Other pupils report that he has given them the incorrect change. He sees that if he wishes to continue to be storekeeper he must learn to make the correct change. Or pupils may be engaged in making furniture. They find that the legs of the tables and chairs are uneven. They see that they must learn to measure more accurately if they are to make strong, usable furniture. Again they cannot keep score in their games because they are inaccurate in their addition, subtraction, multiplication. In comparing the scores of individual pupils with the standard scores for their grades, they see that they are below standard in certain types of work, and if they have any personal pride or group spirit they will work to bring their scores up to standard.

List of Needs. Each pupil should be encouraged to keep a personal list of the things in arithmetic upon which he needs to work. When he has mastered one group of these difficulties, let him report to the teacher. She may test him on this group of facts, and if his work is satisfactory he may be allowed to go on with other difficulties.

Cards containing the facts to be mastered in any one grade, arranged according to their difficulty, may be placed in the class library or on the work table. The pupils may study the facts on card number one, and when each one can recite these facts perfectly to the teacher his name may be placed on the back of the card and he may be allowed to progress to card number two. In this way the pupil is practicing with satisfaction and the teacher knows exactly which children know the various combinations. From time to time cards containing review combinations should be placed out for practice. Inventory tests should be used frequently to let the pupils see how they are progressing and what they are forgetting.

Group Drills. Group drills add variety to the work and can be used very effectively. These drills should be enjoyable, demand efficiency, and be democratic. It has been found that short periods with concentrated attention are better than long periods.

In many cases drill to the extent of overlearning is necessary. For example, in learning a new process the computation in general should be made very easy in order that the attention may be centered on learning the new step. This has been illustrated in the steps in learning division. In teaching carrying in multiplication we should begin with easy multipliers, such as $\begin{array}{r}56\\ \times 2\\ \hline\end{array}$ and $\begin{array}{r}225\\ \times 5\\ \hline\end{array}$

In addition of mixed numbers small whole numbers and easy fractions should be used; for example $\begin{array}{r}3\frac{1}{2}\\ 2\frac{1}{4}\\ \hline\end{array}$

Pupils may know $\begin{array}{r}3\\ \times 6\\ \hline\end{array}$ but fail to get $\begin{array}{r}934\\ \times 6\\ \hline\end{array}$ correct. Hence it is necessary to give drill on $\begin{array}{r}3\\ \times 6\\ \hline\end{array}$ in many different situations, such as $\begin{array}{r}433\\ \times 6\\ \hline\end{array}$ and $\begin{array}{r}342\\ \times 6\\ \hline\end{array}$

There are many facts or processes which are used so frequently in everyday life or in connection with learning other processes that they may profitably be brought to greater strength; for example, the easier addition combinations, the multiplication facts, especially of 5 and 10, ¼, ½, U. S. money, and interest computed at 3% or 6%.

Many of our newer textbooks have most scientifically arranged drill exercises, the use of which should help to improve both the accuracy and speed of our pupils.

Importance of Motive. The stronger the motive the pupils feel for attacking a new situation in arithmetic or for fixing something that has already been learned, the more efficient will be the work. When the pupils feel the need for arithmetical work, attention will be given more cheerfully, dynamic interest will be aroused, learning will go on more rapidly; for there is less danger of divided attention, and the results will be more satisfying because the pupils know what they have accomplished. Since this is true it behooves the teacher to look for pupil experiences the carrying on of which necessitates the learning of facts in arithmetic. He should, however, choose the most valuable ones.

1. The pupils may buy lunches from the cafeteria and decide whether their lunches are properly balanced by calculating the per cent of protein, carbohydrate, fat, and starch found in each article.

2. The pupils may assist in the school banking. Some may serve as receiving tellers, bookkeepers, and accountants, while all may have practice in making out deposit slips and checks. All may check their bank books and see if the interest has been calculated correctly, and the correct balance recorded.

3. The older pupils may weigh themselves, calculate the per cent underweight or overweight, graph the weights of the class, and compare them with the normal weights.

4. The pupils may plan school gardens, lay off the gardens, buy the seeds and plants, gather and sell vegetables, and keep an accurate account of expenses, sales, profits or losses.

5. They may plan school parties.

6. The class may organize itself into a miniature city and carry on the various types of business which the individuals and groups may select.

A short review can often be used to show pupils that they need to learn something new. Graphing the results of a standard test and putting this up before the class will often cause the pupils to feel a need of more accurate work.

Games. Games based on the instinctive tendencies of mastery, competition, curiosity, desire for approval, and joy in mental activity may be used very successfully in motivating arithmetic. If games are used, all the children should have a chance to take part; they should not lead to noise and confusion; they should create a keen desire to learn; and group rather than individual rivalry should be encouraged.

IX. Difficulties Due to Individual Differences

Individual Differences in Ability. Pupils may vary greatly in their ability to cope with situations that call for a knowledge of arithmetic. If teachers do not take these individual differences into account and provide for them, they cannot hope to attain the highest degree of efficiency in their teaching.

Differences in ability to meet arithmetical situations successfully may be due to any one of several causes or to a combination of causes.

1. There may be inborn differences in the pupils' original natures. Even if this is the cause we are not excused from making the very best possible use of the abilities which these pupils do possess, and every attempt should be made to furnish situations that will call for arithmetical activities which they are capable of carrying on.

2. Physical defects which may be removed, such as inability to see the board upon which the arithmetic work is placed or to see accurately the figures in the book, may cause differences to appear. An examination of the eyes of such pupils in order to provide suitable glasses will usually remove this difficulty.

3. If pupils have missed a great deal of time from school, they may have fallen behind temporarily in their work. In this case they should be given individual help by the teacher or by some member of the class who is known by the teacher to be capable of giving the necessary help.

4. There may be a great difference in the quality of learning which has preceded. Some pupils will do extremely well in one thing and very poorly in another. In this case the first step is to find out what each pupil needs and begin there. Do not pile up difficulties and increase differences among children by going on with new and more difficult work when they lack the fundamental background.

5. Lack of interest in arithmetic frequently results in poor work in this subject. If this is the cause of the difference, try to arrange situations in which these pupils are interested, and in which a knowledge of arithmetic will be necessary in order to carry on the activity thus suggested. For example, a girl is interested in sewing or domestic science. She finds it necessary to measure accurately, to increase or reduce recipes which call for multiplication, division, or finding fractional parts. She is willing to learn the arithmetic in order to carry on this activity in which she is interested.

As great as the differences are in the purely mechanical work they are much greater in problem solution. This can be provided for by giving problems varying in difficulty. Encourage the quicker and brighter pupils to work all the problems. Encourage the slower pupils to work as many and as difficult problems as they can.

The following suggestions may prove helpful in handling individual differences:

1. Give standard tests; score for both speed and accuracy, for the purpose of determining the range of abilities. When this has been done, diagnose the difficulties, group the children according to similarity in difficulties, prepare drill cards on combinations needed, and allow each group to work at its own rate of speed. When one group of difficulties has been mastered, allow the groups to pass on to other difficulties. If the difficulty is general, take it up with the class.

2. Give sufficient work to keep the bright children busy, and have a minimum amount for the slower ones.

3. Allow each pupil to work through the problems in the text at his own rate of speed. These may be worked in a notebook and the teacher may check them at intervals. Supplementary texts and texts of varying difficulty may be used for different members of the class. While this individual work is going on, the teacher will have time to give help to those who need it most.

4. Have the pupils from the fourth grade on keep individual graphs showing changes in their scores from day to day. These graphs will stimulate the pupils to compete with their own scores.

5. Present a new step to the class as a whole. Then select those who have not grasped it and give them more help. Individual help may still have to be given to a few. Pupils who understand the step may do something else or help those who need help.

6. Pupils who are up to the grade standard and do not need to spend time in arithmetic may be allowed to work on something else which they need while the slower ones are catching up.

X. Failures Due to Lack of Tests of Teaching

How to Check the Results of Teaching. Teachers frequently fail to check the results of their arithmetic teaching, and hence do not know when they have reached a fair degree of efficiency. This can be done by means of written tests, daily recitations, and standard tests. The standard tests have the greatest value. They inform the teacher of both the relative and the absolute ability of each pupil; they inform the pupil of his own relative and absolute ability; they set objectives and economize time.

XI. The Content and Arrangement of Textbooks

Textbooks. It is true in the teaching of arithmetic as in all other teaching that, other things being equal, the greater the amount of usable material which the teacher has at her disposal, the more efficiently she will be able to handle her subject. A study of the arrangement and content of new arithmetic textbooks as they are placed on the market will be very helpful to the teacher. She may find that some are excellent in the selection and arrangement of drill exercises, others are most helpful in suggesting the best order of developing topics, others are rich in problem material. From each she can receive valuable aids and devices for teaching.

No teacher should attempt to use any arithmetic textbook in her class until she is thoroughly familiar with the organization and content not only of the book she is using, but of the whole series of which this book is only one part. This is necessary for the following reasons:

1. Many textbooks give too little practice on certain facts and processes. The teacher should know this in order to know just where to supplement the practice.

2. They may give too much practice where nothing worth while is gained. Some of this may be eliminated if the teacher is conscious of it.

3. They may not give a sufficient amount of practice when the bond is first formed.

4. They may not distribute the practice wisely. After the establishment of a bond, the time between the practice periods should be gradually increased, and the amount of practice given at each period should be decreased.

5. The text may not provide for grading the steps so that the formation of one bond facilitates the formation of the next. U. S. money aids in learning decimals and may be used as an introduction to decimals.

6. It may be necessary to change the figures in the problems in order to meet the demands of the present time. Problems are much more real to pupils if the figures in them are such as they meet every day. With the rapid fluctuation in prices we cannot expect textbooks to contain up-to-date prices. The pupils may be asked to correct prices which appear to be out of date.

7. The teacher should know whether the answers given in the text are absolutely correct, certainly for the work of her grade. At least she should know how to work each problem.

8. She should know whether the type, spacing, and the like are designed so as to relieve all unnecessary eye strain. In general we should use 12 point type in Grades 3 and 4, 11 point in Grades 5 and 6, and 10 point in Grades 7 and 8. Sometimes the illustrations and diagrams are so complicated and trying on the eyes that no pupil should be expected to use them.

9. Some texts offer many opportunities for working examples, and recording the answers on paper without the time and strain involved in copying the examples. If the text does not provide such opportunities, the teacher should mimeograph drill exercises of this type.

10. It is necessary for the teacher to familiarize herself with the arithmetical terms used. If these terms are unfamiliar to the pupils, they must be explained in order to prevent difficulties that may arise from misunderstanding.

XII. Arithmetical Opportunities in Pupil Activities

Illustrative Example. Teachers frequently fail to see arithmetical opportunities in the activities in which the pupils are engaged. The following is an illustration of how arithmetic as well as other subjects of the curriculum were called for in carrying on a pupil-initiated activity.

Real Interest in Arithmetic Stimulated by a Paper Drive. Statement of the Situation. Early Spring, "Clean Up Week" in the City.

The week preceding "Clean Up Week" the papers were full of it, the health department got out dodgers which were distributed throughout the city, the children were urged to help. At a meeting of the Civic Club in the fifth grade such questions as the following were brought up for discussion. Many of them could not be answered immediately and an interesting investigation followed.

Why do we have Clean Up Week?

Why is the spring an especially good time of the year to have it?

What must we clean?

Why is it necessary to put the different kinds of trash in so many different receptacles?

Will it be wise for us to burn trash in our own yards?

If we do what should we be very careful about?

What about piling the trash in the alleys?

Why doesn't the city want us to do this?

Could we make use of any of this trash?

The following suggestions were made:

1. Old clothes could be given to the Salvation Army.
2. Old rags, auto tires, inner tubes, and scraps of iron could be sold to the junk man.
3. Pictures could be cut from magazines for our scrapbooks. (These children were making scrapbooks for the Crippled Children's Hospital.)
4. We could sell the papers and magazines and make some money for our room

The teacher as a member of the club asked the pupils what they thought of the whole school working together in collecting the papers and magazines. She told of another school that had engaged in a similar undertaking and how much they had made. A motion was made and carried that they suggest the idea to the whole school. A representative was appointed to present the matter to the school at the next school assembly. The matter was presented at the next assembly and the school heartily endorsed it. The principal offered to give a dish of ice cream or a ticket to the school movies each day to the boy and girl in each room bringing the most papers, and to give a prize to the class bringing the largest total amount.

Monday morning the drive started very enthusiastically, the fifth grade particularly being interested because they felt they had started the whole thing. Each class was responsible for its own papers. They were to be securely tied, weighed, and sent to the basement. Some of the older boys assisted the kindergarten, first, and second grades.

The needs arising were as follows:

1. Papers must be weighed in order to determine which pupils brought the most.

2. In weighing the papers they had to know ounces and pounds.

3. They also wanted to know number of ounces in ½ pound, and ¼ pound. Thus fractional parts came up.

4. Record was kept each day of the total amount brought by each class, who had to add these amounts in order to find out how much they had brought for the week. This involved addition of mixed numbers and reduction or addition of denominate numbers.

5. Each class wanted to know how much was brought by the whole school so they sent a committee to every other class for the record of the amount brought in. In order to find this out addition of the pounds and ounces, or pounds and fractional parts was required. (Some classes reported in one way; some in the other.)

6. The total amount collected was about 25,000 pounds. Paper sold for 60 cents per hundred. They next needed to find how much they had made.

7. This class alone brought about 3,500 pounds. What part of the money was theirs?

8. The teacher said that they collected in all 12½ tons. Then they wanted to know what a ton was. They were sent to their arithmetics to find out. Then they wanted to know how many tons their class had brought.

The papers were hauled from the building each afternoon; but as a different committee was appointed each day to carry the papers to the basement, they had an excellent opportunity of getting in mind the relation of amount in pound to bulk, in connection with papers and magazines. This helped greatly in visualizing large amounts like tons.

During this drive the pupils did not forget the pictures for the scrapbooks. Each child was responsible for looking over the magazines he brought and tearing out the pictures. These were turned over to a committee who cut them out nicely and put them in a box until they were needed for the scrapbooks. It was decided that it was better to dispose of the old clothes, scraps of carpet, furniture, rags, and auto tires from their own homes. This was decided on for hygienic reasons, and because of the difficulties involved in hauling to school.

The week following the sale of the papers the class was presented with a check for $21.00. This immediately stimulated the question, "What shall we do with it?" Many suggestions were offered, among them:

1. Have a party. This was voted down after a short discussion. They decided that if they had a party the money would all be spent and they would have nothing to show for it.

2. Buy a picture. They decided that this was too much money to put into a picture.

3. Buy some books. This was decided upon. They wrote for catalogues. While they were waiting for the catalogues they took a trip to the Public Library. The librarian took them to the children's reading room, and showed them the books which she thought would be nice for them to buy. To help them in selecting books they decided to have each child give a three-minute report on his favorite book. This gave excellent language training and also gave the teacher a chance to offer suggestions and to guide them in their choices.

The selection of the books from the catalogues, figuring out the cost and the numbers of books they could get for the money, called for more arithmetic.

Some one suggested that possibly they could get the books cheaper downtown. A committee was appointed to investigate and report. The committee found out that one of the department stores was having a sale of children's books and that they could get twelve of the books they wanted at 65c each. This raised the questions:

How much would the 12 cost?
How much would they have left?

They found that the other books could be purchased in the city just as cheaply as from the catalogues, but the individual copies varied in price. The committee reported on the prices, and the class figured out just which books and how many they could buy.

The books arrived and were piled on the front seats, for there was no place to put them. This did not worry the children. They were too interested in the books. Finally some one suggested that they ought to put up some shelves. This called for the following considerations:

1. Where to put the shelves. Different places were suggested and one decided on. The boys took this as their job, assisted by the manual training teacher.

2. How large to make them. Measurement was needed to get the shelves to fit the space and the books.

3. How to make the shelves more attractive. They bought stain for the shelves. They had to decide about how much was needed and the cost.

The girls decided to make a curtain to protect the books from dust and measured the amount needed, the depth of hems, and calculated the cost at so much per yard. A committee of girls then made the curtain and hemmed it in sewing class. When the shelves were finished each group which had spent money presented a bill correctly made out to the treasurer of the club funds. He paid the bills, received a receipt, and filed it in his book.

During this activity the following subjects of the curriculum were called for:

1. Civics, by helping in the "Clean Up."
2. Hygiene, by discussing, reading, and asking people for information as to:

   Why have Clean Up Week?
   Why have it in the spring?
   Why is the health department so interested?
   Why is it best not to bring the old clothes and rags to school?
   Why should we wash our hands carefully after we have handled the papers?
   What else do we need to keep clean beside our city? Why?
   How were our school and grounds helped?

3. Language:

   Discussions—Talk to the assembly asking the aid of the school.
   Book reports.
   Writing for catalogues.

4. Arithmetic:

   Table of weight, addition of denominate numbers, reduction. Feet and yards in table of length.
   Addition of mixed numbers.
   Addition, subtraction, multiplication with U.S. money.
   Making out of bills.
   Filling in order blank for use of the committee.

5. Reading:

   The desire for reading was stimulated.

In addition to the above the children learned to take pride in their city and school, to be helpful, to coöperate, to use good judgment, to depend upon themselves, to ask for help when needed and to be more interested in school affairs.

Additional Projects. Some additional projects, the carrying out of which will result in valuable learning, are:

1. Gardening.
2. Real estate firm—developing a section of the city.
3. Building a house and furnishing it.
4. Chicken raising.
5. Keeping graphs of arithmetic scores from day to day.
6. Working out a budget.
7. Speeding up service at the cafeteria.
8. Banking in connection with school deposits.

9. Project in developing feeling of appreciation of a new school building, grounds, and equipment by figuring fairly accurately, or estimating the cost to the city.
10. Animal food products.

In this discussion, I have made no attempt to present anything new, but have tried to show how several principles which we accept theoretically may be actually carried out in real classroom situations.

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PART II

JUNIOR HIGH SCHOOL MATHEMATICS

REORGANIZED MATHEMATICS IN THE JUNIOR HIGH SCHOOL

By HARRY C. BARBER
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I. Grades Seven and Eight

Conditions Which Make for Progress. So long as we are mentally alert we desire to improve our work. Our schools to-day are shot through with this urge to remedy their defects, to get to the bottom of their problems, to improve their practice in every department and in every phase.

Our teachers are eager, adjustable, and willing to learn. They are flocking to summer schools. If they do not understand the new things they are asked to do, or why they are asked to do them, they proceed to find out. If they are told that the old course was narrow and that it was disjointed—the ninth grade course having little connection with the earlier courses in method or content; if they are told that there is an algebraic method of work and a geometric source of material which are important in grades seven and eight, they set to work to discover what these criticisms mean. The great number of teachers who are asking to be shown what algebra and geometry are really all about, constitute on the one hand a rather serious arraignment of the past teaching of these subjects, and on the other great promise for the future.

As a result of these conditions new programs of study have developed in nearly all junior high school branches, programs which in the main appear to meet the new and broader demands being made upon the schools of to-day. In mathematics the new unit of instruction is already well under way and its success seems assured; but it still needs to be better understood both in its underlying philosophy and in the details of its execution as these details become settled by classroom experience.

The Broader Aims. Those who are about to teach the new mathematics should have first of all a broad vision of what elementary mathematics is, what it does for modern civilization, and what it should be expected to do for the pupil. The quantitative side of life, which is the field of mathematics, includes everything which we count or measure and everything which has form and position. Exact quantitative knowledge is one of the important bases of the material progress of our civilization. Mathematics should serve as an agency for the pupil's exploration and interpretation of the world about him.

It should serve, too, as a means for the exploration of the pupil's own mind, for the discovery of his abilities and aptitudes. Teachers should think of the mathematics class as the place where vitally important habits and attitudes are developed, and not merely as the place where verbal problems are untangled or mechanical processes drilled upon. They should provide for the gradual growth of the pupil in insight, in power, and in the understanding attitude of mind. (See footnote on page 136.)

Teachers should broadly conceive the duty of the profession to the public. It is all well and good for teachers of mathematics to learn how their subject is applied in this industry or in that office, but if leaders of mathematical education cannot, through the school, bring about gradual improvement in the practice of the community, they are unworthy of the trust that is imposed in them. Furthermore, we must win public respect and support for mathematical instruction if we are to retain the five periods a week to which we are entitled. The surest way to do this is to achieve the broader aims of the instruction so that we can convince teachers, administrators, and laymen of the value of the new mathematics in modern life.

Let the teacher, then, in his reading, thinking, and daily experience steadily broaden his concept of his subject and of its value as a means of education.

Achieving the Aims. Some General Principles. If we are to interpret, for the child, modern civilization in its quantitative aspect, we must cover a fairly broad field of subject matter. What we want of course are the most important elementary notions which will help in this interpretation and which are used and ought to be used in the world about us. This principle is a most important guide in the selection of the mathematical subject matter for these grades.

If we are to establish, for the pupil, correct habits and the understanding frame of mind, we must choose at every turn the procedure best adapted to that end. We must overcome the temptation of text and teacher to tell the pupil as quickly as possible what to do and then to set him to work repeating it again and again. Instead we must proceed to set up a situation which seems real to the pupil, a situation in which a merchant, clerk, mechanic, bookkeeper, artisan, or someone else might find himself, and then lead the child to see his way through, and help him to build up the best method of attack. When he meets another similar situation he is again to reason his way out with whatever help is necessary, and so on. We must repeat the explanation rather than the rule. In the end the pupil will respond correctly *because he understands.*

The work of each year should begin with new and interesting material designed to convince the pupil that mathematics is worth while. To succeed in this at the outset is to go a long way toward success in the course. Then there should follow situation after situation, each appealing to the child as real and worthy of his attention and each introducing or reviewing an important mathematical lesson.

For pupils who are pursuing vocational courses, many of the situations may center around the chosen vocation, but extreme narrowness should be avoided, interest should be developed in the "way the other fellow uses mathematics," broad training should be sought and *cul de sacs* avoided.

Care should be taken to see that important processes, methods, and items of information are used over and over again in the situations studied.

Teaching by means of introductory problems, or what I have called *situations*, should be clearly differentiated from mere problem solution. The solution of verbal problems is only a small part of what we are aiming at, although it has often been treated in books on the teaching of arithmetic and by textbook makers as though it were the whole thing. In fact, the textbook maker, being hard put to it, has constructed verbal problem after verbal problem, more and more involved, until many teachers have come to consider the untangling of these words their biggest aim.

Some drill on the fundamental mechanics is necessary in these grades. Just how much, depends upon many conditions; but at any rate the development of intelligent understanding and of the understanding habit of mind is more important than the drill during this period of beginning adolescence. We are not attempting to make skilled computers.

When the pupil finds himself stumbling or erring in the use of any fundamental process which arises in the course of the regular work, an attempt should be made to interest him in increasing his skill with that process and he should be provided with interesting and economical means for so doing.

These broader aims can be achieved, and it has been my experience that the more the teacher thinks about them, understands them, and is able to attain them, the more satisfaction she finds for herself, for her pupils, and for the public.

Some Specific Illustrations. The remainder of this discussion is concerned with various practical problems of teaching mathematics in grades seven and eight. The conclusions stated are the outcomes of experiments in putting into effect the principles discussed above.

1. *Accuracy.* It is important to take a broad view of the campaign for accuracy in the mathematics class. Accuracy depends largely upon correct habits, but drill narrowly conceived is not the only means of attaining it. Where an automatic response is required, as in the use of multiplication tables, many mechanical repetitions are necessary, but where an understanding is the aim the procedure is entirely different, reclarification is the rule rather than mere repetition.

The location of the decimal in multiplication and division, which causes so many errors and which brings down upon the school the condemnation of the business man, is most effectively handled not by mechanical rule but by the development of number judgment. The habits to establish are these: examine every number concerned in order to determine its approximate size; always make a preliminary estimate of the size of the product or the quotient; compare the result, when obtained, with the estimate; check the work.

The campaign to develop number judgment should be persistent. Appeal should be made to the pupil to depend upon his common sense, to know how large his results ought to be; not to depend upon mechanical rules for placing the point, but to think about the size of numbers as practical computers do. The practical sense developed by this campaign will be found to be one of the notable outcomes of the year's work, and it will engender the habit of expecting to be accurate.

Pupils who are very slow or very inaccurate in the use of fundamental processes should be given diagnostic tests. If the disability appears to be due to a remediable cause, the teacher should attack that cause directly by the remedy indicated and *not* by hit or miss drill; if it is due to weakness in some part of the process, drill on that detail should be encouraged; if, however, it is due to slowness of mental reaction, all pressure or condemnation should be avoided. No effort should be made to bring the slow-reaction pupil to any predetermined rate of speed.

2. *The metric system.* The metric system should be brought into service several times in these grades, at least until the pupil is reasonably familiar with centimeters, grams, and liters. In this respect we owe it to the pupil and to the public to create a desire for its adoption. The United States should fall into line with the rest of the world in adopting decimal subdivisions of all units. The only difficulty in the way is the attitude of the public. It is the duty of the teacher, through the school, to enlighten the public and so to modify this attitude.

3. *Ratio.* The word *ratio* is the name for a simple and extremely useful idea. It must be admitted, however, that a reading of the traditional discussion of ratio and of proportion would not be likely to convince anyone that the idea was either very simple or very useful. The frequent misuse of the idea or the failure to use it in the business or scientific world, to say nothing of magazine and newspaper articles, is evidence that we ought to bring up a generation who will understand ratios better and use them more advantageously.

The first idea about a ratio for the pupil to get is that a ratio is a number. The teacher will understand this from the consideration of the ratio of the circumference of a circle to its diameter. This ratio is 3.14, approximately. It is not necessary to say 3.14 to 1. We say merely that the ratio is 3.14. A trigonometric ratio is another illustration of the same thing. There are many others.

When we say that the ratio of two numbers is 4, we mean that one number is *four times* the other. Here we have the second idea about ratio. A ratio tells *how many times.* The pupil should use ratios as multipliers over and over again until this simple idea becomes thoroughly fixed. It will be of great assistance to him in his future thinking. Ratios used as multipliers are very common in applied mathematics, where they are called ratios, factors, coefficients, and other names. Ratios which are called conversion ratios, such as those used to change inches to meters, kilograms to pounds, and miles to feet, are good illustrations; so is specific gravity in the eighth grade, provided that the teacher is familiar with elementary science and can make the subject simple and real.[1]

Next the pupil should discover that in order to find a ratio he divides. He should know, of course, that ratios are usually expressed in decimal form rather than as common fractions. It is needless to say that the first two ideas about ratios should be very well in hand before the third one is introduced. For example, in the exercise: "Mr. Jones bought 28 feet of hose for \$3.64. I need 50 feet. What should it cost me if I can get it at the same rate?" Here the pupil knows that the answer is either $^{28}/_{50}$ of \$3.64 or $^{50}/_{28}$ of \$3.64. His judgment should tell him which.

Proportion should be postponed until much later. It arises naturally in connection with similar figures. A proportion should be treated as a fractional equation.

4. *Algebraic analysis.* Algebraic analysis is the best method to use in the solution of many kinds of problems. By the use of one letter to represent a quantity which otherwise must be described by one or perhaps several words, we save much time and writing in setting down a complete explanation of a solution. This clarifies our thinking and affords opportunity for practice in neat and systematic arrangement. The symbols, too, enable us to write whole sentences or rules most compactly in the shape of formulas. It is surely a mistake to let the pupil pass these grades without some knowledge of so simple and convenient a symbolism.

Sometimes a principal who formerly taught the traditional mechanical algebra misunderstands the present movement to teach algebra in grades seven and eight. It is merely the simple conveniences of algebra which are wanted, not the meaningless manipulations. Manipulative algebra does not square at all with the fundamental principles which we have laid down and are discussing. To teach it is the surest method of giving the pupil the wrong slant in regard to algebra and its utility. To be sure, we *can* add, subtract, multiply, and divide these literal symbols and we can prearrange them into expressions which are factorable, but we

[1] An understanding of high school science is an almost indispensable aid to the teacher who is trying to interpret mathematics and to understand its place in modern life and in modern education.

do not need to do any of these things in order to make use of our algebra, and doing them gives a most distorted idea of what algebra can do.

5. *Geometric facts.* Some geometric facts are very interesting and they enable us to do quite surprising things. They aid materially in the understanding and appreciation of the world about us. Geometric forms and their measurement can and should be used to give concreteness to the numerical work of these grades. In grade seven the use of the protractor should be well mastered, and in grade eight the use of the compasses. Geometry in these grades does not mean demonstration. What we are after is space concepts, concreteness, and the introduction to the elements of a new field of thinking.

6. *Approximate numbers.* The idea of approximate numbers should become familiar to the eighth grade pupil. He should come to understand that numbers which arise from measurement are approximate; that in computing with approximate numbers the results can be no more accurate than the data; that computations based on three-place tables must not be relied upon to more than three figures; and that in multiplying the approximate distance 28.3 ft. by π he should use 3.14, not 3.1416.

7. *Statistics.* The pupil should understand the great convenience of arranging some kinds of work in columns with proper labels. He should understand the convenience of tabulation. The words *average* and *median* should be familiar to him. All this can be done in the course of the regular work and not as a topic by itself.

8. *Per cent.* The notion of per cent should be led up to by a careful review of hundredths, and per cents should be persistently related to hundredths. It is best to begin with what used to be called the second case, because this shows at once the practical utility of the per cent idea. The Massachusetts state course of study says:

> Begin the work in per cent with the comparison of fractions, as in the standing of ball teams, school attendance, etc. This comparison cannot be conveniently made by the use of common fractions unless these fractions have the same denominator; so we use decimal fractions. When these are expressed as hundredths they are called per cents. In this connection bring out clearly and repeatedly the fact that a fraction is an indicated division. Teach the pupil to perform the indicated division and to give his answer to the nearest per cent, and later, to the nearest tenth of a per cent. If

one team wins 5 of 13 games it wins 5/13 of the games. 5/13 means $5 \div 13$ or .38 or .385, which is 38% or 38.5%. This introduction to per cent justifies the use of per cent in the mind of the pupil.

The next step is to find a per cent of a number. This presents only one difficulty, that of thinking of a per cent as a decimal fraction and so writing it. 38% of 13 may be written $.38 \times 13$ or $\frac{.38}{13}$. Since .38 is a little more than 1/3, the answer will be a little more than 1/3 of 13 or about 5. This step will check the work of the preceding paragraph.

The indirect case, where the base is to be found, is best explained by an equation. "5 is 18% of what number" translates directly into $5 = \frac{18n}{100}$ or $5 = .18n$.

Considerable time can be saved by the avoidance of the tangled verbal problems which formerly encumbered this part of arithmetic. The time saved can be much more profitably spent on the types of work we have been discussing.

II. Ninth Grade Algebra

The Challenge. William James, the American philosopher, said that algebra was a low form of cunning. Now Professor James was not an idle critic of education but a true friend and guide. Furthermore, he lived in the modern age in which more people are demanding education, and in which there is a more insistent search for fruitful fields of education than ever before. Yet he gave to algebra very poor educational standing and called it a low form of cunning.

His is a part of a very notable current criticism of algebra which asks, "Why should girls study algebra?" which claims that this subject or that subject "is surely more valuable than algebra," and which comes from commissioners of education and other leaders who ought to know. Even the layman who studied the subject a generation ago will usually, when called upon to defend algebra in the presence of teachers of mathematics, fall back upon the value of mastering a hard, exacting, and disagreeable task,—the Mr. Dooley idea—"Teach them anything so long as they don't like it."

It is quite clear, of course, that the algebra Professor James was aiming at is not the kind of which many of us are now thinking and which we are beginning to teach. It is the kind which Mary studied. In reply to her father's question about her success in high school, Mary replied that she was getting on all right except in algebra and she did not understand that. This formula was repeated night after night for some time but finally the phrase about the algebra was dropped. Her father said, "You understand your algebra all right now, don't you?" "Oh no!" replied Mary, *"but you don't have to understand it. You only do it the way they show you."*

It belongs to the period when we overestimated the value of the ability to manipulate algebraic symbols, and when we made a routine of teaching this manipulation; when we overestimated the capacity of the child and made the blunder, always fatal to good education, of driving the child so far into the subject as we could get him to perform the operations we showed him, instead of only so far as we could get him to understand and so get the intellectual benefit of the learning, and the attitude of understanding.

The Reply. Things have changed—they are changing—and in our reply to Professor James we may now call attention to:

1. The value of mathematics in modern life.
2. The admission on our part of certain errors in the algebra teaching of the past, in content and in method.
3. The philosophy underlying the current modifications of algebra teaching.
4. Some specific instances of modification of algebra which are growing out of this new philosophy.

Value in Modern Life. We may think of mathematics broadly as the methods used by the human race in dealing with its quantitative situations. It is a unique intellectual achievement: in itself a distinct field of education. Either you understand its methods of quantitative thinking or you do not. Either you are educated in this field of human accomplishment or you are not.

As to the relative importance of this field of thinking: measurement and exact knowledge underlie the modern advance of our civilization. Consider in this relation Kepler's "To measure is to know" and all of our scientific advancement since Kepler's time. Here again, either you have the habit of measuring and of acquiring exact knowledge or you have not.

In the field of mathematics, algebra plays a notable part. It supplies the language and technique. It gives a method of stating and studying the relationship of quantities. Thus broadly conceived, algebra may lay claim to educational values of the greatest significance.

The Massachusetts state course of study in junior high school mathematics says:

> The pupil should be led to think of mathematics
> 1. As a tool.
> 2. As an interesting field of knowledge.
> 3. As a mode of human thought.
> 4. As an indispensable aid to the progress of civilization.
>
> He should see that man's mastery of nature depends upon exact quantitative knowledge, that is, upon mathematics, and that this is true whether in the days of the building of the pyramids and of the Egyptian rope-stretchers; or in the days when Columbus sailed an unknown ocean; or in the days of modern structures, engines, radio, and automobiles.

This line of thought, if followed out, will be sufficient to convince thoughtful people, whatever their prejudices based upon the experiences of their youth may be, that the quantitative field has fundamental value as a department of education. Whether we have always taken hold by the handle most likely to help us realize this value is another question.

Our Admission of Past Errors. We might as well begin by making certain admissions for the good of our souls.

It is true that we have formerly included in the first course in algebra much material so hard that we have been forced to teach it by routine and rule and to lose consequently much of the educational value.[3]

It is true that we have been sometimes so crowded for time as to allow ourselves to be forced into indefensible methods. Witness the well-nigh universal comment of algebra teachers: "I should like

[3] A subject or a method may be said to have educational value:

1. If it helps the student in the exploration of his own mind: the more important the guidance afforded, the greater the educational value.
2. If it helps the student to understand the world: the more important the segment of human knowledge considered, the more directly the new understanding functions, the greater the educational value.
3. If it establishes correct habits and attitudes: the more important the habits and attitudes developed, the greater the educational value.

to do this or that because it is more interesting and more valuable but there is no time." "Crowding" has in fact lent some edge to the criticism of Professor James.

It is true that we have exaggerated sometimes the importance of the manipulation of symbols, at the expense of the thought values, and of the educational values in the sense in which we are using that term.

Much has been included in the trick cases of factoring, in the removal of parentheses, and elsewhere, which leads to exactly what Professor James means by "low form of cunning."

We have been too abrupt in our introduction of signed numbers and their use and have consequently fallen into the teaching of a species of juggling which may readily be included in Professor James's definition of our subject.

It is true too that the notion has been prevalent that anyone can teach first year algebra. On the contrary, this period during which the pupil is beginning to think in a general way about quantitative relations is a most critical one and calls for careful and expert guidance. We should be more careful in selecting the teachers for this part of the work.

It is needless perhaps to go further with our confessions.

The New Philosophy. The new philosophy underlying the changes in the teaching of elementary algebra is a part of the world-wide advance in philosophic and educational thinking, which is probably one of the outstanding achievements of the passing and of the present generation, and in which Professor James himself had no small part. (See for example the literature of pragmatism, and the work of such men as Professors Dewey and Bergson.)

The new philosophy may best be characterized by the words *common sense*. No further attempt to define it can be made here. In education it says that we should work with nature rather than against her; that we should put the child and his interests at the center; that the best mental training comes from the pursuit of subjects *interesting* and *useful* to the child. under the guidance of good teaching.

Concerning the teaching of elementary algebra, the new philosophy says that the value lies not so much in the "know-how" as in the attitude established and in the intellectual experience gained. For most pupils algebra has not very much value unless it is so taught as to develop the understanding attitude and habit of mind.[8] The methods used throughout must be primarily those which aim at teaching an understanding; which repeat the explanation rather than the rule.

We must not attempt to lead the young people so far into the subject that our only recourse is to drill until they can perform the various dodges we have shown them. (Professor A. M. Whitehead, the English philosopher, says that algebra tends to become a dodge for doing this and a dodge for doing that.)

"Some guardian angel of the children should stand at the door of the algebra class and whisper to the teacher, 'Do not hurry.'" The new statement of the algebra requirements of the College Entrance Examination Board definitely aims to reduce the crowding and to allow time for the thought values.

Our thesis can, of course, be developed at length because it has so many stimulating corollaries. I shall attempt, however, only one more point, namely, that every algebra teacher should be familiar with the field of the natural sciences. I have found that in building up in teachers' minds a concept of the place of algebra in education, the lack of a knowledge of science and of the scientific attitude of mind is one of the greatest handicaps. In this connection it is interesting to read the discussion of the union of pure mathematics and applied mathematics in the presidential address of Professor E. H. Moore to the American Mathematical Society in 1902, which is reprinted in the *First Yearbook* of the National Council of Teachers of Mathematics (1926).

Some Resultant Changes. The task of reconstructing algebra to meet the demands of the new philosophy and psychology is no inconsiderable one. However, it has been carefully thought out and is already well under way. Most of the rest of this paper is concerned with the application to the old algebra of the new psychology. The resultant changes outlined are not theoretical ones, neither are they untried; but they have their sanction in years of patient experiment.

1. *Formulas and equations.* Formulas and equations are the fundamentals of elementary algebra and their study should con-

[8] By the understanding attitude of mind we mean that attitude which, when confronted by a new situation, seeks an understanding of it and attempts to reason a way out—constructing, testing, rejecting hypotheses and methods—rather than that which *asks* for the rule or the procedure. This is an attitude which can be developed, but only if the teaching aims intelligently at this end and adapts its content and method thereto.

stitute the bulk of the first year work. From them grow the related ideas and processes which occupy the remainder of the time. At the outset the pupil should be confronted with the question: What do formulas and equations do for us and how do they do it?

Either formulas or conditional equations used for the solution of verbal problems may be made the starting point, but formulas seem to serve the purpose better because they are so obviously useful, and they connect so definitely with mensuration, arithmetic, and practical experience. On the other hand, the problems needed to introduce the equation method of solution are so simple as to be likely to defeat the extremely important aim of making algebra appear important and useful from the outset.

2. *Approximate computation.* The mensuration formulas and other formulas give opportunity to teach the ordinary facts about approximate numbers which the intelligent artisan and the student of science need to know. Numbers which arise from measurements, and numbers which are rounded off, as, for instance, the numbers in most numerical tables, are approximate, and computation with them must follow the simple rules of approximate computation or else become absurd. To use prearranged cancellations in order to avoid practical computation is not only puerile but misses the opportunity to teach these simple, useful, and very important properties of numbers. Nor should it be thought that the artisan, mechanic, and computer are the only ones to suffer loss by this practice. It proves to be true that dealing with approximate numbers and rounding off the results so that they agree in accuracy with the data, is the best kind of preparation for the use of numerical tables and for the understanding of incommensurables and of those misfit numbers which we call irrational.

3. *Logarithms.* It is possible to teach logarithmic computation in the ninth school year but it is most unwise to do so. The pupil, fresh from the eighth grade, needs a better understanding of our number system and how it works. He needs more practice to give him facility and confidence in ordinary numerical computation. He needs all this and more before he is ready for the short cuts made possible by tables of logarithms.

It is very important for practical reasons, to teach the use of tables to all ninth grade pupils, but this can be done by the use of tables of squares and of square roots, and the trigonometric tables if they are used—as they may well be. Surely simpler tables than the logarithmic should be used in teaching the first ideas of the use of tables.

4. *Language and manipulation.* The algebra reorganized in accordance with the new psychology relegates to the proper position the study of the algebraic language and of manipulation. The pupil is learning to use formulas and other equations. In so doing he meets from time to time new symbols and new processes. When he needs them he investigates and masters them. This is an application of the law of the "felt need."

Problem solution naturally becomes the vehicle for the introduction of each new operation. A situation is set up in which the new symbol or the new process is needed, and hence met and mastered. Translations from "words to algebra" and vice versa naturally arise and are handled *in direct connection with the problems and the formulas.* Useless processes naturally find themselves omitted because the need for them never arises. In this connection consider, for instance, most of the cases of factoring, the square roots of polynomials, involved parentheses, unusual fractions, and so on. These are, of course, precisely the things condemned by Professor James and other thoughtful critics of our subject. We are already answering these critics by omitting these parts of algebra, which, as nearly everybody now sees, were worse than a means of wasting time.

The pupil comes to see in the symbols help, not mystification, common sense, not arbitrary rule. He reads an algebraic expression in order to see what its symbols of operation tell him to do. He does not "remove a parenthesis" (absurd phrase) but operates upon the terms in the parenthesis as the symbols tell him to do. Thus he builds up a habit of reading the directions and carrying them out, of observing how the symbols are tangled and proceeding to untangle them.

5. *Transposition.* When we gave to thought values the first place in algebra teaching and made the development of right quantitative habits and attitudes the primary consideration, we automatically ruled out transposition. Transposition is a substitute for thought at just the moment when what we most want is thought. It defeats our major aim. It makes, in fact, manipulation the aim. It creates the juggling habit and the attitude of do-so-because-they-say-so. Instead of transposing, the pupil naturally adds if he has too little or subtracts if he has too much.

At first thought transposition appears to be too great a convenience to be thrown overboard, but a little experimenting will convince any thoughtful teacher that it is not so much a convenience as a handicap. We have here, in fact, a thoroughgoing test of a teacher's conversion to the new psychology. The longer transposition is postponed, the more thorough the conversion. When it is gone entirely, then is the teacher really putting first the understanding habit of mind.

Any system which would permit transposition in grades seven and eight is its own condemnation.

6. *The place for signed numbers.* This is a moot question. The complexity of the task of teaching signed numbers is now well recognized. The new psychology clarifies the situation. It tells us to introduce signed numbers *as they are needed.*

Historically, the minus sign has three meanings: first, subtraction; second, a shortage or a suspended subtraction which cannot yet be performed; third, opposite direction. The difficulties of teaching signed numbers have come from introducing these numbers long before they were needed and from trying to teach all three meanings at once. If the extension of the meaning of the minus sign is made gradually and if the *first two meanings* are used to explain all of the operations—as they can easily do—the pupil will come through with the understanding attitude of mind; otherwise he can scarcely be expected to do so.

It is noteworthy that at least half of elementary algebra can be completed before the pupil need be confronted with the multiplication of polynomials.

7. *The function idea.* The notion of a function which has been much discussed recently in connection with elementary algebra, arises naturally enough from the study of the equations and formulas. The fact that in similar figures, with increasing dimensions, the surfaces grow more rapidly than the lines, and the volumes faster still, is a most practical notion and can be made the vehicle for teaching just what the pupil needs to know about the functional relation. It is also helpful in studying the formula $C = \pi d$ to ask, "What is the effect on C if d is increased?" and so forth. What is wanted here is a simple and practical although slow-growing notion, and not a difficult or complex idea which needs much verbal explanation.

8. *Quadratics.* The arbitrary subdivision, "algebra to quadra-

tics," ought to be broken down because the solution of simple quadratics is simpler and more important than the complicated manipulations included in the old assignment; because the beginner ought to get glimpses of the solutions and the graphs of equations with more than one root; because in the solution of quadratics, factoring takes on some utility; and because solution by formula provides a typical situation in which a formula is useful and it affords the best kind of drill in the use of a formula.

9. *Trigonometry.* The unit of work in numerical trigonometry is of much more significance than the material which it displaced because: (1) It makes use of a simple and much used type of equation and thus connects with the heart of algebra. (2) It affords a drill in a practical kind of computation which has been admittedly much neglected in this grade in the past. (3) It teaches the use of tables. The pupil should not leave this grade until he can use simple tables and make linear interpolations.

Conclusion. The obstacles in the way of the reform which is proposed and, in fact, already in progress, are not due to any inadequacy in its underlying philosophy or psychology. The more one studies these foundations the more profit, enlightenment, and satisfaction one finds. They are, in fact, the basis of a thoroughgoing revision of the program not only in mathematics but also in history, in science, and in language.

The difficulties are not those of presenting a course which can be defended before educators and the thinking public as of conspicuous value to commercial, industrial, general, and college pupils. They do not lie in the adaptation of the ninth grade course to that of grades seven and eight.

They are not due to our fear lest we too much soften the requirements. "The purpose which I have named last, that of securing keener interest on the part of the pupil, seems to me more important than the other two. Sneers at easy ways, and at attempts to amuse cannot becloud the memories of your own youth; if you worked hard it was not because you were hammered into it, but because you found *interest* in subjects that repelled others not less able than yourselves. Many that failed could have succeeded; no searcher of statistics can find out what the world has lost by their rejection of mathematics." [4]

[4] From the pungent pen of George W. Evans, for more than a generation now a leader of our thinking about mathematical education.

On the other hand, the obstacles are found in our own natural inertia. The beginning teacher naturally falls into the habit of teaching as he was taught. The busy executive thinks of algebra in terms of his own study of the subject. The parents of our pupils studied the old algebra. Any person who has been away for some time from the teaching of mathematics and the literature of the subject will find not a little to wonder at in the modernization which has already occurred. Even in appearance the new algebra is different from the old because of the rearrangement and the change of emphasis. The change of balance is tremendous. Teachers over eighty years of age will do well to avoid the new algebra entirely because of the jolt it will afford. Perhaps no subject in the curriculum is more changed by the new psychology. Even the drill has changed because each process, being useful, finds its drill in its recurring use and application, and hence in repeated clarifications.

Another obstacle is the indiscreet use of *tests of mechanical mastery*. At our present stage of progress, these tests may produce just the opposite of good results. They make the worse appear the better method. They make manipulation, not thought, the goal. The quickest way to master the trick should be precisely what we are *not* seeking. In that path lies the condemnation of our former ways.

A Catechism. We may now state *a catechism for the friends of good algebra teaching who are at a loss in the present crisis.*

Is the algebra I teach useful and interesting?

Have I found the most valuable mathematics for the children of this grade?

Am I getting the thought values or is mine a narrow course in manipulation?

Can I defend its content and method before leaders of education? Does it meet Professor James's criticism?[5] Do I have other defense for it than Mr. Dooley's—"Teach them anything so long as they don't like it"? Am I bringing up a generation of young people who will defend algebra on some better basis than this?

Do I equip my pupils with the habits and attitudes most significant for their lives in modern civilization?

Am I skillful in the new type of drill, for the upper school years, which depends upon the repeated *use* of a process at proper intervals

[5] It must not be assumed that Professor James was unconscious of the educational values of algebra when well taught. There is much in his writings to refute such a notion.

of time, and upon repeated reclarification; which repeats the explanation rather than the rule; and which minimizes mechanical repetition?

Is my teaching of algebra keeping pace with the new program in other subjects? Am I reading the literature which leads me to see more and more value in the algebra I teach?

Am I as a teacher in a private school putting more or less value into the algebra I teach than is the teacher in the public school?

Am I as a graduate or a teacher in a women's college doing more or less to aid the progress of algebra than the men's colleges are doing?

EFFECT OF VARYING THE AMOUNTS OF ARITHMETIC, GEOMETRY, AND ALGEBRA STUDIED IN THE JUNIOR HIGH SCHOOL

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Survey of Progress. Our sluggish treatment of arithmetic in Grades VII and VIII, as well as our excess of zeal for algebra in Grade IX, was in no small measure responsible for the coming of the junior high school. In recent years we have, accordingly, reduced somewhat the burden of algebra in the ninth grade, and have sought to brighten the earlier grades with patches of intuitive geometry. We have made significant changes also in our point of view. As a result, we like to believe that we detect a quickened interest on the part of our pupils in mathematics.

What are we doing to capitalize this interest? Probably not as much as we should. For while most of the mathematics texts prepared for use in junior high schools show a better choice of materials for presentation and better motivation also of the different topics than did the texts of twenty years ago, nevertheless almost all these newer texts reveal a marked hesitancy to give substantial amounts of subject matter under the different topics treated. My experience with seventh and eighth grade pupils from all walks of life for the past twelve years would lead me to believe that the newer texts are not making the most of their opportunity. I believe, and I think I can show, that these texts might be even more mathematically stimulating, in breadth as well as in depth. Why should we not recognize that certain stray bits of mathematical lore—quite useless in themselves—are so captivating and intriguing in content as to be extremely valuable in arousing interest in certain other topics which are more important but less alluring? And why should we so skillfully whet the appetites of our pupils for a given subject if in the end we allow them only a nibble?

Let us then consider whether we are making at present the best use of the pupils' time, and see if by varying the allotment of the different subjects to the different grades we cannot arrange to give the pupils more substantial and significant amounts of mathematics without encroaching on the other activities which should properly claim their time and attention.

Algebra. Let us begin with algebra. Where should we first teach it in substantial amounts, and how much should we give? In many private schools it has long been the practice to begin substantial instruction in old style algebra in the eighth grade, beginning in February, March, or April and covering the work on equations and the fundamental operations with negative numbers by the end of the year. In the ninth grade the attack was renewed, beginning with factoring and fractions and following the usual course up to quadratics by the end of that year. This was only one degree better than the practice in the public schools during the same era: it at least allowed more time in which the pupils might assimilate the bewildering mass of new ideas and habits. And this for us is an important point. For, while I have no desire to argue for a return to the old style algebra, the experience of these private schools shows that by taking a somewhat slower pace than was the custom in the public schools, even the old style algebra could be learned without too great agony, and a substantial part of it in the eighth grade, at that. Even if we were finally to decide against giving much algebra in the eighth grade, this experience would show that pupils of that age are capable of doing substantial work in mathematics of some sort or other; and it is their due that they should have it.

If, then, we have at our disposal a new presentation of algebra more vital and less difficult than the old, we need not hesitate on grounds of difficulty, at any rate, to devote at least half the time in the eighth grade to a serious study of this subject. Whether we should or not, is an even more difficult question. I shall hope to show, however, that distinct advantages accrue from a serious study of the problem, equation, formula, graph, and fundamental operations with positive and negative numbers in the first half of the eighth grade; and among these advantages I would mention the possibility of reducing the time devoted to arithmetic in the seventh and eighth grades together to the equivalent of but one year without impairing the work in the technique of arithmetic, and with distinct improvement in the ability to solve problems in arithmetic. All this can be done in the public schools, in ordinary classes, taught by ordinary teachers.

Results of an Experiment. In December, 1922, I consented to supervise—though somewhat informally—the work in mathematics in the seventh and eighth grades in one of the public schools of Cambridge, Massachusetts,—the Agassiz School. The work in those grades was devoted mainly to arithmetic, with about a month of formal algebra at the end of the eighth year in which the emphasis was on the fundamental operations. The work in drawing in those grades included also some geometric designs. Through the courtesy of the School Committee and Michael E. Fitzgerald, Superintendent of Schools, and with the kind coöperation of the headmaster and teachers concerned, we were able to make certain changes in the course of study, beginning in September 1923.

We agreed to begin in Grade VII with a half year of intuitive geometry, using the second part of Wentworth-Smith-Brown, *Junior High School Mathematics, Book I,* as a text. And in the eighth grade we planned to devote the first half year to algebra, covering the first seven chapters of Durell and Arnold, *First Book in Algebra.* The second half of each grade was to be devoted to arithmetic, using the text supplied by the school system (Walton and Holmes, *Arithmetic, Book IV*). The argument in favor of thus enriching the work in these two grades by substantial doses of intuitive geometry and algebra at the expense of almost a year of arithmetic rested on the assertion that the intuitive geometry would be valuable for its own sake and would also arouse interest in the other work in mathematics; and that this increase in interest, plus the power and insight into the processes of arithmetic which the algebra would contribute, would balance any loss in attention to the arithmetic as such. This has proved to be the case, as the following figures show.

	C. A.	I. Q.*		Peet-Dearborn Test			
				I Probs.	I Adj'd	Aver. II-V	Adj'd Aver.
Standard Medians	....	...	VII	41	..	53	..
			VIII	56	..	65	..
1923 Medians	13-7	112		54	48	67	60
1924c	13-8	110		66	60	66	60
1925	13-4	116		54	47	70	60
1926	13 6	116		71	61	63	58
1924w	14-1	96		50	52	51	53

* By Dearborn Intelligence Test II C, D. P.E. of individual I.Q.'s = ± 9.

Interpretation of Results. The above table shows the scores which successive classes obtained on the Peet-Dearborn Arithmetic Test at the end of the eighth grade. In the absence of information as to the probable error of scores on that test I have included in the table the standard median scores in problems (I) and in technique (the average of scores on tests II-V, covering the four fundamental operations) for Grade VII as well as Grade VIII in order to show the effect on these scores of one year of training in arithmetic. It will be noted that the class of 1923 is the control class with which the records of the later classes may properly be compared. To facilitate this comparison I have divided each median score by the median I.Q. of the class, and have called the quotient the "adjusted score." This method of adjustment is based on the assumption that gains in intelligence are commensurate with gains in arithmetic as revealed by scores on certain tests; and there is, of course, no reason to suppose that this is so. The chances are, however, that any error so introduced is far less than many of the other errors to which these data are subject. For with only 40 (!) pupils in each class there is little point in taking these figures too seriously. They may be expected, however, to reveal marked gains or losses.

Intuitive Geometry. I like to interpret these figures as showing that the intuitive geometry in Grade VII adds so much of interest to the work of that year as to compensate for the reduction in the time allotted to arithmetic; and similarly for Grade VIII. To be sure, the class of 1926 shows a slight falling off in technique; but their teacher in the seventh grade allowed her enthusiasm for the intuitive geometry to carry her well beyond the middle of the year before she changed to arithmetic (for which I cheerfully assume the blame, as a result of my somewhat casual supervision of this work). This shows, moreover, that when we have cut down the work in arithmetic to a half year in Grade VII and a half year in Grade VIII, we have reached the danger point. In fact, we may even have passed it. For surely we have no reason to be proud of our present showing in the technique of arithmetic, which at best only equals the average of work done in other schools. On the other hand, we cannot expect any improvement if we simply go back to the former dreary program of nothing but arithmetic in these two grades. We shall probably be wiser if we first of all try to improve our methods of teaching and drilling on the fundamental operations. We can, however, derive considerable satisfaction from the marked improvement in problem work in arithmetic, effected no doubt by the serious work in algebra during the first half of Grade VIII.

The figures for 1924w are appended to show the result of the eighth grade program on a special section of 23 children, 18 of whom were transferred from another school to the Agassiz School in September, 1923. They apparently were not the cream of the other school, if one may judge from their I.Q.'s. We are unwilling to accept, therefore, any large share of responsibility for their poor showing in technique, in which they appear to be fully a year behind. In problems in arithmetic, however, they did remarkably well, which was hardly to be expected of a class of such little promise. I like to attribute this gain to the half year's work in algebra which these pupils had at the beginning of the eighth grade, their only year in the Agassiz School.

Importance of Eighth Grade Algebra. It should be emphasized that this work in algebra is no timid exhibit of a few formulas and graphs not seriously to be relied upon in later work in the ninth grade, but is of considerable substance, as a glance at the first seven chapters of the Durell-Arnold text will show. It should also be emphasized that this is by no means the upper limit for algebra in the eighth grade, but that even more could be done were we not obliged to consider also the legitimate demands of arithmetic and other subjects.

This is borne out by an experience of ten years and more ago at the Horace Mann School for Boys in New York City, where for a term of years the boys covered a full year of algebra—up to quadratics—in the eighth grade, and completed the subject of algebra in the ninth grade. That was not an experiment by one teacher with a single class or two. It was rather the policy of the school for years, and affected all classes. This policy was not of my making; I had rather more to do with shattering it. But despite its defects it shed light on the capacity of eighth graders for algebra, and the degree of retention of that body of material over the summer vacation. These eighth grade classes proved able to take the usual algebra up to simultaneous linear equations and did so uniformly, year in and year out, though not without effort. They showed, however, that they had been given more than they could assimilate—as one might easily surmise—for it was not uncommon for the ninth grade teacher to spend two months reteaching the work of the previous year before going on to the new material; that was, moreover, the usual and expected practice. Despite the fact that these pupils were able to understand and do good work with the factoring involved in algebraic fractions, and simultaneous equations in Grade VIII, they could not retain very much of what they had learned in the last few months of the year.

Amounts of Algebra Given. It would appear then that from considerations of efficient learning we should give less than a year of algebra in the eighth grade. We have already seen that to give as much as a half year of algebra at the beginning of the eighth grade has a favorable effect on the subsequent work in arithmetic, and we have discovered no ill effects from reducing the allowance for arithmetic to half a year in Grade VII as well as in Grade VIII. Such a program means of course that the children in these two grades learn nearly twice as much mathematics as formerly, but with no increase in effort other than what results naturally from a marked increase in interest. It is worthy of note that no complaints are heard from either pupils or parents; the teachers like it and comment favorably.

There are no figures available as yet to show that these pupils do better work in algebra in the ninth grade as a result of the earlier work in the eighth grade; such figures when obtained would undoubtedly serve only to corroborate the obvious. The change in attitude which these pupils show toward their later work is highly significant, however. Whereas formerly the algebra of the ninth grade was a bugbear and a source of low marks and much discouragement, it is rare indeed to hear the pupils even mention it now. They accept it as a matter of course and get good grades in it.

Suggested Procedure. One reason for the higher marks is that the present ninth grade course in algebra in Cambridge starts the subject afresh. There is good reason for this—as in so many other school systems—because at present so few pupils enter the ninth grade with any appreciable knowledge of algebra. It would seem unnecessary, however, for the textbooks to perpetuate this program, which can only result eventually in discrediting the work of the earlier grades and so render it ineffectual. It would seem to be unfortunate to present a good treatment of the equation, formula, and graph in a book designed for the second year of the junior high school, and then to begin the subject of algebra all over again in the following volume designed for the third year, implying in effect that the algebra taught in the eighth grade did not amount to much after all, and that to secure a really good foundation it was necessary to rebuild from the ground up. While it may be necessary for the next few years to wait patiently until teachers in the eighth grade have perfected their technique of teaching algebra, it would seem in the meantime to be wise not to shatter their slight store of confidence by refusing to build upon the results of their efforts.

Trigonometry. If our fear of a substantial offering of algebra in Grade VIII has proved groundless, we may with greater confidence face the question of trigonometry in Grade IX. Such a suggestion still staggers many; they undoubtedly feel that there must be some connection between the difficulty of the subject and the length of its name. Or possibly they remember that they never studied it until they went to college—if then even—and so are perfectly sure that it is unfit for children in the ninth grade. In either case the modern view of the subject is not understood.

As in the case of algebra, one way to meet such objections is to show that the subject can be taught successfully in the eighth grade; another way is to teach in the ninth grade twice as much as any one seriously thinks of proposing for that grade. I have done both.

In connection with my summer course for teachers of junior high school mathematics, I conduct a demonstration class in which most of the pupils (about 35) have just finished the seventh grade. For several years now I have shown these pupils how to solve a right triangle by means of tables of tangents, sines, and cosines, devoting to this work sometimes two and never more than three periods. Despite this meager allowance of time many of the class show a surprising grasp of the subject (surprising, that is, to any one who thinks this subject is hard) and most of the rest show a vague half knowledge which needs but little more time for drill and individual guidance before they, too, would show mastery. I go only far enough to convince (as I hope) my audience of teachers that it can be done and then turn at once to another topic.

Several years ago at the Horace Mann School for Boys it was the established practice in the ninth grade to complete quadratics by the middle of the year and then devote the second half year to logarithms, the slide rule, trigonometry, elementary statistics, and advanced arithmetic. In the work in trigonometry we regularly taught the solution of the oblique triangle by means of the Law of Sines and the Law of Cosines, using logarithms wherever applicable.

Four teachers took turns doing this. None of the pupils offered objection, and only one parent; and we converted him.

It is not my purpose to show that trigonometry is best begun in the ninth grade; in fact, I am convinced that it can be begun even earlier. My object is rather to show that substantial amounts of trigonometry—more than most people might guess—can be given to ninth grade pupils with good results. It is strictly a high school subject and the pupils like it.

Demonstrative Geometry. As for demonstrative geometry in the ninth grade, I have tried that too, following in general the outline and methods suggested by those who advocate work of this sort for this grade. I did not make a success of it, however, possibly because my heart was not in it. From my own personal observation at the time I should say that those pupils could have used that time to better advantage in some other way. I believe, however, that by means of a wholly different approach it is still possible to show pupils in the ninth grade the true nature of a proof, and that something of value of this sort can be done in as short a time even as six or eight weeks. I am ready and interested to make trial of this in the near future.

What Pupils Can Learn. In all the foregoing it is my contention that pupils in the junior high school grades can learn a great deal more mathematics than is commonly supposed to be the case. Whether teachers in general can teach as much mathematics as their pupils can learn is quite another question. I believe that they can. In order to do so they should become thoroughly familiar with the various subjects which they are to teach and then should go even further to gain the necessary background and perspective. All this they can do under adequate guidance. It is not fair, however, to expect a teacher to do this so long as she is responsible for the proper teaching of several other subjects besides, each of which doubtless has its own exacting demands if the pupils are to receive their due. If the former eighth grade teacher has a liking for mathematics and likes to teach it, it would seem wise to encourage her to master this subject and to expect her to teach seventh and ninth grade classes in mathematics also, shifting some of her history or English classes to other teachers who prefer to teach those subjects and are glad to be relieved of mathematics. In this way the pupils would get better instruction in all subjects. This need not, and should not, mean as thorough departmentalization as obtains in the bigger senior high schools; for the pupil needs to be introduced gradually to a school so organized.

Importance of Trained Teachers. We cannot expect to improve the course of study in mathematics in the seventh, eighth, and ninth grades to any great extent, however, until the mathematics in those grades is taught by teachers who know mathematics and like to teach it. Even if the bulk of the work in the seventh and eighth grades were to remain largely arithmetic, there would still be need for teachers who know the subject they are teaching. Of course the child is even more important than the subject, but for that very reason I would shield him from the faulty and uninspired teaching which is still to be found, even in these days.[1] Every wrong impression as to the nature of mathematics and every wrong attitude toward that subject which poor teaching begets means simply another injury done to the child, an injury which is lasting and difficult to mend.

For this the teaching is to blame, but not the individual teachers. There is no dearth of good teachers for the junior high school grades. So long, however, as we expect these teachers to be equally proficient at imparting all the different subjects in the curriculum, we are asking them to do the impossible.

There is nothing convincing about a lame presentation of a subject imperfectly understood. Why should we continue to ask teachers who have no liking for mathematics and less proficiency in the simplest exercises which that subject demands, to misrepresent themselves, and mathematics also? For while we profess that the bulk of the instruction in mathematics for the junior high school years shall be concerned with the material which every well-informed citizen should have at his command, it by no means follows that the present generation of teachers for those grades possess this information, not to mention the extended background necessary for the proper teaching of it to a class of children.

One of the arguments for enriching the course of study in the seventh and eighth grades is to get rid of the great waste of time which accompanies the presentation of arithmetic alone in those years. But merely to enrich the course of study will not accomplish this end. For imperfectly trained teachers can also waste the time of a class in geometry or algebra: they are so ill at ease in

[1] A mass of evidence can be adduced to support this point, but it would be out of place in this discussion.

the subject itself that their teaching is labored and halting, and often wrong. They should, on the contrary, be so familiar with elementary algebra, plane and solid geometry, trigonometry and the psychology of learning, that they can present the mathematics of these grades with the light def' touch of the master who delights in his work. When so presented mathematics will appear to the pupils to be within their powers and, inviting their attention, it will no longer be a formidable and forbidding chore. Without this command of subject matter a teacher can neither catch nor impart the true spirit of the work in mathematics for these grades.

The Proper Goal. However the course of study is enriched, so long as it be in depth as well as in breadth, it would seem to be too early yet to be sure that less time is needed for adequate learning of the new than was thought necessary for the old. It would be well first of all to endeavor to attain the goal established by the National Committee's report, namely to teach more thoroughly the basic principles of the subject and to give a broader appreciation of the place of mathematics in life around us; for it was to attain that very end that time was set free by the reduction of the requirement. Unless some other subject or activity is clearly pinched for time and can prove its right to time formerly allotted to mathematics, it would be wise to make sure that that time is not still needed for mathematics before we gayly give it away; for the measure of that need is not yet.

General Mathematics. So far I have said nothing about general mathematics, partly because I am not sure that I know just what it is. If I understand the proponents of general mathematics aright, it is more than a revision of the course of study in which old landmarks are ignored; it is rather an attitude which regards mathematics as greater than the sum of all its parts, and in which the interrelations between the parts are of more significance than the parts themselves. I find it easy to subscribe to this; and while I venerate the old familiar landmarks, I would not hesitate to set up new ones which might serve us better. I believe, however, that I set greater store by these lines of demarcation, wherever located, than do the proponents of general mathematics, who, I believe, would abolish them altogether. I can see that it is often helpful to view a subject as a whole before proceeding to a detailed study of certain parts of it; I can see also that the results of this detailed study should be reviewed in the light of the general setting. I cannot lose the conviction, however, that the detailed and thorough study of the part itself has great cultural value also. I would not sacrifice depth for breadth, nor breadth for depth; I would keep both. After all, the argument is mainly an academic one. For the "landmarkers" have learned from the "fusionists" that a realignment of boundaries is not so fearsome an undertaking as they had at first imagined. And the "fusionists" despite their own protestations have, in fact at least, preserved a number of boundaries or thought-tight compartments out of deference to the learning processes of their pupils. It is perfectly possible for a "landmarker" and a "fusionist," working quite independently, to draw up for their respective schools courses of study which are very much alike. This has happened to my certain knowledge. Each then contends that the two courses are different because permeated by different spirits. There is probably some truth in this. It is clear, then, that the "fusionists" are slightly more venturesome than the "landmarkers" (new style); it is equally clear that the "landmarkers," profiting by the experience of the "fusionists," would seem for the moment to have a little more of the truth.

Conclusions. The following ten conclusions seem to me to be reasonable:

1. We need at least an aggregate of one year of arithmetic in Grades VII and VIII.
2. As ordinarily taught two years of arithmetic in these grades is no better than one.
3. To get greater returns from the arithmetic in these grades we need greater interest and new methods of drill rather than more time.
4. Algebra in the first half of Grade VIII improves the ability to solve problems in arithmetic.
5. A full half year of algebra can be profitably given in Grade VIII.
6. A whole year of algebra in Grade VIII is too much.
7. There is time enough for at least half a year of intuitive geometry in the junior high school grades.
8. It is possible to teach some numerical trigonometry in Grade VIII if desired.
9. It is possible to teach the solution of the oblique triangle in Grade IX, using logarithms.
10. For effective teaching of mathematics in the junior high school we need a certain degree of departmentalization with adequate supervision.

PROBLEMS IN THE FIELD OF JUNIOR HIGH SCHOOL MATHEMATICS

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I. What to Teach

Survey of Textbooks. A hasty inspection of the mathematics textbooks prepared during the past five years for the junior high school grades gives evidence that the fundamental problem "what to teach" is still with us. This is especially true when the seventh and eighth grade program of study is considered. The other problem of "how to teach," with its related problems of "how to care for individual differences," "mastery of the fundamental processes," "co-ordination with the senior high school," "educational prognosis," "time allotments," "teacher preparation," and the like, cannot be discussed without having a definite notion of "what to teach."

The best evidence of the present trend or trends along the lines of "what to teach" is furnished by a brief survey of the contents of the recent texts prepared for the junior high schools. Such a survey reveals substantial agreement on the question of content for the ninth year. There is evidence that the indictment contained in the memorandum addressed to the General Education Board by a committee appointed by the Mathematical Association of America no longer holds for a large number of progressive school systems. Formal algebra of the type referred to in this memorandum is passing from our schools. The gist of the memorandum is expressed in the following passage.[1]

The Algebra Situation. "The situation that needs to be met may best be illustrated by the case of algebra. Our elementary algebra is, in theory and symbolism, substantially what it was in the seventeenth century. The present standards of drill work, largely on non-essentials, were set up about fifty years ago. The few lines of application of algebra to nominally practical questions found

[1] Schorling, Raleigh: *A Tentative List of Objectives in the Teaching of Junior High School Mathematics with Investigations for the Determining of Their Validity*, p. 2. George Wahr, Ann Arbor, Michigan, 1925.

in our college entrance examinations are mere variants of problems that are centuries old, and that often represent only remotely any real conditions of to-day. A considerable number of teachers, in both the secondary schools and the colleges, believe that the amount of time spent by pupils on abstract work in difficult problems in division, factoring, fractions, simultaneous equations, radicals, and the like is excessive; that such work leads to nothing important in the science, and adds but little to facility in the manipulation of algebraic forms.

"It is urged by many friendly critics that instead of giving the students a good all-round idea of what mathematics means and its general range of application our present secondary school courses are too abstract, often uninteresting, except to the mathematically inclined, and not as valuable as they might be as an aid to college in general, or to life work for those who enter at once into their careers."

Content of Seventh and Eighth Grades. In view of the above quotation, attention may be focused on the content prepared for the seventh and eighth grades. An analysis of four of the recent texts, selected because of their wide appeal, reveals two distinct trends. One stresses social arithmetic and the other favors the mathematics of science, art, and industry. For purposes of discrimination the one type of text may be called "Social" and the other "Mathematical," although both employ sound mathematical principles. The analysis of the four texts selected gives the following facts.

Exhibit A. Mathematical Type. Here we have

1. Space devoted to Measurement, Geometric Forms, Intuitive Geometry, and so on. 10 chapters (213 pages).
2. Space devoted to Social Arithmetic—Banking, Insurance, Stocks and Bonds, Buying and Selling, Home Budget, and so on. 8 chapters (157 pages).

Exhibit B. Mathematical Type. In this case we have

1. Item 1 as above, 65 units (174 pages).
2. Item 2 as above, 27 units (133 pages).

Exhibit C. Social Type. Here we find

1. Item 1 as above, 3 chapters (12 pages).
2. Item 2 as above, 22 chapters (310 pages).

Exhibit D. Social Type. In this case we find

1. Item 1 as above, 31 units (40 pages).
2. Item 2 as above, 185 units (190 pages).

NOTE: Space devoted to review of fundamental processes, tests, graphing, and charting was not included because both types of texts cover those subjects in a satisfactory way. The names of the texts are withheld because of a lack of time in which to survey the many texts at hand. The findings, however, may be verified by a similiar study of radically different books on the market.

Results of Analysis of Textbooks. The above rough analysis of four texts gives an indication that two extremely different types of texts, as far as content is concerned, are being used in American schools. In the so-called "Mathematical" type of text appears a preponderance of chapter headings such as: "How to Construct, Measure, and Use Angles"; "Geometric Forms"; "How to Find Areas and Perimeters of Plane Figures"; "How to Find Volumes and Surfaces of Solids"; "How to Measure Lines and Angles"; "Finding Inaccessible Distances"; and "Positive and Negative Numbers in Mathematics."

In the so-called "Social" type of books such chapter headings as the following appear more frequently: "Simple Accounts and Business Forms"; "Banks and Banking"; "The Meaning and Nature of Insurance"; "How our Food is Produced and Distributed"; "Keeping Household Accounts"; "What the Investor Should Know about Stocks and Bonds"; "How Our Taxes are Collected and Used"; "The Cost of an Education"; and "Clothing the Family."

The above illustrations point quite conclusively to the fact that textbooks are being prepared by authors differing widely in their opinions of what constitutes a junior high school mathematics program. We even find one author writing two types of books to supply the demands of those concerned with the selection of instructional material. The problem of "what to teach" is indeed with us. How shall it be solved? It is the purpose of the writer to discuss this question with reference to the junior high school program in a large industrial city. The question will be considered primarily from the point of view of junior high school aims. No attempt is made to settle the problem, but to encourage wide discussion.

Junior High School Aims. From the list of aims commonly accepted for the junior high school there appear to be three to which the mathematics program can contribute the most. They are:

a. To continue training in the fundamental processes.
b. To train pupils for vocational life.
c. To train pupils in their social-civic duties—worthy home membership and citizenship.

The first of these aims seems to receive its share of emphasis in the newer texts. There are, however, indications that this problem needs to be discussed from the point of view of "how to teach." This will be done at a later point. The proposal to train pupils for vocational life through a plan of general instruction designed for all raises many questions. This need for vocational training is very important, but should training for social-civic duties as they relate to worthy home membership and citizenship be considered of greater importance? The writer believes it should and therefore will adhere to this viewpoint in the following discussion. In support of this view a suggested course for junior high schools will be outlined.

As a starting point Professor Upton's [2] analysis of the mathematics needs for citizenship under the headings of "practical usage in the bread and butter sense" and "to appreciate the life that is going on around them, though they themselves may take no active (direct) [3] part in it" will be used. The ideas expressed by Professor McMurry [4] in his strong plea for more social arithmetic in the following significant paragraphs express the writer's point of view:

"Yet these stories are not introduced primarily because of this valuable aid to the processes. On the contrary they would be the cream of arithmetic, the leading objects of the study.

"They would deal with such fields as industry, transportation, commerce, health, and government and would lead not only to knowledge of many live issues in these fields, but also to attitudes, ambitions, and convictions intimately related to conduct as a citizen."

Harap [5] has given support to the same point of view in his *Education of the Consumer*. The figures presented by John T. Flynn [6]

[2] C. B. Upton, *The Reorganization of Mathematics in Secondary Education*, 1923, p. 308.

[3] Direct is the interpretation of the writer.

[4] Frank McMurry, "The Question that Arithmetic Is Facing and Its Answer." *Teachers College Record*, June, 1926, p. 881.

[5] Henry Harap. *The Education of the Consumer*. Macmillan Co., 1924.

[6] John T. Flynn, "Who Owns America?" *Harper's Magazine*, May, 1926, pp. 753-62.

in his article entitled "Who Owns America?" through which he statistically demonstrates the fact that we are changing from a population of independent owners to salaried workers and wage earners are also interesting. This also is in keeping with the idea that much of the mathematics for the junior high school, the last school for the large majority of our population, should deal with problems of consumption. The need of this large majority as seen by those concerned with the social problems of living seems to be for an increasingly greater emphasis upon training which will enable people to cope more effectively with problems of consumption and with problems of participation in the affairs of the community.

A Proposed Junior High School Program. The employment of bread and butter and appreciation needs as fundamental aims for a mathematics program may suggest the thought that what is bread and butter training for one pupil will be appreciation for another. This will always be true so long as we are dealing with the great mass of children attending our public schools. We shall constantly be forced to think in terms of the larger group and plan extra courses for special groups when it has been determined that such procedure is advisable. The real problem before us is to decide what the larger group needs are from the point of view of mathematics instruction to meet bread and butter needs and to give a background of appreciation as equipment for citizenship. This does not mean that the development of the power to do mathematical thinking should be neglected. The pupils of junior high school age are, on the average, mature enough to adopt better ways of thinking and of expressing relationships than those learned in the lower grades. There is no reason why a citizenship program should interfere with mathematics training; it is simply a question of the content to be used as the medium of instruction.

In outlining a mathematics program for the seventh, eighth, and ninth grades under the headings of bread and butter and appreciation needs three major topics, *Consumption, Production* and *Facilitating Agencies,* are suggested. These major topics involve the activities of a well-rounded life because to be good citizens we all, by the very nature of things, must consume, we must produce in some way or other, and we must coöperate with our fellow men in the social, civic, and industrial affairs of the community. The discussion will center around these three larger topics.

Consumption. The business of living is the most important business in which we engage. Whether or not we are useful citizens, happy and contented, depends very much upon our ability to strike an intelligent balance between our income and our outgo. The earning of a living usually requires continued activity along one line, but the spending of these earnings usually carries one into a large number of fields of activity. It is often said that the efficiency expert finds it harder to make both ends meet in handling his home budget than to operate a large industry involving a thousand times as much money. The factors involved in the business of consumption and spending one's income are almost too numerous to mention. Budget makers have classified expenditures under the headings of food, clothing, shelter, service, advancement, savings and investments. In our complicated system of living each of these items has innumerable subdivisions. Then why shouldn't a large part of our junior high school mathematics be based upon the all-important topics of consumption and the spending of our incomes?

This part of the program could be centered around the home budget idea. The problems of the consumer and average citizen in obtaining his food, clothing and shelter, pleasure and enjoyment, and in making his investments and savings could provide the basis for study and discussion. It is obvious that the mathematical needs in studying these problems would be many. It is also obvious that these problems would raise many social and economic questions. For instance, in the study of shelter, the question of owning and renting would naturally arise. The advantage of one over the other cannot in all cases be estimated in dollars and cents. It would be necessary to infuse some of the social reasons why one is more desirable than the other. Further, one could not study the matter of investing a part of the income without considering the different types of investments from the point of view of income, security, and availability of funds. The social content cannot very well be separated from what is strictly mathematical.

Food and Clothing. A summary treatment of the topic of consumption will give a more comprehensive idea of what this larger unit of instruction might include. As consumers, the amounts of income, without reference to the details of obtaining this income, would be the first consideration. The problems raised by a study of food in relation to consumption would include finding and comparing costs, planning for specified periods, methods used in buying and paying for foods, marketing of foods, preparation of foods, relative values of foods, and the like. The difficulty would lie in limiting the number of lines of study which are important to the consumer in spending an approximate proportion of his income most advantageously for food. The study of clothing from the point of view of cost, durability, comfort, and style could be attacked in much the same way as that of food.

Shelter. Housing conditions since the war have forced many people to revise their budgets because it has become necessary to spend an increasingly larger proportion of the family income for shelter. Conditions have become so acute in some states that it has been necessary to pass laws regulating the activities of landlords. On the other hand, every possible stimulation has been given to home owning. It matters little in the last analysis whether people rent or own from the point of view of expenditure; it costs to own as well as to rent. Pupils should be given an opportunity to study the cost of ownership. This would entail matters of taxation, insurance, land and home contracts, mortgages, notes, depreciation, upkeep and the other concerns of the home owner. It would also be hard to limit the amount of time and space devoted to this topic, although it is necessary to do so.

Service. The topic of service or operating expenses is closely related to the one of shelter. Budget makers include under this heading expenditures for such things as heat, light and water source, equipment and replacements, repairs, telephone, laundry, household supplies, furniture, allowances, gifts, entertaining, cigars, candy, dentist, physician and drugs. The consumer certainly needs to know how to plan for the list of expenditures. It is this department of the budget which causes the most trouble. A long list of problems dealing with the economical methods of obtaining these services may be raised.

Advancement. The subject of advancement, i.e., expenditures for church contributions, charity, vacations, amusements, instruction, books, magazines, travel, automobile upkeep, is a subject to which little attention has been paid in our schools. Perhaps the average layman would not be so startled by the announcement of large donations to community funds and drives if he figured his own on the basis of per cent of total income. We live in a very unscientific age in which the average citizen does very little exact thinking. He is more or less confused and dumbfounded by the bigness of things about him. The other topics under advancement are equally important.

Savings and Investment. The problems of savings and investment go hand in hand. Many budgets are well managed up to this point. Unscrupulous promoters and operators, in spite of all warnings, are still able to fleece the great American public. There is need for giving our boys and girls some protective mathematics. We can at least acquaint them with some of the dangers. The facts that the consumer should know in order to deal intelligently with banks, trust and loan companies, and individuals would make up this part of the course.

Mathematics in Home Budgeting. Where does mathematics instruction enter into a study of home budgeting? A survey of the actual mathematical needs will show that a thorough mastery of percentage, decimal fractions, common fractions, graphing and charting, ratio and proportion, as well as some practice in elementary accounting, would be needed to meet the home budget problems of the consumer. What would be better than the practice of using the above-named mathematical tools in situations within the comprehension of the pupils?

Production. Again the matter of bread and butter learning and appreciation needs as equipment for citizenship enter into our discussion. We are confronted with the ancient problem of arranging a course of mathematics instruction which is good for most pupils. How much should the mass of pupils know about production to become good homemakers, good citizens, and good workers in whatever kind of work they engage? We know offhand the approximate percentage of our population which will engage in work requiring some understanding of the mathematics used in production. We know further that to be intelligent consumers it would be well for all people at least to be able to understand and check the mathematics used by the producer. Most of us also agree that the best cure for group differences is for each group to learn more about the other groups. A study of production would be valuable if it gave pupils a sympathetic understanding of some of the problems of the producer. The demand that junior high school courses provide vocational guidance is further reason for stressing matters relating to production.

Remuneration. Production can be considered under three headings: remuneration, services, and industry. The topic of remuneration could include a study of the amounts of remuneration and methods of payment, from hourly wage, monthly salary, fees, and the like to incomes from business investments. Comparisons of earnings by people in different lines of work could be made. It should always be borne in mind, in this study, that individuals and groups of individuals are entitled to a legitimate income and payment for services well rendered. We should at the same time impress pupils with the fact that the ability to render services depends upon such things as native ability, special aptitudes, and training. It is not intended that the mathematics classes should become civics and vocational guidance classes. Opportunities, however, for making natural and effective connections between this type of instruction and what is strictly mathematical should not be overlooked. The writer takes the position that the value of mathematics instruction depends upon the correlations and meanings which can be given to it.

Services. The second subtopic under production, that of services, offers a way out for the mathematics people who attempt to justify the more technical phases of mathematics instruction on the basis of everyday needs. In order to know enough about production to be intelligent consumers, useful homemakers, and good citizens, individuals must at least have sufficient mathematical training to understand and check the planning and execution of the common types of work. For instance, an individual wishing to have a basement excavated should be able to determine whether a certain contract price is better than a per cubic yard price. Certain notions of spatial relationships are an aid in reading blueprints, drawing to scale, making layouts and measurements. An elementary understanding of ratio and proportion is useful in dealing with problems involving mixtures of such things as concrete, cement, garden sprays, paints, and the like. This list of illustrations could be extended indefinitely. In many cases the argument could be made that people without mathematical ability are daily performing the tasks indicated. The answer is that people with mathematical training can do these tasks with greater efficiency and satisfaction.

Industry. The average citizen would also profit by a quantitative study of the distribution of the fruits of production. The problem of "from producer to consumer" is becoming a vital part of the larger problem of production. Closely related to this is an overview of production, represented by a study of industries, such as auto manufacturing, fruit growing, fishing, copper mining, wheat raising, and the like. A numerical study of the industries would strengthen the general notions obtained from social science study.

Duty of Mathematics Teachers. The topic of production could be extended without limit according to our desires. If we believe that boys and girls should receive instruction which will help in the selection of life's work, which will increase their ability to construct and plan, which will give a larger viewpoint of the part played by the individual and groups of individuals in production, and which will make more intelligent consumers in dealing with those who perform our work, we are forced to bring the work-a-day world to the attention of pupils. Those entrusted with the problem of mathematics instruction must, in carrying through this program, do the things which they can do better than the other departments of the school. It is evident that the subject matter of mensuration and spatial relations, the trigonometric functions of the right triangle, equations and formulas, graphic representation, and positive and negative numbers would receive special emphasis in the study of production. The technical school people were the first to awaken to the fact that the early mathematics training, demanding more than arithmetical ability, could be linked up with the situations within the experiences of the pupils. Their example is gradually being followed by those planning instruction of a general type.

Facilitating Agencies. In addition to being producers and consumers we are all members of organized society. There has grown up in organized society a series of facilitating agencies by means of which we carry on our group enterprises. These agencies now form a complicated part of our social structure. The average layman finds it hard to know enough about these agencies to participate intelligently in playing his part in the game. We have been forced to pass immigration laws because we have found the number of unintelligent was becoming too large for the safe functioning of our governmental agencies. In the face of all this no one can dispute the contention that increased emphasis must be placed upon the phase of education which aims to make the masses more intelligent members of organized society. The question for mathematics people to decide is the extent to which a study of facilitating agencies shall be considered in mathematics courses. As in the case of production, those items might be included which can be handled better by using mathematics as the core subject rather than some other subject in the curriculum.

The study of facilitating agencies might include a consideration of the social, political, and economic phases of group organization from the point of view of incomes and expenditures for the same. This would mean a study of local, state, and national taxation and other sources of income. The problem may be attacked as one of the consumer; for example, how the consumer contributes to the different governmental sources of income.

The expenditures to maintain the operation of the facilitating agencies would involve local, state, and national budgets. They would involve expenditures for education, judiciary functions, charity, good health, recreation, public lighting, fire and police protection, good roads, and the like. In every community the facts necessary for such a study are available.

The quasi-public utilities represented by the agencies of trade, transportation, and communications should also receive attention. The magnitude of these, the services rendered, and the relation to the average citizen are important. Most of this study would deal with graphic representation and specific case studies of the agencies related to the lives of the pupils.

The contention is often made that the layman cannot understand the very complicated social, economic, and political organization of which he is a part. However, when city departments want to "sell" their budgets to the voters they find methods of expressing service costs in terms quite understandable. Quasi-public utilities have adopted educational programs aiming to give the consumers facts relating to their operations. If the public and quasi-public agencies can do these things, it is reasonable to expect that school people can utilize the same methods.

An attempt has been made here to crowd a whole course of instruction into a few paragraphs. It has been necessary to omit a large number of illustrations and give no consideration of units of instruction, grade arrangement, and exact content. The exact mathematical content has been suggested. However, it might be added that the mathematical values from the study of social material would be derived from the methods of work introduced. For example, the mathematical basis for the study of insurance, taxation, and interest is now the use of the formula. It would be by the inclusion of the higher and better ways of thinking and doing that pupils would receive instruction in mathematical thinking. The units of work dealing with production would in themselves provide for a great deal of the elementary mathematics now included in unified and general mathematics courses.

A study of consumption, production, and facilitating agencies from the point of view of satisfying bread and butter and appreciation needs would not materially change the mathematical content of many of the present junior high school texts. The shift in "what to teach" would be in the point of view taken in presenting the material of instruction. The utilization of mathematical methods of computation would provide an easy transition to a study of more formal mathematics in the ninth grade. Learning through concrete situations would provide a better rather than a poorer background for the senior high school mathematics courses.

If the mathematics group is to keep apace with the universal demand for socialization of school work, as evidenced by social biology, social science, social art, and what not, further recognition must be given to the problem of striking a proper balance between mathematical content and social experience. There seems to be little justification for seventh and eighth grade texts devoted to a strictly formal treatment of "measuring lines and angles," "positive and negative numbers," "geometric forms," and the like.

II. How to Teach

The Fundamental Processes. The results [7] of the Schorling-Clark-Rugg Inventory Test reveal such facts as:

(a) For the problem "Does 4896 ÷ 10 equal 4,896 or 48.96 or 4896 or 489.6?" the percentage of correct responses was 35.5 from a group of 3,198 beginning seventh grade pupils.
(b) The problem "Does 1.2 × 5 equal 6.0 or .60 or .060 or 60?" was correctly solved by only 63.7 per cent of the same group.
(c) For "What do you do to find ⅔ of ⅝? Do you add? If not, what must you do?" only 54.9 per cent of the beginning seventh graders made the correct answer.

The writer gave a mixed fundamentals test to a group of approximately 5,000 seventh grade pupils. The results, in part, are shown on the following page.

[7] Schorling, Raleigh. *A Tentative List of Objectives in the Teaching of Junior High School Mathematics with Investigations for the Determining of Their Validity*, pp. 21-29. George Wahr, Ann Arbor, Michigan, 1925.

Test Items	Percentage of Correct Responses
1. 7.38 — 3.946	57
2. $2\frac{1}{2} \times 4\ 2/3$	53
3. Divide 30.96 by 1.44	50
4. Find 16 2/3 of 24	44

Significance of Test Results. To these results might be added others equally good or bad. In the face of these we may seriously question the theoretical consideration that the fundamental operations are taught in the first six grades to a point of mastery demanding from the junior high school only a brief review and practice to reach the standards set by business and industry. This consideration may hold for the more gifted group, but the common complaint of classroom teachers is that no small part of the entering seventh graders need more than a brief review and practice exercises. They point to the fact that the mass of material taught in the fifth and sixth grades has produced much half learning. To overcome this, nothing short of reteaching is necessary. How may this best be done? Some suggest a further curtailing of the obsolete arithmetic material of the first sixth grades while others insist on keeping the fundamental skills in use through wise applications in Grades VII, VIII, and IX.

The authors of the several standardized test lesson series suggest a plan for testing, diagnosis, remedial work, and practice until standards have been reached. In the operation of this plan, speed seems to be uppermost in the minds of a great many teachers and pupils. Diagnosis and remedial work are slighted because of either misconception of the purpose of standardized tests or lack of time in which to give individual attention to those needing it. The result in many classrooms is a continued practice of errors.

The authors of a recent series of Arithmetic Work-Books [8] have sensed the persistent abuse of drill devices. They therefore have not left the diagnostic and remedial features to chance but have included in the pupil books explanations such as teachers should give. Good results, however, depend upon the ability of pupils to read and understand the explanations given.

In several school systems using standardized drill exercises a plan for divorcing speed from the ability to manipulate correctly has

[8] Knight, F. B., Ruch, G. M., and Studebaker, J. W., *Arithmetic Work-Books*. Scott, Foresman & Co., 1926.

been employed. According to this plan the difficulties are first taken one at a time with whole classes during the regular drill period of from ten to fifteen minutes daily. Those failing on a short inventory test of a difficulty are retaught to the point that the correct method has been mastered. Following this systematic review for purposes of reteaching only, the ground is retraced for the purpose of attaining the speed standards. The disadvantage of the plan lies in the fact that a few pupils do not need reteaching. This number in the average and under-average classroom situations, however, is generally small; so small that it is safe to predict that between fifty and seventy-five per cent of the pupils of such classrooms will fail to correctly solve problems involving multiplication of mixed numbers and placing of the decimal point in the division of decimal fractions. The test results given on pages 165-66 substantiate this statement.

The reasons for stressing the fundamental processes with integers, common and decimal fractions are important enough to give the matter serious consideration. Most important of these reasons is the complaint of the business and industrial world that our schools are satisfied with low standards of attainment as evidenced by the performances of American boys and girls. These complaints refer to a failure to do the simple computations needed in everyday life. They are directed for the most part at the group of pupils who leave school when the law permits. This group represents in many cases pupils finding the school curriculum too difficult for them. Whatever the methods used, the junior high school cannot afford to send boys and girls into business and industry poorly equipped to meet the simpler mathematical needs of everyday life.

III. Individual Differences

Methods of Handling Individual Needs. The school literature of the day abounds with schemes and methods for caring for varying abilities of the junior high school group. In actual practice these amount to little more than an administrative organization based upon a psychological classification of pupils. Textbooks in the main do not provide graded material for the different ability groups.

On what basis can the units of mathematics instruction be organized so as to give to all pupils valuable training commensurate with their abilities? A plan by which differentiation is made between computational and informational content is now being used in the Detroit junior high schools. Exact data cannot as yet be given to fully prove the superiority of this arrangement over other plans. Classroom teachers, however, are satisfied that the social situations are not above the level of understanding for even the duller pupils. Freed of the responsibility for difficult manipulation work they maintain that the duller pupils profit from a study of social topics often thought to be beyond the comprehension of seventh and eighth grade pupils. To illustrate the type of differentiation to which reference is being made, a unit from the Detroit Course of Study in Intermediate School Mathematics is here reproduced.

Unit XIV

FORMS OF REAL ESTATE INVESTMENTS ABOUT WHICH WE SHOULD KNOW

Pupil Goals

A. GENERAL OBJECTIVES

1. To realize importance of real estate as a basis of taxation and wealth.
2. To know some of the precautions to be taken in buying real estate, i.e., entering into contracts, nature of titles, desirability of property, good judgment in buying.
3. To learn something about the conditions of ownership.
4. To appreciate the fact that unusual earnings in real estate are in the same class as large dividends from stocks and bonds.

B. SPECIFIC OBJECTIVES

Minimum Standards. The pupils should:

1. Have some information about the subdivision of property.
2. Understand the functions of real estate firms.
3. Know how a legitimate purchase or sale of real estate is transacted.
4. Know how the titles to property may be examined.
5. Know that contracts or deeds are evidences of ownership or equity.
6. Know how payments on purchases are made.
7. Know important considerations to be made in buying real estate, such as size of lots, pavements, restrictions, school, church, recreation, and transportation facilities, taxes, type of surroundings, standing of selling agency, and development of property.

Additional Standards

8. Know the difference between buying property as an investment and buying for saving purposes.
9. Know how to figure the cost of home owning.
10. Know how to determine profits and losses on real estate investments.
11. Know some of the methods of determining possible growth and development.
12. Know what first and second mortgages are.
13. Learn the different methods of financing home and property owning.

Suggestions for Lesson Plans

C. INTRODUCTORY MATERIAL

Material for Minimum Achievement Group

1. Teacher and pupils may trace the growth of the city.
2. The value of the school property now and when the school was built may be determined.
3. The value of homes in the neighborhood when built and at the present time may be considered.
4. The extent of subdivisions in and about Detroit may be discussed.

Material for Greater Achievement Group

5. Pupils may tell how undeveloped property is bought and sold, including the cost per acre, cost of development, selling and final sale price on lots.
6. The difference between buying a permanent home and buying a house as investment proposition may be considered.

D. ACTIVITIES

1. Pupils may draw a diagram of a plot of land and divide it into lots of given dimensions.
2. The buying and selling of real estate may be dramatized by organizing the class into real estate company and customers. Agreements may be entered into, contracts made, payments figured, etc.
3. A collection of the forms used in closing deals, contracts, etc., may be made and pasted in notebooks.
4. The precautions to be taken in buying property may be listed.
5. The class may dramatize the procedure in buying or selling a house and lot under the following conditions:

D. ACTIVITIES—*Continued*

5*a*. A man buys a house and lot for $10,000. He makes a down payment of $3,000. There is a mortgage of $4,000 on the house. The seller agrees to sell on a contract and the buyer agrees to assume the mortgage when it is due.

5*b*. A man buys a house and lot for $10,000. He makes a down payment of $3,000. A mortgage is arranged for at the bank and notes are signed. Payments are made and the deed transferred.

E. PROBLEMS

1. What is meant by an option? A down payment? Monthly payment? Interest? Principal?
2. How much do real estate men receive for selling property?
3. What is mortgage? What are the kinds of mortgages?
4. What are land contracts?
5. What is meant by discounting contracts and mortgages?
6. What is meant by foreclosure?
7. What services do development companies render to the community?
8. Why do people not shop for real estate in the same way as they shop for clothing, furniture, and the like?

TYPE PROBLEMS

Minimum Group

1. A house cost $3,800 to build; the lot cost $1,000. To obtain 10% gross income on the investment what should the monthly rent be?
2. How much would the owner obtain clear for the rent of the house each month if his expenses for one year were as follows: Taxes $125, upkeep $30, water bill $12, and street assessment $25?

Additional Group

1. The house can be bought for the cost plus $400, or on the installment plan by paying $65 per month for 8 years. A person could also borrow money at 6% interest and pay cash for the house by paying $1,250 plus interest each year. Which would be the better, buying for cash or on the installment plan?

The above arrangement of a teaching unit was reached through a tryout of a tentative organization which was revised on the recommendation of teachers giving the plan a trial. The plan represents an attempt to judge the work from two points of view, complexity of social situations and difficulty of mathematical computation. For the duller groups both elements were made as simple as possible, thereby permitting all pupils to study the same social topics. The matter of deciding what work should be attempted with the individual classes is left entirely to the teacher because no method of mathematical classification has been devised through which groupings for mathematics instruction might be made.

IV. Coördination between the Junior and Senior High Schools

Present Lack of Coördination. Lack of coördination between junior and senior high schools has in many places operated against the success of junior high school plans for mathematics instruction. This has been due in part to the fact that college entrance is, for most colleges, based upon credit received for work completed during the ninth, tenth, eleventh, and twelfth grades. The tendency in some centers seems to be in the direction of considering for college entrance only the character of work done during the last three years of the senior high school. Until this becomes a fact in practice most junior high schools will find it necessary to complete one year of algebra of the college entrance type during the ninth year. Such a contract may seriously interfere with development of the junior high school mathematics program.

Lack of Proper Records. A second problem of coördination between junior and senior high schools relates to records which pass from one school to the other. Pupils who pass from the lower school to the higher and concerning whom the higher school has no information are almost in the same category as pupils who enter the higher school on transfer from other cities and states. Mere certification of graduation from the junior high school does not materially aid the senior high school group in the classification and programming of pupils.

The senior high school needs to receive records from the junior high school indicating the ability ratings, subject marks, and the teachers' personal estimates of the pupils entering it. In the field of mathematics the latter can be given if the junior high school mathematics course is considered a tryout course for further mathematics work.

Prognosis. Although it is dangerous to pass judgment upon pupils, the newer type of ninth grade mathematics does give school officials some basis for predicting future success along the lines of mathematics study. The writer has dealt with a plan whereby the junior high school mathematics teachers make lists for the senior high school to which their pupils will go, dividing pupils into recommended, doubtful, and not recommended groups for further mathematics study. These recommendations cannot be complied with in all cases, but they do give the senior high school people something definite with which to work. This scheme has brought about a satisfactory relationship between the two schools.

Summary Statement. The following points summarize the point of view herein presented:

1. The writer has taken the stand that the question of "what to teach" in the junior high school mathematics course should deal primarily with the problem of fitting the mass of pupils attending the junior high school for successful performance of social-civic duties as they relate to worthy home membership and citizenship.

2. To remedy the conditions of under-learning common to the mastery of the fundamental processes and the abuse of practice lesson devices, a plan of reteaching the type difficulties in a systematic way was suggested.

3. The problem of individual differences was discussed from the point of view of differentiating within the units of instruction as outlined in the course of study.

4. For a better coördination between junior and senior high schools a plan was submitted which makes definite recommendations with regard to future mathematics courses, based on the ninth grade.

OBJECTIVES IN THE TEACHING OF JUNIOR HIGH SCHOOL MATHEMATICS

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I. THE PURPOSE OF THIS STUDY

General Purpose. No work is ever completely efficient unless the object in view is clearly understood. The great object in studying mathematics in the junior high school is to give to pupils some idea of the general nature and uses of business arithmetic, intuitive geometry, practical algebra, and the simplest part of trigonometry, together with a knowledge of the meaning of a demonstration. All this requires on the part of the teacher the recognition of subsidiary objects which are generally classified as objectives. In this section we shall consider certain important and typical objectives.

Classification. These objectives may be listed like the words in a dictionary; they may be classified according to their probable psychological sequence; they may be arranged according to the branches of mathematics commonly considered; they may be set forth with no regard to order; or some effort may be made to list them in the order which experience has shown will be most helpful to teachers even though there is some duplication of statements and some sacrifice of logical sequence. In this section the last of these five plans will be followed.

It should, however, be repeated that the list is purposely incomplete. It would be a very simple matter to give a thousand or more objectives, but such a list would be lacking in emphasis and would be so wearisome in its details as to repel instead of assisting the teacher who is seeking to obtain a clearer view of the purposes in teaching mathematics in the junior high school. Every thoughtful teacher will mentally add to the list, and some teachers will, and should for their purposes, eliminate such items as fail to commend themselves to their best judgment.

II. Great Central Objectives

Mathematical Objectives. Certain objectives are strictly mathematical, while others are of a more general nature. The former may be classified as follows:

A. AN INTRODUCTION TO THE GENERAL RANGE OF ELEMENTARY MATHEMATICS

1. *The application of arithmetic to business problems.* This means that the pupils should be shown the uses of arithmetic that any well-educated citizen is expected to understand, but not the technicalities of special branches like banking, bookkeeping, or machine-shop practice.

2. *Intuitive geometry.* This means such a knowledge of shape, size, and position as people need to have for purposes of general information.

3. *The algebra of the formula, graph, directed number, and equation.* This means that these four concepts are so important in elementary science, in simple mensuration, and in ordinary business that everyone should know something about them.

4. *The general nature of trigonometry.* This means that all pupils should have some idea of how distances and heights are measured by this simple device. It does not mean that any of the difficult parts of trigonometry are to be studied in the junior high school.

5. *The significance of a demonstration.* This means that every pupil should have the privilege of seeing the nature and of understanding the significance of a mathematical proof. This is best done by a few theorems in geometry. It does not mean that the pupil is to take a difficult course in demonstration, but that he should know the force of the word "demonstrate."

B. SOME APPRECIATION OF THE POWER OF MATHEMATICS

1. *In ordinary life.* This means that the pupil should appreciate something of the power of computation, of its application to common measurements, of the power of the formula to "do things," and of the value of the graph in everyday business.

2. *In related fields of knowledge.* This means that the pupil should see that, without mathematics, there could be no modern

business, no engineering, no machinery beyond the simple lever and the wheel, no currency, no insurance, no travel in the modern sense of the word, no great buildings, and no sciences. He cannot be brought to see all the applications of mathematics in these fields, but he can be led to see enough to show him the enormous power of the subject.

3. *In further mathematical work.* This means that we cannot progress in mathematics without the elements upon which we depend. Men do not become great merchants, bankers, or scientists without a knowledge of the mathematics of the junior high school, acquired there or in other ways. Great tunnels are planned, great bridges are built, and great ships sail the seas, only as the result of the knowledge of higher mathematics, and this knowledge is possible only after the earlier mathematics has been mastered.

C. THE INCREASE OF CERTAIN POWERS

1. *Using symbols.* This means that, although the pupil has long been using number symbols (1, 2, 3, . . . ; I, II, III, . . .) and such signs as $+$ and $-$, the power to use mathematical symbols must be extended to include other forms (a, $\sqrt{b}$, $-n$, x^4, . . .) if any further progress is to be expected.

2. *Analyzing relations.* This means that the pupil should be trained to select the essential facts from among those which are not essential to the solution of a problem; that he should connect these in a logical manner; and that he should so use them as to attain a result that he can depend upon as accurate. This power is fundamental in all scientific and economic fields.

3. *Constructing graphs.* This means that the use of graphs has now become so common that everyone should have some idea of how to represent simple statistics by a bar graph or a curve-line graph, some of which work they may have done earlier in the grades. It does not mean that they should all know how to construct more complicated types.

4. *Interpreting graphs.* This means that people generally need to know how to interpret or to find the meaning of such graphs as are ordinarily seen in newspapers, magazines, government reports, and popular scientific journals and books. These include not only bar and curve-line graphs, but circular graphs and other types that are too difficult for the pupils to construct.

D. FOSTERING THE STUDY OF MATHEMATICS

1. *For the improvement of more advanced mathematics.* This means that such pupils as show a taste for the subjects should be discovered and then be encouraged to pursue their studies farther. Most parts of advanced mathematics have practical applications and all are concerned with the discovery of truth. The possible use of all this work cannot be foretold. It does not mean, however, that pupils should be forced to study any of the more advanced parts of mathematics for which they have no taste.

2. *For mental pleasure.* This means that mathematics offers mental pleasure to a great many pupils. There is the same reason for gratifying this taste as for gratifying a taste for literature, music, or the fine arts. It does not mean that any prolonged attempt should be made to compel a pupil to like what is repugnant to him.

General Objectives. As already stated, certain objectives are of a general nature instead of being strictly mathematical. The attainment of these objectives, however, is facilitated by the study of mathematics. The objectives themselves may be classified as follows:

A. ESTABLISHING CERTAIN HABITS

1. *Neatness and method.* This means that the pupil's written work should be neat, uniform in method of presentation, and clean in general appearance. It should suggest to him the desirability of neatness and cleanliness, and of the methodical arrangement of all his work. The very nature of mathematics lends itself to this type of habit formation, this being the one science which demands with great emphasis these several features.

2. *Thinking.* This means that mathematics should afford and does afford an unusual opportunity for concentration and for the play of constructive imagination.

3. *Moral conduct.* This means that mathematics affords a constant opportunity of displaying honesty to one's self. The pupil should constantly ask, "Is this unquestionably correct? Have I checked this operation? Am I honest to myself in being positive of this result before I proceed farther and cause myself trouble through my error at this point? Can I rely upon myself and then lead others to rely upon me?"

4. *Character.* This is an outgrowth of the preceding paragraph. It means that mathematics constantly affords opportunities to encourage a reverence for truth, for absolute accuracy of statement, for beauty of form, and for the recognition of the unity of "the good, the true, and the beautiful."

B. EXERCISE IN FUNDAMENTAL MODES OF THOUGHT

1. *Simplicity of language.* No science lends itself to the cultivation of simplicity of language to the extent found in the case of mathematics. The whole range of this branch of knowledge is characterized by succinctness of expression and the elimination of complex phraseology. In elementary mathematics there has been a special effort made in recent years to avoid the use of unnecessary technical terms and of pedantic expressions of all kinds. The effect of all this upon the pupil's style of speaking and of writing cannot fail to be salutary.

2. *Accuracy in reasoning.* In the junior high school the pupil becomes more conscious of the fact that one of the essential features of mathematics is the accuracy of its processes and of the reasoning employed in its solutions. Nowhere in the entire curriculum is such accuracy essential, and nowhere else does he receive such training in this phase of mental activity.

3. *Originality in thought.* This is found in connection with all branches of knowledge. A pupil may show originality in literature, music, historical interpretations, and science just as well as in mathematics. It forms a definite objective in every line of thought. It is therefore of necessity a definite and important feature in the junior high school work in mathematics. The pupil should be encouraged to use all the originality he can in attacking solutions of whatever problems are presented. To compel a pupil to solve a problem as the teacher directs or as the textbook solves it is to stifle a pupil's originality. If his plan is not as economical as some other, that may be shown him afterwards.

III. PSYCHOLOGICAL SEQUENCE OF OBJECTIVES

General Purpose. Having considered the great central objectives in the teaching of junior high school mathematics, we shall now suggest a list of certain important detailed objectives, arranging them with some attention to their psychological sequence.

A. APPRECIATION OF MATHEMATICS AS A USEFUL ART

1. *Power of expressing data systematically.* This means that, from certain facts or data ("things given") that are available for the solution of a problem, we shall be able to select the relevant data, tabulating them in the most systematic way by which to express the facts of the case and to make the interpretation simple.

2. *Scrutinizing these data in solving problems.* This means that we should be careful to find whether the one who gathered the facts was reliable; whether he has used the best methods (graphical or otherwise) for arranging them; and whether by further examination we cannot make some improvement upon his work.

3. *Organizing these data as an aid to memory.* This means that, by means of graphs or otherwise, we should learn how to develop methods for making certain data stand out prominently in our minds whenever it is important that such data be immediately recalled. For example, it may be necessary to remember certain relative values which, when represented by curvilinear graphs, will be more easily remembered than the numbers upon which they are based.

4. *Succinctness of mathematical statements of laws as formulas.* This means that the pupil should appreciate the great value of the formula as a practical substitute for the long rule that was formerly used. For example, to state that $A = \frac{1}{2}bh$ is to use an expression that is far more succinct and useful than the old rule for finding the area of a triangle.

5. *The equation as an aid in using formulas.* This refers to the fact that, from the interest formula $i = prt$, we can readily obtain a formula for p, r, or t, by the ordinary methods of very simple equations. Therefore, from this single formula we can easily find three others, this being much better than the old plan of learning four long rules. The pupil should see that this is the chief value of an equation,—not the solving of a long list of mere number puzzles.

B. APPRECIATION OF MATHEMATICS AS A SCIENCE

1. *Significance of symbolism.* This means that the pupils should see what a powerful tool we have in the symbols of arithmetic and

algebra, as in the significance of the expression $a^2 + b^2 = c^2$ and $a^2 = c^2 - b^2 = (c + b)(c - b)$.

2. *Interlacing of branches.* This means that there is no part of mathematics that is not related to all the other parts. For example, the equation $a^2 + b^2 = c^2$ is a statement that is related to arithmetic, algebra, geometry, and trigonometry.

3. *The relation of mathematics to allied subjects.* This means, for example, that the subject of latitude and longitude in geography is closely related to intuitive geometry, arithmetic, trigonometry, and analytic geometry; that a subject like physics could hardly exist without arithmetic, algebra, geometry, and trigonometry, and that it also makes extensive use of higher mathematics; and that all the other sciences are constantly indebted to mathematics of various kinds.

4. *The relation of formulas to general truths.* This means that many laws which are not directly concerned with algebra, for example, are now expressed in algebraic language. For example, in any crystal the number of faces plus the number of vertices is equal to the number of edges increased by 2; that is, as a formula we have $F + V = E + 2$.

5. *The eternal verities of mathematics.* This means that a correct mathematical statement is true "yesterday, to-day, and forever." A human law may change, every building must some time decay, all life is continually changing but $(a + b)^2$ is always equal to $a^2 + 2ab + b^2$.

6. *The universality of functional relationships.* This is a common but unnecessarily difficult way of saying that everything in this world depends upon something else, just as the area of a circle depends upon the length of the radius, the length of the diagonal of a square depends upon the length of a side, and the distance traveled is a function of the rate and the time. For example, the height of a tree depends upon the kind of tree it is, and also, for a certain period, upon its age.

7. *The value of mathematics for its own sake.* The French have an expression *l'art pour l'art* ("art for art's sake"). This means that, while painting may be useful on a barn, a painting of a landscape may be beautiful and may be admired for its artistic qualities. So geometry has various uses, but we may like it even more for its beauties, its standards of truth, and its succinctness of statement.

8. *Relation to the Infinite.* This means that, as the pupil progresses, he will see through mathematics an approach to some conception of the Infinite that he can never see in any other field of study.

9. *Recreational side of mathematics.* This means that mathematics has its recreational side if only we will seek it out. It is no mere chance that a "magic square" like this one exists. In this case each row, each column, and each diagonal has 15 for its sum. This magic square is probably the oldest record that we have of mathematics in Eastern Asia. All mathematics is a kind of game to those who know how to play it.

8	1	6
3	5	7
4	9	2

10. *Connection with art.* This means that mathematics enters into all great architecture; into a large part of decorative art, as in symmetry, spiral forms, ellipses, and regular polygons; into music; and even into painting, as in the study of perspective.

11. *Rhythm of mathematics.* This means that through all mathematics there runs a rhythm. A child likes to say "5, 10, 15, 20, 25, 30." or "2, 4, 6, 8, 10, 12," and the pupil in algebra takes pleasure if encouraged to do so, in the expansion of $a + b$ to the various simpler powers.

12. *Relation to Nature.* This means that, if the idea is suggested to him, the pupil will tend to take an interest in seeing how constantly mathematics enters into natural forms. The cross section of a banana, an apple, or the seed of a rose will serve to initiate him into a search for such relations.

C. APPRECIATION OF THE HISTORIC GROWTH OF MATHEMATICS

1. *Mathematics a moving stream.* The history of mathematics as a separate study is not desirable in the junior high school, but the teacher will add greatly to the interest in the subject by referring to the fact that mathematics started as a tiny brook in the remote mountains of the Past and that it has increased in power as it has watered the lands all through the centuries.

2. *Significance of our numerals.* All pupils should see the power of our numerals to readily express large numbers; they should know that these numerals are only about a thousand years old in Europe, and that they probably came from India by way of Arabia; and

they should also know that there have been many other systems, out of which we still keep only the Roman.

3. *Growth of the fraction.* This means that pupils should know that, for some centuries, our "common fraction" was, as its name says, common all through Europe and America, but that to-day the decimal is the more often used, as in fractions of a dollar. They should know that we now use only halves, thirds, fourths, and eighths in most of our everyday business. The decimal fraction, for practical use, is only about a hundred fifty years old, although it was known considerably earlier.

4. *Displacement of compound numbers.* It is a valuable thing for the teacher and an interesting one for the pupil to know that, until very recently, such compound numbers as 10 mi. 32 rd. 4 yd. 2 ft. 8 in. were taught and used. About all that we have left in common use to-day is found in cases like 2 ft. 8 in., 1 lb. 4 oz., and 2 hr. 45 min. The compound number has been almost entirely displaced by the common and decimal fraction.

5. *Merging of decimals and per cents.* This means that there is no mathematical difference between 6% and 0.06, and that nowadays per cents are treated as a part of decimals, or immediately in connection with them.

6. *Systems of measure.* This means that the pupils should see that the world discards continually materials that have outgrown their usefulness. A hundred years ago people used candles, they traveled long distances on horseback, they measured land by roods, they measured cloth by ells, and they used the "long hundredweight." We now use fewer measures, and a large part of the highly civilized world uses the metric system. For general information we need to know about the meter (radio wave-lengths), the kilowatt (electric meters), the gram and the cubic centimeter (medicine), the kilometer (international sports), and, in general, about the metric system in comparison with the older and more difficult one which we generally use.

7. *Change from rule to formula.* This means that we should all see the value of the change made in the last few years in changing the emphasis from the long and often difficult rule to the brief and simple formula.

8. *Development of symbols.* This means that we could have no formulas without symbols, and that our ordinary symbols of algebra were invented to meet world needs about three centuries ago, thus

revolutionizing mathematics and making possible such work as that of the junior high school.

9. *Growth of applied mathematics.* This means that the world has always been applying its mathematics, and that these applications have grown with world needs. No one thought, when trigonometry began to be well known in the schools about three centuries ago, that it would be used to-day to find the size of our universe or that it would be made so simple that we could teach it in Grade VIII or Grade IX.

10. *Great names and great periods.* This means that, just as we know who William the Conqueror was, so we should know that Pythagoras was also a great conqueror, but in another field, and that his story is quite as interesting as that of the Norman who subdued England; that it is a good thing to know about Washington as the father of his country, and also about Euclid, who was the father of elementary geometry; and that the Elizabethan period means much to the world, but that the great period in which geometry developed in Greece means much more in the history of civilization.

D. ATTITUDES OF MIND TO BE DEVELOPED

1. *Responsibility for accuracy.* It is not merely a habit of mechanical accuracy that we develop, but also a feeling of personal responsibility for accurate reasoning and accurate results.

2. *Satisfaction with thorough work and precision of statement.* It is sometimes asserted that our great industries tend to make men but little more than parts of a machine and that the pride of achievement is lost. This is not entirely true; indeed the amount of truth in the statement is probably less than we think; but at any rate it is not true in the intellectual l[illegible] of the school. There is such a thing as just pride and hones[illegible] satisfaction with a good piece of work, and nowhere can this be better fostered than in the study of mathematics.

3. *Commonsense estimates of results.* This means that a pupil should early cultivate an attitude of mind that leads him to make a sensible guess as to a result before he begins his solution of a problem in arithmetic There is no better rough check upon the accuracy of his wo

4. *Dissatisfaction with vague results.* For example, when called

upon to interpret the meaning of a graph a pupil should feel dissatisfied if he has not something reasonable and positive to offer; and when he solves a problem, he should not be satisfied with a result that is not clearly expressed.

5. *Recognition of irrelevant data.* It is one of the valuable features of geometry, even of the intuitive kind, that a considerable number of things are known about a figure, but that out of these we need to use only a few to prove what we wish or to compute a required length or area. The attitude of mind that leads us directly to the rejection of unnecessary facts is a valuable asset to anyone.

6. *Discrimination between the true and the false.* This means that in mathematics we constantly meet with the question, "Is this true or is it not?" It is a healthy attitude of mind to develop,—that of weighing the two sides of every question of this kind.

7. *Desire to analyze a complex situation into its components.* This means that we should cultivate the habit of thinking of the details that make up the mass. For example, a graph may rise violently between two points; in interpreting the graph we need to state the reasons for this rise, and a large number of possibilities suggest themselves. It is a good attitude of mind that leads us immediately to analyze this mass of possible causes into its component elements and be able to select the most powerful ones.

8. *Self-reliance in attacking a problem.* There is a great difference between self-conceit and self-reliance. Mathematics should cultivate that attitude of mind that leads a pupil to say, "I not only *can* do this thing, but I *will* do it."

9. *Desire to search out the truth.* An old philosopher once said, "There is no pleasure comparable to standing on the vantage ground of truth." It is a great asset to feel this and to have an attitude of mind favorable to the search for those things which are true. Mathematics is one of the foremost sciences in fostering an attitude of mind that always searches out the truth,—not merely the probable, but the actual.

10. *Constant seeking for applications of mathematics in daily life.* Such an attitude of mind will go far to awaken and to maintain an interest in all lines of mathematics that the pupil studies. It will be a constant source of surprise and interest to find out how many such applications we can see if we but search for them.

11. *Interest in developing skill in mathematics.* This means

that in teaching we should endeavor to develop a similar spirit to that which leads a boy to take pride in his skill in tennis, a girl in basket-ball, and an adult in golf. We can hardly hope to do this to the same degree as in the case of ordinary outdoor sports, but we can approach the same objective.

12. *Desire to generalize.* This means that we should seek to establish that attitude of mind which leads to the discovery of general laws governing a range of special cases. This is illustrated in the case of the sum of the angles of a triangle. What is the sum in the case of a 4-sided figure? one of five sides? one of six sides? —and so on. What is the general law? Does this hold in the case of a 2-sided polygon? There are few things in geometry that give greater pleasure than the joy of generalizing.

E. IDEALS TO BE CULTIVATED

1. *Devotion to truth.* This means that mathematics is concerned not only with proving that a statement is true, but in discovering truths for ourselves. In the preceding paragraph a question was raised about a 2-sided polygon, and the pupil who discovers that it really exists, that the sum of its interior angles is $0 + 0$, and that it obeys the laws of other polygons, has made a step toward devoting himself to the search for truth.

2. *Originality in action.* This means that one ideal which every pupil should seek to cultivate is that he may depend only upon himself. If this is done early in mathematics, a pupil will soon come to feel an independence that will lead him to pay little attention to book proofs and to depend more and more upon his own originality.

3. *Neatness in solutions.* This means that good taste should be an ideal in our written work as it should be in our dress and in the furnishing of our rooms. People of good sense and refined taste do not care for a vulgar display of wealth in such matters but they appreciate neatness. In our work in mathematics slovenliness in writing, in computing, and in the drawing of figures usually means slovenliness in thinking.

4. *An appreciation of our relation to the universe.* This means that it is chiefly through mathematics that we attain some grasp of this relationship. It is mathematics that tells us much of what we know of the infinitesimal as it is typified in the electrons in an

atom, and it is mathematics that reveals to us the grandeur of the infinite as typified in the space about us. Fortunate the class that has a teacher filled with such ideas and ideals and who is able to pass these on to the pupils without wasting time or making drudgery out of a great epic.

5. *Respect for one another as small portions of the Infinite.* This means that we all have a feeling of infinity and of ourselves as part of some stupendous whole, and that mathematics cultivates this tendency. Such ideals make for the brotherhood of man, and pupils are apt to appreciate this if the fact is not made so dull as to become mere drudgery.

6. *Regard for the beautiful.* This means that, for example, the ideal of the beautiful exists in intuitive geometry as really as it does in nature or in the fine arts. It is a dull class that cannot be inspired to admire the beautiful in this subject, if only the teacher also admires these attributes, but does not talk too much about this fact.

7. *Loyalty to the family, the community, and the state.* This means that thrift is not the acquiring of wealth for ourselves alone, but that it is inspired by a feeling of loyalty to the family, a feeling that old age must not make itself a burden; that insurance is not a burden, but that it is an evidence of loyalty to the community, each contributing to another's loss; and that taxes are simply an evidence of loyally repaying the state for protecting us and our property, for educating us, for looking out for public health, and for providing good roads and pure water.

8. *Respect for a good reputation.* This means that, in the study of banking and commerce, of investments, and of whatever has to do with care and honesty, there should stand before the pupils the ideal of the good citizen, the man with a reputation that makes him trusted by his fellow men.

IV. Important Concepts of Elementary Mathematics

General Purpose. The purpose of considering these concepts is to allow the teacher to make certain that the greater ones have been satisfactorily brought to the attention of the pupils. It is not the purpose to consider at this time the important abilities to be cultivated, these being presented later

Classification. In order to aid the teacher who is completing the work in some special branch, or who, in a course in general mathematics, desires to see if the important concepts in each topic are recognized, these concepts are here classified according to special subjects. It must, however, be understood that various concepts included in one branch are also needed in one or more other branches. In all cases it is expected that the concepts shall be understood but not necessarily defined; and if defined, that the definitions will not in general be memorized.

A. IMPORTANT CONCEPTS IN BUSINESS ARITHMETIC

Account, bank
Amount of a commission
  of a debt
  net
  of a note
Annuity
Approximation
Assessment
Average
Bank
Bill
Bond
Broker
Brokerage
Budget
Calorie
Capital
Certificate, postal savings
Checks, on operations
  bank
  traveler's
Collateral
Collector of taxes
Commercial paper
Commission
Compensation, workmen's
Corporation
Coupons
Credit, letter of
Creditor
Debtor
Deposit slip
Discount, bank
Discount, on bills
  on a note
Dividends on stock
Draft
Drawee
Drawer
Drawing, scale
  working
Duty on imports
Face of a bond
  of a note
  of a policy
Gain
Graphs
Income
Indorsement
Insurance
Interest, simple
  compound
Invoice
Loss
Lumber measure
Maker of a note
Margin
Maturity of a note
Money order
Mortgage
Note, at a bank
  with collateral
  demand
  interest-bearing
  joint
  negotiable

Note, promissory
Overhead
Par
Payee
Payments, partial
Per cent
Policy
Post Office
Premium
Price list
    marked
    selling
Principal
Proceeds
Profit
Quotation, stock
Rate of commission
    of discount
Rate of income
    of insurance
    of interest
    of tax
Receipt
Revenue, customs
    internal
Share of stock
Statement, bank
    store
Stock in a corporation
Tariff
Taxes
Trade acceptance
Valuation, assessed
Value, face
    par

B. IMPORTANT CONCEPTS IN INTUITIVE GEOMETRY. SEE ALSO E

Altitude
Angle, acute
    bisector
    of depression
    of elevation
    obtuse
    size of an
    straight
Angles, adjacent
    alternate
    complementary
    corresponding
    equal
    interior
    made by a transversal
    supplementary
    of a triangle
    unequal
    vertical
Arc
Area
Axis of symmetry
Base of a polygon
    of a solid
Bisector of an angle
    of a line
    perpendicular
Center of a circle
    of a regular polygon
    of a sphere
Central angle
Circle
Circumference
Compasses
Complement
Cone
Congruence
Construction of a figure
Cube
Curve
Curve surface
Cylinder
Degree, angular
Diagonal of a polygon

B. IMPORTANT CONCEPTS IN INTUITIVE GEOMETRY—*Continued*

Diagonal of a solid
Diameter
Direction
Distance
Ellipse
Equality
Figures, congruent
  geometric
  similar
  symmetric
Formula
Height of a plane figure
  of a solid
Hemisphere
Hexagon
Hypotenuse
Length
Line, broken
  curve
  horizontal
  segment
  slanting
  straight
  vertical
Lines, equal
  oblique
  parallel
  perpendicular
  proportional
Measurement
Midpoint
Octagon
Parallel
Parallelogram
Pentagon
Perimeter
Perpendicular
Pi (π)
Plane
Point
Polygon
Position
Prism
Proportion
Protractor
Pyramid
Quadrilateral
Radius of a circle
  of a cone
  of a cylinder
  of a regular polygon
  of a sphere
Ratio
Rectangle
Rectangular solid
Rhombus
Root, square
  cube
Ruler
Scale drawing
Section
Sector
Semicircle
Side of an angle
  of a polygon
Similarity
Size
Solid
Sphere
Square
Straight angle
  line
Supplement
Surface, curve
  plane
Symmetry
Transversal
Trapezoid
Triangle, acute
  equilateral
  isosceles
  obtuse
  right
Triangles, congruent

Triangles, similar
Vertex of an angle
  of a cone
  of a polygon
Vertex of a solid
  of a triangle
Volume

C. IMPORTANT CONCEPTS IN BEGINNING ALGEBRA

Abscissa
Addition
Aggregation
Axiom
Binomial
Cancellation
Checks on operations
Coefficient
Constant
Coordinates
Cube of a number
Degree
Denominator
  lowest common
Dependence
Dependent variable
Division
Elimination
Equations, checking
  of condition
  degree of
  equivalent
  fractional
  inconsistent
  indeterminate
  linear
  literal
  numerical
  quadratic
  simultaneous
Exponent, fractional
  integral
  negative
  positive
  zero
Factor, literal
  monomial
Factor, numerical
Formula
Fraction, algebraic
  as an exponent
  proper
  signs of a
  terms of a
Fractions, clearing of
Graphs, bar
  broken-line
  circular
  curve-line
Identity
Independent variable
Index of a root
Known quantity
Members of an equation
Monomial
Multiple
  lowest common
Multiplication
Negative exponent
Numbers, algebraic
  directed
  negative
  positive
Numerators
Operations, with directed numbers
  signs in
Ordinate
Origin
Parentheses
Polynomial
Power
Product
Proportion

C. IMPORTANT CONCEPTS IN BEGINNING ALGEBRA—*Continued*

Quadratic equation
  trinomial
Quotient
Radical
Ratio
Reciprocal
Reduction of fractions
Root, cube
  of an equation
  index of a
  square
Satisfy an equation
Scale, algebraic
Sides of an equation
Signs
Solution, checking a
  of an equation
Square
Substitution
Subtraction, signs in
Symbols
Terms, of a fraction
  lowest
  of a polynomial
  of a ratio
  similar
Transposition
Trigonometry. See D. following
Trinomial
Unknown quantity
Variable
Variation
Zero
  as an exponent

D. IMPORTANT CONCEPTS IN BEGINNING TRIGONOMETRY

Cosine
Cotangent
Functions
Indirect Measurement
Sine
Tangent
Trigonometry

E. A FEW OPTIONAL CONCEPTS IN BEGINNING DEMONSTRATIVE GEOMETRY IN ADDITION TO THOSE ALREADY LISTED UNDER B (INTUITIVE GEOMETRY) TO BE GIVEN IF REQUIRED IN THE JUNIOR HIGH SCHOOL

Angle, central
  exterior
  inscribed
  oblique
Axiom
Chord
Converse
Corresponding parts
Demonstration
Distance, between parallel lines
  from a point to a line
Foot of a perpendicular
Inclination
Intercept
Lines, concurrent
Locus
Median
Oblique lines
Postulate
Problem
Proof, nature of a
Pythagorean Theorem
Side, included
Sides, adjacent
Solution, nature of a
Tangent
Theorem

V. Abilities in Arithmetic

General Purpose. One of the most notable changes in the teaching of arithmetic in the last quarter of a century is that which relates to the ability of pupils to do certain specified and important things in the way of computation and to solve certain types of problems. It is the purpose of this discussion to bring into prominence only the great features of arithmetic which demand specific abilities. In addition to these it is desirable but not necessary to acquire the ability to use the slide rule.

Classification. The abilities needed in arithmetic may be classified under five general heads, as follows:

A. Fundamental Operations

Ability to perform accurately the fundamental operations with respect to

1. Whole numbers.
2. Decimals extending to ten-thousandths.
3. Fractions with denominators 2, 3, 4, 8, and 16; less commonly, 5, 6, and 12; possibly, 10.

Ability to express a ratio as

4. A common fraction.
5. A per cent or a decimal.

Ability to use the short cut in multiplying a number by

6. 10.	8. 50.	10. 1000.	12. $16\frac{2}{3}$.	14. $66\frac{2}{3}$.
7. 25.	9. 100.	11. $12\frac{1}{2}$.	13. $33\frac{1}{3}$.	15. 75.

Ability to use the short cut in dividing a number by

16. 10.	17. 25.	18. 50.	19. 100.	20. 1000.

Ability to express in decimal and per cent form the fractions

21. $\frac{1}{2}$.	23. $\frac{2}{3}$.	25. $\frac{3}{4}$.	27. $\frac{3}{8}$.	29. $\frac{7}{8}$.
22. $\frac{1}{3}$.	24. $\frac{1}{4}$.	26. $\frac{1}{8}$.	28. $\frac{5}{8}$.	30. $\frac{1}{6}$.

Ability to

31. Find a square root to the nearest hundredth.
32. Check all operations.
33. Estimate results in advance.
34. Use such tables as those of square and cube roots.

B. PER CENTS

Ability to find

1. Any required per cent of a number.
2. The net price of an article on which a discount is given.
3. What per cent one number is of another.
4. A number of which a certain per cent is given.
5. Per cents of increase or decrease.

Ability to use the fractional equivalents of

6. 12½%.	8. 62½%.	10. 25%.	12. 75%.	14. 33⅓%.
7. 37½%.	9. 87½%.	11. 50%.	13. 16⅔%.	15. 66⅔%.

Ability to understand the meaning of such expressions as

16. 2½%.	18. 3¾%.	20. 5%.	22. 100%.	24. 1.25%.
17. 3.5%.	19. 5.75%.	21. 0.5%.	23. 125%.	25. 12.5%.

C. DENOMINATE NUMBERS

1. Ability to know and to use the common tables of measure.
2. Ability to add or subtract in such cases as the one here shown, the work being generally limited to feet and inches, yards and inches, or pounds and ounces.

5 ft. 8 in.
2 ft. 9 in.

3. Ability to multiply or to divide in such simple cases as that of 8 ft. 6 in. by 3.

Ability to perform such simple reductions as that of

4. Feet to inches.
5. Yards to inches.
6. Pounds and ounces to ounces.
7. Inches to feet or yards.

D. STATISTICS AND STATISTICAL GRAPHS

Ability to make such statistical tables as those relating to

1. Height of pupils.
2. School attendance.
3. Weather reports.
4. Growth of a plant.

Ability to make and read such tables as those referring to

5. Height in relation to weight.
6. Weight in relation to age.

Ability to draw, with or without squared paper, graphs in the form of

7. Bar diagrams. 8. Broken lines. 9. Curve lines.

Ability to do each of the following:

10. Find the average in a group of numbers.

11. Interpret graphs of various kinds, including those of Ex. 7-9, and also circular graphs and pictorial graphs such as are found in magazines and newspapers.

12. Criticize graphs that give a false impression of the statistics concerned.

13. Decide upon the best type of graph for a given set of statistics.

14. Locate points with respect to two perpendicular axes, OX and OY, using x for the abscissa and y for the ordinate.

15. Compare two or more statistical graphs drawn with respect to the same axes.

16. Understand and use directed numbers in connection with graphs.

17. Select a proper scale for a graph.

E. BUSINESS FORMS AND DEVICES

Ability to do each of the following:

1. Keep a personal account book.
2. Keep a cash account for the home.
3. Make a budget showing probable receipts and expenses for the year.
4. Make out a deposit slip for a bank.
5. Write and indorse checks.
6. Make out a bill of goods.
7. Check such a bill as a grocer might send.
8. Find the discount on a bill of goods.
9. Read a gas, electric, or water meter.
10. Write a promissory note.
11. Find the date of maturity of a note.
12. Find the time between the date of a note and the date of maturity.
13. Find the interest on a note.
14. Discount a note, (1) when no interest is specified, and (2) when it is interest-bearing.
15. Balance a bank statement of deposits (credits) and withdrawals (debits).
16. Find the commission on the sale of property.

17. Distinguish between stocks and bonds as forms of investment.

18. Understand the meaning of newspaper quotations of the prices of stocks and bonds.

VI. Abilities in Geometry

General Purpose. Geometry is such an extensive branch of mathematics that it is necessary to limit its scope in the junior high school. The limitations to be imposed were considered on page 174. It is now proposed to set forth certain of the most important abilities to be developed within the boundaries thus determined.

Classification. For this purpose we shall first consider the general abilities which we can reasonably hope to develop, and shall then take up those that relate more particularly to certain important details. The following classification represents the features which the teacher will wish to emphasize.

A. GENERAL ABILITIES

Ability to do each of the following:

1. Use the common measures of length, area, and volume.

2. Use the metric measures of length, limited to the kilometer, meter, centimeter, and millimeter; area, limited to the squares of the units of length; and volume, limited to the cubic meter, cubic centimeter, and cubic millimeter. The decimeter will naturally be mentioned, but as a unit of linear measure it is not so important as the others.

3. Read a line segment lettered in either of two convenient ways.

4. Read an angle lettered in any one of three convenient ways.

5. Measure a line segment with a ruler, the result being accurate to the nearest tenth of an inch, or to the nearest millimeter, depending upon which scale is used.

6. Measure a line segment by using dividers (compasses) to transfer its length to a ruler.

7. Find approximate distances on the floor or out of doors by means of pacing.

8. Use squared paper for the purpose of finding the length of line segments transferred to it by the dividers (compasses), and the area of plane figures drawn upon it to scale.

9. Add one line segment to or subtract it from another.

10. Measure an angle by means of a protractor.

11. Given two similar figures, use proportion to compute any side when sufficient data are known.

12. Measure the height of an object by shadow reckoning, using proportion.

13. Measure the distance to an inaccessible object by means of a scale drawing or else by proportion.

14. Locate a place by using a horizontal and a vertical axis, as in longitude and latitude.

B. DRAWING AND CONSTRUCTING

Ability to do each of the following:

1. Distinguish between drawing (either freehand or with the ruler and protractor) and construction (with only the ruler and compasses).
2. Understand a scale drawing of a simple plan of a building.
3. Understand the meaning of a map drawn to scale.

Ability to draw figures of the following kind, using the ruler, protractor, and, if convenient, a draftsman's triangle:

4. A line segment of given length.
5. An angle of a given number of degrees.
6. A right angle.
7. A line parallel to a given line.
8. A square with the side of a given length.
9. A rectangle of any convenient size.

Ability to perform the following constructions with a ruler and a pair of compasses:

10. Construct a circle with a given radius.
11. Bisect a given arc.
12. Bisect a given angle.
13. Bisect a given line.
14. Construct a right angle.
15. At a given point on a line construct a perpendicular to the line.
16. From a given point outside a line construct a perpendicular to the line.

17. Divide a given line segment into a given number of equal parts.

Ability to construct the following figures, using only a ruler and a pair of compasses:

18. An equilateral triangle having a given side.
19. An isosceles triangle having the base and one of the equal sides given.
20. An angle equal to a given angle.
21. A line parallel to a given line.
22. An angle equal to the sum of two given angles or to the difference between two given angles.
23. A regular hexagon inscribed in a circle.
24. A square inscribed in a circle.
25. An equilateral triangle inscribed in a circle.
26. A triangle having two sides and the included angle given.
27. A triangle having two angles and the included side given.
28. A triangle having the three sides given.
29. A square having its side given.
30. A rectangle having two adjacent sides given.
31. The center of the circle of which an arc is given.
32. Angles of 30°, 45°, and 60°.
33. The perpendicular bisectors of the sides of a triangle.
34. The bisectors of the angles of a triangle.
35. The perpendiculars from the vertices of a triangle to the opposite sides.
36. The medians of a triangle.
37. A copy of a given geometric design.

C. MAKING CORRECT INFERENCES

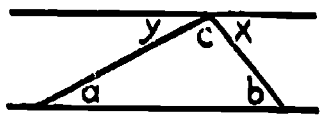
Ability to make correct geometric inferences with respect to such simple cases as the following:

1. The congruence of triangles.
2. The alternate angles formed by a transversal cutting two parallel lines.
3. The sum of the interior angles of a triangle.
4. The parallelism of two perpendiculars to the same line.
5. The similarity of triangles having the three angles of one respectively equal to three angles of the other.

D. PROVING GEOMETRIC STATEMENTS

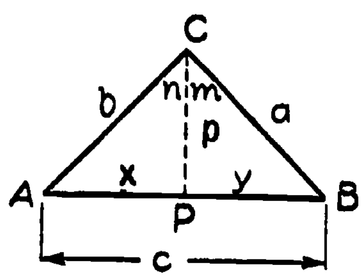
Based upon the inferences made above, under C, or upon simple proofs based upon the axioms and postulates of elementary geometry, to prove a few propositions leading to some celebrated theorem, the following being a sample of the sequence to be used:

1. The sum of the angles of any triangle is 180°. Based upon the assumption that alternate angles are equal. Hence $a = y, b = x$, and so $a + b + c = x + c + y = 180°$.



2. The Pythagorean Theorem.

Based upon these steps:



$$\angle C = 90° \text{ (given)},$$
$$\angle s \text{ at } P \text{ are } 90°,$$
$$\angle B + \angle m = 90°,$$
$$\angle B + \angle A = 90°,$$
hence $\angle B + \angle m = \angle B + \angle A$

and $\angle m = \angle A$

Similarly $\angle n = \angle B$

Therefore the three angles of ΔAPC are respectively equal to the three angles of ΔCPB, and to those of ΔACB. Therefore all three triangles are similar. Therefore

$$\frac{c}{a} = \frac{a}{y}, \text{ whence } a^2 = cy;$$

and

$$\frac{c}{b} = \frac{b}{x}, \text{ whence } b^2 = cx.$$

Adding,

$$a^2 + b^2 = cy + cx = c(y + x) = c^2.$$

That is, in any right triangle

$$a^2 + b^2 = c^2$$

The carrying out of the proof will require about a week, but at the end the pupils will have the satisfaction of having proved one of the great theorems of the world, and they will have an idea of the meaning of geometric proof.

These two propositions are merely typical. Others, with other sequences, may be used.

VII. ABILITIES IN ALGEBRA

General Purpose. It should be reiterated that the great objective in elementary algebra is the ability to use formulas. This means that pupils should be able (1) to evaluate a formula, and (2) to derive one formula from another. The second of these is somewhat less clearly (for the pupil) stated as the ability to "change the subject" of the formula. The pupil should understand that a formula is a shorthand statement of a rule. He should also see that a simple formula such as he uses may be represented by a graph, realizing that a rule is a translation of a formula, and a graph is its picture. The rest of elementary algebra is an elaboration of these principles. It includes equations (useful in deriving one formula from another), operations with algebraic expressions (useful in solving equations), and other features set forth in the following classification.

Classification. The classification of these abilities is based upon this modern view rather than upon the ancient one of beginning with operations of doubtful value and for which, at the time, the pupil could see no use.

A. FORMULAS

1. *Ability to discover certain rules and to translate these into formulas.* For example, the pupil should be able to discover that $a^2 a^3 = a^5$, to infer the rule, and then to write the formula

$$a^m a^n = a^{m+n}$$

2. *Ability to translate formulas into rules.* For example, he should be able to read the rule from the formula

$$(a+b)^2 = a^2 + 2\,ab + b^2$$

and to apply this to finding the square of a number like 72.

3. *Ability to evaluate formulas.* This means that, for example, given the formula

$$C = 2\pi r,$$

he should be able to find the value of C when $r = 15$, using $^{22}/_7$ as the value of π.

4. *Ability to derive one or more formulas from a given formula.* This means that, for example, from the interest formula $i = prt$, the pupil should be able easily to derive formulas for p, r, and t.

5. *Ability to represent by a graph any simple formula.* This means that the pupil should be able to make out a table of values of the letters in a formula and to represent the formula by a curve based upon these values. The temperature formula, where $F = \frac{9}{5}C + 32$, represents the limit of difficulty for work of this kind.

6. *Ability to understand the dependence of one quantity upon another.* This means that, in the formula just given, F depends upon C for its value. Every formula represents some such dependence; in fact, all our acts, our thoughts, our successes, our failures, and our standing in school depend upon something; nothing happens in our lives without the influence of something else.

7. *Ability to work with ordinary simple formulas.* This means the ability to use formulas of the following types:

Interest	$i = prt$
Amount at simple interest	$A = p + prt = p(1 + rt)$
Amount at compound interest	$A = p(1 + r)^n$

B. THE GRAPH OF A MATHEMATICAL LAW

1. *Ability to understand and use directed numbers.* These are needed in the study of graphs.

2. *Ability to represent points by means of the usual coordinate system.* This includes the appreciation of the relation of a point to a number pair (x, y), an understanding of the four quadrants formed by the axes, and the use of a table of corresponding values for the variables, such as

If $x =$	-2	-1	0	1	2
then $y =$	0	1	2	3	4

3. *Ability to understand the types $y = kx$ and $xy = k$.* This means that the pupil must understand the direct variation of y as x *varies in the equation* $y = kx$, k being some constant; and the inverse variation, as in $xy = k$. He must be able to represent each case graphically, and must see that functional relations are better visualized by the graph than by the equation.

4. *Ability to compare graphs drawn with respect to the same axes.* This means that, if we draw two related graphs on the same sheet, we can thus better compare the laws which they represent than by means of the equations.

5. *Ability to interpolate and exterpolate.* This means that, by the graph, we can usually approximate a value between two other values (interpolation) and we can predict the approximate values when the curve is slightly extended (exterpolation).

6. *Ability to interpret the graph of a simple formula.* This means that the pupil should be able to construct the graph of any simple formula, and then proceed to interpret it. In every case we draw the graph only for special values of all but two variables. For example, if $A = lw$, we give some special value, say, 2, to one of the letters, say, w. We thus have $A = 2l$, and we then proceed to draw the graph.

7. *Ability to interpret intersections with the axis.* This means that, for example, in the equation $y = x^2 - 3x + 2$, the curve will cut the x axis at the points 1 and 2; that is, if y is 0, we have $x^2 - 3x + 2 = 0$, and the values of x are then 1 and 2. We interpret these intersections, therefore, as the values of x in this equation. In the case of the equation $y = x^2 - 3x + 9$, the curve will not intersect the x axis; that is, the equation $x^2 - 3x + 9 = 0$ has no real roots.

8. *Ability to understand the relations between the variables in certain special cases.* This means that the graphs should show this relationship in such equations as

$$y = x^2,\ y = x^3,\ y = x^4, \text{ and } y = 2^x.$$

9. *Ability to read a compound-interest graph.* The formula is $A = p\ (1 + r)^n$, and we draw the graph for certain special cases, as that in which $p = 1$ and $r = 0.06$. We then have $A = 1.06^n$, which brings the formula under the preceding paragraph.

10. *Ability to use the graph of $y = x^2$.* This means that the pupil should construct the graph and then actually use it for finding approximate square roots like $\sqrt{1.5}$, and for finding squares.

11. *Ability to use the graph of an equation like $y = 3x + 1$.* This means that the pupil should construct the graph and see that, when $y = 0$, the value of x is found where the line cuts the x axis. He should see that this is a special case of the general type, $y = ax + b$. As an example of the importance of this work, let the pupils construct a graph of the temperature formula which shows the relation of the Fahrenheit to the Centigrade thermometer, $F = \frac{9}{5}\,C + 32$, or $F = 1.8\,C + 32$.

12. *Ability to use and to interpret the graph of an equation like*

$y = 2x^2 + x - 10$. This means that the pupil should construct the graph and see that it cuts the x axis in the points 2 and $-2\frac{1}{2}$; that is, that the roots of the equation $2x^2 + x - 10 = 0$ are 2 and $-2\frac{1}{2}$. It should be observed that this is a special case of the type

$$y = ax^2 + bx + c,$$

the latter representing a family of equations formed by giving to a, b, and c any values we choose.

13. *Ability to discover a maximum or a minimum in a graph.* This means that we can find the greatest or the least value of y in the equation of the preceding paragraph by examining the curve and locating its highest or its lowest point.

14. *Ability to read values from a graph.* This means that the pupil should acquire the ability to read quickly and either precisely or to the required degree of approximation the values of x corresponding to assigned values of y, and vice versa.

15. *Ability to use graphs in allied fields of science.* This means that the pupil should develop the ability to make use not only of the formulas in intuitive geometry, but of the simpler ones in physics as given in any modern textbook in that subject, of those in business that are simple enough, and of such other formulas as he may see in reading about the radio or in his work in elementary science.

16. *Ability to distinguish between the significance of the graph of statistics and that of a mathematical law.* For example, a statistical graph shows a probable tendency, while the graph of a mathematical law shows a certain one.

17. *Ability to express a ratio graphically.* This means that, given $\frac{x}{y} = 4$, we can write the equation $x = 4y$ and can then construct the graph. From this graph we can find the value of x that goes with any special value of y, and vice versa. In the same way we can study the graph of the general case, $\frac{x}{y} = k$, as already stated in item 3 above.

C. LINEAR EQUATIONS IN ONE UNKNOWN

1. *Ability to translate a verbal problem into an equation.* This involves two somewhat distinct abilities illustrated by the following problems:

(1) What is the cost of 8 pencils at k cents each? The answer is expressed by the equation $C = 8k$.

(2) What is the number which, when doubled and then increased by 5, is equal to 12? The equation is, $2n + 5 = 12$.

2. *Ability to translate an equation into words.* This is the reverse of the preceding objective. It means that equation $x^2 + 1 = 10$, for example, may be translated as follows: Find a number whose square increased by 1 is 10.

3. *Ability to solve equations of the type $ax + b = c$.* This means that the pupil should be able to solve any linear equation of the form $2x + 7 = 19$. It involves the ability to use each of the first four axioms if necessary.

4. *Ability to understand the significance of the graph of an equation of the type $y = ax + b$.* This means that such a graph may be used for solving an equation like $3x - 7 = 8$, this being one of the family of equations of the type mentioned.

5. *Ability to solve linear equations containing common or decimal fractions.* The types of equations selected should be those relating to the necessary formulas of mensuration, science, or business.

6. *Ability to use equations in solving problems.* As far as possible, the problems should be such as relate to elementary science or to simple business conditions. Since the technicalities of science and business, however, are often beyond the experiences of the pupils, problems of a similar type but of artificial content must often be used.

7. *Ability to interpret artificial numbers in a result.* This means that such artificial numbers as fractions, certain roots, and negative numbers sometimes have a meaning in the solution of a problem and sometimes do not. For example, consider this problem: "The number of times I picked up a pencil to-day is the value of x in the equation $2x + 3 = 2$. What is the number?" The result is that $x = -\frac{1}{2}$, which means nothing in the problem: (1) because I cannot pick it up half a time, and (2) because I cannot pick it up a negative number of times. The interpretation therefore is that the problem, as stated, has no solution.

D. DIRECTED NUMBERS

1. *Ability to represent directed numbers graphically.* This means that the pupils should be able to construct an algebraic scale upon which they represent all types of numbers known to them,—posi-

tive, negative, integral, fractional, and surd (that is, roots like $\sqrt{2}$), thus extending the general notion of number. The "imaginary number," like $\sqrt{-1}$, does not form a part of this system.

2. *Ability to use directed numbers practically.* This means that, in practical problems, such numbers as $-\$1000$, $-\frac{1}{2}$ in., and -75 lb. usually have a meaning, and that this meaning should be recognized and understood.

3. *Ability to add or to subtract directed numbers.* In either case the numbers may be written either in a column or in a row, some of the numoers being positive and some negative, or all being of the same kind.

4. *Ability to multiply by a positive or by a negative number.* This means that the pupil should develop the ability to perform such multiplications as $2\frac{1}{2} \times 6$, $3 \times (-7)$, $(-2) \times 3$, and $(-3\frac{1}{2}) \times (-2\frac{3}{4})$, the numbers being arranged either in a column or in a row. It carries with it the inverse order, illustrated by the case of $3 \times (-2)$.

5. *Ability to divide by a positive or by a negative number.* This operation, taught as the inverse of No. 4, includes the cases $4 \div \frac{1}{2}$, $6 \div (-2)$ and $(-6) \div (-2)$, together with the inverse case $(-6) \div 2$.

6. *Ability to use directed numbers in graphs.* This has already been mentioned in connection with graphs (page 199).

7. *Ability to use directed numbers in formulas.* This means that such ideas as negative weight, negative direction, and negative pressure now enter into many practical formulas and their significance must be understood.

8. *Ability to remove either one or two sets of parentheses.* This means especially the case in which negative numbers are involved or in which a negative sign precedes the parentheses.

9. *Understanding of the double use of the signs + and −.* This means that the pupil should recognize that these signs are used both as signs of quality (as when we speak of a positive or a negative number, say $+2$ or -7), and as signs of operation (as in the case of $9 + 3$ or $9 - 5$).

E. OPERATIONS ON POLYNOMIALS

1. *Addition, subtraction, multiplication, and division.* This means that these four operations, known as "the fundamental operations," are to be performed with polynomials in the manner de-

scribed under directed numbers. It does not mean, however, that much attention will be paid to this kind of work, in itself quite unimportant, except with easy polynomials such as will actually be needed in some type of applied problem.

2. *Multiplication of a binomial by a monomial.* This represents the most important type of multiplication beyond the case of monomials only. It should be effected with or without using parentheses, and should be illustrated geometrically by the use of a rectangle.

3. *Multiplication of any polynomial by a monomial.* This is much less important in practical work than the preceding case.

4. *Multiplication of a binomial by a binomial.* This represents the most important type of multiplication of a polynomial by a polynomial. It should be illustrated geometrically in certain instances by the use of a rectangle of which the base is $a + b$ and the height is $x + y$. This should lead into the development of these formulas:

$$(1)\quad (a + b)(x + y) = ax + ay + bx + by$$
$$(2)\quad (x + a)(x + b) = x^2 + (a + b)x + ab$$
$$(3)\quad (a + x)(a + x) = (a + x)^2 = a^2 + 2ax + x^2$$
$$(4)\quad (a + x)(a - x) = a^2 - x^2$$

with their variants in cases like $(x + a)(x - b)$, $(x - a)(x - b)$, $(a - x)^2$, and $(ax + b)(cx + d)$.

5. *Division of a binomial by a monomial.* This should be looked upon as the inverse of No. 2 and should include both integral and fractional forms.

6. *Division of a polynomial by a monomial.* This is of less practical value than the preceding case, but should be treated in the same way.

7. *Division of a polynomial by a binomial.* This is of value, at present, only in leading up to a case in factoring that is sometimes helpful in solving equations. The limit of difficulty should be a case like $(x^2 - x - 12) \div (x - 4)$.

8. *Using symbols of aggregation.* This should be limited to cases that are needed in connection with simple formulas.

9. *Understanding the value of factoring.* This means that the pupil should recognize the two chief values of factoring—a subject that often seems valueless. These are (1) that it enables us to transform one formula into another which can be more easily evaluated, and (2) that it can sometimes be used to advantage in

solving equations. For example, if $A = \pi R^2 - \pi r^2$, this is more easily evaluated for $R = 7.8$ and $r = 2.2$, by writing it $A = \pi (R + r)(R - r)$. Furthermore, we can easily solve the equation $x^2 + 4x - 21 = 0$, when we see that $(x - 3)(x + 7) = 0$. However, this is not so important as the former.

10. *Understanding factoring as the inverse of multiplication.* For example, because $(2x + 3)(5x + 2) = 10x^2 + 19x + 6$, we see that the factors of $10x^2 + 19x + 6$ are $2x + 3$ and $5x + 2$.

11. *Ability to remove a monomial factor.* For example, to see that $ax^2 + bx = x(ax + b)$. This is a valuable thing to do in the evaluation of a formula or in deriving one formula from another.

12. *Ability to factor the general quadratic trinomial.* This means the factoring of expressions in the form $ax^2 + bx + c$. For example, $2x^2 - 11x - 21 = (2x + 3)(x - 7)$. As a special case, a may equal 1, giving cases like $x^2 + 3x - 28 = (x + 7)(x - 4)$. For practical work the importance of this general type is much exaggerated, being used only in cases of fractions or equations specially made up to illustrate its use.

13. *Ability to factor expressions like $a^2 - b^2$.* This case has value in simplifying certain formulas as shown in No. 9.

14. *Understanding why division by 0 is not permitted.* This understanding is necessary when, in checking, a divisor becomes zero.

15. *Understanding the significance of complete factoring.* That is, if required to factor $x^4 - a^4$, it is not sufficient to write $(x^2 + a^2)(x^2 - a^2)$. We should write $(x^2 + a^2)(x + a)(x - a)$.

16. *Ability to apply the laws of exponents.* These laws are expressed as follows:

$$a^m a^n = a^{m+n},\ a^m / a^n = a^{m-n},\ (a^m)^n = a^{mn},\ (ab)^n = a^n b^n.$$

In simplifying formulas they are of great value.

17. *Understanding of zero, negative, and fractional exponents.* The zero exponent is of no practical value in our work at present, but the negative and fractional exponents are often seen in formulas.

18. *Ability to check all results.* This is one of the essential things in algebra. Every time a formula is derived, the work should be checked. The most convenient check is the one in which small numerical values are substituted for the letters.

19. *Understanding of the generality of algebra.* This means that the pupil should be led to see that, for example, the identity

$(a+b)^2 = a^2 + 2ab + b^2$ includes an infinite number of special cases such as the following:

$$(3+2)^2 = 3^2 + 2 \cdot 3 \cdot 2 + 2^2,$$

or

$$5^2 = 9 + 12 + 4 = 25.$$

F. FRACTIONS

1. *Understanding that a fraction means division.* This means that there are various ways of considering a fraction like $\frac{3}{4}$. We may think of it as three of the four equal parts of unity; as a fourth of 3; as the ratio of 3 to 4; or as an indicated division of 3 by 4. While all of these lead to the same result, it is simpler in algebra to think of the fraction $\frac{a}{b}$ as indicating the division of a by b.

2. *Understanding the principle of signs.* This means that the pupils should be so trained as to have no hesitancy in seeing that

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b}$$

3. *Ability to reduce a fraction to lowest terms.* This means that the pupil should simplify all formulas as much as possible in this way. To take a simple illustration, we know that the area of a circle may be expressed either as πr^2 or as $\frac{1}{2}rC$, whence

$$\tfrac{1}{2}rC = \pi r^2,$$

and hence

$$C = \frac{\pi r^2}{\frac{1}{2}r}.$$

Now it is manifestly undesirable to leave this fraction in such an awkward form, and hence it should be reduced to lowest terms and simplified as much as possible. It is, however, of no practical value to devote much time to reducing such fractions as

$$\frac{x^2 + 6x + 9}{x^3 + 9x^2 + 27x + 27}$$

They almost never enter into practical algebra.

4. *Ability to perform other reductions.* This means that there should be a moderate amount of work requiring such reductions as the following:

$$\frac{ax+b}{x}=a+\frac{b}{x}$$

and

$$3a-\frac{b}{4a}=\frac{12a^2-b}{4a}$$

The purpose of such work is to simplify formulas or to get them into a form better suited to use in deriving other formulas. The reduction to a L.C.D. is necessary in connection with simple fractions having monomial or even binomial denominators. The reduction or operating with fractions like

$$\frac{x^2-2x-15}{x^3-125} \text{ and } \frac{x^2-9}{x^3-5x^2-9x+45},$$

however, is useless from the standpoint of practical algebra or of the theory of the subject. However unimportant such work may be with respect to reduction, addition, or subtraction, it is, if possible, of even less practical importance in multiplication and of still less in division. Such cases are merely made up to illustrate a principle; they so rarely arise in a practical problem as to have only a theoretical value.

5. *Ability to simplify such complex fractions as may be needed in working with practical formulas.* This requires, in elementary work, no more difficult cases than the following:

$$\frac{a+\dfrac{b}{c}}{a^2+\dfrac{b^2}{c^2}} \text{ or } \frac{\dfrac{a+b}{a-b}}{\dfrac{a-b}{a+b}}.$$

6. *Understanding of the connection between arithmetic and algebraic fractions.* It means a great deal to see how such relations as

$$\frac{a}{b}+\frac{x}{y}=\frac{ay+bx}{by}$$

express laws that we continually use in arithmetic. For example, the above is a formula that applies to all such additions and subtractions as $\frac{2}{3}+\frac{3}{8}$ or $\frac{5}{8}-\frac{3}{5}$, to take two cases of greater difficulty than those ordinarily seen in practice.

7. *Ability to check all results.* For example, in the above formula let $a=b=x=y=1$, and we have

$$\frac{1}{1}+\frac{1}{1}=\frac{2}{1}=2,$$

and so there is probably no error.

G. FRACTIONAL EQUATIONS

1. *Ability to clear an equation of fractions.* This means that in any equation in which either numerical or literal fractions enter we should be able, if desirable, to write an equivalent equation without any fractions. It is not, however, always desirable to do this, as was formerly the case when pupils were afraid of fractions. There is no sense in clearing of fractions an equation like

$$\tfrac{3}{4}x+21=\tfrac{5}{8}x+29.$$

Any pupil who has progressed thus far in school should be able to solve it without using pencil and paper. Do not be afraid of the expression, "clear an equation of fractions," but be sure it is understood. After the meaning is plain it serves a good purpose.

2. *Ability to solve numerical equations containing fractional coefficients.* These may be either common or decimal fractions, as in the following cases:

$$\tfrac{3}{4}x=7, \quad 0.25x=4-x, \quad x+6\%x=12.$$

3. *Ability to solve fractional equations.* This means, in general, equations with monomial or binomial denominators. Even in these simple forms, we do not often need the binomial denominator in more than one fraction in our work with elementary formulas. This work is, for our present purposes, usually limited to cases like this: (1) Solve for p the equation

$$A=p(1+rt);$$

(2) Solve for s the equation

$$\frac{sr}{a}=r^n-\frac{a-s}{a},$$

each of which leads to formulas that we use in algebra.

4. *Ability to derive one formula from another.* We have frequently met with this case, and it is mentioned here again because the work very often requires the use of fractional equations of a simple type.

5. *Understanding the generality of a literal equation.* This applies to integral as well as to fractional equations. Any literal equation permits of an unlimited number of numerical equations of the same family and having the same form of solution. For example, if

$$ax + \frac{a}{b} = b,$$

then

$$x = \frac{b - \frac{a}{b}}{a}$$

$$= \frac{b^2 - a}{ab};$$

therefore, without solving again, we may substitute in the formula and see that the equation

$$3x + \tfrac{3}{4} = 4$$

has for its root

$$x = \frac{16 - 3}{12} = \frac{13}{12}.$$

6. *Ability to solve for a constant an equation like $y = ax + b$.* This means that it is sometimes desirable to express the value of a or b in terms of the other three letters. For example,

$$a = \frac{y - b}{x}.$$

7. *Ability to evaluate formulas involving fractions.* This is the most important part of our present work in fractions, and the simplest.

8. *Ability to check all results.* This usually involves either the substitution of an integral expression or a fraction in an equation of the first degree, or the substitution of a number in place of letters.

H. RATIO, PROPORTION, AND VARIATION

1. *Understanding a ratio as an abstract quotient.* This means that the quotient of any number divided by another of the same denomination is the ratio of the first to the second. For example,

$$\frac{3 \text{ ft.}}{4 \text{ ft.}} = \frac{3}{4}, \quad \frac{4 \text{ in.}}{2 \text{ in.}} = 2, \quad \frac{\$5.25}{\$0.25} = 21.$$

2. *Understanding a proportion as an equality of ratios.* This means that a proportion is merely a fractional equation and hence it should be treated as such. For example, if we have

$$\frac{x}{a}=\frac{b}{c},$$

we may solve the equation and find that

$$x=\frac{ab}{c}.$$

There is no reason for speaking of means, extremes, antecedents, or consequents, as was formerly the custom. Such words are merely useless lumber in our modern structure.

3. *Understanding that a ratio, written in fractional form, is subject to all the laws of fractions.* The old way of writing a ratio as $a : b$, is objectionable in our work, except as it must be known before a pupil reads books on physics. These books are generally very conservative and they fail to use the much simpler fractional form now generally taught in algebra.

4. *Understanding variation as related to ratio.* This means that if x varies directly as y, we have the ratio

$$\frac{x}{y}=k,$$

some constant (k) therefore being the ratio of x to y. If, however, x varies inversely as y, then

$$\frac{x}{\frac{1}{y}}=k, \text{ some constant,}$$

or

$$xy=k,$$

or

$$x=\frac{k}{y},$$

which is also a ratio. Hence, in any case, a variation involves a ratio. Indeed, if it were not for the convenience of the term "variation," it might easily be abandoned. The relation of two variables should be considered graphically as well as symbolically; that is, we should consider with the pupils the graph of the equation $x=ky$ (where k is a known number) as well as the fact that k is the ratio of x to y.

5. *Ability to solve problems in variation.* This means at this time that the pupils should be given simple and genuine problems in both direct and inverse variation, even if the subject is not treated in the textbook in use.

6. *Understanding more fully the idea of function.* This means that if x varies directly as y, then

$$x = ky, \text{ where } k \text{ is a known number,}$$

and hence that x depends upon y for its value; that is, x is a function of y.

I. SIMULTANEOUS LINEAR EQUATIONS

1. *Understanding types.* This means that, by the aid of graphs, the pupils should come to appreciate the significance of simultaneous, inconsistent, and equivalent equations in two unknowns. Such equations may be represented respectively as follows:

$$\begin{array}{ccc} x + y = 5 & x + y = 5 & x + y = 5 \\ 3x - y = 3 & 2x + 2y = 7 & 3x + 3y = 15 \end{array}$$

T. graphs of these three will illustrate the types.

2. *Ability to choose the best method.* This means that there are three convenient methods of solution. These are (1) addition, (2) subtraction, and (3) substitution. They are advantageou.ly used in the following sets respectively:

$$\begin{array}{ccc} x + y = 5 & 3x + y = 9 & x + 3y = 11 \\ 3x - y = 3 & x + y = 5 & y = 3 \end{array}$$

3. *Ability to solve applied problems.* This means that the pupil should be able to use his knowledge of simultaneous equations in practical work. It will be foun.l, however, that this field is rather limited unless we introduce fictitious problems or else presuppose more technical knowledge of science, industry, or commercial affairs than the pupil has. This is evident when we come to examine most of the current textbooks.

4. *Ability to check all results.* The only complete check is the substitution of the roots in both the original equations, showing that each reduces to an identity.

J. POWERS AND ROOTS

1. *Understanding necessary terms.* This means the understanding of the meaning of the terms *power, root, principal root, real number, rational number, irrational number, imaginary number, exponent,* and *index* of a root. There is also some value in the term

base. It is not necessary to memorize the definitions of such terms. This will be understood through use.

2. *Seeing the practical need for roots.* It is of little value to learn about roots unless we are going to use them. It is, therefore, well to give the formula $A = \pi r^2$ to be solved for r. The result is $r = \sqrt{A \div \pi}$. We therefore see the desirability of knowing (1) that the reciprocal of π (that is, $1/\pi$) is 0.301, and (2) that we then need to find the square root of $0.301A$.

3. *Ability to find the square root of a number.* This means that a pupil should know (1) how to use a square-root table; (2) how to find a square root without a table, either by the formula for $(a + b)^2$ or by some such method as "trial and error." He should also know how to find an approximate root from the graph of $y^2 = x$, or $y = \sqrt{x}$.

4. *Ability to find the square root of a polynomial.* This means to find the square root of such polynomials as $x^2 + 2xy + y^2$ and $x^2 + 6x + 9$, the sole purpose being to aid the pupil in understanding arithmetic square root. The work may be extended to the case of

$$x^2 + 2xy + y^2 + 2xz + 2yz + z^2,$$

but there is no practical need of continuing farther.

5. *Understanding how far to carry a square root.* This means that, in the case of a surd like $\sqrt{2}$, there is no end to the possible number of decimal places. When directed to carry to three decimal places, this means to the nearest thousandth. The value of $\sqrt{2}$ to three decimal places, or to four significant figures, is 1.414. The term "significant figures," means any one of the figures 1, 2, 3 . . . 9, and also 0 whenever it is known that the place it occupies is really a zero. For example, in the number 205, the 0 is significant; but when we say that the distance to the sun is 92,000,000 miles, we mean that we are not certain of more than two figures, so that the zeros are not significant in the ordinary use of the term.

6. *Understanding common exponential symbolism.* This means that the pupil should understand not only that $a^2 = aa$, $a^{-n} = 1/a^n$, and $a^0 = 1$, but that $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^m a^n = a^{m+n}$, $a^m/a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, and $(a^x b^y)^n = a^{nx} b^{ny}$.

7. *To apply square root.* This means that square root is to be applied in the formulas and in simple problems relating chiefly to the Pythagorean Theorem.

8. *To solve radical equations.* This means that the pupil should be able to solve such radical equations as are needed in ordinary formulas. To prepare for this work, abstract cases may be given of no greater difficulty than $\sqrt{2x+1} = 2x - 1$.

K. QUADRATIC EQUATIONS

1. *Ability to construct and interpret the graph of a quadratic function.* This means to construct and interpret the graph of a function like $y = 2x^2 - 3x + 1$. It should be constructed, interpreted for the case of $y = 0$, and used to find the actual or approximate roots of the equation $2x^2 - 3x + 1 = 0$. The pupil should see that this is a special case of the general quadratic function $y = ax^2 + bx + c$, and that the latter represents an entire family of such functions.

2. *Understanding of the terms "complete" and "incomplete."* This means, these terms as applied to a quadratic equation. The older terms "pure" and "affected" have gone out of general use.

3. *Ability to solve a quadratic equation by factoring.* This means that if $x^2 - 2x - 63 = 0$, then $(x + 7)(x - 9) = 0$, whence either $x + 7$ or $x - 9$ may equal zero, and so $x = -7$ or 9. Any quadratic equation can be solved by factoring if we extend "factor" to include irrational numbers; but the operation usually becomes too difficult for elementary pupils. The only equations that they can solve in this way are simply made up for this purpose, not representing any genuine applications of algebra. It should be understood that quadratic equations are not always required in the junior high school. Their introduction is optional.

4. *Ability to solve by completing the square.* There are two practical ways of quickly solving any quadratic. The first is the method of completing the square, this requiring the memorizing of the fact that, in the equation $x^2 + px = -q$, we must add to both sides (members) the square of half the coefficient of x. In the case of the quadratic $ax^2 + bx + c = 0$, we may, if we wish, reduce it to the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

or

$$x^2 + \frac{b}{a}x = -\frac{c}{a},$$

and then we may complete the square as before.

The second method is by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which we obtain if we solve the equation just given. This requires that we memorize the formula.

Each method therefore requires memorizing. For practical purposes there is little choice. The actual numerical work is the same in one case as in the other.

5. *Ability to find the maximum or minimum value of a quadratic function.* This may be done in two convenient ways within the reach of the pupil at this time. Suppose, for example, we have

$$y = x^2 + 6x - 2;$$

then we may draw the graph and find from it that the smallest value of y is -11.

The second plan is to observe that

$$x^2 + 6x - 2 - y = 0.$$

Solving for x,

$$\begin{aligned} x &= -3 \pm \sqrt{36 + 8 + 4y} \\ &= -3 \pm \sqrt{44 + 4y} \end{aligned}$$

Evidently the smallest that $4y$ can be is -44; for, if it were smaller, we should have a negative number under the radical sign, which would give us an imaginary number. Therefore the smallest that y can be is -11.

6. *Ability to solve applied problems.* There are only a few types of genuinely real problems that are simple enough for pupils in Grade IX and that require quadratic equations. Such as we have may be found in the better kind of modern textbooks.

7. *Ability to check the solutions.* In simple cases this is best done by substituting the values of the supposed roots in the equation. If the substitution is difficult, the best check is to observe that, in the equation $x^2 + px + q = 0$, the sum of the roots is always $-p$ and the product is q. For example, the two roots of the equation $x^2 - 7x + 12 = 0$ are 3 and 4, because $3 + 4 = -(-7)$ and $3 \times 4 = 12$. Such a check is easily applied.

VIII. ABILITIES IN NUMERICAL TRIGONOMETRY

General Purpose. Trigonometry can be made very difficult or very easy, and so can every other subject. For the junior high school it should not be as difficult as the relatively useless part of algebra that it replaces. The general purpose in teaching it is to show the pupils the significance of indirect measure—of measuring the distance across a river, for example, without going from one side to the other. If presented simply, as a practical subject, it is both easy and interesting besides having a value that is unquestionable. It is the basis of all measurements of land on the earth's surface, of the size of the earth itself, of the distances to other heavenly bodies, and of a good part of our great pieces of engineering.

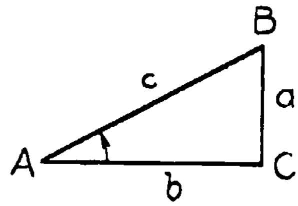
It should be the purpose of the school to show the significance of all this in the life of to-day, using only those functions and formulas that are necessary for this purpose.

Classification. The subject is taught for so short a time in the junior high school that the classification must necessarily be limited in extent. The following topics should be considered:

A. NECESSARY FUNCTIONS

1. *Four important functions.* In trigonometry there are several possible functions of an angle. Of these the beginner needs only four,—sine, cosine, tangent, and cotangent. We could easily reduce the number even more, but these four are desirable in the introductory course.

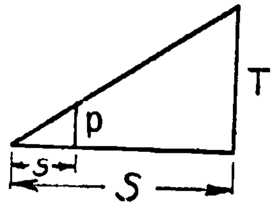
2. *Definition of functions.* It is possible to define these functions as lines, but it is more convenient to work with them at present if we define them as ratios, as follows:



$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}, \quad \cot A = \frac{b}{a}.$$

3. *Introduction to the functions.* It is advisable to introduce the functions by beginning with the tangent. For example, to take a case that always interests pupils, if a tree T feet high casts a shadow S feet long at the same time that a post p feet high casts a shadow s feet long, we have the proposition

$$\frac{T}{S} = \frac{p}{s},$$



whence $$T = \frac{p}{s} . S.$$

That is, we can find the height of the tree by multiplying the length of its shadow (S) by the ratio $\frac{p}{s}$. Thus we can find the height of the tree indirectly; that is, not by climbing the tree and measuring with a tape.

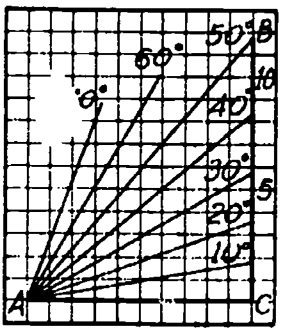
The ratio $\frac{p}{s}$ is called the *tangent* of $\angle A$; that is,

$$\tan A = \frac{p}{s}.$$

We then can find the height of a in the first of these figures by observing that

$$a = b \tan A.$$

4. *Ability to find the functions.* It is possible to find the tangents of angles of various sizes by means of squared paper, as here shown. In practical work, however, we use tables which have been computed for us by methods of higher mathematics and which give the tangents of the angles correct to three or more decimal places.



On a piece of squared paper we lay off angles of 10°, 20°, 30°, . . . at A, as here shown. Then taking $AC = 10$, we may draw CB. The tangent of each angle is then the number of spaces cut off on CB, divided by 10. We can thus find approximate values of the tangents as follows:

$$\tan 10° = \frac{1.8}{10} = 0.18 \qquad \tan 30° = \frac{5.8}{10} = 0.58$$

$$\tan 20° = \frac{3.6}{10} = 0.36 \qquad \tan 40° = \frac{8.4}{10} = 0.84$$

The pupil should do this work for the tangent, sine, and cosine, and thereafter should use the tables.

5. *Ability to use any convenient tables.* For the beginner, either three-place or four-place tables may be used. Take whichever the textbook offers.

B. APPLICATIONS

1. *Ability to apply the subject to solving right triangles.* There are more interesting and valuable applications of simple trigonome-

try than of any other part of elementary algebra, perhaps excepting formulas. The work should be applied to actual outdoor measurements, such as finding the height of a tree, the school building, or a church spire.

2. *Instruments needed.* If the school has a transit, this should be used. If it has not, the pupils can easily make, as they do in very many schools, simple instruments for measuring angles, using a pointer turning on a paper protractor.

3. *Ability to check.* All work in mathematics should be checked. A desirable and convenient check in trigonometric measurements is achieved by comparing the results in class.

IX. Desirable Information in Arithmetic

General Purpose. Arithmetic is practically endless in its applications. If we wished to open up new lines of problems we might introduce the arithmetic of the machine shop, of the carpenter shop, of automobile manufacturing, of foreign exchange, of chemistry, and of hundreds of other branches. These are all technical, however; that is, each relates to some special department rather than to the needs of people in general. The question that must be considered first of all in making up a course of study relates to the kind of general information that all people need. This, therefore, is a paramount question in arithmetic as well as in other subjects.

Classification. While this desirable information varies somewhat according to important local industries, the main branches are the same in all schools. They may be classified in various ways, and for an extended study of the subject the teacher is referred to other sources. For the teacher of mathematics in the junior high school, who wishes a brief statement of the subject from the standpoint of the upper grades, the following is suggested as a workable scheme.

A. FUNDAMENTAL PROCESSES

Reading numbers. This may be limited to billions, the trillions being rarely met with to demand attention in schools. As the pupils proceed, they will perhaps find that in science they will need much larger numbers, but that these are rarely called by name. In actual practice a scientist will write 2.35×10^{15} for the number

23,500,000,000,000,000, but he will seldom try to read the number except as 2.35 times 10 to the 15th power.

2. *Degree of accuracy in writing results.* This means that the pupil should understand the significance of the expression "correct to hundredths," "correct to four decimal places," and "correct to five significant figures," as explained on page 212.

3. *Degree of accuracy in measurement.* This means that the pupil should understand that there is no such thing within his mental grasp as an absolutely accurate measurement, but that in certain activities of modern life a very high degree of approximation is demanded.

4. *Economy of operation.* This means that the pupil should not only know how to perform the ordinary operations with whole numbers and with both common and decimal fractions, but he should know how to use such short cuts as are really valuable.

B. APPLICATIONS OF PER CENTS

1. *Meaning of per cent.* This means that the pupil should understand that percentage is merely a part of decimal fractions; that 6% is merely another way of writing 0.06 or $6/100$; and that if it were not for business customs we could get on very well without the symbol % and the term "per cent."

2. *Computation of profit on the cost.* If an article is bought for $1, and the cost of selling is 60¢, the total cost is $1.60. If the rate of profit is to be 20% on the total cost, the profit is 20% of $1.60, or 32¢, and the selling price is $1.92. This is the old method and is still used in small business. In large business the term "margin" is replacing the word "profit." It is the difference between cost plus expenses of selling, and the selling price.

3. *Computation of profit on the selling price.* In the preceding case, if the profit is to be 20% on the selling price, then $1.60 must be 80% of the selling price. Therefore the selling price is $2 and the profit is 40¢. This is the modern method and is used in large business houses.

4. *The meaning of discount.* This refers to discount on sales and discount on notes, the mathematics being the same in each case.

5. *Ability to use the six-per-cent method.* This applied to both interest and discount on a note, the two being mathematically the same.

6. *Meaning of commission.* This refers to commission on sales of produce, land, or any other salable property.

7. *Signficance of the simple-interest formulas.* This means the formulas, $i = prt$ and $A = p\ (1 + rt)$.

C. PROBLEMS OF THE HOME

1. *Personal account.* This means the knowledge of how a personal account is kept. The account should show all receipts, all expenses, and the balance at the end of each week or month.

2. *Household account.* This means the same as the preceding, only applied to the household. It is such as a housewife might keep.

3. *Budget system.* This means the knowledge of how to prepare a budget for an individual or a household, showing probable receipts and expenses for a year. In this way we are led to plan to save a certain amount each year so as to prepare for the "rainy days" of life.

4. *Personal inventory.* This means that at the close of each year all adults should "take stock," writing down the amount of property that each one owns. This will encourage saving a reasonable sum each year.

5. *Typical home problems.* This means that the pupil should have a reasonable amount of information relating to food values, food costs, clothes values, clothes costs, discount sales, heating, lighting, decorating, sewing, and repairs.

D. UNITS OF MEASURE

1. *Counting.* This means that the pupil should know the meaning of such terms as dozen, score, and gross. Such terms are old but are still used, although the score is not very common and the gross is used chiefly in wholesale purchases.

2. *Length.* This means particularly the common units of inch, foot, yard, and mile, with intimate knowledge of the rod in rural communities. It is interesting to mention the "light year," used in astronomy, and meaning the distance that light travels in one year. There are stars that are hundreds and even thousands of light years away.

3. *Weight.* This means a knowledge of the ounce, pound, and ton. The long ton is not of great importance in these grades except in mining communities and a few special localities.

4. *Liquid units.* These are the pint, quart, and gallon. The gill is becoming obsolete except in the buying of cream in certain cities.

5. *Dry units.* These are the quart, peck, and bushel, with some mention of the pint.

6. *Square units.* These are the squares of the units in No. 2, together with the acre and, in rural communities, the square rod.

7. *Units of time.* These are the second, minute, hour, day, week, month, year, decade, and century.

8. *Angular units.* These are the second, minute, degree, right angle, straight angle (180°), and circumference (360°). Decimal parts of the degree are replacing minutes and seconds.

9. *Kitchen units.* Among these are the teaspoonful, tablespoonful, and cupful. A pint of water weighs approximately a pound.

10. *Foreign money.* It is desirable to know the value in our money of such foreign units of value as are substantially fixed. These are approximately as follows: £1 English = $4.87, 1 German gold mark = 24¢, 1 Japanese yen = 50¢, $1 Mexican = 50¢, 1 Swiss franc = 1 Spanish peseta = 19.3¢. The subject has little importance to the great mass of our people under the disturbed conditions of foreign exchange, and is usually omitted at present.

11. *Metric system.* The chief units needed are the meter, kilometer, centimeter, millimeter; the squares of these measures; the cubic meter, cubic centimeter, and cubic millimeter; the liter; and the gram and kilogram, with a knowledge of the significance of the prefixes "centi-" and "milli-" in centigram and milligram.

12. *Convenient equivalents.* While not to be memorized, it is desirable to know of the following approximate relations: 1 bu. contains 1¼ cu.ft.; 1 gal. contains 231 cu.in.; 1 cu.ft. of water weighs 62½ lb.; 1 T. of hay occupies 500 cu.ft.; 1 T. of hard coal occupies 35 cu.ft.; 1 T. of soft coal occupies 42 cu.ft.; 1 cu.ft. of brickwork contains, with the mortar, 22 bricks.

E. CONCERNING THE STORE AND THE SHOP

1. *Bills.* The general nature and meaning of a bill of goods.

2. *Making change.* The common method of making change; additive subtraction.

3. *Cash registers.* Their nature and purpose.

4. *Inventory.* Necessity for taking it once or twice a year.

5. *Invoice.* Its nature as compared with a store bill. Discount or discounts allowed.

6. *Bill of lading.* Its purpose and general nature.
7. *Contract.* The general nature of a contract.
8. *Parcel post and express.* How to ship goods in small quantities.
9. *Transportation problems.* How to ship goods in large quantities.
10. *Commission on sales.* Although already considered under per cents, the subject is also of importance in this connection.
11. *Overhead.* Meaning of "overhead," or the cost of doing business, and the necessity of adding this to the net cost in computing the total cost of goods in a store or a shop.
12. *Elements of a payroll.* How a payroll is made out and how the money is drawn from the bank to allow for the precise change in making payments.

F. CONCERNING BANKING

1. *Post-office banks.* Significance of the Postal Savings System, which is in effect a government savings bank.
2. *Savings banks.* Their purpose; how business is done with such banks; the meaning of compound interest.
3. *Commercial banks.* How money is deposited; how it is withdrawn: pass book, checks, drafts; borrowing money; collateral.
4. *Transmitting money or its equivalent.* Checks, drafts, money orders. Recognition that the term "draft" is rapidly being replaced by "check."
5. *Trade acceptances.* A new form of draft accompanying a shipment of goods.
6. *Exchange.* The cause of the great fluctuation in foreign exchange in recent years. The reason why the subject of foreign exchange is not at present desirable for general school instruction, except in connection with countries in which the money is on a gold basis, like ours.
7. *Building and loan associations.* In localities where these have been successful through a series of years and are conservatively managed, their nature should be explained. If possible this should be done by an official of the association.

G. COMMUNITY ARITHMETIC

1. *Benefits from the community.* This means that the schools should make clear the benefits which everyone receives from the

nation, the state, the city, the village, and the community in general. These include free schools, free roads, parks, water supply, street cleaning, street lighting, sewerage, and the like.

2. *Our duty to the community.* This means that the schools should lead the pupils to take an interest in the general nature of the expenses of all the above branches of community government, should show the justice of the levying of taxes, and should show our duty in paying such taxes. After all, it is our government and we should put the best men in office and should pay what the government needs, this government including our local community as a part.

3. *Typical community interests.* This means that the pupils should be informed concerning the cost of good roads, of public buildings, of street lighting, of water supply, and the like, and that they should then solve a reasonable number of local problems relating to this work.

4. *Insurance.* This means that the social benefits of insurance should be made clear, many joining to help those who are in need through ill health, accident, fire, or the loss of members of the family. The three or four leading types of life insurance policies should be understood; the nature of fire insurance should be explained; the workmen's compensation laws should be discussed, and accident and casualty insurance should receive attention. The pupils should understand the meaning of a policy, of the rate (premium), and the face.

H. THRIFT AND INVESTMENTS

1. *Savings.* This means some attention to the importance of saving in a country where a large majority of our people are dependent upon others when they are sixty years old. Everyone should save while he is able. A bank account started early and maintained is likely to grow year by year.

2. *Safe investments.* It is the duty of the schools to warn all pupils against reckless investments. The difference in income between 5% and 8% is only $3 on a hundred per year, but the former is likely to be absolutely safe while the latter is not. Never gamble in stocks; the cards are stacked against you.

3. *Difference between stocks and bonds.* People with even relatively small amounts of money to invest are in the habit at present

of buying stock in some large corporation, like a railroad, or of buying a bond of some such corporation or of a city, a state, or the national government. The distinction between stocks and bonds should therefore be understood, and their relative desirability should be explained.

4. *Information about stocks.* The pupils should understand such terms as preferred stock, common stock, certificate of stock, dividends, par value, above and below par, brokerage, and stock exchange.

5. *Information about corporations.* The pupils should understand the general nature of a corporation, the method of electing directors and officers, and the way in which corporations are financed and managed.

6. *Investments in bonds and mortgages.* The difference between a bond and a mortgage should be understood, and how one is security for the other. The question of the desirability of a bond and mortgage as an investment should be considered.

I. STATISTICS AND GRAPHS

1. *Relative value of a statistical table and a graph.* This means that each has a definite place in business life: that one is more precise in details while the other is more vivid in presentation.

2. *Types of graphs.* The pupil should be familiar with the bar graph, curve-line graph, circular graph, and pictorial graphs of various kinds, such as are seen in current journals and magazines.

3. *Interpretation of graphs.* As already stated, this is more important than the drawing of graphs. It means that the story which the graph tells should be clearly understood.

J. TITLES OF SPECIAL PROJECTS SOMETIMES SUGGESTED FOR STUDY

1. *Transportation*
2. *Forestry*
3. *Automobiles*
4. *Gardening*
5. *Farm tractors*
6. *Pottery*
7. *Room decorating*
8. *Poultry raising*
9. *Machinist's work*
10. *Iron industry*
11. *Paving*
12. *Dairy farm*
13. *Plastering*
14. *Silo capacity*
15. *Papering*
16. *Mixing concrete*
17. *Contents of hay stacks*
18. *Spraying*

It should be clearly understood, however, that the "project" in general represents a very artificial type of problem. It may seem real to the teacher but be very unreal to the pupil. A girl of twelve is not likely to be very much interested in paving, nor a city boy in the contents of haystacks. A "project" is of value only when each pupil has a genuine interest in it, and even then it needs to be weighed carefully before it is allowed to take any time from the work in computation.

X. Desirable Information in Geometry

General Purpose. Whether or not there is any demonstrative geometry taught in the junior high school, a considerable range of desirable information will naturally be secured. For example, pupils will come to know the simple facts of congruence by intuition, and similarly those relating to the shapes and sizes of figures and to position. These facts may, therefore, properly be considered as desirable information easily within the reach of pupils of these grades. Even if only a brief allowance of time can be given to a few simple demonstrations and exercises, the chief objective will be attained; that is, the pupil will obtain a fair idea of what is meant by a mathematical demonstration.

Classification. This information may be classified as follows:

A. SIMPLE INSTRUMENTS

1. *Meaning of intuitive geometry.* This means geometry without scientific demonstrations and allowing the use of any drawing instruments we wish.

2. *Leading concepts.* These are points, lines, surfaces, and solids. We use instruments for drawing lines, either straight or curve, and we use lines for locating points. We do not use instruments for establishing surfaces or solids, except as this is done indirectly by drawing lines.

3. *Primary instruments.* The instruments primarily used are the ruler (or straight edge) and compasses. These are sufficient for elementary geometry.

4. *Additional instruments.* For convenience, in intuitive geometry, we supplement the primary instruments by others. For example, we use a T-square in drawing parallels, a plumb line for establishing a vertical, a level for establishing a horizontal, a com-

pass for measuring horizontal angles, and a transit for measuring both horizontal and vertical angles. With all these instruments the pupil should be acquainted, at least through pictures.

B. SHAPES OF FIGURES

1. *Regularity of figures.* This means a knowledge of what is necessary to make a regular plane or solid figure.
2. *Symmetry of figures.* This means a knowledge of symmetry with respect to a point, a line, or a plane.
3. *Similarity of figures.* This means a knowledge of the similarity of plane figures and of solids.
4. *The standard types of figures.* This means a knowledge of the various kinds of triangles, of quadrilaterals, of prisms, and the like. In the junior high school it does not include the oblique cone, prismoid, or other types not commonly seen by pupils.

C. SIZE OF FIGURES

1. *Methods of measuring lengths.* This means the common methods of determining lengths and of checking the work. It includes both direct and indirect measurements; that is, by tape lines, by rulers, by compasses, and by other scales, and also by trigonometry.
2. *Methods of measuring areas.* This means the common methods of dividing the figure into rectangles or triangles, or approximately so; the method of drawing to scale and then measuring the area of the drawing; the finding of a volume by measuring the amount of water necessary to fill an irregular container, or the amount of water that will run over when an irregular solid is slowly lowered into a full container.

D. POSITION

1. *Point referred to axes.* This means the ability to locate a point with reference to two axes, just as we locate a place by latitude and longitude.
2. *Point referred to angle and distance.* This means the ability to locate a point in a certain direction and at a certain distance from a known point.
3. *Point referred to two points.* This means the ability to locate

a point when its distances from two fixed points are known. Several possible solutions are to be considered in all cases of loci.

4. *Point referred to a line and a point.* This means the ability to locate a point when its distances from a fixed line and a fixed point are known.

5. *Point referred to distances from planes and points.* This means that loci with respect to planes and spherical surfaces are to be considered informally.

E. CONGRUENCE THEOREMS

1. *First congruence theorem.* This refers to the case of two sides and the included angle. All such cases may be taken with or without proof, according to the ability of the pupil or the class.

2. *Second congruence theorem.* This refers to the case of two angles and the included side.

3. *Third congruence theorem.* This refers to the case of the three sides.

F. THEOREMS ON ANGLES AND TRIANGLES

1. *Equality of vertical angles.* This refers to the vertical angles formed when two straight lines intersect. It may be taken as a postulate if desired.

2. *Isosceles triangle.* This refers to the equality of the base angles of an isosceles triangle.

3. *The converse of No. 2.* This refers to the equality of the sides opposite two equal angles of a triangle. Whether or not the preceding case (No. 2) is proved, this may properly be taken at first as a postulate.

4. *Angle sum.* This refers to the fact that the sum of the angles of any triangle is 180°. It should be extended to figures of four, five, and six sides and should include the cases of both regular and irregular figures.

G. PARALLELS AND PARALLELOGRAMS

1. *Condition of parallelism.* This refers to the case of equal corresponding angles or alternate angles formed by a transversal cutting two lines, the lines then being parallel.

2. *Converse of No. 1.* This means that, if the lines are given

parallel, the corresponding angles or the alternate angles will be equal.

3. *Opposite angles of a parallelogram.* This refers to their equality.

4. *Consecutive angles of a parallelogram.* This refers to their being supplementary.

5. *Opposite sides of a parallelogram.* This refers to their equality.

H. SIMILARITY

1. *First condition of similarity of triangles.* This refers to the case in which two angles of one triangle are respectively equal to two angles of another.

2. *Second condition of similarity of triangles.* This refers to the case of one angle and the proportionality of the including sides.

3. *Results of similarity.* This refers to the fact that the angles are respectively equal and the sides are respectively proportional.

4. *Pythagorean Theorem.* This may, as heretofore suggested, be easily proved by similarity.

PART III

SENIOR HIGH SCHOOL MATHEMATICS

MATHEMATICS IN THE SENIOR HIGH SCHOOL

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I. Aims

Aims of the Elementary School. The twelve years of school life preceding the college period are variously divided, but the general plan is coming to be (1) the elementary school, Grades I-VI; (2) the junior high school, Grades VII-IX; and (3) the senior high school, Grades X-XII. With the pre-school work, the subdivisions of the elementary school, and the various types of technical schools this discussion is not concerned.

In the field of mathematics the great objective in the elementary school is the ability to compute accurately to the extent demanded in actual business. This carries along with it the application of computation to such problems as are really found in daily life, and the limitation of this computation to such work as most people have occasion to do. The great mathematical objective, however, is computation.

Aims of the Junior High School. The great objective in the junior high school is a survey of the nature of mathematics as far as the abilities of the pupils in general permit. There are various great departments of knowledge, more or less related, such as literature, history, language, art, natural science, mathematics, religion, and contemporary civilization. Every one has a right to know the general nature of these departments. Mathematics is one of them, and the central objective of the junior high school is to let pupils know more about the applications of arithmetic, and to have some idea of the meaning of algebra, trigonometry, and the geometries (intuitive quite fully, and demonstrative as to its general nature).

The junior high school therefore should offer the opportunity for making a general survey of the meaning of the great branches of human knowledge. It should not fear to be superficial, for its mission is to show a wide surface rather than a narrow depth. It should be at the same time a period in which interest should be awakened, knowledge extended, and a diagnosis made of the pupil's abilities and tastes. At its close each pupil should have shown his parents, his teachers, and himself what is his natural bent of mind and what it will probably continue to be. If he has no taste for or ability with respect to mathematics, and does not expect to enter a college or a technical school, he should no longer be required to pursue the subject, and similarly with respect to the other great branches. His mind has been given its chance; it should now be given its choice. As to what is a reasonable choice is a matter to be determined by his parents, his teachers, and himself. It is not likely to be such as to lead to any narrow specialization in the senior high school, but it is quite probable that it will lead him to cease studying any special subject—say natural science, foreign language, mathematics, drawing, or the manual arts—for which he has no taste and in which he has shown no ability.

Aims of the Senior High School. The senior high school should therefore so arrange its curriculum in mathematics as to meet the needs of the pupil who likes mathematics and who wishes to pursue it, or at least who likes it well enough to study it for the purpose of continuing in further work for which it is a necessity,—in particular, any of the sciences.

The aim is still in the best sense superficial, as must necessarily be the case with most pupils of the age of those in the senior high school; but the superficiality is less marked, for the pupil now begins to get below the surface. For example, he now has his first opportunity of coming into close contact with real mathematical thinking, this being in the field of geometry. His geometry thus far has been that of childhood, of the intuitive stage of mental development; it now becomes that of manhood, of logical reasoning, of seeking to know causes, of criticizing his own beliefs, and of knowing instead of surmising.

In algebra his work thus far has also been childish; if poorly taught, he has learned a certain amount of the mechanics of the subject, much of which (like most of the work in the "four operations," or in factoring, or in the treatment of fractions) he will never use again; but if well taught, he has learned the real uses of the simpler parts of the subject in the fields of elementary science. None of this work requires any great amount of mathematical reasoning; it is like the intuitive stage in geometry. Now, however, he enters upon the study of a more advanced type of algebra, and here he begins to look beneath the surface. He no longer needs to spend much time upon the simple manipulation of elementary algebraic expressions; the uses of easy equations in connection with the simpler formulas of science are well known to him, and he is able to apply graphs to the study of certain elementary functions. The problem now is to apply mathematical reasoning to those parts of algebra that are needed in the more advanced work in science and in mathematics,—notably with respect to the equation, to exponents and radicals, and to the relation of algebraic to geometic forms as illustrated in the work of graphs.

These and other phases of mathematics will be considered in detail later in this discussion.

The Foundations upon Which to Build. What has been said with respect to the elementary school and the junior high school shows the foundations upon which we can safely build. Briefly stated, these are as follows:

1. *Ability to compute.* This is the mechanical part of number work. Such computation can be performed upon any one of various types of calculating machines, and is so performed at present in most lines of business and science, particularly where there is a large amount of work to be done. It can also be and is often performed by means of tables of various kinds, some of which the pupil already knows how to use. It will always be necessary to be able to compute accurately with the pencil and paper; in fact, the technique of computation must be understood in order to use calculating machines and tables. The future of computation in our best schools, however, will probably see courses in the use of these mechanical aids, just as courses in typewriting are becoming common and just as the slide rule (a form of computing machine) is often seen in classes in the junior high school.

2. *Intuitive geometry, including elementary mensuration.* We are already relegating the measurement of areas and volumes to the junior high school, where it properly belongs, and are thus allowing more attention to be given to demonstration when the study of geometry is taken up in Grade X.

3. *The algebra of the formula, graph, directed number, and the equation.* This carries with it a certain amount of manipulation of algebraic forms, but happily a great deal of the work on the "four operations," factoring, and fractions has, as already mentioned, given place to something of greater value.

4. *The elements of numerical trigonometry.* This means that

the pupil should know, preferably as part of his algebra, at least three trigonometric functions (the tangent, sine, and cosine) and their applications to simple problems of mensuration. The study of the right triangle is all that need be undertaken in this connection.

5. *Some idea of the meaning of a demonstration.* This is a desirable part of the work of Grade IX for two reasons: (*a*) that the pupil who leaves school at the close of this grade should know what it means to demonstrate; and (*b*) that every pupil should have the opportunity of showing whether or not he can pursue mathematics farther to any advantage. Not until he reaches the demonstration stage does he enter the real domain of mathematical thought.

Upon these foundations, then, what superstructure can be built that will best serve the purposes already mentioned?

II. Content Material

Mathematics in Grade X. If the pupil is to pursue his work in mathematics beyond the intuitive stage represented in Grades I-IX, he must now begin to build upon a more substantial foundation. To the mathematician the science rests upon the foundation of demonstrative geometry. Not until this subject is begun does the pupil really appreciate the significance of mathematics. It is here that rigorous logic begins to be applied; it is here that he first appreciates the step, "If that, then this." Here he comprehends the significance of the further chain of reasoning illustrated by the statement, "I can prove A if I can prove B; I can prove B if I can prove C; but I can prove C, and hence I can retrace my steps and prove A." By these two acquisitions he is inducted into the domain of mathematical thought, and this is the appropriate work of Grade X.

Teachers are sometimes moved to ask why algebra and demonstrative geometry cannot be more closely related in the junior high school. The answer is that their purposes and their methods are about as little connected as French and arithmetic; there are points of contact, but they are very few. The pupil of to-day is not prepared for a purely geometric treatment of proportion, and so we are justified in depending upon algebra for this part of geometry; we are also justified in using such forms as a and b to represent angles and lines; but the amount of algebra involved is slight.

On the other hand the bonds between intuitive geometry and algebra are numerous, and these subjects, together with trigonometry, are closely related.

Demonstrative geometry may well, therefore, be the central feature of the work of Grade X, algebra and trigonometry being brought in whenever there is a real advantage in so doing, and in any case being reviewed along with geometry so as to keep them fresh in mind.

Amount of Geometry Necessary. It was formerly the custom to teach plane geometry for a year or more, this being followed by solid geometry for at least a half year, and from the standpoint of the mathematicians this was a good plan. It gave a more thorough and extensive knowledge of the subject than we obtain when we limit the field. When, however, we consider the age of the pupils at present, and the age in which they live, we must recognize the fact that very legitimate claims can be advanced by other important branches of human knowledge which hardly existed in the nineteenth century.

The proper question for us to ask to-day is whether the legitimate claims of geometry as a method rather than as a body of facts can be met in less time than was formerly allowed, and the answer that will probably appeal to any unbiased mind is in the affirmative. Many teachers of geometry say that "the ground" of plane and solid geometry cannot be covered in a year, forgetting that it is the method of geometry rather than the body of facts that is the purpose of the study. While it is, mathematically speaking, desirable to be familiar with a considerable body of material, the method of geometry, the ability to reason geometrically, the power to transfer this reasoning to non-mathematical regions,—this can be quite satisfactorily acquired in a single year. A good textbook in geometry will supply the material for a more extended course, but the school may properly decide to omit such topics as inequalities and many of the constructions of plane geometry, and to limit materially the work in solid geometry, especially in the field of spherical polygons of all kinds. It should be recognized that we do not teach geometry for the purpose of acquiring the important facts of mensuration or for "developing spatial imagination"; each is accomplished fully as satisfactorily in intuitive geometry.

If it is recognized, therefore, that the aim is the method of geometric reasoning, then a single school year serves the purpose and permits also of cumulative reviews of the important parts of junior high school mathematics as well as of the geometry which was studied in the tenth year.

Whether or not such a course meets the needs of those who propose to pursue certain technical lines is one which demands attention. It cannot be maintained that it is not essential to those who are seeking to understand mathematics as a science or who give promise of being leaders in mathematical research in subjects like electricity or aerostatics; but for the one who will probably reach only to the level of a shop foreman in an industrial plant of some kind it may well be that a course in the mathematics of the workshop will be much more profitable. Since, however, we are at present concerned with the general type of school, this is not the place for considering the needs of the technicians.

Differences in Needs of Pupils. Thus far the mathematics suggested has met the needs of nearly all classes of pupils, the single exception being that in which the needs of the technician appear, as referred to in the preceding paragraph. It should be observed, however, that at the close of the work of the junior high school the pupils are beginning to show their aptitudes in the various possible roads which are open to them. Those who wish to go to college and who show the necessary ability to profit by the work there offered will need a more diversified type of mathematics than those who are going into commercial work; and those who expect to specialize in science will need more and higher mathematics than those whose tastes lead them into literary fields. The modern psychological tests give hope that the real aptitudes of young people can be discovered earlier than was formerly the case, and that they may begin their preparation for their life work as early as the senior high school, doing this not by means of any narrow specialization but by emphasizing those phases of a subject like mathematics that are particularly within their range of interest.

For this reason some of the senior high schools will offer commercial courses that require a great deal of commercial algebra; others will offer technical courses that require two or three years of machine shop mathematics or of the mathematics used in certain special trades; while others, and by far the largest number, will offer the mathematics needed by the pupil who is to enter college and which will prepare him for the study of science, economics, mathematics, and various other subjects that go to make up a broad culture course. Such a course will naturally remain the standard for those who do not go beyond the high school as well as for those who proceed to college.

Since it is not possible in the space allowed to this section to explore all the possible needs of different classes of pupils, it is now proposed to consider a few of the branches of mathematics that may properly enter into a modern course of a general nature.

Mathematics in Grade XI. When we come to the question of mathematics in Grades XI and XII we enter upon a domain that is only slightly explored in this country. In this respect we are where we were fifteen years ago with respect to the junior high school. These years have developed a fair consensus of opinion as to the latter type of school, and the next few years will do the same for the senior type.

Just as we have, however, many instances to-day of the unwarranted use of the term "junior high school" in cases in which the first two years offer nothing but arithmetic, and the last one offers only the older type of algebra, so we shall have many "senior high schools" that are of the same old type with the new name,—the mathematics consisting of reviews for examinations of one kind or another, plus trigonometry and higher algebra.

Following the principle that mathematics should be a privilege rather than a burden in the senior high school, being open only to those who wish to study it, who show an aptitude for it, and who will need it for work in science, economics, or pure mathematics, we may now consider some of the possibilities for Grade XI.

First, in any of the probable lines of his future activity the type of student under consideration will need more trigonometry than what he learned in his algebra of Grade IX or earlier. A satisfactory course in plane trigonometry, built with a view to his previous introduction to the subject, can be given in a half year. Such a course will prepare the student to continue his work by subsequent reading in case he needs to cover more ground in order to carry on investigations in any special science,—say mechanics, electricity, or hydrostatics. It will even permit of a little work in spherical trigonometry, a subject that is usually rather barren unless and until the student is nearly ready for a course in geodesy or one in astronomy. Such a course in plane trigonometry is easily within the reach of selected students in Grade XI, and experience has shown that they handle the subject quite as successfully as is done by freshmen in college.

The objectives of this course in trigonometry have been laid down by the College Entrance Examination Board, in a modern spirit, substantially as follows:

1. Review of the four most important functions—the sine, cosine, tangent, and cotangent.

2. Extension of the functions to include the secant and cosecant, and the computation of five of these ratios from any one.

3. Circular measure for angles—radians.

4. Fundamental formulas and identities, as follows:

a. Ratio formulas such as

$$\tan x = \frac{\sin x}{\cos x}.$$

b. Pythagorean formulas, such as

$$\sin^2 x + \cos^2 x = 1.$$

c. Addition theorems not only for $\sin(x + y)$ and $\cos(x + y)$, but also for $\tan(x + y)$.

d. Double angle formulas; that is, formulas for $\sin 2x$, $\cos 2x$, and $\tan 2x$.

5. Solution of simple trigonometric equations such as

$$6 \sin x - \cos x = 2, \cos 2 x = \sin x, \tan (x + 30^\circ) = \cot x.$$

6. Logarithms.

7. Law of Sines and Law of Cosines.

8. Solution of right and oblique triangles.

9. Application to mensuration, including simple work in surveying and the measuring of heights and distances.

Second, what is known as intermediate algebra may properly occupy the second half of the year, thus allowing for the use of trigonometry in case it is desired to introduce the study of the complex number. The objectives for this course have been carefully studied by the College Entrance Examination Board and are substantially as follows:

1. Numerical and literal equations of the second degree, with the interpretation of their graphs.

2. The binomial theorem for positive integral exponents, with its application to compound interest.

3. Arithmetic and geometric series, including the infinite decreasing geometric series.

4. Simultaneous linear equations in three unknown quantities.

5. Simultaneous equations including one linear equation and one quadratic equation, or two quadratic equations of the types easily solved by quadratics.

6. Exponents and radicals, including fractional, zero, and negative exponents, the rationalization of the denominator in such fractions as

$$\frac{a+\sqrt{b}}{c+\sqrt{d}},$$

and the solution of such equations as

$$\sqrt{3-2x}-x=30.$$

7. Logarithms, unless these have been sufficiently taught in the preceding work in trigonometry.

Mathematics in Grade XII. Not considering at this time such technical courses as commercial or machine-shop mathematics, or courses devoted to preparing students for examinations of various kinds, but considering only such work as is best for the general knowledge of the student, we enter a field that has long been cultivated in Europe but which has only recently been opened in this country. The work proposed will seem to many teachers as beyond the powers of the pupils, but it has been and is being given without the slightest difficulty in various schools in this country, the pupils finding it interesting, and valuable and being able to use it easily in elementary work in physics and mechanics.

If it were suggested that we should teach analytic geometry and the calculus in the high school there would be those who would recall their own work in these subjects in the sophomore year in college, and would often lament at the dullness and aridity of the teaching. They would then say that such subjects are entirely unsuited to the students in Grade XII. What they should say, however, is that the type of teaching they had in college is unsuited—not the essential parts of the subjects themselves.

The fact is, we already teach a certain amount of analytic geometry in every modern high school in this country, namely, when we teach the graphs of equations. Like all other subjects, they may be made too hard for the pupils or they may be made so simple as to be perfectly suited to the kind of student that will be permitted to continue his mathematics in Grade XII. It is, therefore, entirely feasible to attain the following objectives in this school year:

1. Straight-line formulas, including the point-slope form, the slope-intercept form, the two-point form, and the two-intercept form.

2. Certain curve-line equations, including the three conic sections, with their graphs, and including some study of the graphs of equations of higher degrees for the purpose of understanding more clearly the nature of the roots.

3. The variation of functions, a subject that naturally follows the study of trigonometry and topics 1 and 2 above.

4. Rate of change of a function, including uniform rate in the case of a linear function, average rate, and instantaneous rate.

5. The application of this work to physics and geometry and including especially extreme values as in the case of maxima and minima.

6. The idea of a limit, both numerical as in the case of an infinitely decreasing geometric series, and geometric as in the case of an inscribed regular polygon.

7. The relation of limit to instantaneous velocity, to instantaneous direction, and to tangent and slope.

8. Differentiation and its relation to maxima and minina, with applications—one of the most interesting topics that the pupil meets during the year.

9. Integration and its application to the mensuration of areas.

The pupils who complete such a course in mathematics will be quite able to read understandingly the elementary parts of works on physics and will have a vision of the meaning of mathematics that is far more significant than that which comes to those who complete the old-time course. That it is too difficult for them is disproved by numerous classes that are doing this very work in various schools in this country, not to speak of the large numbers who have been doing as much and more for many years in the secondary schools of the leading countries of Europe.

Instead of such a radical departure from the conventional course, the work of the first half of this grade may, however, be devoted to advanced algebra with the following leading objectives:

1. Theory of equations.
   a. An equation of the *n*th degree has *n* roots if it has one root.
   b. Factoring of polynomials in one variable, and the remainder theorem.

c. Symmetric functions of the roots of an equation.
d. Complex roots.
e. Approximate solution of a numerical equation.

2. Determinants of the second and third order, with simple transformations and applications.
3. Complex numbers.
4. Permutations, combinations, and probability.

Mathematicians would generally prefer such a course to the former, the reason being that it is less superficial and furnishes a more substantial basis upon which college mathematics can be built. From the standpoint of general information, however, and of utility in reading modern scientific literature, the other is the more valuable. Indeed, in the appreciation of mathematics which it gives, it is not improbable that it will in the long run tend to make the better mathematician.

THE MATHEMATICS CURRICULUM IN THE SENIOR HIGH SCHOOL

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I. DESIRABLE OUTCOMES

A New Point of View. To rediscuss a subject that has been taken up for so long a period and from so many different angles, and to do it in such a way as to make any fresh contribution, is a difficult assignment. However most of this discussion in the past has been either from the standpoint of the mathematician or less frequently from that of the school executive. It may be possible for one who has been in both positions, but whose chief concern for many years has been the study of children and the attempt to adapt the school to their needs, to attack the problem more definitely from the standpoint of developing individuals instead of putting emphasis on either the subject or the school organization.

Possible Gains from the Study of Mathematics. It seems necessary as a preliminary to consider to some extent the possible gains from the study of mathematics in the last three years of secondary school work. For convenience, a classified list has been made in which are included gains that either have been claimed or are conceivable as coming from the subject. I do not presume to say that the list is complete or that all the ends are obtainable in general. A case might easily be made for changing some subheadings from one class to another or for completely changing the method of analysis. For the purpose of this discussion the details of arrangement do not seem particularly vital. The headings are, of course, not exclusive; as a matter of fact, many of them are very closely related indeed. Some parts of C, for example, are possible subheads under B1 and B2, but are listed under C for emphasis on their particular values as well as on their habit-forming sides.

A. Actual utilitarian value of the subject matter itself.
   1. For direct practical application by individuals.
      a. By those whose occupations make use of the subjects studied.

   b. By people in general.
2. For use in future work in mathematics.
   a. In advanced study.
   b. In teaching.
3. For use in scientific fields.
   a. In occupational fields.
   b. In advanced study.

B. Value for mental training of a somewhat general kind.
1. As general training of the mind which it is hoped will carry over into other fields by increasing mental efficiency.
2. As specific training to meet similar problems in mathematics and to prepare one to do thinking in closely related situations.
3. As practice in the use of a fundamental method of thought with a definite and easily handled type of material.

C. Development of habits and influence of attitudes.
1. Developing the habit and method of grasping and analyzing new situations.
2. Learning to think through a problem by means of logical steps.
3. Acquiring a habit of demanding reasons for conclusions, and of weighing evidence, leading to willingness to examine critically one's own mental processes.
4. Acquiring a mastery of certain logical procedures, such as that of the exhaustion of possibilities.
5. Developing an independence of thinking, leading to drawing conclusions for one's self.
6. Strengthening habits of attention and concentration through practice in situations requiring them.
7. Increasing the likelihood of making accurate and concise statements, both oral and written.
8. Learning to appreciate the leading part mathematics has taken in the development of mankind.
9. Learning to enjoy the use of the mind; as one writer has said, "to play with it."
10. Realizing the satisfaction that comes from achievement.
11. Learning to generalize, and developing an appreciation of the universality and the impersonality of formulas and scientific laws.
12. Reaching an appreciation of absolute truth with the possible

inspirational, almost spiritual, response to mathematics and the universe that may come to some pupils.

13. Development of the esthetic and imaginative part of one's nature.
14. Being affected by many other influences on attitudes, as those toward intellectual effort, toward those in authority in relation to the pupil, the school subjects, the teachers, and the community, brought about by the work assigned, the method of handling it, the pupil's own efforts, and other such factors.

D. Knowledges and correlations.

1. Knowledge of nomenclature, facts, formulas, and uses of mathematics, regarded as a convenient familiarity which might add to contacts and give understanding of allusions in a field likely to be more or less frequently met.
2. Knowledge of the simple, fundamental laws of logic.
3. Mastery of a symbolic language that is generally used.
4. Some mastery of English in slightly different situations than are commonly taught. This would include the translation from words to symbols and vice versa as well as training in concise and absolutely accurate expression.
5. Increased understanding and love of nature.
6. Some correlation with foreign languages from which many of its terms are derived.
7. Possible interweaving with history and use in social science.
8. Better preparation for the arts, both representative and musical.
9. Many interrelations with various sciences.

It must be granted at once that many of these gains are seldom achieved because of teaching that does not realize their value or does not know how to attain them. Nevertheless, those that appear possible, and most if not all of them seem to be, should be weighed and considered in planning the curriculum.

Utilitarian Value. Emphasis on mathematics in the past has centered most definitely on A and B, with more emphasis perhaps on B. In regard to A, even the most ardent mathematicians have been forced to confess that there are wide differences among different groups of pupils. Practical application is a sound aim for those who will actually use the mathematics in their occupations.

Even here, however, short-cut methods, tables, and the like have proved so usable that only a small portion of the usual college preparatory type of mathematics course would need to be taught if this were the only value. For the general public, second year algebra, geometry, and the more advanced subjects have so little definite use in life that such use alone cannot be made a justification for the time spent on them. Where the mathematics is to be used in more advanced work, in teaching, or for application in sciences, there is a justified demand for its inclusion.

General Training. The first heading under B has been the subject of more discussion than almost any other aim in education. It does not seem that general training with no plan for carry-over has as great value as an intellectual strengthener as has sometimes been claimed. Nevertheless, elements common to mathematics and other fields of thinking can undoubtedly be carried from one field to another, and it is possible so to plan the work that applications of types of thinking can be compared and related in various fields with resultant general benefit. There seems no question that real training value can be gotten from the courses in question if the requisite ability on the part of teachers and pupils is present.

That one can acquire definite techniques of mathematical thinking which are of the greatest value in going on with mathematics, and which can be made use of in somewhat similar situations in other lines of thought, is somewhat inherent in what is stated in the preceding paragraph and seems beyond question.

Geometry, in particular, furnishes a very definite and comparatively easily handled body of material by which elementary logic and sound methods of thought can be illustrated and practiced, especially if illustrations from other fields are correlated with those in geometry. I, personally, believe that this aim is an exceedingly important one.

Habits and Attitudes. The third group, C, is worthy of much more study than has ever been given to it. The importance of habits and attitudes has never been sufficiently realized, and much of the psychology concerned is yet in its beginning. Improvements in teaching seem likely to center largely about their consideration.

Several of the earlier headings under C are closely related to B3, and, of course, also apply to B2. In a sense, they are all general training, but for the teaching to be most effective, it is important that they should be in the mind of the teacher as definite separate aims. The one least understood is the last, yet in all probability every act of the teacher or undertaking of the pupil has influences on attitudes that may be far reaching in their effects.

Knowledges and Correlations. Headings under D, while of some importance, are not exclusive to the more advanced subjects of mathematics, but, with the exception perhaps of 2, may easily be reasonably well covered in junior high school courses. They would hardly serve as a basis for determining the curriculum of later years, though they should somewhat influence the content.

Individual Differences. But before considering what these possible aims might indicate, it is necessary to examine the material that comes to the senior high school. There is a continually increasing appreciation of the tremendous variation in types of children. While it is true that children of very low intelligence quotients are unlikely to reach this stage of schooling, nevertheless there is a range of general abilities, as determined by the standard intelligence tests, that extends from below average to the most extraordinary abilities. In addition to this, the children of approximate equality in general or average ability will vary tremendously in their response to various fields of thought and action. Of two pupils of the same intelligence quotient, one may show a decided weakness in rote memory, but some ability in mathematical thinking, while the other may exactly reverse this situation. Recent estimates made by mathematicians and psychologists of the probable intelligence quotient necessary to a pupil's reasonable mastery of first year (9th grade) algebra, range from 108 to 112, with the most common, as well as the average, estimate at 110. If this is approximately correct, it indicates to some degree the intelligence quotient necessary for undertaking the usual senior high school algebra, geometry, and trigonometry, except that it may be possible for some minds to be stimulated by a certain visual concreteness in geometry, or by the ease of application of trigonometry, beyond the response that would be aroused by the symbolic language of algebra. The fact that these are all divisions of mathematics does not mean, as is quite commonly assumed, that a pupil is equally responsive to them or is able to think equally well in any two even if the response happens to be of the same intensity.

Wide Range of Requirements. It appears, therefore, that there should be a reasonably wide range of mathematical requirements to suit the varying ability of the pupils. It should, of course, be understood and emphasized that as far as we know at present there is no advantage in trying to force a pupil through mathematical thinking beyond that for which his mind is fit, and that many disadvantages can and do come from such attempts. This is not the place to discuss the conclusions of modern psychology in regard to the limits of improvement possible to various types of mind, and in various kinds of thinking. Neither can we take up the abnormalities of behavior and attitudes arising from demands a child cannot possibly fulfill. A teacher of mathematics, however, should not fail to keep in touch with the literature on this subject, and, by its conclusions, should be influenced not to become discouraged or impatient, but to analyze and study individuals, and to demand all—but not more than all—that each pupil can give.

The problem would then seem to be that of arranging a minimum course that is suited to, and of value for, all who enter the senior high school, with a reasonably wide range of possibilities for those who are likely to gain from a more technical study of mathematics.

The Method of Presentation. Another question would modify plans for the courses, although it is to some extent a technical one concerning the method of presentation. The question of fusing the mathematics subjects was debated for many years before the recent junior high school mathematics courses were planned and quite widely put into effect. The matter seems not to have been finally settled by the experiments already undertaken. Some advocates of fused courses are suggesting that similar methods be carried into senior high school subjects. Others feel very definitely that the experiment has not succeeded and that it has failed because the pupil's scattering of attention over the various subjects of mathematics has failed to build up habits or technical mastery on which more advanced work can be solidly founded.

Unification of Topics. It seems unquestionably true that school subjects have been too much divided into compartments, thus bringing about a situation that does not to any comparable degree exist in life. Through the elementary school and to quite a degree in the junior high school it is possible to confine the child's study to a few fields within each of which limitations are as far as possible done away with, and even between which the walls are easily broken down when opportunity to combine the fields makes it advisable. This situation is helped by the fact that many of the skills and knowledges studied are fundamental, and are best mastered in connection with their application. For example, arithmetical skills become usable more easily if they are used in other subjects, in games, and in life situations, naturally and frequently during the acquiring period. There is, therefore, an added incentive to the school to break down artificial restrictions on the interrelating of subjects. My own opinion, however, is that in the upper classes there is not the same disadvantage in introducing pupils to distinct branches of human knowledge in which particular methods of attack and particular techniques can be used, even though those fields of knowledge may be closely related and may from the logical viewpoint of the adult seem to present possibilities for economy by combination. There is no economy if such a combination prevents the formation of habits and attitudes of mind that will prove valuable, or if confusions arise through the attempt of the limited adolescent mind to adjust itself to the logical arrangements that seem logical only to mature experience. For example, I regard elementary algebra on the one hand and plane and solid geometry on the other as quite definitely different fields, although they have many opportunities to reënforce each other and to attack the same problem from different angles. I am quite certain that each has certain values which cannot as easily be gotten through the other, and that for those pupils who take both there is a real advantage in the mastery of each as a separate subject, provided there is no neglect of comparisons and of opportunities for anticipating the possibilities of eventually welding them into a single even more powerful mathematical tool.

Correlation. At one time I carried out, with my own classes and with the help of other teachers in my department of mathematics, an experiment in trying to get all possible correlation values between plane geometry and intermediate algebra. The two subjects divided the days of the week, with certain days assigned to each as a major interest, but with the understanding that any degree of flexibility proving to be of advantage would be permissible. A single mark was to be given on "Mathematics," and the course was to have the usual total amount of time for its completion. The plan was eventually given up, because both pupils and teachers felt a definite lack of continuity and consequent breaking of interest and failure to master modes of thought. An interesting side light comes from the fact that it proved difficult, if not impossible, to mark fairly (unimportant as many of us believe the marks themselves to bel) because of the differences in ability in the two subjects shown by many pupils. What single mark could fairly indicate that a pupil was doing very well indeed in geometry, but was failing in algebra, or vice versa?

Condensing Subjects. Another suggestion that has important support concerns not fusing, but condensing. It would make plane and solid geometry a one-year subject, emphasizing the parts most likely to be applied and assuming or touching lightly that which is only a necessary part of the logical sequence. This takes for granted that the test of value is future use. If there is any part of senior high school mathematics that has importance as training material and as an influence on attitudes and habits, it is geometry. While the time at present devoted to it seems a reasonable allotment, there is all too little allowance for the "below the surface" investigations and extensive explorations that can be made such an invaluable part of it. I should be very sorry to see any condensation for most of those who take it.

Calculus in the Senior High School. Again we have the suggestion that the elements of calculus might well be taught in the last year of the secondary school. If this were done in a selected group that planned to go further with mathematics, it might well prove feasible. The fact that it is done to some extent in other countries seems to me no incontrovertible argument in respect to our conditions and needs. It is certainly unfair to compare too positively methods that can be used with the highly selected secondary school groups in some European countries and those being tried with the many divergent mental types that are welcomed in the schools of similar advancement in the United States.

Perhaps the testimony of many men of fine minds—even mathematical ones!—that they never really understood what calculus was about until increased maturity brought the light might be considered an argument, although that too is influenced by many factors.

My own belief has always been that the extent to which fourth year mathematics classes should take up college algebra beyond the usual requirements, analytical geometry, and calculus should be a flexible matter, governed by the possibilities of the classes and the abilities of their teachers. Some classes are ready for, and can really grasp, mathematical procedure far beyond other classes of the same age and preparation. A good teacher can open up a fascinating range of possibilities with resulting inspiration for future study, without skimming the cream from the most directly interest-arousing parts; while a poor teacher will either spoil the subject for future study or will load the immature mind with half understood facts and theories. In other words, no attempt should be made to force all teachers to teach the calculus in the high school.

II. Possible Future Curriculum

Aims. In consideration of the various factors that have been discussed, what seems the possible future curriculum for the three senior high school years?

At the end of this time a pupil should have achieved from among the mathematical aims at least the following two:

1. A mastery of the knowledges and skills most important in life.
2. Whatever of the mental training, the habits, and the attitudes the subject can give that pupil in its share of his time.

Mathematics for Everybody. But most of the skills and knowledges *universally needed*, including knowledge sufficient for life contacts with matters of mathematical interest, are included in arithmetic and the other mathematics that is studied before senior high school. Is it mastered there, and can that part therefore be considered as finished? I think not, and therefore I wish that all students *could* be required as a prerequisite to graduation to take a course in mathematics in common use, or of such value that it should be in common use, among those having no particular mathematical contacts. This course might be given at any year, but would seem of most value perhaps, especially to those whose schooling is finishing, if it were given the last year or at the earliest the year before. In content, it would naturally overlap the mathematics of the junior high school, but with a more mature method of attack. It would review operations, shortening them when possible and emphasizing the use of tables, would start each boy and girl in life with a simple, already practiced, method of keeping personal and household accounts, and would explain away the mysteries—perhaps social as well as mathematical!—of taxes, insurance, investments, personal bank relations, and the like. It would also review and renew mathematical knowledge likely to be convenient, and would make the reading of the daily paper more easily understood by occasional analysis of its mathematical allusions and expressions.

It is possible that it might discuss gambling from a common sense mathematical viewpoint, that the earning power of money saved through thrift would be emphasized, and that many unforeseen opportunities would arise to make mathematics a help to those taking the course. One of its most important "by-product" aims should be, with many of the children, to help to put money in its proper place in relation to life. It should be understood in terms of return given for labor or other considerations, and should be valued too much for waste, but too little for overemphasis.

I do not know that there would be anything really new in such a course, or how greatly it would eventually differ from that given to pupils in commercial courses of various kinds for whom the vocational element has such importance. I do know that a good share of our boys and girls who take the usual college preparatory mathematics, or who take a general course with little or no senior high school mathematics, need such training in order to handle their own affairs efficiently. It could and should be correlated with shop, art, homemaking, and any other subject admitting of it, but it should relate itself above all else with the mathematical thought and needs of the community outside the school.

How many periods a week would be required for such a course, I do not know. I believe it could be given in fewer periods to those taking the usual mathematics than to those who are not in such classes. It might well prove that a rearrangement of time distribution in junior and senior high school would make it possible to put more algebra in the lower classes by saving some of the time now given to the subject matter of such a course as this, and as a result would enable the school to transfer some time from secondary algebra to this general course.

The Beaver Country Day School is trying to develop a course of this kind, though it does not seem feasible at present to require it of all pupils.

Habits and Attitudes. If the habits and attitudes under C are as important as they seem to be, each pupil should have some opportunity to develop them. It is true, of course, that no one subject can make exclusive claim for providing any particular kind of training. Some educators feel that attitudes and mental self-activity are all-important—as indeed they are—but that the particular field of activity is of comparatively little importance. The late Dr. Charles W. Eliot, on the other hand, has called the study of music the most efficient mental training of all. Nevertheless, mathematics does present definite and easily tested opportunities for logical analysis, organization, and the cultivation of such thought habits as are listed under C. I am confident that the pupils whose natural ability fits them for it, and whose course plans admit of it, will gain very greatly by the study, under good teaching, of at least algebra and plane geometry.

It seems necessary, however, to provide a rather wide range of opportunity for pupils studying these subjects. If the recommendations of the National Committee are taken as a minimum course, pupils of varying degrees of ability can be encouraged to read along supplementary mathematical lines, to go to various textbooks for different viewpoints, to attempt a wide range of problems in algebra and originals in geometry, and where it seems worth while to bring back to the group the results of their work.

Flexibility in Courses. Mathematics should no more be confined to the limits of a textbook than should English or history. More than most school work it needs guidance and stimulation to show the pupils its possibilities beyond the routine requirements. Therefore, the name "algebra" or "geometry" on a schedule should be simply an indication of the major field of interest and the minimum requirement to be covered, not a complete definition of the work of the year.

Logical Thinking. May it not also be possible, perhaps in connection with the "mathematics for use" course already discussed, to give to pupils not taking algebra and geometry an elementary course in logical thinking, not entirely confined to mathematics but drawing many simple illustrations from it as well as from other fields? The formula could be emphasized, the simpler laws of logic could be discussed and used, and much of general value might well come from it.

I have repeatedly tested classes about to begin geometry on their instinctive accuracy of reasoning. Only a small percentage have any objection to assuming the negative or converse of a true statement, and all need to become conscious of logical procedure. For many years, all geometry classes of my own school have started with a very short course on logic, but those who do not study geometry need it even more.

The Regular Course in Mathematics. For the regular course in mathematics, there seems at present little choice in topics. The report of the National Committee is too recent, and was too thorough, to encourage tampering freely with the content of algebra and geometry, at least until longer trial has raised questions of serious importance.

As to the order of subjects, I should like to make a suggestion. There seems to me to be very real advantage in having secondary algebra follow elementary algebra, with plane geometry coming in the second, instead of the first, senior high school year. There is then a continuity in the algebra that makes for greater mastery and better established habits. Also, if geometry is to give its fullest benefit in the development of logical procedure, it should be taken by pupils who are fairly mature and so are ready to study it to advantage. This order has been tried in public and private schools, and seems to those using it to have proved of advantage.

Technical Courses. The courses discussed here take no account of technical training such as is given in vocational courses of various kinds. Their main aim is, naturally, that of direct use in occupations. Nevertheless, whatever they lack of the two minimum requirements already mentioned should be compensated for in some way, probably by enrichment and broadening of existing courses, with other than utilitarian aims in mind.

I do not feel that I have kept sufficiently in touch with recent developments in shop mathematics, commercial courses, and others of similar nature to justify my discussing details regarding them.

Place of Present Demands. It seems timely to say at this point that no school curriculum, in whole or in part, can afford to base its planning too completely on present demands for specific use. It is not possible to foresee conditions in the world for any considerable time, and, therefore, general preparedness, open-mindedness, and resourcefulness may prove to most of the population more important than overspecialization.

Review Course. In the last year of the senior high school many of the pupils now take no mathematics. Others review algebra and geometry, or take up some or all of solid geometry, trigonometry, and advanced algebra. For those going on with mathematics later, the review course has real value. It should have value also for those who will take no more of this subject and should have greater importance than it now has for those who are to go on. For this course too often is simply a review of what has gone before, with no widening of the horizon, no greater insight into generalization, no opening of vistas into new possibilities for exploration.

It is true that college examinations loom ahead for at least some members of any such class. This fact should not, nevertheless, exclude everything but narrow college preparation and more or less routine drill from the work of even the college candidates. Certainly those not planning for college have a right to all the advantages that can come to them from their present courses. It is possible to arrange a course that will so unify, give meaning to, and extend the mathematics that has gone before that it will become almost a new subject, although it will at the same time serve as an effective review, and an agent for strengthening habits and influencing attitudes.

Elective Courses. I have little to say in regard to the so-called "advanced courses" that are elective. Solid geometry extends the practice of plane geometry and deals in knowledge of three-dimensional space. If the plane geometry has been well taught, solid geometry presents little difficulty to most pupils, although an occasional one finds that visualization in three dimensions troubles an imagination that could meet the demands of two dimensions fairly well. Similarly, however, it presents few advantages that have not already been available, particularly if the three-dimensional concept has been brought in from time to time, as it well may be. This subject seems to me to merit inclusion only in the schedules of pupils particularly interested in mathematics, or required to take the subject for direct preparation.

Trigonometry is an extension of geometry which, besides giving a new method of attack, produces new symbols that can be manipulated algebraically. Its immediate applicability makes it of particular interest to some pupils if it is not taught in an over-abstract way. It does not seem that its main purpose can be a mastery of logic, as is true in geometry, for the major part of its problems requiring analysis can be fitted into a few forms or procedures, and practice beyond the point necessary to master these key forms becomes repetition of little value. Nevertheless, I believe trigonometry to be the most important of the three subjects in question for a pupil who is not to continue mathematics beyond the high school.

Advanced algebra seems to be an elective of value principally to a limited group who need it as preparation for advanced work. although it offers much of interest to one mathematically inclined.

Conclusions. Although to me mathematics has always presented greater scope for mental activity and greater opportunity for genuine enjoyment than has any other subject, yet my study of pupils of various ages, which includes eighteen years of use of standardized tests and fourteen of giving individual mental tests, has convinced me that for a good share of the secondary school pupils, a well-balanced course suited to each individual is perhaps the most important consideration. In each course mathematics should have the place it can earn by its satisfaction of the demands and needs of the pupil in question. Yet there is so much of tremendous importance in other subjects, such as social science, the natural sciences, handwork, art and music, that is often underestimated and neglected, that we cannot afford to be too dogmatic and too grasping in stressing the unique advantages of our subject and demanding the maximum time for its study. Rather we should be incessantly criticizing our own claims and testing our own results; for if we fail to achieve results commensurate with the time given us, and to influence permanently and constructively the pupils under us, inevitably other subjects will, and should, crowd us little by little out of the school day. Each subject, mathematics or any other, is a means to educational ends, not primarily an end in itself. If any one proves not to be the best means, even those who love it must be willing to see it relegated to the past.

The burden of proof, as always, remains on those who teach the subject!

A REORGANIZED COURSE OF STUDY FOR SENIOR HIGH SCHOOL MATHEMATICS

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I. Introduction

Reorganization. Reorganization of senior high school courses in mathematics becomes expedient, if not obligatory, with the dominant emphasis in the modern school upon social objectives and the prevailing acceptance of the principles of educational psychology. The ideals which have guided the organization of the junior high school curriculum operate with equal significance in the senior high school. During the past ten years, however, junior high school activities have occupied the center of the stage; senior high school problems, not being vigorously agitated by reformers, have for the most part remained in obscurity. The demand for modified procedure is felt within the senior high school when it becomes necessary to reconcile formal academic work with the lively undisciplined multitudes of children produced by activities programs or project curricula of our thriving junior high schools.

University departments of education and their laboratory schools are far ahead of general classroom practice. The important and highly significant messages of Perry, Moore, Dewey, and Nunn have not yet penetrated to the far corners of the country nor indeed to the hearts of all cities. There are many high school teachers of mathematics who have never heard of the Report of the National Committee on the Reorganization of Mathematics in Secondary Education, nor did their school librarians order a copy when it was to be had for the asking. It is not uncommon for the state university admissions circular to prescribe objectives and for the authorized textbook to furnish an adequate course of study. Where classroom teachers have the deciding vote in the choice of textbooks, preference is usually shown for the orthodox old-fashioned type of text. Again, some who say they have tried the newer methods and a textbook in general mathematics have abandoned both and are frankly reactionary and antagonistic to any change in traditional methods. We must face these facts with optimism and continue with renewed effort our attempts to develop an improved professional attitude in the new generation of teachers now graduating from teacher-training institutions.

The National Department of Superintendence now maintains at Washington a Bureau of Curriculum Development. The larger cities have in the office of the City Superintendent of Schools a "Department of Research and Guidance." These organizations have a very effective scheme of teacher coöperation and proudly display the pictures of the army of teachers of the nation at work on their courses of study. Whether or not they are the creative artists responsible for their final reports, teachers who participate in this committee work become more critical of their teaching, analyze objectives and the organization of subject matter, and make a study of methods and teaching outcomes. Probably one of the greatest values of such committee work is the mutual benefit derived from exchange of practical ideas and of very definite, concrete, and detailed illustrations of classroom procedure which embody and interpret the spirit of modern educational theory. With this thought in mind, I am reproducing the outlines of a course of study in mathematics for senior high schools prepared for the public schools of Oakland, California. I have omitted bibliographies and certain other material. The committee responsible for the report was composed of the mathematics department heads in the five senior high schools of Oakland: Edward J. Albrecht, Fremont High School; Walter A. Stafford, Oakland High School; Jean Tuttle, Roosevelt High School; Mary W. Tyrrell, Technical High School; Gertrude E. Allen, University High School, Chairman.

II. General Statement of Objectives in Mathematics

The National Committee. The National Committee for the Reorganization of Mathematics in Secondary Schools formulated the following statement of aims which cover in a general way the entire field of work in mathematics:

"The primary purposes of the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quantity and space which are necessary to an insight into and a control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual."

All branches of pure mathematics from the infant's experience of simple counting in the nursery to the student's study of the highest branches of analysis at college should be slowly and continuously developed by the pupil himself under the wise help and guidance of parent and teacher as the appropriate instrument for the solution of problems presented in the individual's attempt to understand and interpret his social and physical environment. The particular mathematical experience which forms the material of the educational process must, at every stage, in both quantity and quality, be appropriate to the present capacity of the individual who is expected to assimilate it. This is not always kept in mind by those who do the teaching.

Quantitative relations are expressed in three ways: (1) arithmetically in tables and numerical values, representing the number concept; (2) algebraically, in formulas, equations, and functions, representing the concept of abstraction or generalization; (3) geometrically in length, area, volume, and graphs, representing the space concept.

As Professor J. W. A. Young says, "The function of arithmetic is to arouse interest in the quantitative side of the world, to give accuracy and facility in simple computation, and to give a working knowledge of a few concrete practical applications. The function of geometry is preëminently to develop space intuition and to train to active and careful thinking."

Again, Professor Nunn says, "The object of algebra is to develop a calculus, that is, a system of symbols and rules for the manipulation of symbols, by means of which the investigation of some definite province of thought or of external experience may be facilitated."

III. Elementary Geometry

A. *One-Year Course.*

Tenth school year, or later in special cases by the advice of the school counselor.

Prerequisite—Elementary Algebra.

Required for qualification for the Junior Certificate at the University of California.

B. *Aims.*

1. To develop space intuition by:
   a. "Laying a foundation of experience upon which to build."
      1) Experiment and measurement.
      2) Constructions.
         a) Geometric drawing.
         b) Making models and crude instruments.
      3) Observation of geometric forms in nature, architecture, and decorative design.
      4) Exercise of spatial imagination.
   b. "Organizing a body of knowledge out of this experience." The definite goal is:
      1) To gain an accurate knowledge of the significant propositions of geometry.
      2) To develop and learn for practical use the essential formulas of mensuration.
      3) To reveal possibilities in further exploration and to find incentives to carry on.
   c. "Applying the resulting knowledge to practical use in the concrete world."
2. To furnish favorable material for exercise in the process of logical thinking.
   a. To develop an understanding and appreciation of the more mature method of deductive reasoning in the field of geometry.
   b. To form habits of exact, truthful statement, and of logical organization of ideas in this field.
   c. To establish and exercise a conscious technique of thinking—using as a basis Professor Dewey's analysis of a complete act of thought as follows:
      1) A felt difficulty.
      2) Its location and definition.
      3) Suggestions of possible solution.
      4) Development of reasoning of the bearings of the suggestions.
      5) Further observation and experiment leading to its acceptance or rejection.
   d. To foster all possible transfer of ability to the solution of new problems both mathematical and non-mathematical.

C. *Outline of the Course.*

The outline of the course which follows is purposely designed with sufficient flexibility to accommodate both the conventional plane geometry course and the new one-year course designated as "The Essentials of Plane and Solid Geometry," which is being given as an alternative to the former type of work. Deviation from the traditional course lies in the conscious effort to build on the concrete experience of the pupils and make abundant applications in practical applied problems. The logical sequence of topics is still maintained throughout, but more abstruse and intricate forms of logical treatment are omitted so that the fundamental theorems of plane geometry are completed earlier. The last five weeks in the alternative course are devoted to mensuration as indicated in the outline. The status of students who go on with the senior course in "Solid Geometry and Supplementary Work" is not affected by this plan.[1]

The sequence of propositions depends in a large measure upon the textbook used by the class. Regardless of the text, it is essential that the teacher *emphasize* the *relative importance* of the *significant theorems, group related theorems,* and *generalize closely related groups* in one theorem when practical. Without interrupting the logical sequence of proving theorems in many topics such as congruence, symmetry, and locus problems, there is an added interest and significance if the statements of relations in the plane are extended to include analogous relations in space. The following topical outline covers the content of a year's course:

1. Congruence and equality.
   a. Congruent triangles,—their significance in trigonometry and in mechanical constructions.
   b. Sum of the angles of a triangle and a polygon.
      1) Importance in the theory of trigonometry.
      2) Isosceles and equilateral triangles and regular polygons.

[1] It seems inadvisable and unnecessary to reprint the standardized syllabus compiled by the College Entrance Examination Board and distributed from 431 West 117th Street, New York. The bulletin as issued by this board (September, 1922), contains a complete list of the propositions starred and unstarred for (1) a year's course in plane geometry, (2) a new one-year course in plane and solid geometry recommended as an alternative for (1), and (3) a one-semester course in solid geometry.

3) Applications in design for a surface covering such as quilt patterns, tiling, carpet, and oilcloth patterns.

c. Right triangle and the theorem of Pythagoras.

1) Deriving theorems for the length of a side opposite an obtuse and an acute angle in a triangle.
2) Different interesting proofs and various important applications of the Pythagorean theorem.
3) Ratio of the side to the diagonal of a square; and of the altitude to the base in an equilateral triangle.

2. Symmetry and similarity.

a. Similar triangles and similar polygons; the right triangle; sine, cosine, and tangent functions of an angle; problems in heights and distances; ratio of the lines in a right triangle formed by dropping a perpendicular from the vertex of the right angle upon the hypotenuse and the constructions depending upon the equality of these ratios.

b. Ratios of corresponding lines, corresponding areas, and the volumes of similar solids.

c. Symmetry,—central, axial, planar. Observations of symmetry in nature, architecture, stage settings, art design, and room furnishings.

3. Form and position.

a. Rectilinear figures and solids bounded by planes.

1) Intersecting lines and intersecting planes; pencil of lines, pencil of planes; concurrent lines in a triangle; the centers of a triangle and the related circles.
2) Perpendicular lines and perpendicular planes; rectangles and rectangular solids.
3) Parallel lines and parallel planes with their related angles, parallelograms and parallelepipeds.

b. Circles.

1) Subtended arcs, angles, and chords.
2) Secants and tangents to a circle with related angles; relation of intersecting lines which meet in a circle.
3) Regular polygons, inscribed and circumscribed; limiting value for the ratio of perimeter to diameter; computing π, the "ratio of the circumference to the diameter," area of regular polygons and circles; informal treatment of the theory of limits.

c. Locus of a point which moves about in space but satisfies fixed conditions; locus of a moving point when positions are restricted to the plane; informal treatment of variables and constants.

4. Geometric construction.

Constructions of more or less complexity to stimulate thought in applying previously acquired knowledge in new situations.

5. Mensuration.

This section applies only to the alternative course designated as "Plane and Solid Geometry" and constitutes the work of the last five weeks.

a. Development of the standard formulas for areas and volumes of solids.

b. Numerical computations involving these formulas applied in miscellaneous problems of practical value.

D. *Recommendations and Comments.*

1. Recommendations.

Special attention is directed to the recommendations of the National Committee concerning solid geometry and the endorsement of these recommendations by the College Entrance Examination Board. They propose a major requirement of one year of plane geometry and a half year of solid geometry; a minor requirement of one year of plane and solid geometry. This minor requirement is to be offered as an alternative of the conventional minor requirement of one year of plane geometry.

Provision for this alternative minor is made in the outline o' this course. It is made in recognition of the fact that a very small percentage of the high school graduates in California ever take a formal solid geometry course while a very large majority take the course in plane geometry. Provision is made for a large degree of flexibility in the course so that each teacher in his own way may try to make the geom' 'y work a part of life instead of a mere school creation. For many students, there is greater educational value in the inward development of *power* to *use* the *subject* in its manifold applications than in the outward insistence upon its logical aspects for the purpose of "mental discipline." The alternative course is being used in many of the Oakland schools and the results are most encouraging both for the students who take no more advanced courses and for those who specialize in mathematics and continue with the formal course in solid geometry.

2. Adaptation to students of varying ability.

Classes are arranged in homogeneous ability groups, as consistently as administrative difficulties permit. Adaptation to the several groups is made by varying the intensity and quantity of original work, and the number of excursions from the main line of work. Speed is essentially the same. It is not profitable to hurry over the ground even with the accelerated groups.

3. Teaching guides.

a. Quotations. For inspiration and helpful suggestions teachers are referred to recent contributions to *The Mathematics Teacher;* to a pamphlet, *The Teaching of Geometry in Schools*, prepared for the Mathematical Association, G. Bell and Sons, Ltd., London; and to a book entitled *Mathematical Education* by Benchara Branford. A few quotations are included here.

"Our view of the curriculum now shapes itself as follows: The school must be thought of primarily not as a place where certain knowledge is learnt but as a place where the young are disciplined in certain forms of activity,—namely, those that are of greatest and most permanent significance in the wider world. The school is a select environment where the creative energies of youth may work toward individuality under the best conditions.

"The curriculum that best fits the present needs, interests, and capacities of boys and girls is quite similar [to] if not identical with the curriculum that best prepares for future needs.

"Systematic exploration of special parts of mathematics is of vital importance to continued growth of the science. . . . It does not, however, by any means follow that the branches of mathematics should be presented to beginners with the formal elaboration which is the inevitable mark of the treatment as separate subjects."—T. P. Nunn.

"It is essential that the boy be familiar by way of experiment, illustration, measurement, and by every possible means with the ideas to which he applies his logic, and moreover that he should be thoroughly *interested* in the subject."—John Perry.

"There should be no abrupt transition from the introductory intuitional geometry to the systematic, demonstrative geometry."—John Dewey.

"There are two stages in the study of geometry,—(1) the heuristic stage in which the chief purpose is to order and clarify the spatial experience which the pupil has gained from his everyday intercourse with the physical world, explore the more salient and interesting properties of figures, illustrate the useful applications of geometry; (2) systematic stage, the chief purpose of which is to organize a logical system when the theorems are arranged in a logical sequence following from a few simple principles."—T. P. Nunn.

Professor Nunn proposes a larger list of postulates for beginners. Take as a *postulate any statement of common acceptance,* reserve for *proof* as a *theorem* some *new fact* which could not be proved by direct intuition, symmetry, or superposition. He suggests as fundamental postulates:

1) A given figure can be exactly reproduced anywhere. (The notion of congruence.)
2) A given figure can be reproduced anywhere on any, enlarged or diminished, scale. (The notion of similarity.)

Under these he would group

a) Equality of vertically opposite angles.
b) Angle properties of parallel lines.
c) Properties of figures which are evident from symmetry.
d) Properties of figures which can be demonstrated by superposition.

"The development of the powers of imagination, abstraction, and reasoning should be made continuously from experience and knowledge gained in daily life. . . . Lead gradually from previous experience and imagination to the new world of abstract thought and ideal truth, or, perhaps, present it as an outcome of and one with that more limited world of which alone he is at first cognizant." —G. St. L. Carson.

"Teachers in the lower schools have never realized that the union of logic and space studies deprived them of one of the most natural and fundamental subjects of instruction; namely, form study. The logical statement of the principles of geometry has blinded modern as well as mediæval teachers to the worth of this subject for younger pupils. Children who have not the maturity required for abstract reasoning have a keen enjoyment in recognizing the basic facts of geometry in the tiles of the floor, the decorations on the wall, the arches and windows of great buildings, the symmetry of their own bodies, and so on continuously."—C. H. Judd.

"Logical organization is the goal and not the point of departure . . . what is important is that the mind should be sensitive to problems and skilled in their solution."—John Dewey.

"Logical rigor is never absolute,—that degree of rigor is best fitted to the degree of maturity of the understanding to which it is to appeal. . . . School geometry of the present day in order to be effective must be characterized by a generous list of assumptions, a shrinking body of propositions, numerous interesting applications, and continued exposure to solid geometry. . . . Everything which is obvious is taken for granted; and argument is used only to introduce the unexpected."—Benchara Branford.

b. Definition of an educated man.

The *New York Times* once defined an educated man as a man who knows when a thing is proved. Geometry teaching may help students to qualify by cultivating:

1) A problem solving attitude of mind.
2) A critical attitude toward data.
3) Skill in analysis.
4) Ability to generalize.
5) An understanding of converse relationship.
6) Recognition of the place of assumptions in all thinking.

7) Ability to detect common fallacies in non-mathematical reasoning resulting from hasty generalizations from insufficient data, begging the question or reasoning in a circle, reasoning from incorrect data, incorrect conclusions from given data, and assuming that converses are always true.

c. Projects.

Several teachers have reported interesting experiments in project work in geometry; for example, in the review and with the conclusion of the work with circles each member of the class prepares a booklet on the "Computation of π." These reports have all the earmarks of a scientific Master's thesis; the constructions are carefully made, the numerical work exact, the historical setting and applications are given. The teachers as well as the students get keen enjoyment in the surprise over the artistic perfection of their technique of bookmaking, the individuality shown in the story of π, and the fanciful flights of imagination in the illustrations and applications. The project was initiated as a record of a computation based upon the use of the sine and tangent ratios referred to the regular inscribed and circumscribed polygons. Since this method is very much more concrete than the one in the text, it is greeted with enthusiasm. Small units of the work are completed each day, the whole extending over a week's time. Teachers report that the regular class test following this work is the most satisfactory one of the semester. Miss Durst's geometry class in the University High School is varying its former plan this semester and compiling one book as a class project. One of the girls has offered to type the material and edit the book, provided her classmates write legibly. Other units of work well adapted for the project books are geometric constructions, mensuration problems, the Pythagorean theorem with different proofs, its history and applications, geometric forms in nature, in machinery, in architecture.

d. Geometry and the inductive approach.

Geometry has been so long regarded as the ideal material for deductive reasoning that we are apt to neglect the value of the inductive approach in teaching children. By developing theorems with the class, we cultivate the attitude of exploration and discovery and it is a source of constant surprise to observe the originality and keenness of young students in their suggestions for experiments and solutions.

e. A formula for proving geometry theorems.

In the proof of a theorem, it is well to require a clear statement of what is given, what is required, and a brief indication of the procedure:—in other words, the starting point, the goal, the road by which one is to travel. Colonel Edwards of the University of California,—his former students number legion,—had a formula for proving geometry theorems. It is his proud boast that this formula is the secret of success of the foremost diagnosticians, financiers, and engineers in California. Try it! "If the conditions which are said to exist, are desired to exist, appear to exist, do exist, then what are the necessary, sufficient, and previously established conditions?"

f. Definitions and use of symbols.

All teachers should read carefully Chapter VII of the National Committee Report and try to abide by the very clear and definite recommendations given regarding a standard vocabulary and the treatment of definitions.

g. Standards of drawing.

Geometric figures should be correctly and neatly drawn for blackboard or for paper work.

h. Periodic summaries and generalizations.

A summary of related propositions and generalizations of related theorems should be made from time to time. For example, we may have the following summary: In similar figures the corresponding lines are proportional, areas are proportional to the squares of corresponding sides, volumes are proportional to the cubes of corresponding sides. Or we may have the following examples of generalization:

1) Theorems on congruent triangles: A triangle is determined if any three independent elements are given, except in the ambiguous case.

2) If a pencil of lines cuts a circle, the product of the segments of one line equals the product of the segments of any other, the segment being measured from the vertex of the pencil to the intersection of the line with the circle.
3) If the arms of an angle cut a circle, the angle has the same measure in degrees as one-half the sum of the intercepted arcs, one-half the difference, or one-half the intercepted arc, according as the lines meet within the circle, without the circle, or on the circle, —and so on *ad infinitum.*

E. *Equipment.*

"Mark Hopkins at one end of a log and James Garfield at the other" are still the most important parts of the equipment. Other accessories may be used to advantage in making the mathematics classroom a well-ordered laboratory,—meter rulers, standards of weight and measure, compasses, protractors, models of geometric solids, a pantograph, slated globes, slide rules, pictures of buildings, bridges, and designs, reference texts, mathematical puzzles, and scientific books on related subjects.

IV. Elementary Algebra

A. *One-Year Course.*

Prerequisite to all other courses in senior high school mathematics.

A detailed discussion of the outline, suggestions for teaching, illustrations, and references may be found in the junior high school pamphlet prepared by our committee.

A large number of students change their minds about the possibility of going to college, or for other valid reasons find it advisable to take the elementary algebra later than the ninth year. Since elementary algebra is not a prescribed subject in the junior high school and is a prerequisite for all other courses in high school mathematics, it is necessary to incorporate it as a fundamental unit of the senior high school course of study.

Consult the junior high school course for adaptation to homogeneous ability class groups.

B. *Specific Aims.*

1. To develop the ability to grasp, interpret, and master the simple problem situations that are of common occurrence in life.
2. To give an insight into the processes which form the basis of the great constructive achievements in the world of engineering, astronomy, navigation, and physical science.
3. To maintain and increase skill in the fundamental operations of arithmetic.
4. To develop an understanding of and skill in the use of literal signed numbers.
5. To cultivate a definite degree of skill in using the ordinary tools of problem solving,—the formula, the graph, the equation, together with the essential technique.
6. To foster habits of accuracy, neatness, and order, and of appreciation of logical continuity of thought.

C. *Topical Outline of the Course for the First Semester.*

1. Problem analysis.
   a. Problems without figures.
   b. Translation from verbal statements to the shorthand of algebraic symbols and from the algebraic symbols to the verbal statement.
   c. Formulas.
      1) Inductive derivation of simple formulas.
      2) Deriving one formula from another.
      3) Evaluation by numerical substitution.
      4) Application to problems and checking.
2. Fundamental operations with literal and signed numbers.
   a. Rationalize the work in developing
      1) Laws of signs,
      2) Laws of exponents.
   b. Routinize the work by using timed practice exercises.
   c. Check results.
   d. Practice in arithmetic computations by selecting exercises with common and decimal fractions for coefficients.
3. Simple equations.
   a. Solution and check.
   b. Analysis and solution of verbal problems.

4. Special products and factors.
   a. Monomial factor.
   b. Difference of squares of two bases.
   c. Quadratic trinomials.
      1) Perfect square.
      2) Product of two binomials with the first terms alike.
      3) General form, "cut-and-try" by cross products.
   d. Equations and problems involving special products and factoring.

D. *Topical Outline of Course for the Second Semester.*

1. Special products and factors.
   a. Review and extension of previous work.
   b. Drill with timed practice exercises to gain automatic facility.
   c. Solution of equations by factoring.
   d. Verbal problems.
2. Fractions.
   a. Fundamental operations with algebraic fractions.
   b. Solution of fractional equations.
      1) Ratio; proportion taught as a fractional equation.
      2) Simple literal equations, formulas.
   c. Verbal problems involving fractional equations.
3. Trigonometric ratios.
   a. Study of similar right triangles by observation and measurement.
   b. Indirect measurements as involved in simple problems of common interest, computation of heights and distances.
   c. Measurement of angles.
      1) Sine, cosine, and tangent ratios.
      2) Use of tables.
4. Graphs.
   a. Construction and interpretation.
      1) Bar graphs.
      2) Circle graphs.
      3) Coördinate graphs.
   b. Popular and scientific uses illustrated by student collections from varied sources.
5. Systems of linear equations.
   a. Solution of equations by the following two methods:

1) Graphic method.
2) Algebraic method.

b. Verbal problems most conveniently solved by systems of equations.

c. Required work restricted to systems of two unknowns. Certain individuals may profitably handle systems involving more than two unknowns.

6. Radicals.
   a. Square root of numbers.
   b. Simplification of such radicals as may occur in the solution of quadratic equations.
   c. Illustration by geometric figures.
7. Quadratic equations.
   a. Solution of quadratic equations by
      1) Factoring.
      2) Completing the square.
      3) Quadratic formula,—solution by the formula is not a fixed requirement but is recommended provided that the majority of the students and the time at their disposal justifies this form of solution.
      4) Verbal problems involving quadratic equations and the interpretation of the two roots in these problems.

V. Algebraic Theory I

A. *One-Semester Course.*

Lower eleventh grade.

Prerequisite: Elementary Algebra and Plane Geometry. May combine with Trigonometry (higher eleventh grade) for major in mathematics. (Two majors of three years each, one of which must be English—California State Requirement.)

B. *Aims.*

1. General—to develop the ability to
   a. Analyze a problem.
   b. Formulate it mathematically.
   c. Interpret and check the result.
2. Specific—To improve the technique of
   a. Understanding the language and symbols of algebra and using them correctly.

b. Acquiring skill and accuracy in the solution of equations and in the important algebraic manipulations which their solution involves.

c. Developing the most important algebraic formulas and memorizing for automatic use the most essential.

C. *Outline of the Course.*

1. Review and extension of the work of elementary algebra.
   a. Four fundamental operations with literal and signed numbers, including synthetic division with a binomial divisor.
   b. Factoring including the following:
      1) General quadratic trinomial.
      2) Factoring by grouping.
      3) Sum and difference of the same *n*th power of two bases.
      4) Cube of a binomial.
      5) Factor theorem—using synthetic division.
   c. Fractions.
      1) Fundamental operations.
      2) Fractional equations.
      3) Solution of verbal problems involving fractions.
2. Quadratic equations.
   a. Solution by formula.
   b. Solution by factoring.
   c. Graphic solution using remainder and factor theorems.
3. Systems of equations.
   a. Linear systems solved by determinants.
   b. Quadratic systems. Postpone more involved forms for further treatment in Algebraic Theory II.
   c. Solution of verbal problems.
4. Formulas.
   a. Construction of formulas.
   b. Constants and effect of changes in variables involved.
   c. Manipulation and interpretation.
   d. Graphic representation.
   e. Application to particular problems.
5. Ratio, proportion, and variation.

Emphasize and illustrate as clearly as possible such uses as are made of these three topics in elementary physics and chemistry.

6. Fundamental laws of exponents.
   Square root and cube root of numbers,—brief attention.
7. Radicals,—fundamental work.
8. Logarithm table and slide rule—demonstration and practice.

VI. Algebraic Theory II

A. *One-Semester Course.*

Higher eleventh or lower twelfth grade.
Prerequisites,—Elementary Algebra and Algebraic Theory I.
Required for freshman course in the College of Commerce.

B. *Aims.*

Same as in Algebraic Theory I. Work ranges over more advanced topics and more difficult phases of topics already treated.

C. *Outline of the Course.*

1. Theory of quadratic equations. Verbal problems and interpretation of roots.
2. Roots and radicals.
   a. Fundamental operations involving radicals.
   b. Square roots of binomial surds.
   c. Radical equations.
   d. Complex and imaginary quantities.
3. Theory of exponents.
   a. Theory of logarithms.
   b. Logarithmic equations.
4. Systems of equations.
   a. Determinants of second and third orders, their applications in the solution of linear systems.
   b. Systems of quadratics,—special methods.
   c. More verbal problems solved by systems.
5. Series.
   a. Arithmetic progressions.
   b. Geometric progressions. Limits as exemplified in the term "sum to infinity."
   c. Other simple series.
   d. Compound interest and annuities; variety of other problems involving series.
6. Binomial theorem.

a. Proof and practice.
b. Applications from biology, and the like.

7. Permutations and combinations.
8. Probability and chance.
9. Supplementary topics.
   a. Solution of cubic, quartic, and numerical equations of higher degree. Applications of graphs and the factor and remainder theorems.
   b. Theory of limits. Zero and infinity.
   c. Trigonometric equations.
   d. Indeterminate equations and indeterminate forms.
   e. Mathematical induction.
   f. Imaginaries.
   g. Cube roots of numbers.
   h. Statistics, correlation, average, maximum, minimum, median, mode, frequency distribution, graphs.
   i. Differentiation and integration of rational integral functions.
      1) Problems in maxima and minima.
      2) Areas under simple curves. Archimedes' method for area of a parabolic segment. Simpson's Rule.

D. *Suggestions.*

It does not seem practical to prescribe an ironclad selection of topics that must be followed in this course as classes differ so widely in ability, in previous training, and in interests for future work. Sections 1-8 in the outline form a minimum requirement for Algebraic Theory II as university entrance requirement. It is usually possible to study several of the topics on the supplementary list; others must be deferred to a college course. It is perhaps wholesome to give even a slight exposure rather than leave the impression that there are no more worlds to conquer.

E. *Adaptation to Students of Varying Ability.*

All courses preceding and prerequisite to Algebraic Theory I are designed for the masses. It would seem for the best interest both of the accelerated students and of the retarded students that classes should be arranged in homogeneous ability groups and that the selection of subject matter, the rapidity of progress, and the methods of instruction should be governed accordingly. This plan is followed in the Oakland schools as far as administrative difficulties will permit.

The purely elective courses of the eleventh and twelfth years, however, are designed for the mathematically inclined and the mathematically capable, and students of that type are entitled to first consideration. If there are many misfits who are not easily encouraged to seek other lines of activity, it is only fair that they be grouped in one section partly so that the work may be adjusted to their needs; partly, and of greater importance at this stage, so that students of superior ability in mathematics may be encouraged to set higher standards of accomplishment and be stimulated to do the best work of which they are capable. Program complexities often prevent such grouping and in such cases it is necessary to see what can be done for the slow student by individual attention, flexible assignments, and judicious omissions. As far as the semester's work is concerned, the course fails to function in the general scheme if the subject content is too severely pruned.

F. *Teaching Emphasis.*

In the days of the Edward Olney algebra textbooks, theoretical demonstration was given main emphasis; in the next generation, when the G. A. Wentworth texts were in vogue, formal manipulation was carried to an extreme; during the present generation, teachers believe that they have been awakened to the light of a new and rational psychology—the spirit of which may be expressed in some such statement as "Knowledge should be acquired as the result of purposeful action and should in turn become a means to action." By this token algebra is regarded as an instrument of material conquests and of social organization. Emphasis is placed on problem solving. Algebraic tools bear a resemblance to Professor Jevons' famous "logical machine" in which the exploration of a field of truth could be carried out by pulling levers and turning handles, yet the student must constantly remember that his original abstractions and the final outcomes must always be interpreted in terms of concrete things in the concrete world.

Standard Practice Exercises and Diagnostic Tests are helpful devices in classroom work and should be a regular part of the teaching equipment. Algebraic work must be a generalization and a continuation of arithmetic work and a conscious effort must be made to improve skill in numerical calculation.

VII. Trigonometry

A. *One-Semester Course.*

Higher eleventh or lower twelfth grade.

Prerequisites: Plane Geometry and Algebraic Theory I.

Suggested for combination with Algebraic Theory I for the third year of the major in mathematics.

B. *Aim.*

The aim in the course in trigonometry is to carry out the project of indirect measurement by the solution of triangles, and to develop the knowledge and skill necessary to do it intelligently and accurately.

C. *Outline of the Course.*

1. Trigonometric functions of an angle.
   - *a.* Natural functions.
     - 1) Trigonometric identities.
     - 2) Use of tables.
   - *b.* Logarithms of functions and the use of logarithmic tables.
2. Development and memorization of formulas essential
   - *a.* In the solution of right triangles.
   - *b.* In the solution of oblique triangles.
3. Practical applications.
   - *a.* Problems from the school texts and selections of supplementary material.
     - 1) Computation by logarithms.
     - 2) Checks by graphical solution and by the slide rule.
   - *b.* Problem projects.
     - 1) Explanation of the transit.
     - 2) Field work with the transit.
     - 3) Solution of original problems from the data obtained in field work.
4. Trigonometric equations and identities.
   - *a.* Addition formulas.
   - *b.* Double-angle formulas.

c. Half-angle formulas.
d. Sum and product formulas.
e. Solution of equations which require the use of these identities.
f. Logarithmic equations.

D. *Suggestions and Recommendations.*

1. Utilize the previous experience of the students. Build directly upon the junior high school work in scale drawing and numerical problems on the right triangle. Carry on with the plane geometry study of congruent triangles, similar triangles, ratios in the right triangle, and the Pythagorean theorem. Exercise the skill gained in algebraic theory by using logarithms and solving equations.
2. Do not permit formalism to obscure the practical value of the subject or dull the interest of the beginner. To this end, many teachers prefer to separate geometric and trigonometric equations from the practical work in the solution of triangles. The necessary formulas can be developed by geometric proof.
3. Reduce to a minimum memorizing of formulas.
4. Require a formulated plan complete in outline before beginning on the details of computation. Insist upon orderly arrangement, legible handwriting and figures, and reasonably accurate drawings. Emphasize economical methods and accuracy in numerical computations.
5. Require pupils to check the results.
6. Require habitual use of a correct mathematical vocabulary.
7. Trigonometry affords an excellent invitation for term projects.
8. With a strong class, there may be an opportunity of investigating the spherical triangle and the possibility of extending the formulas of the plane triangle to the solution of the spherical triangle. Point the way also to calculations based upon solar observations, astronomical problems, and navigation problems.

E. *Equipment.*

1. Surveyor's transit, stadia rod, and steel tape.
2. Slide rules.
   a Large slide rule for class demonstration.
   b. Students' slide rule for individual use.

3. Engineer's manual with seven-place computation tables.
4. Reference texts for supplementary work.

VIII. Solid Geometry and Supplementary Topics

A. *One-Semester Course.*

Prerequisites: Plane Geometry and Algebraic Theory I and II. A variable fraction of a semester is devoted to solid geometry and the remainder of the time to introductory work in analytic geometry, descriptive geometry, or the calculus. Selection of supplementary material to be determined by the class and the teacher concerned.

B. *Aims in Solid Geometry.*

1. To exercise further the spatial imagination of the student.
2. To give him a knowledge of the fundamental spatial relationships and the power to work with them.
3. To give him abundant practice in the solution of practical mensuration problems, thus correlating the work with the arithmetic and algebra.

C. *Outline of the Course in Solid Geometry.*

1. Propositions relating to lines and planes, and to dihedral and trihedral angles.
2. Propositions relating to the mensuration of the prism, pyramid, and frustum; the right circular cylinder, the cone, and frustum, based on an informal treatment of limits; the sphere, and the spherical triangle.
3. Spherical geometry.
4. Similar solids.
5. Numerical computation in practical problems.

D. *Suggestions.*

The following suggestions are embodied in the recent report of the National Committee on Mathematical Requirements. They are here submitted as an authentic guide:

"It is felt that the work in plane geometry gives enough training in logical demonstration to warrant a shifting of emphasis in the work on solid geometry away from this aspect of the subject and in the direction of developing greater facility in visualizing spatial relations and figures on paper, and in solving problems in mensuration.

"For many of the practical applications of mathematics, it is of fundamental importance to have accurate space perceptions. Hence it would seem wise to have at least some of the work in solid geometry come as early as possible in the courses, preferably not later than the beginning of the eleventh school year. Some schools find it possible and desirable to introduce the more elementary notions of solid geometry in connection with the related ideas of plane geometry. Desirable simplification and generalization may be introduced into the treatment of mensuration theorems by employing such theorems as Cavalieri's and Simpson's, and the Prismoid Formula, but rigorous proofs or derivations of these need not be included. Beyond the range of the mensuration topics indicated above, it seems preferable to employ the methods of the elementary calculus."

IX. Introductory Work in the Elementary Calculus

A. *Supplementary to and a Part of the Solid Geometry Course.*

B. *Aim.*

To acquaint the student with the operation of one of the most powerful and attractive of mathematical tools.

C. *Outline of the Work as Suggested by the National Committee on Mathematical Requirements.*

1. The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or the slope of a curve.
2. Applications of derivatives to easy problems in rates and maxima and minima.
3. Simple cases of inverse problems; for example, finding distance from velocity, and the like.
4. Approximate methods of summation leading up to integration as a powerful method of summation.
5. Applications to the simple cases of motion, area, volume, and pressure.

D. *Informational Facts on the Calculus.*

The following notes on the calculus as a high school subject are adapted from the National Committee Report.

It has been said that the calculus is that branch of mathematics which school boys understand and senior wranglers fail to comprehend. Boys can be taught the subject early in their mathematical career and there is no part of their mathematical training that they enjoy better or which opens up to them wider fields of useful exploration. The phenomena must be first known practically and then studied philosophically.

Work in the calculus should be largely graphic and may be closely related to that in physics: the necessary technique should be reduced to a minimum by basing it wholly or mainly on algebraic polynomials. No formal study of analytic geometry need be presupposed beyond the plotting of simple graphs.

It is important to bear in mind that while elementary calculus is sufficiently easy, interesting, and valuable to justify its introduction, special pains should be taken to guard against any lack of thoroughness in the fundamentals of algebra and geometry. No possible gain could compensate for a real sacrifice of such thoroughness.

It should also be borne in mind that the suggestion of including elementary calculus is not intended for all schools nor for all teachers or all pupils in any school. It is not intended to connect in any direct way with college entrance requirements. The future college student will have ample opportunity for the calculus later. The capable boy or girl who is not to have the college work ought not, on that account, be prevented from learning something of the use of this powerful tool. The applications of the elementary calculus to simple concrete problems are far more abundant and interesting than those of algebra. The necessary technique is extremely simple. The subject is commonly taught in secondary schools in England, France, and Germany, and appropriate English texts are available.

The relations between algebra and geometry are especially emphasized in France, and the relations between mathematics and physics receive special emphasis in Germany, Holland, and Switzerland.

When a European boy has completed his twelfth school year, he has had the opportunity of studying more mathematics than is offered in any ' the secondary schools of the United States. He has had mo. practice in applying his mathematics in physics, cosmography, and mathematical geography than is the case with the American boy. The simultaneous study of several mathematical subjects has resulted in a more complete mastery of each. He sees the unity of mathematics in a way that is seldom true of the American boy. He can use his arithmetic and algebra in the solution of geometric problems and his arithmetic and geometry in the solution of algebraic problems much better than the American boy. He has some knowledge of analytic geometry and of the infinitesimal calculus. The frequent drills and reviews so common in the European schools have furnished him a large number of mathematical facts and formulas that he can use more readily than his American brother. Mathematics to him is an interesting and fruitful subject, because he has learned to appreciate something of its deeper significance.

E. *Text and Reference Materials.*

The following have been selected from the bibliography of the Course of Study as the most helpful:

Author	Title	Publisher
Brewster, C. W.	Common Sense of the Calculus	Oxford (Clarendon Press)
Goff, R. R.	Simplified Calculus for High Schools	High School, New Britain, Conn.
Kinney, J. M.	Calculus in the High School	Mathematics Teacher, October, 1923
Longley and Wilson	Introduction to the Calculus	Ginn and Co.
Palmer, C.	Practical Calculus for Home Study	McGraw-Hill Book Co.
Rosenberger, N.	Place of Elementary Calculus in Senior High School Mathematics	Bureau of Publications, Teachers College, New York
Thompson, S. P.	Calculus Made Easy	Macmillan and Co.
Granville, W.	Differential and Integral Calculus	Ginn and Co.
Woods and Bailey	Elementary Calculus	Ginn and Co.

X. Introduction to Descriptive Geometry

A. *Supplementary to and a Part of the Solid Geometry Course.*

Alternative with topics of the calculus or analytic geometry.

B. *Aims.*

1. To cultivate the spatial imagination.
2. To furnish a more adequate background for later work in college.

C. *Topical Outline of the Work as Arranged by Mr. Albrecht.*

1. Explanation of the method of orthographic projection. Comparison of the first and third angle projection.
2. Blackboard problems in projections of point and line; braces of lines and planes.
3. Lines of intersection of two planes.
4. Lines parallel and perpendicular to lines and planes.
5. True length of a line.
6. Angles made by lines with reference planes.
7. Explanation of conventional ray of light used for casting shadows. Trigonometric computation of the true angle which a ray of light makes with reference planes. Geometric verification by construction; orthographic projection of ray.
8. Conic sections and developments.

D. *Texts and Reference Materials.*

Author	Title	Publisher
Faunce, L.	Descriptive Geometry	Ginn and Co.
Miller, H. W.	Descriptive Geometry, Revised, 1925	John Wiley and Sons
Randall, O. E.	Elements of Descriptive Geometry	Ginn and Co.
Randall, O. E.	Shades and Shadows and Perspective	Ginn and Co.
Tracy and North	Descriptive Geometry	John Wiley and Sons
Tracy, J. C.	Mechanical Drawing	American Book Co.

XI. Introduction to Analytic Geometry

A. *Supplementary to and a Part of the Solid Geometry Course.*

Alternative with topics of the calculus or of descriptive geometry.

B. *Aims.*

1. To give opportunity for the increase of skill in numerical computation, in logical analysis, and in the fundamental processes of algebra, geometry, and trigonometry.

2. To clarify and extend the concept of functionality and its graphic representation.
3. To emphasize the essential unity and interplay of the different branches of mathematics.

C. *Outline of the Course.*

1. Graph of an equation.
   a. Rectangular and polar coördinates. Graphic solution of familiar algebraic equations.
   b. Straight lines.
      1) Various forms of the equations of a straight line.
      2) The angle between two lines.
      3) Equation of an angle bisector.
      4) Distance and area formulas.
2. Essential properties of the conic sections.
   a. Development of the equation from the definition of locus for the circle, ellipse, parabola, and hyperbola.
   b. Construction of their graphs by the location of points and by mechanical devices.
   c. Investigation of the properties of these curves by study of their equations and graphs.

XII. Analytic Geometry

A. *One-Semester Course.*

Higher twelfth grade.

Prerequisites: Plane Geometry, Algebraic Theory I and II, and Trigonometry.

B. *Aims.*

1. General.

To strengthen the foundations for advanced work by affording opportunity for review, extension, and application of the preceding work in the high school course.

Designed primarily for college preparatory students, but not for advanced credit at the university.

In the interests of the non-college preparatory student, the work is practical, interesting, and thorough, and is designed to give some idea of the nature of advanced mathematics which he would not otherwise obtain.

2. Specific.
   a. To investigate the properties of the conic sections by the development and study of their equations and to make concrete applications in the analysis and solution of practical problems.
   b. To give opportunity for the increase of skill in numerical computation, in logical analysis, and in the fundamental processes of algebra, geometry, and trigonometry.
   c. To clarify and extend the concept of functionality and its graphic representation.
   d. To emphasize the essential unity and interplay of the different branches of mathematics.
   e. To open the door of the calculus and reveal something of its purpose and methods.

C. *Outline of the Course.*
1. Same as section C, subdivision 1 above.
2. Same as section C, subdivision 2 above.
3. Concrete problems.

   Problems selected from different textbooks giving interesting applications in natural phenomena, such as orbits of a planet and the path of a projectile; architectural designs such as bridge structures, arches, amphitheaters; mechanical devices such as springs, gears, and reflectors; statistical curves.
4. General equations of the second degree.
   a. Test for the nature of the conic section represented.
   b. Transformation by rotation and translation of axes.
   c. Graph of an equation.
5. Superficial view of other plane curves with especial attention to those of historic interest in the trisection of an angle and the duplication of the cube.

XIII. Social and Economic Arithmetic

A. *One-Semester Course.*

Elective in either the lower or the higher twelfth grade.

B. *Aims.*
1. General: To make a direct contribution to training for
   a. Intelligent citizenship.

   b. Worthy home membership.
   c. Vocational efficiency.
2. Specific.
   a. To stimulate an active interest in quantitative thinking in the domain of private, vocational, and civic affairs.
   b. To cultivate right ideals of thrift and promote efficiency in the management of home and personal finances.
   c. To gain a better understanding of business and industrial procedure.
   d. To improve skill and accuracy in ordinary computation.

C. *Outline of the Course.*
1. Personal and home interests.
   a. Budgets—personal and family.
   b. Expense accounts—personal, household, and business.
   c. Thrift in time, money, and natural resources.
      1) Time schedules, regular habits, punctuality.
      2) Conservation of natural resources; conservation of health and energy; food values; the cost of wastefulness; provision for illness and old age.
      3) Units of measure and their uses—metric system and the foot-pound system; standard time; international date line; applied problems.
      4) Insurance—life and property.
      5) Comparison of values—buying for cash or charge accounts, owning or renting; contract purchases or mortgages; conservative investments or "get-rich-quick" concerns.
      6) Estimating costs—getting an education; furnishing a home; feeding and clothing a family; travel; amusements; owing an automobile.
2. Business and vocational interests.
   a. Business law, procedure in modern business activities.
   b. Organization, individual proprietorship, partnership, corporations.
   c. Banking, savings and commercial accounts, deposit slips, pass books, checks, drafts, discount, interest, foreign money and exchange, travelers' checks, letters of credit.
   d. Investments, simple and compound interest, different ways of investing money, marks of a good investment.

3. Civic affairs.

   Public utilities, civic improvements, government enterprises, expenditure, taxes, duties, customs, income tax.

4. Practice in computation.
   a. Drill in the fundamental operations for automatic mastery with whole numbers and fractions of common occurrence.
   b. Oral work and mental calculations. Preëstimating the answer and approximating results.
   c. Short-cuts and rapid calculations.
   d. Use of standardized tests and practice pads.
   e. Checking results.

5. Our national thrift creed.

   "Make a budget, keep a record of expenditures, have a bank account, carry a life insurance, make a will, own a home, pay your bills, invest in government securities, share with others."

D. *Special Demand and Justification for the Study of Arithmetic in the Senior High School.*

The president of the Mathematical Society during the period of the World War remarked in his retiring address that there was a great future for mathematics in our country,—especially accurate arithmetic. Our fellow citizens and patrons complain bitterly and truthfully that great numbers of our high school graduates are inefficient and unreliable in the solution of simple numerical problems of common occurrence in everyday life. Our own colleagues of the science departments report universally that the large majority of students are handicapped in their study of physics and chemistry because they cannot add, subtract, multiply, and divide with any degree of accuracy and understanding. The truth may be humiliating, but we might as well face it honestly and if we can't fit all children to the perfect school, we might try the experiment of fitting the school to the needs of the imperfect children. This course is particularly designed for those students who comply with the *California law* but *graduate from high school without taking mathematics*, and also for those who take the minimum requirements for the junior certificate at the university.

XIV. STANDARD TESTS AND PRACTICE EXERCISES

Examinations. Examinations in the past have always meant pass or not pass, success or failure in gaining admission to the university or to a civil service or professional group. In the Oakland high schools, it is customary to promote students on the basis of their accomplishment throughout the term as recorded on the quarterly report cards, and not on the basis of a final examination; the high schools are accredited, and therefore recommended students may enter the University of California without examination. Since so much responsibility is placed upon the judgment of the teacher, he is grateful for adequate objective standards which may serve as a check and a guide for both students and teacher.

A complete bibliography of Standardized Tests in Mathematics for Secondary Schools, with descriptions and evaluations, is given by Professor Upton in Chapter XIII of *The Reorganization of Mathematics in the Secondary Schools.* Each type of test suits the purpose for which it may be designed to assist in separating students into homogeneous ability groups; to measure student achievement; to measure the effectiveness of teaching method; to diagnose student difficulties; to furnish practice exercises for drill in developing speed and accuracy. Until quite recently, there were few standardized tests in high school mathematics. Several of the Oakland classes have assisted in the process of standardizing tests which are now available in printed form. With the coöperation of the Department of Curriculum Development, Research and Guidance, of the Oakland Public Schools, it will be possible to obtain standard tests of proven worth for practical service in classroom work.

XV. MATHEMATICS CLUBS

Nature and Function. Mathematics clubs have demonstrated their value as organized student activities and have found a place in the Oakland high schools. They provide a worthy medium for self-expression for gifted students and inspire them with higher ideals for continued effort. Thus they serve as one means of capitalizing mathematical talent for the benefit of society and for the personal satisfaction of the individuals concerned. The activities of a mathematics club include both work and play. They are determined by the special interests of the members and are initiated by the students themselves.

The students organize their club with due formality, elect officers, invite a faculty adviser, appoint committees, fix qualifications for membership, choose a name, and arrange for meetings and programs. The time preferred for meetings is the regular activities period or another free school period. Occasionally a Saturday morning field trip, an evening session, a dinner party, a play, or a school assembly program may be arranged.

In the *First Yearbook of the National Council of Teachers of Mathematics* there is an article on the "Recreational Values Achieved through Mathematical Clubs in Secondary Schools" by Marie Gugle and Others, Columbus, Ohio.

XVI. Projects in Senior High School Mathematics

Use. Teachers of senior high school mathematics choose to make a very liberal interpretation of the project method. They have indeed a very advantageous position for assuming this privilege. No mathematics is required for graduation from a California high school; no mathematics is required for admission to the University of California; ergo, since there is no outer urge, it must be an inner urge that causes a continually increasing number of students to initiate for themselves the "wholehearted, purposeful activity" of studying trigonometry, for example, in the social setting of an Oakland high school.

In the narrower sense of the term project, the teachers believe that it is impossible to develop any degree of skill or any extensive knowledge of mathematics by attention which is only incidental to some material enterprise, such as building a railroad or bridge, or something equally foreign to the children's activities. There are several small projects closely connected with the classroom work, such as miniature theses on graphs, the computation of π, the Pythagorean theorem, geometric constructions, geometric originals, the fourth dimension, the trisection of an angle, the Einstein Theory, the calculus, mathematical instruments, geometric forms in nature, the mathematics required for an engineer, and the like. Larger projects are the mathematics club and standardized tests, both described in detail in another section.

In the broader sense of the term, every course in senior high school mathematics is a large unit project; more particularly is this true of the strictly elective courses of the eleventh and twelfth school years. Teachers have the opportunity here of keeping the activities on the high level of an ideal project.

Developments in recent years have disclosed a worthy and vitally necessary project for mathematics teachers; namely, to "sell" their subject to the makers of high school curricula and to the principals of high schools before these officials will permit any mathematics to be written upon their clean slates. To a group so strongly entrenched behind a blind faith in "formal discipline" and an arbitrary law of "prescribed subject," the new ruling was something of a shock, but the results in the main are very wholesome.

Professor Charles N. Moore, of Cincinnati, in an address to mathematics teachers expressed very happily the plan of this teacher project. "We must demonstrate that man was led to the pursuit of mathematical knowledge by his eager desire to understand the universe and to control the forces of nature, that he found this knowledge essential for the higher developments of trade and commerce and all the other varied developments that have had a place in the creation of our present day civilization, in short, that the progress of the world is now and always has been bound up with the development of our knowledge of mathematics. . . . Let us make plain to our disciples that the quest for mathematical knowledge is one of the most important and most fascinating portions of the great adventure, of man's eternal effort to penetrate further into unknown regions and master them for his possession and use, and they will be convinced that while mathematics may be a difficult subject, it can never be a dry subject. . . . When they study mathematics, they are acquiring knowledge that real live . . . people find highly necessary to use in human activities of fundamental importance."—*Mathematics Teacher*, December, 1922.

XVII. Summary

Unity of Mathematics. While there is no radical revolution indicated in the accompanying course of study, there is evidence throughout of a gradual and more effective adaptation of the subject to meet the changing requirements of society and a more enlightened understanding of the way children learn. The aim has been to select and ar. nge the material on a psychological rather than a strictly logical basis. In general, the changes are characterized by an early introduction of certain subjects that have formerly been reserved for advanced students, by the extension of electives upward, by the lowering of the fences between the different fields of mathematics, by the elimination of purely artificial difficulties, and by the consistent emphasis on the applications in the world outside of school. While the core of each term's work is a separate subject, every one must be made conscious of the essential unity in all mathematics work. Let him use arithmetic and algebra and trigonometry throughout the geometry work. In this way mathematical education may be a process of continuous, consistent growth and development.

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