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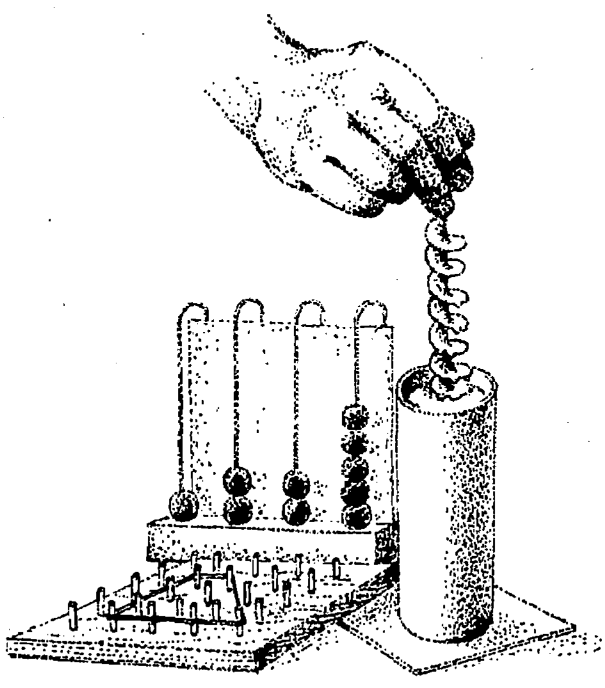
ABSTRACT

A collection of articles from the ARITHMETIC TEACHER is presented which are about practical, classroom-tested ideas for the instruction and use of teacher-made instructional aids. These entries deal only with manipulative type aids. They have been selected for the clarity of purpose and relationship to contemporary topics in the elementary school mathematics curriculum. The articles provide sufficient information and specifications so that teachers can construct the aid and include directions or examples relative to using the aid for instruction. The organization of topics is based on major strands of elementary school mathematics: whole numbers, numeration, integers, rational numbers, geometry, and measurement. (JP)

Teacher-made Aids for Elementary School Mathematics



NCTM

Teacher-made Aids for Elementary School Mathematics

Readings from the *Arithmetic Teacher*

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Introduction

The *Arithmetic Teacher* has been noted through the years as a professional journal that presents many practical, classroom-tested ideas for the consideration of its readers. Among these ideas are many valuable suggestions for the construction and use of teacher-made instructional aids.

The initial screening of articles for possible inclusion in this book of readings revealed that there are three closely related categories of instructional ideas: (1) manipulative aids, (2) activities, and (3) games and puzzles. In this book, we have limited our selections to those articles that relate primarily to the first category. For our purposes, a manipulative aid is an object that contributes to the learning experience of a pupil by providing him an opportunity to be physically and mentally involved.

Other factors that were significant in selection were (1) that the article have a clear purpose and be related to a contemporary topic in the elementary school mathematics curriculum; (2) that the article provide sufficient information and specifications so that a teacher can construct the aid; and (3) that the article provide directions or examples relative to using the aid for instruction.

In this book, we have organized our selections on the basis of major strands of elementary school mathematics—whole numbers, numeration, integers, rational numbers, geometry and measurement. We hope you will envision multiple applications of each teaching aid. With an open mind and a little creative effort, you will be able to think of many ways to adapt the suggested ideas or improve on them.

It is our hope that this publication will serve as a meaningful and useful resource book to teachers of elementary school mathematics.

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Teaching Aids: What, How, and Why

The articles in this opening chapter offer the reader much food for thought in the appropriate selection and use of teaching aids in the mathematics classroom.

In the opening article, Reys provides an excellent discussion of criteria that should be considered when selecting manipulative materials for teaching mathematics. He considers both the pedagogical and the physical aspects associated with the selection of aids. Reys summarizes the use of manipulative materials and then provides a stimulating discussion of some dos and don'ts for teachers who plan to use manipulative materials in the teaching of mathematics.

Bernstein also offers a set of principles for the selection and use of teaching aids for elementary school mathematics. Each of his principles is amplified with examples of actual teaching aids.

In the concluding article of this chapter, Fennema examines the *why* of mathematics teaching aids. The role of concrete models at various levels of the learning process is discussed, and much of the relevant research is summarized.

Considerations for teachers using manipulative materials

ROBERT E. REYS

An associate professor of mathematics education at the University of Missouri—Columbia, Robert Reys is involved in developing mathematics laboratory activities as an integral part of mathematics preparatory programs for both preservice and in-service elementary teachers.

Classroom teachers of mathematics are witnessing an unprecedented period of proliferation in *manipulative materials*. Commerical catalogs list a great variety of available materials; professional journals carry many advertisements claiming that this device or that aid will provide a panacea for learning a certain mathematics topic; and professional meetings are frequently inundated with exhibits displaying new manipulative materials. This influx of newly available materials has precipitated many problems. The wide range of quality found among various materials has made the problem of selection much more difficult. It has made it impossible to list all available materials and discuss the merit—or lack of merit—of each. It has created doubts in some teachers' minds about the educational value of the materials. It has raised additional teacher-oriented questions such as "What are some guidelines for selecting manipulative materials?" "What materials should be used?" "What are some dos and don'ts of using them?"

During the decade of the sixties several fine articles appeared discussing considerations in the selection of learning materials (Berger and Johnson 1959; Bernstein 1963; Davidson 1968; Hamilton 1966; Spross 1964). The present article is limited to a discussion of manipulative materials as opposed to other teaching aids. Furthermore, it is addressed specifically to classroom teachers in an effort to provide some current rationale, as well as guidelines, for the selection and use of manipulative materials.

What are manipulative materials?

The use of the term *manipulative materials* raises one fundamental question, namely, "Just what are manipulative materials?" In this context manipulative materials are objects or things that the pupil is "able to feel, touch, handle, and move. They may be real objects which have social application in our everyday affairs, or they may be objects which are used to represent an idea" (Grossnickle, Junge, and Metzner 1951, p. 162). Hence, not all teaching aids or instructional materials are manipulative materials. Suffice it to say here that manipulative materials appeal to several senses and are characterized by a physical involvement of pupils in an active learning situation.

Rationale

In teaching mathematics we are primarily concerned with concept formation as opposed to the memorization of facts. The mental processes involved in concept formation are much more complex than those associated with the memorization of a mass of isolated details. There is little disagreement among contemporary psychologists regarding the role of concept formation in the learning of mathematics. However, there are several existing theories about how to best foster proper concept formation. The results of recent psychological investigations into the ways children learn mathematics by men such as Jerome Bruner, Zolton Dienes, Robert Gagné, Jean Piaget, and Richard Skemp are beginning to have an influence on mathematical pedagogy. In short, more is known today about the way children learn mathematics, and the general nature of the mathematics they are capable of learning at various stages, than has previously been known. Ironically, it is still not known precisely how children learn, but the efforts of researchers are continually providing new evidence to suggest (and oftentimes refute) various learning theories. Since learning is an individual matter and invariably dependent on numerous factors, some of which are quite elusive, it is highly unlikely that a comprehensive learning theory that is completely satisfactory to all people will evolve in the foreseeable future.

A comparison of prominent learning theories will not be made here, but it seems appropriate to identify the following statements, subscribed to by most learning psychologists:

1. Concept formation is the essence of learning mathematics.
2. Learning is based on experience.
3. Sensory learning is the foundation of all experience and thus the heart of learning.
4. Learning is a growth process and is developmental in nature.
5. Learning is characterized by distinct, developmental stages.
6. Learning is enhanced by motivation.
7. Learning proceeds from the concrete to the abstract.
8. Learning requires active participation by the learner.
9. Formulation of a mathematical abstraction is a long process.

This list is not exhaustive, nor are the statements independent. In fact, they are closely interrelated. Suffice it to say that the above statements form the basic foundation underlying the rationale for using manipulative materials in learning mathematics.

Many prominent mathematics educators have strongly urged greater use of manipulative materials in teaching mathematics. The rationale for this emphasis seems educationally sound. Unfortunately, research studies in this area have "not been conclusive in either supporting or refuting the value of manipulative aids" (Beougher 1967, p. 31). Most of the questions cited by Brown and Abell (1965, p. 548), such as "Are there certain manipulative devices that lend themselves better to different methods of instruction?" and "Will a device help one child and hinder another?" are yet to be answered. One can only hope that quality research focused on manipulative materials and mathematics learning will provide some objective evidence relevant to the issues. In the meantime, classroom teachers are still faced with the problem of selecting and using manipulative materials in their classroom.

Selection criteria

The rapid increase in available commercial materials has made the job of selection not only difficult but also more crucial as the market is flooded with products. There are many criteria to consider in developing and procuring manipulative materials. In order to simplify this discussion, only important criteria in two basic categories, namely, pedagogical and physical, will be considered. The proposed criteria are not exhaustive, nor is any hierarchy of importance suggested by the order in which they are discussed. Although some considerations are more significant than others, the relative importance attached to each criterion should be determined by the teacher. Any final evaluation of manipulative materials should weigh strengths and weaknesses against the educational potential.

Pedagogically there are many criteria to consider in selecting manipulative materials. One of the most important considerations is whether or not the materials serve the purpose for which they are intended.

Furthermore, do these materials do something that could not be done as well or better without them? Since mathematics is mental, do the materials develop the desired mental imagery?

The following criteria should be included in any list purporting to identify pedagogical considerations in the selection of manipulative materials:

1. *The materials should provide a true embodiment of the mathematical concept or ideas being explored.* The materials are intended to provide concrete representations of mathematical principles. Therefore it is important that, above all else, the materials be mathematically appropriate.

2. *The materials should clearly represent the mathematical concept.* Concepts are embedded so deeply in some materials that few, if any, pupils extract relevant ideas from their experience with the materials. This problem is further compounded by materials that have extraneous distractors, such as bright colors, which actually serve as a hindrance to concept formation. These experiences result in an "I can't see the forest for the trees" complex. This is, of course, not all bad, as it requires pupils to cull out extraneous data, yet in many cases such materials serve more as a deterrent to correct concept formation than as an aid.

3. *The materials should be motivating.* There are many factors that ultimately contribute to motivation. Several of these, such as attractiveness and simplicity, will be discussed later. Materials with favorable physical characteristics will frequently stimulate the pupil's imagination and interest.

4. *The materials should be multipurpose if possible.* That is, they should be appropriate for use in several grade levels as well as for different levels of concept formation. Ideally, the materials should also be useful in developing more than a single concept. Such wide applicability is frequently achieved by using a portion or subset of materials. For example, logic or attribute blocks have much multiapplicability through the careful selection and use of pieces.

This requirement should not preclude the procurement and use of materials designed exclusively for embodying one concept. In fact, if use of certain materials results in concept formation that is otherwise impossible, then such items should be considered. In other disciplines, such as science and physical education, considerable funds are spent on devices that teach a single concept. Shouldn't mathematics teachers have a similar opportunity? Besides, using materials (even those designed for one specific function) often suggests additional topics or concepts that might be explored.

5. *The materials should provide a basis for abstraction.* This underscores the importance of the requirement that materials correctly embody the concept. In addition, caution should be exercised to ensure that the concept being developed is commensurate with the level of abstraction needed to form the mental image. Care must also be taken to ensure that the level of abstraction is commensurate with the ability of the student to abstract.

6. *The materials should provide for individual manipulation.* That is, each pupil should have ample opportunity to physically handle the materials. This may be done individually or within a group, as circumstances dictate. Such manipulation utilizes several senses, including visual, aural, tactile, and kinesthetic. In general, the materials should exploit as many senses as possible. Compliance with this generalization is particularly important with younger children.

Physical criteria are important, since many sources of information available to teachers, such as commercial catalogs and brochures, describe physical features of the materials. A careful scrutiny of physical criteria would be helpful in initally screening manipulative materials. Among the physical characteristics to consider in selecting manipulative materials are the following:

1. *Durability*

   The device must be strong enough to withstand normal use and handling by children. If and when maintenance is needed, it should be readily available at a reasonable cost.

2. *Attractiveness*

   The materials should appeal to the child's natural curiosity and his desire for action. Materials in themselves should not divert attention away from the central concepts being developed. Nevertheless there are certain qualities —such as aesthetically pleasing design; precision of construction; durable, smooth, and perhaps colorful finish— that are desirable. Nothing can be more distracting than pieces of a tangram puzzle that do not fit properly or a balance beam that doesn't quite balance.

3. *Simplicity*

   The degree of complexity is of course a function of the concept being developed and of the children involved, but generally the materials should be simple to operate and manipulate. Although the materials may lend themselves to a host of complex and challenging ideas—for example, the attribute or logic blocks— they should be simple to use. In an effort to construct and use simple devices, there is the inherent danger of oversimplifying or misrepresenting the concept. In all cases care must be taken to ensure that the device properly embodies the mathematical concept. In addition, the design of materials should not require time-consuming, mundane chores such as distributing, collecting, and keeping an extensive inventory record of a large number of items.

4. *Size*

   The materials should be designed to accommodate children's physical competencies and thus be easily manipulated. Storage is an important consideration directly related to size—no device should take up more than a reasonable amount of storage space. Suitability of size is also important in preventing misconceptions or distorted mental images within the child's mind.

5. *Cost*

   The index used to assess the worth of materials must ultimately weigh their use against cost. In this context, cost is used in a broad sense. Thus cost must include the initial expenditure and maintenance and replacement charges based on the life expectancy of materials under normal classroom use. The teacher-related cost, a function of the time required to learn how to use the materials effectively, is an item of utmost importance. It is not uncommon for someone other than a classroom teacher to order mathematics materials, but without proper planning, orientation, and preparation, it is ludicrous to expect teachers to use new materials effectively with their pupils. Therefore, *any purchase of new materials should be accompanied by a planned program designed to familiarize the teachers with these materials.* As a result, any cost estimate for manipulative materials should reflect the teacher-education phase as well as the expenditure for materials.

Teachers are often confronted with the dilemma of whether to use commercial or homemade manipulative materials. Many manipulative materials are relatively easy to make and can often be produced by the teacher and/or pupils. There are many priceless, intangible by-products, such as additional mathematical insight and increased motivation, that result directly from classroom projects. Nevertheless, one should weigh production costs for homemade materials, including labor, materials, and so on, against the cost of similar commercial products. Quality, of course, is another consideration. Frequently there is a marked difference in quality between homemade and commercially produced materials.

It would be ideal if manipulative materials could meet all the aforementioned criteria. Finding such materials would be tantamount to finding a "fish that runs fast and flies high." Consequently the search continues. It is hoped, however, that these criteria will provide teachers with some guidelines for both the selection and the use of manipulative materials.

Using manipulative materials

There have been several fine lists summarizing uses and functions of teaching aids. Many such lists apply specifically to manipulative materials. Among the most common uses of manipulative materials are the following:

1. To vary instructional activities
2. To provide experiences in actual problem-solving situations
3. To provide concrete representations of abstract ideas
4. To provide a basis for analyzing sensory data, so necessary in concept formation
5. To provide an opportunity for students to discover relationships and formulate generalizations
6. To provide active participation by pupils
7. To provide for individual differences
8. To increase motivation related, not to a single mathematics topic, but to learning in general

From this list it should be evident that manipulative materials may be used in a variety of ways. It should also be noted that the mere use of manipulative materials does not ensure that they are being used properly. Manipulative materials must be used at the right time and in the right way if they are to be effective. Failure to select appropriate manipulative materials and failure to use them properly can destroy their effectiveness. Some specific dos and don'ts for teachers who plan to use manipulative materials follow:

1. Do consider pedagogical and physical criteria in selecting manipulative materials. A prerequisite for effective use of manipulative materials is their appropriateness. The physical criteria for manipulative materials as well as the pedagogical considerations should not be taken lightly.

2. Do construct activities that provide multiple embodiment of the concept. It is difficult, and often foolhardy, to abstract or generalize from a single experience. Thus the pupil should be presented with different situations manifesting the concept or structure to be learned. For example, in developing the concept of three, children might examine sets with three elements for one activity. The number line, balance beam, and Minnebars might also be used to provide different embodiments for the same concept. The case for multiple embodiment has been ably presented by Dienes. Although the idea is pedagogically sound, it has yet to receive widespread use by classroom teachers.

3. Do prepare in advance for the activity. Be sure you, as the teacher, use the manipulative materials in the complete activity before they are used by pupils. As you make this trial run, you should consider questions such as: What prerequisite skills are needed before these manipulative materials are introduced? Are the directions clear, and can they be easily followed? Are there an adequate number of leading questions? Are the manipulative materials commensurate with the level of the pupils and appropriate for the mathematical concept? What are some potential problem areas, and how might they be alleviated?

4. Do prepare the pupils. The type of preparation depends on both the manipulative materials being used and the age of the pupils. Above all else, be sure the pupils are ready to profit from experience with the materials. Care should be taken to provide the necessary directions for beginning the activity. One must guard against telling pupils precisely what to do with the materials, as this might sterilize the learning experience. On the other hand, sufficient direction should be provided to prevent mass confusion, which may quickly lead to discipline problems.

5. Do prepare the classroom. Check to ensure that all required materials are on hand. Also be sure they are operative, accessible, and available in sufficient quantity. The arrangement of the classroom furniture should be examined to ensure that it is suitable for the planned activities.

6. Do encourage pupils to think for themselves. The use of manipulative materials in an informal situation provides an ideal climate for creativity, imagination, and individual exploration. This atmosphere encourages pupils to think for themselves. However, in order to get students to begin and then continue to think for themselves, it is imperative that the teacher provide encouragement of and show respect for pupils' ideas. A teacher's dismissal of a student's idea as being trivial, incorrect, worthless, and so on, will repress future ideas.

7. Do encourage group interaction. Discussion within, as well as among, groups can be intellectually stimulating. Encourage students to communicate with their peers and teacher. The importance of having this opportunity to tell one's thoughts, observations, and ideas cannot be overestimated. As pupils grow older this freedom to express personal ideas is accompanied by a responsibility to defend or at least support a position, should the need arise. Some teachers fear that one student will dominate a group of peers. This may sometimes happen; however, the careful selection of group membership can keep this problem at a minimum.

8. Do ask pupils questions. It is often essential that certain points be called to the pupils' attention. Sometimes big ideas are missed completely. Other times one child may divert group attention to some minor or obscure point. In either case, you, as the teacher, must be prepared to ask pertinent leading questions.

9. Do allow children to make errors. Some may view this as heresy. However, children must have an opportunity to be wrong or to make a mistake. Often greater [illegible]ng and more lively discussion follow an incorrect answer than a correct one. Besides, the natural learning process is characterized by much trial-and-error learning. To do otherwise, that is, to attempt to eliminate incorrect answers or faulty speculation, is to create a highly artificial learning situation.

10. Do provide follow-up activities. Discussion, correlated readings, reports, and projects, as well as replications of activities, enhance the prospects of learning. Searching questions forcing pupils to further analyze and synthesize their results can be very productive, as they encourage students to "pull together the loose ends." They might be followed by additional questions that require extrapolation from these activities and encourage speculation on the outcome of other related events.

11. Do evaluate the effectiveness of materials after using them. Immediately upon the completion of an activity, it can be very helpful to note particular problem areas, strengths, weaknesses, and suggestions and to define areas of needed improvement as well as possible areas of modification. A continuous reevaluation of manipulative materials ultimately results in better materials as well as more effective use of them.

12. Do exchange ideas with colleagues. Many new functions of manipulative materials result from actual classroom use. Sometimes pupils either consciously or unconsciously propose additional uses. At times, informal exploration with manipulative materials by either teacher or children suggests new activities, which adds to the reservoir of potential uses for this set of manipulative materials. A mutual exchange of ideas among teachers allows each to profit from the experience of the others. Perhaps you remember the fable: If I have a dollar and you have a dollar and we exchange dollars, we both still have a dollar. However, if I have an idea and you have an idea and we exchange ideas we both now have two ideas.

Now for some teacher don'ts!

1. Don't use manipulative materials indiscriminately. Care must be taken to ensure that these materials properly embody the mathematical concept being developed. Be sure the materials and concept are commensurate with your objectives and the pupils' level of development.

2. Don't make excessive use of manipulative materials. They should be used only when they represent an integral part of the instructional program and when the program could not be achieved better without the materials. One exception to this might be manipulative materials that are directed more toward recreation. There are instances where the traditional curriculum fails to reach many pupils. Often the recreational aspect of manipulative materials has attracted the attention of these youngsters and eventually paved the way to more academically oriented activities. Some teachers fear that excessive use of manipulative materials will lead to overdependence on physical representations. There are cases where the manipulative materials are used as "crutches." However, most pupils will gradually stop using the materials when they have reached a higher level of development. Signs of boredom from the children may indicate excessive use of manipulative materials; or may suggest the need for raising additional questions or extending the concepts being explored with the manipulative materials.

3. Don't hurry the activity. Once the concept has been developed, most children are eager to explore other ideas. However, every pupil should have ample opportunity to use the manipulative materials, thereby convincing himself of the principle or formulating the concept. Hurrying through the activity may impose unnecessary pressure on some pupils as well as creating a very artificial learning situation. Few individuals learn well when they are rushed. Some children may formulate the concept within minutes, whereas other children may require several days or perhaps months. Rushing children as they use manipulative materials does not solve the problem but rather compounds it.

4. Don't rush from the concrete to the abstract level. This is a sequel to the previous suggestion. Perhaps the most frequent error in using manipulative materials is the speed at which children are rushed from the concrete stage to the symbolic level. There seems to be some myth that you can't learn mathematics unless you are actually writing something, that is, working with symbols. This is, of course, nonsense! Most good mathematics at the primary level is done without symbolization. In fact, if serious consideration were given to Piaget's research, nearly all mathematics in the primary grades would be at the concrete stage. It must be noted that symbolization occurs quite late in concept formation. Symbols are reserved for describing or making a record of the concept or mathematical principle. Hence, they can only be properly used after the concept has been abstracted. Since the process of learning a mathematical abstraction is time consuming, it is ludicrous (at least with most elementary children) to use manipulative materials for one or two days and then move directly to the symbolic level. The wrong kind of experience may result in the children's viewing manipulative materials as toys or entertainment, in no way related to mathematics.

5. Don't provide all the answers. In working with manipulative materials pupils acquire experience in abstracting from a set of phenomena or a body of data. As each child is actively involved in this process, conflicts frequently arise. One pupil has one answer, another child has a somewhat different result. Often the first reaction of the teacher is to settle the issue by providing the correct answer. It is difficult to resist the temptation to tell the correct answer, but resist the teacher must! To do otherwise is to discourage individual thought, squash natural curiosity to search for other solutions, promote dependence on the teacher rather than independence, and preclude further discussion of the problem, as

everyone now knows the correct answer. On the other hand, you may decide to ask some leading questions; you may have the pupils explain their solution; you may wish to have them replicate the activity using the manipulative materials; or you may pursue some other alternative. Regardless of the option selected, the teacher must refrain from serving as the purveyor of truth and source of all knowledge. Remember that to children and adults alike, "the art of being a bore consists in telling everything."

Conclusion

Perhaps the best one can do is identify those materials that best meet the criteria and then concentrate on developing effective ways of using them. This requires several steps. First, the desired learning must be clearly identified. Then manipulative materials that will aid in the learning process need to be selected. The third step requires that these materials be integrated into an organized learning sequence, so that pupils progress from the simple and concrete to the complex and abstract. Only in this way can manipulative materials be an integral part of the mathematics education program.

Remember that manpulative materials are not to be considered a substitute for teaching, i.e., something one uses in lieu of teaching. There is not now, never has been, and, it is hoped, never will be a genuine substitute for a good teacher who knows how and what children need to learn and when they need to learn it!

References

Beougher, Elton E. *The Review of the Literature and Research Related to the Use of Manipulative Aids in the Teaching of Mathematics.* Pontiac, Mich.: Special Publication of Division of Instruction, Oakland Schools, 1967.

Berger, Emil J., and Donovan A. Johnson. *A Guide to the Use and Procurement of Teaching Aids for Mathematics.* Washington, D.C.: National Council of Teachers of Mathematics, 1959.

Bernstein, Allen L. "Use of Manipulative Devices in Teaching Mathematics." ARITHMETIC TEACHER 10 (May 1963):280–83.

Brown, Kenneth E., and Theodore L. Abell. "Research in the Teaching of Elementary School Mathematics." ARITHMETIC TEACHER 12 (November 1965):547–49.

Copeland, Richard W. *How Children Learn Mathematics: Teaching Implications of Piaget's Research.* New York: Macmillan Co., 1970.

Davidson, Patricia S. "An Annotated Bibliography of Suggested Manipulative Devices." ARITHMETIC TEACHER 15 (October 1968):509–24.

Dienes, Zolton P. *Building Up Mathematics.* London: Hutchinson Education, 1960.

Hamilton, E. W. "Manipulative Devices." ARITHMETIC TEACHER 13 (October 1966):461–67.

Grossnickle, Foster E., Charlotte Junge, and William Metzner. "Instructional Materials for Teaching Arithmetic." In *The Teaching of Arithmetic*, Fiftieth Yearbook of the National Society for the Study of Education, pt. 2. Chicago: University of Chicago Press, 1951.

Multi-Sensory Aids in the Teaching of Mathematics. Eighteenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications, Teachers College, Columbia University, 1945.

Spross, Patricia. "Considerations in the Selection of Learning Aids." ARITHMETIC TEACHER 11 (May 1964):350–53.

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Use of manipulative devices in teaching mathematics

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The use of manipulative and pictorial material in the teaching of mathematics has been an accepted procedure for many years. The modern teacher is surrounded with methods books discussing how to make these devices, and with commercial catalogs offering a tremendous variety of these materials. How does one navigate this maze and make wise decisions concerning the expenditure of professional time and energy and of school funds? We believe that from the experience of the profession certain principles regarding the selection and use of such material have evolved.

There should be a direct correlation between the operations which are carried on with a device and the operations which are carried on in doing the same mathematics with paper and pencil.

For example, consider the sketch of the abacus shown in Figure 1, illustrating the sum of 23 and 34. Note that the effect in the columns of the abacus correlates directly with the effect in addition with the symbols representing the numbers under consideration.

Now consider the sketch (Fig. 2) and the accompanying example (Fig. 3). In this example we are showing 17 plus 74. Note that in the sum of the first column it is necessary to "carry" because the sum is greater than 10. The operation as shown on the abacus is precisely the same as that shown by paper and pencil, that of exchanging ten beads in the ones column for one in the tens column. By contrast one version of the ancient Chinese abacus calls for five beads in each column, each with a value of one, and two beads above a divider, each with a value of five. The sketches (Figs. 4 and 5) illustrate the same example, 17 plus 74, added on such

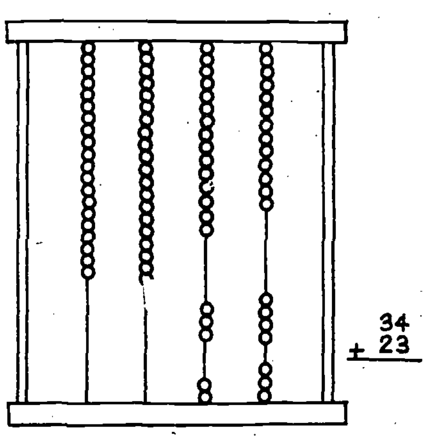


Figure 1

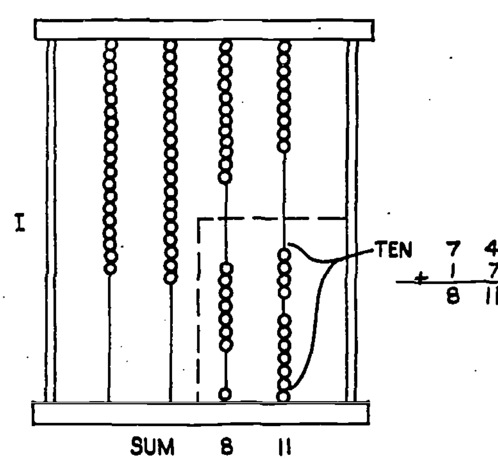


Figure 2

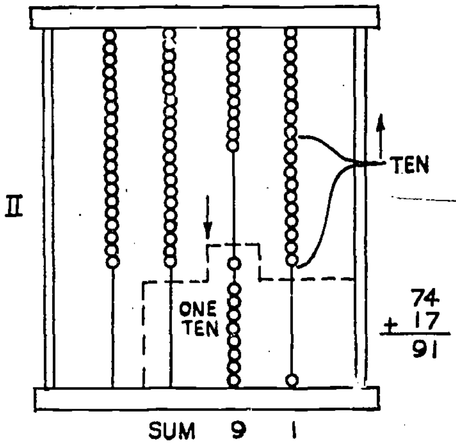


Figure 3

an abacus. The lack of relationship between the abacus operation and the paper and pencil operation will lead to confusion.

The use of the manipulative aid should involve some moving part or parts, or should be something which is moved in the process of illustrating the mathematical principles involved.

The grouping of objects in the early grades with the use of bottle caps, tongue depressors, or the myriad other innovations which have assisted teachers in the past, or the use of the same materials in the intermediate grades to act as symbolic representations of the elements of word problems (i.e., 4 bottle caps represents 4 cans of soup), would be excellent illustrations of this principle. The child then has the opportunity to move these items around on his desk, and the motion effect helps to make the problem more concrete than a simple pictorial representation of the example. The current emphasis on and discussion of discovery methods in teaching reveals another facet of this important principle. The availability of manipulative objects in less structured learning situations gives the child the opportunity to explore relationships and discover principles by manipulating the parts of the aid. Mathematics teachers often help students to discover that the same problem can be approached and solved in a variety of ways. When this is true of an example most of the different approaches can be illustrated with varying manipulations of the appropriate aid.

The use of manipulative and pictorial materials should exploit as many senses as possible.

We know that human beings learn through all the senses and that among the most important of these in the learning process are the visual, aural, tactual, and kinesthetic senses. We believe that the kinesthetic and tactual senses in particular are too often ignored in current practice. When a first-grade child uses an abacus, counting frame, or other similar aid to solve a problem, he is often observed moving his fingers over the beads or

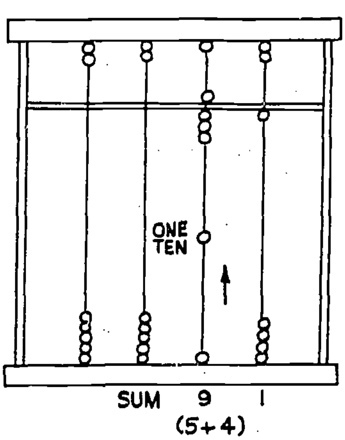


Figure 4. To add 4 in the ones column, it is necessary to deduct one and add a bead ...ating 5.

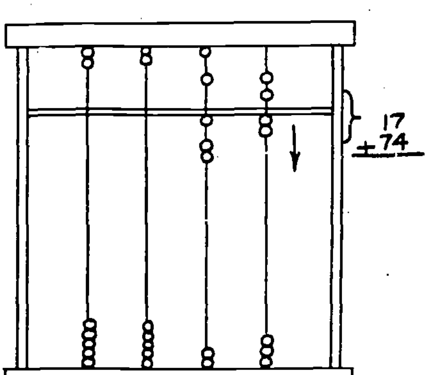


Figure 5. Two beads representing five ones apiece are exchanged for one ten bead.

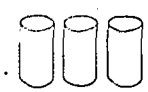
touching the objects as he counts them. This is an important part of the learning process. It is therefore desirable that in the use of such materials the teacher have enough available so that all the children have an opportunity to handle and manipulate it. It is also desirable that the materials be made in bright colors, with contrasting colors to stimulate the visual sense where appropriate. There is some danger of color dependency, which is to be avoided.

A student should have his own aid, or ample opportunity to use one many times.

Consider the situation where the teacher demonstrates a principle with a manipulative device, goes through the explanation possibly a second time, and then assigns the children a series of seat-work problems all of which are done with paper, pencil, and symbolic representations only. Contrast this situation with one in which each child has the opportunity to handle an aid and use it as he does his seat work. In such a situation, the child may manipulate the aid for many dozens, even hundreds, of examples so that the principle illustrated just once by the teacher becomes increasingly clear in the child's mind.

In general, learning may proceed from using physical models, to using pictures, to using only symbols.

Let us consider the example of three cans of soup, under the following headings:

Physical model	*Picture*	*Symbol*
3 actual cans of soup	A photograph of 3 cans of soup	The printed words, "three cans of soup"
or 3 empty soup cans	*or* A schematic diagram, viz.,	*or* "3 cans of soup"
or 3 cylinders		*or* "3s"

The use of the physical or manipulative aid should be permissive for the child rather than mandatory.

Some children are able, after brief study, to operate on "an abstract level," and forced use of multisensory aid in their mathematical work will only slow them down and may create bordom or resentment. On the other hand, denying such children the availability of the aid because they do not need it defeats opportunity for exploration and discovery of the new patterns mentioned above. Other children may need to use the concrete assistance rendered by the aid for a period of time longer than one would expect for the so-called "average" child. Teaching experience has indicated that students tend to give up the use of manipulative devices when they no longer need them.

A device should be flexible and have many uses.

The hundred board is a case in point. This familiar device has been used in many classrooms to illustrate some features of number structure for our number system with base ten. When some parts of the board are covered it may be used to illustrate number systems to other bases. The board has also been used to illustrate some of the principles involved in decimal and percent calculations. Cost is always a major factor in the choice of devices. (Our definition of cost includes the time that teachers and children may put into constructing home-made devices.) Therefore the finished aid should be convenient to use and store, and should be fairly durable. Obviously such materials as are consumed in use are not included in this principle.

The principles elicited indicate that manipulative devices need not be complicated, and apply in a variety of teaching formats, i.e., large group or small group instruction, individual work, prescribed exercises, cooperatively planned projects, independent discovery work, and so forth. Students may be involved in producing the aid, or gathering material for it. The teacher must decide whether the investment of such time or energy is worthwhile relative to the gains achieved in terms of carefully chosen goals.

While it is obviously impossible to apply all of these criteria to all possible teaching aids in all situations, they do provide some sensible guide lines enabling teachers to decide which materials are worthy of the investment of professional energy and school financial resources. It is hoped that these criteria will be helpful in the professional decisions facing us in this area.

Models and mathematics

ELIZABETH H. FENNEMA

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Although there is reasonable agreement about which mathematical ideas should be taught in the elementary school (Begle 1966) and that these ideas should be taught meaningfully (Dawson and Ruddell [a] 1955), there is little agreement on how learning environments should be structured to facilitate this learning. One reason for this lack of agreement is an inadequate recognition of the role that concrete and symbolic models can and should play in facilitating the learning of mathematical ideas.

Most, if not all, mathematical ideas that are taught in the elementary school can be represented by at least two types of models that aid in making the ideas meaningful to the learner.[1] A *concrete model* represents a mathematical idea by means of three-dimensional objects. A *symbolic model* represents a mathematical idea by means of commonly accepted numerals and signs that denote mathematical operations or relationships. To understand how the same idea can be represented by these two diverse models, consider multiplication of whole numbers (interpreted as the union of equivalent disjoint sets). A concrete model that demonstrates a specific instance of the idea $4 \times 3 = 12$ would be 4 groups of 3 counters—buttons, stones, pencils—the total of which is 12. The counters can be manipulated, grouped, and handled in a variety of ways to show that 4 groups of objects with 3 in each group always have a total of 12 objects. Meaning comes directly through manipulation of the objects. A symbolic model that represents the same idea is $3 + 3 + 3 + 3 = 12$. The threes can be added to obtain twelve, but 3, +, =, and 12 are merely marks (symbols), and when such arbitrarily chosen symbols are used, their only relation to reality is what the child has learned before. If this model is meaningful, it is because the learner is able to attach some meaning to the symbols.

The ability to use symbols to solve problems and the ability to learn more advanced principles are instructional objectives common to all elementary school mathematics programs, although these objectives often are not explicitly stated. Symbols help in generalizing ideas so that they apply to diverse situations, and they promote the transfer of learning (Bruner et al. 1966). However, the use of symbols is not innate; it must be carefully developed

1. A third type of model, the *pictorial*, can represent mathematical ideas and is commonly used in many elementary school mathematics textbooks. It shares attributes of both concrete and symbolic models. Because the theoretical rationale and empirical evidence regarding the usefulness of pictorial models in improving learning are less clear-cut than those of the other two models, pictorial models are not a direct concern of this paper. However, inasmuch as pictorial models provide a bridge or intermediate step between concrete and symbolic models, as the roles of the latter two are made clearer the of the pictorial model will be clarified also.

by using both concrete and symbolic models in a carefully controlled manner throughout the elementary school years. The route to developing this ability is not a direct one because children of seven through eleven years of age are able to use symbols effectively only after they have experienced the ideas to be learned through action—through the manipulation of concrete models. This means that most children in the elementary school years often require the use of concrete models to make symbolic models meaningful.

The idea that experiences with concrete models should precede experiences with symbolic models is neither new nor unique. Van Engen stated in 1949 (1949, p. 397) that the "meaning of words cannot be thrown back on the meaning of other words. When the child has seen the action and performed the act for himself, he is ready for the symbol for the act." However, this idea is meeting increasingly greater acceptance. Major theoretical support for the use of concrete models before symbolic models in teaching mathematical ideas is provided by Piaget, who has proposed a comprehensive theory of cognitive development that encompasses individual growth from birth to maturity. According to Piaget's theory, schemas (mental structures) are formed by a continual process of accommodation to and assimilation of the individual's environment. This adaptation (accommodation and assimilation) is possible because of the actions performed by the individual upon his environment. These actions change in character and progress from overt, sensory actions done almost completely outside the individual to partially internalized actions that can be done with symbols representing previous actions, to completely abstract thought done entirely with symbols. Thus development in cognitive growth involves first the use of physical actions to form schemas and then the use of symbols to form schemas. Learners change from a predominant reliance on physical action to a predominant reliance on symbols. It should be noted, however, that except for the very early months of a child's life, learning involves both physical actions and symbols that represent previously performed actions.

Translation of this cognitive-development theory to instructional practice indicates that learning environments for children at various developmental levels should include both concrete and symbolic models of the ideas to be learned, with special attention given to ensure a major emphasis on those kinds of experiences that represent the predominant type of orientation (concrete or symbolic) most appropriate to the development of schemas at the various developmental levels. It follows that the learning of children who are relatively more dependent on physical action (at an early level of cognitive development) is made meaningful by a learning environment with a predominance of physical experience (concrete models) and that the learning of children at a more advanced level of development is made meaningful by a learning environment with a predominance of symbolic experience (symbolic models).

Although empirical evidence concerning the use of the two models in learning specific ideas is meager and often inconclusive (Fennema 1969), there is a set of studies that support the theoretical position set forth in this paper. The largest subset of this group of studies concern one particular set of concrete models, the Cuisenaire rods and their prescribed teaching method (Gattegno 1964). Most of these studies were similar in design: two groups were equated in some ways; the control group was taught in a traditional way with traditional materials (often defined no better than this) and the experimental group was taught with the Cuisenaire materials as prescribed; and learning by the two groups was evaluated in some way. Table 1 is a summary of these studies.

Several conclusions can be drawn from these studies. The Cuisenaire materials were more effective than traditional ma-

Table 1
Summary of some studies comparing the effectiveness of Cuisenaire and traditional materials

Author[1]	*Grade level*	*Test used*	*Significant difference in favor of Cuisenaire or traditional materials*	*Mathematical content*
Aurich (1963)	First	Standardized achievement	Cuisenaire treatment	Total range of first-grade work
Hollis (1964)	First	Standardized achievement	Cuisenaire treatment	Total range of first-grade work
Crowder (1965)	First	Standardized achievement	Cuisenaire treatment	Total range of first-grade work
Nasca (1966)	Second	Standardized achievement	Neither treatment	Total range of second-grade work
		Cuisenaire achievement	Cuisenaire treatment	
Passy (1963)	Third	Standardized achievement	Traditional treatment	Computation and arithmetic reasoning
Haynes (1963)	Third	Author constructed	Neither treatment	Multiplication
		Standardized achievement	Neither treatment	
Lucow (1963)	Third	Author constructed	Neither treatment	Multiplication and division

[1] Omitted from this group of studies is Brownell's study (1966). It does not fit the paradigm of the other studies, and the results were ambiguous.

terials at the first-grade level (Aurich 1963; Hollis 1964; Crowder 1965). With older children the results are ambiguous. Passy (1963) found that the traditional method was more effective than the Cuisenaire method, whereas Nasca (1966), using a test that included more advanced principles than those in standardized tests, found that the Cuisenaire method was more effective. In two studies (Haynes 1963 and Lucow 1963), significant differences were not detected. In spite of the inadequacies of these studies, there is some indication that children learn better when the learning environment includes a predominance of experiences with models suited to the children's level of cognitive development.

Additional studies involving other concrete models have been reported. In these studies, summarized in table 2, traditional teaching (usually undefined) was compared with teaching supplemented by the use of concrete models.

Here again, first-grade children (Lucas 1966) appear to benefit when learning is aided with concrete models representing mathematical ideas. Some studies (Dawson and Ruddell [b] 1955; Norman 1955; Howard 1950) indicated that children in the middle grades learned better when learning was facilitated with concrete models. The use of concrete models with older children (Mott 1959; Spross 1962; Price 1950; Anderson 1957) appears to neither improve nor hamper learning of mathematical ideas.

Collectively, these data tend to support the hypothesis that a learning environment embodying representational models suited to the developmental level of the learner facilitates learning better than a learning environment that ignores the developmental level of the learner. Specifically, learning of mathematical ideas is likely to be facilitated by a predominance of concrete models in the early grades and a gradually increasing proportion of symbolic

Table 2
Summary of studies comparing effectiveness of learning mathematical ideas facilitated by concrete or symbolic models

Author	*Grade level*	*Concrete model*	*Test used*	*Significant difference in favor of:*	*Mathematical content*
Lucas (1966)	First	Dienes Attribute Blocks	Standardized achievement and author constructed	Dienes treatment for conservation of number and conceptualization of mathematical principles; traditional for computation and solving of verbal problems	Identified in Piagetian terms: multiplication of relations and addition-subtraction relations
Ekman (1966)	Third	Counters	Author constructed	Neither treatment at end of instructional period; concrete-model group on a retention test	Addition and subtraction algorithms
Dawson & Ruddell ([b]1955)	Fourth	Many diverse models	Author constructed	Concrete-model group	Division of whole numbers
Norman (1955)	Third	Concrete and semiconcrete models (number lines, drawings, and counters)	Author constructed	Neither treatment at the end of instruction; concrete and semi-concrete-model group at the end of two weeks	Division of whole numbers
Howard (1950)	Fifth & Sixth	Concrete and semiconcrete models	No information available	Neither treatment at the end of instruction; concrete-model group three months later	No information available
Mott (1959)	Fifth & Sixth	Many multi-sensory aids	Standardized achievement	Neither treatment	Measurement
Spross (1962)	Fifth & Sixth	Concrete aids that had cultural significance	Standardized achievement	Neither treatment	Total range of fifth- & sixth-grade work
Price (1950)	Fifth & Sixth	Multisensory aids	No information available	Neither treatment	Division of fractions
Anderson (1957)	Eighth	Various visual-tactual devices	Author constructed	Neither treatment	Area, volume, & Pythagorean theorem

models as children move through the elementary school.

Although the evidence, both theoretical and empirical, appears to indicate that the ratio of concrete to symbolic models used to convey mathematical ideas should reflect the developmental level of the learner, it should not be inferred that either model alone can suffice at any level in the elementary school. Piaget placed children up to twelve years of age in the concrete-operational stage of cognitive development. Children in this stage are capable of learning with symbols but only if those symbols represent actions the learners have done previously. The function of the classroom learning environment is to ensure that the children perform or have performed those actions that represent all the mathematical ideas that are to be taught in the elementary school. This will mean that as children enter elementary school, the learning of most mathematical ideas will need to be facilitated through concrete representation both because the developmental level of the children indicates that this is the appropriate learning style and because the experiential background of the children is meager. At upper levels the reverse will be true. The learners' experiential background will be much richer and their learning orientation will indicate that their use of symbols is more adequate. However, when new ideas are introduced, regardless of the grade level, care must be taken to ensure that appropriate actions (experiential background) have taken place.

A study reported by Fennema (1969) emphasized the importance of this point. Eight randomly assigned groups of second-grade children were introduced to multiplication. Half the groups learned with a symbolic model that reflected actions previously done. That is, $2 + 2 + 2 = 6$ was used to represent $3 \times 2 = 6$. The other half of the groups learned with a concrete model, the Cuisenaire rods. Three two-rods that measured the same length as one six-rod represented $3 \times 2 = 6$. At the end of a three-week instructional period, the learning transfer of children who used the symbolic model measured at a higher level than the learning transfer of children who had used the concrete. These children were at a developmental level in which a predominance of concrete models would normally be more effective. However, because the children had previously performed the actions necessary for the understanding of multiplication they learned better with symbolic models that permitted generalization of learned ideas to unlearned ones.

The best indicator of which representational model should be used with a certain child to explicate a specific idea is the performance of the child. If children are given freedom, they will select the model that makes the idea most meaningful to them. Concrete models when first used are intriguing, and children prefer using them because of their novelty and because they make mathematical ideas meaningful. However, concrete models are inefficient and cumbersome to use in problem solving. Children soon become aware of this and prefer to use symbolic models as a means of learning when symbols are truly meaningful. Therefore an effective teacher will carefully observe children and attempt to determine which models are most meaningful and acceptable to the children concerned. At the same time, children should have freedom to choose from alternative types of models whenever they are called upon to solve a problem.

If concrete and symbolic models are used in the way suggested, meaningful learning of mathematical ideas is more likely. Children will be better able to apply their learning to new situations, and comprehension of the highly abstract content of advanced mathematics will be more easily acquired.

References

Anderson, G. L. "Visual-Tactual Devices and Their Efficacy: An Experiment in Grade Eight." ARITHMETIC TEACHER 4 (November 1957): 196–203.

Aurich, Sister M. R. "A Comparative Study to Determine the Effectiveness of the Cuisenaire Method of Arithmetic Instruction with Children of the First Grade Level." Master's thesis, Catholic University of America, 1963. Reproduced by the Cuisenaire Company of America.

Begle, E. G. "Curriculum Research in Mathematics." Paper based on a talk given at the Wisconsin Research and Development Center for Cognitive Learning, University of Wisconsin—Madison, 1966.

Brownell, W. A. "Conceptual Maturity in Arithmetic under Differing Systems of Instruction." *Elementary School Journal* 69 (1968): 151–63.

Bruner, J. S., et al. *Studies in Cognitive Growth.* New York: John Wiley & Sons, 1966.

Crowder, A. B. "A Comparative Study of Two Methods of Teaching Arithmetic in the First Grade." Doctoral dissertation, North Texas State University, 1965.

Dawson, D. T., and A. K. Ruddell [a]. "The Case for Meaning Theory in Teaching Arithmetic." *Elementary School Journal* 55 (1955): 393–99.

———[b]. "An Experimental Approach to the Division Idea." ARITHMETIC TEACHER 2 (February 1955): 6–9.

Ekman, L. G. "A Comparison of the Effectiveness of Different Approaches to the Teaching of Addition and Subtraction Algorithms in the Third Grade." Doctoral dissertation, University of Minnesota, 1966.

Fennema, E. *A Study of the Relative Effectiveness of a Meaningful Concrete and a Meaningful Symbolic Model in Learning a Selected Mathematical Principle.* Technical report no. 101. Madison: Wisconsin Research and Development Center for Cognitive Learning, 1969.

Gattegno, C. *For the Teaching of Elementary*

Mathematics. New Rochelle, N.Y.: Cuisenaire Co. of America, 1964.

Haynes, J. D. "Cuisenaire Rods and the Teaching of Multiplication to Third Grade Children." Doctoral dissertation, Florida State University, 1963.

Hollis, L. Y. "A Study to Compare the Effects of Teaching First Grade Mathematics by the Cuisenaire-Gattegno Method with the Traditional Method." Doctoral dissertation, Texas Technical College, 1964.

Howard, C. F. "Three Methods of Teaching Arithmetic." *California Journal of Educational Research* 1 (January 1950): 25–29.

Lucas, J. S. "The Effect of Attribute-Block Training on Children's Development of Arithmetic." Doctoral dissertation, University of California —Berkeley, 1966.

Lucow, W. H. "Testing the Cuisenaire Method." ARITHMETIC TEACHER 10 (November 1963): 435–38.

Mott, E. R. "An Experimental Study Testing the Value of Using Multisensory Experiences in the Teaching of Measurement Units on the Fifth and Sixth Grade Level." Doctoral dissertation, Pennsylvania State University, 1959.

Nasca, D. "Comparative Merits of a Manipulative Approach to Second-Grade Arithmetic." ARITHMETIC TEACHER 13 (March 1966): 221–26.

Norman, M. "Three Methods of Teaching Basic Division Facts." Doctoral dissertation, State University of Iowa, 1955.

Passy, R. A. "The Effect of Cuisenaire Materials on Reasoning and Computation." ARITHMETIC TEACHER 10 (November 1963): 439–40.

Price, R. D. "An Experimental Evaluation of the Relative Effectiveness of the Use of Certain Multi-Sensory Aids in Instruction in the Division of Fractions." Doctoral Dissertation, University of Minnesota, 1950.

Spross, P. M. "A Study of the Effect of a Tangible and Conceptualized Presentation of Arithmetic on Achievement in the Fifth and Sixth Grades." Doctoral dissertation, Michigan State University, 1962.

Van Engen, H. [a] "The Formation of Concepts." In *The Learning of Mathematics: Its Theory and Practice*. Twenty-first Yearbook of the National Council of Teachers of Mathematics. Washington, D.C.: The Council, 1953.

——— [b]. "Analysis of Meaning in Arithmetic." *Elementary School Journal* 49 (February–March 1949): 321–29; 395–400.

2

Whole Numbers

The study of whole numbers extends throughout the mathematics curriculum of the elementary school from initial experiences in counting and finding the cardinal number of a set through work with the algorithms for addition, subtraction, multiplication, and division. Inherent in much of this study is the development of an understanding of the characteristics of our decimal system of numeration. Since this work with decimal bases and place-value concepts is of such importance, we have included in a separate section (chapter 3, "Numeration") the articles dealing primarily with these ideas.

As children's mathematical concepts are developed along the concrete-abstract continuum, we note that the hundred-board is an excellent manipulative aid that has a broad range of applications. Volpel offers a comprehensive and practical discussion of this device and its many uses. This article should provide a stimulus for the reader to explore multiple uses of the many teaching aids that are presented in this book of readings.

The cardinal and ordinal uses of number are among the early number concepts taught to young children. Fitzsimons provides some interesting ideas for teaching the cardinal number of a set, and Patterson suggests some activities with a picture line which involve pupils informally with both cardinal and ordinal number work. Experiences with the picture line should serve as excellent preparation for future work with the number line. A brief, but clever, suggestion for using a "number clothesline" to teach basic addition facts is offered by Orans.

The numeral-number association is another very important instructional task in the primary grades. Armstrong and Schmidt give a good description of a set of inexpensive materials that they constructed for the purpose of teaching the numeral-quantity association to trainable-level mentally retarded pupils. Many of these materials could also be used effectively to teach the same concepts to any child at the early stages of his mathematical awareness.

Some early experiences with basic addition facts and missing addends may be provided by work with the number-line balance suggested by Borgen and Wood. Becker's "elevator numbers" combine addition practice and fun in a puzzle format.

For those children who have difficulty understanding a table of basic multiplication facts, Schrage suggests the construction of an array matrix. This should be very helpful for the average-to-slow pupils. Shafer contributes some interesting thoughts about the possibilities of using a tape recorder to provide some of that much-needed individualized drill and practice in building skill with the multiplication facts.

Teaching children to apply their mathematical knowledge continues to be one of the trouble spots for teachers. The concluding article in this chapter focuses on one technique for teaching problem solving. Sharff suggests the use of number-picture problem situations that involve the pupils in the creation of problem situations as well as in finding solutions. She has some very interesting ideas and illustrations.

The Hundred-Board

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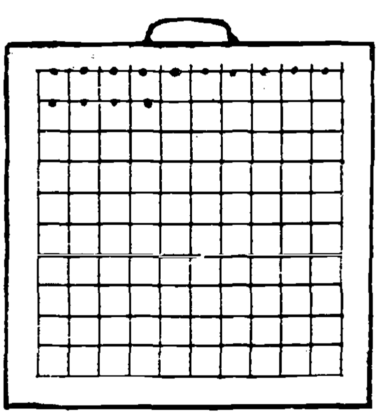
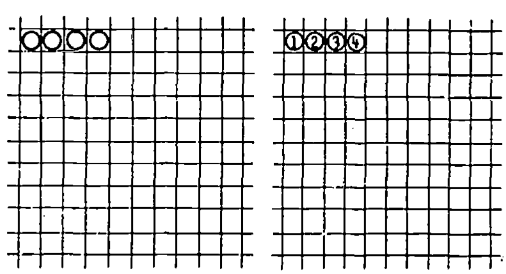
EVERY TEACHER OF ARITHMETIC should have a *hundred-board.* The writer believes it is one of the most efficient teaching aids for use in the development of understandings of concepts and operations with number quantities. If a teacher could select but one gadget for use in the arithmetic classroom he should choose the hundred-board because of its many uses with regard to both the scope and grade placement of material.

The hundred-board can take one of several forms. The simplest home-made board consists of a square of one-half inch plywood large enough to be ruled off into 100 smaller squares each 2 inches by 2 inches, with a 2 inch border around the outside. A finishing nail should be driven at the center of the top line of each small square. The finished product should then be stained and equipped with a handle to facilitate handling, hanging, and storing. In this form it can be used on a table, stand, easel, tripod, or in a chalk tray.



Variations of the above can be effected by use of pegs or hooks placed at the exact center of each small square, by increasing its size so that it can be set on the floor for easy handling by small children, or can be constructed with feet so that it can be used while resting in a horizontal position on a desk or table.

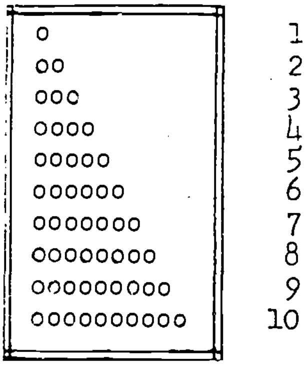
The teacher should have a set of numbers from 1 to 100 and a set of one hundred unnumbered objects which can be hung on the board. The numbers can be purchased commercially or prepared by numbering oak-tag labels, department store price tags, or poker chips punched for hanging. The teacher should use objects which permit easy handling and which at the same time are attractive, such as spools, curtain rings, rubber washers, identification tags, one-inch pieces of plastic hose, and similar objects.



In the paragraphs which follow the writer will show a few uses of the hundred-board.

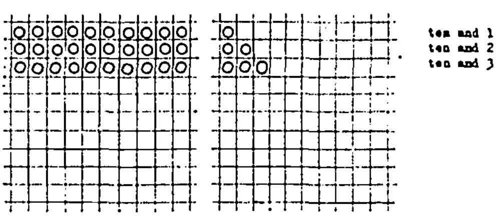
Number

The hundred-board, initially, can be used to show the cardinal aspect of numbers. To show that two represents 1 more than one, that three represents 1 more than two, etc. we present the pictures of these objects. Thus when a child learns to say "4" he should associate that name with that things. This is "fourness." The meaning of the number expressions from one to ten should be shown on the board as follows:



Now after a child has learned the meaning of "ten" he can count tens. Reference to a hundred-board filled with numbers clearly reveals groups of tens and the child should be taught to count them as tens. It should be as easy for him to comprehend 1 ten, 2 tens, 3 tens, 4 tens, etc. as it is for him to learn to count 1 marble, 2 marbles, 3 marbles, . . . or 1 case, 2 cases, 3 cases, etc.

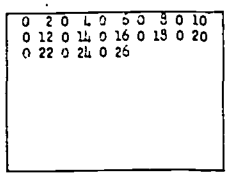
Next we introduce the "teen" numbers with the help of a second hundred-board. Each row of the hundred-board represents a ten, so if a filled hundred-board is placed beside the one pictured above the child ought to visualize number quantities like ten and 1, ten and 2, ten and 3 and the other numbers up to ten and 10 or 2 tens. If we encourage the child to continue to count in this manner, in terms of groups of tens and ones, he will readily learn and, we hope, comprehend the meaning of the numbers up to 10 tens or one hundred. In this way the child will learn to say expressions like "3 tens and 5," "3 tens and 6," "3 tens

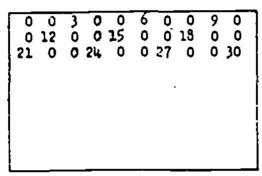


and 7" before he is told that such number expressions can be shortened to thirty-five (35), thirty-six (36), and thirty-seven (37). When the child knows that four full rows of discs represent 4 groups of ten discs he can be shown how to express this quantity symbolically as 40. An extension of this concept should enable him to replace all the discs with others numbered to 100.

We can use the numbers on the hundred-board for teaching the child to count by twos. We begin by having all of the numbers turned over (face down). We ask the child to skip the first disc, number 1, and turn over the next disc. We skip one again and turn over the next one, we skip one again and "face-up" another. If we continue doing this until we reach 50 or 60 the child should observe the pattern of endings and should observe that full columns are turned down and full columns are exposed. Those numbers exposed are called "even" numbers and all of them end in either 2, 4, 6, 8, or 0.

The same procedure might be followed when teaching children to count by threes, fours, or fives. If all of the discs are turned down, then we skip two and turn over one, skip two and turn over one, etc. when counting by threes. The numbers exposed will be 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, etc. The pattern of endings does not begin to repeat until after we have shown ten threes.





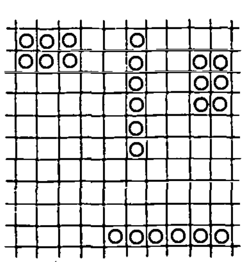
Another technique to follow in learning to count by twos or threes is to stack all the numbers in an ordered row or pile, take them one by one, place them in rows of 2 or 3 as the case may be. One does not have to replace all 100 numbers on the board but enough to enable him to understand the concept of counting by groups and to learn the respective patterns. The number board then will look like this:

1	2	21	22
3	4	23	24
5	6	25	26
7	8		
9	10	etc.	
11	12		
13	14		
15	16		
17	18		
19	20		

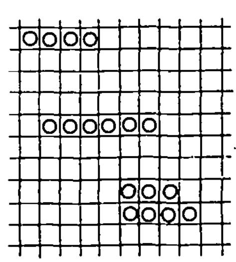
1	2	3	31	32	33
4	5	6	34	35	36
7	8	9	37	38	39
10	11	12			
13	14	15	etc.		
16	17	18			
19	20	21			
22	23	24			
25	26	27			
28	29	30			

The hundred-board is useful for picturing the various patterns of number quantities. The quantity 6, for instance, may be shown on the number board in several patterns. If children re-arrange discs (a constant number of discs) *they will* discover numerous facts concerning the given number. This activity develops insight into number quantities and number relationships.

Groups of different sizes may be pictured on the board and children challenged to determine the largest group. If they see groups containing 4, 6, and 7 spools they ought to be able to deduce that 6 contains (2) more than 4, that 7 contains (1) more than 6, and that 7 contains (3) more than 4. Conversely they will learn that 4 and 6 are smaller than 7 by 3 and by 1 respectively. The comparison phase of counting can be taught through the use of the hundred-board.



Groups of 6



Groups of 4, 6, and 7

Addition

The hundred-board is very useful in helping children discover basic addition facts. Since addition is a "putting together" operation we want to know how many we have altogether if we put 3 discs with a group of 4 discs. We show 4 discs on the top row and 3 discs on the second row and then combine them into one group. We discover that when we put 1 (of the 3) with 4 we will have 5, another 1 will make 6, and the last 1 will make 7. Thus 4 discs and 3 discs are 7 discs. Children will discover that there are many addition facts whose sums are less than 10—the combining of two small sets of discs will often leave the top row incomplete. Also, in learning the meanings of the numbers 1 to 10 they will have discovered how many more are needed to complete the row of 10. For instance, the last row of discs needs 0, the next to last row needs 1 so $9+1=10$, the row containing 8 needs 2 so $8+2=10$, etc. In working with the complete set of 10, children ought to learn the sub-sets which add to 10. The board showing the sum of 4 discs and 3 discs should gradually have the following appearance:

0000 becomes 00000 then 000000 and then 0000000.
000 00 0

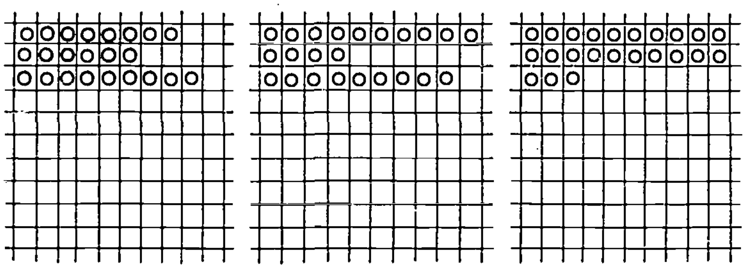
Manipulations of this type suggest to the students that addition represents an increase, a movement to the right—a forward motion.

oxxxxxxxxx	1 + 9 = 10
ooxxxxxxxx	2 + 8 = 10
oooxxxxxxx	3 + 7 = 10
ooooxxxxxx	4 + 6 = 10
oooooxxxxx	5 + 5 = 10
ooooooxxxx	6 + 4 = 10
oooooooxxx	7 + 3 = 10
ooooooooxx	8 + 2 = 10
ooooooooox	9 + 1 = 10
oooooooooo	10 + 0 = 10

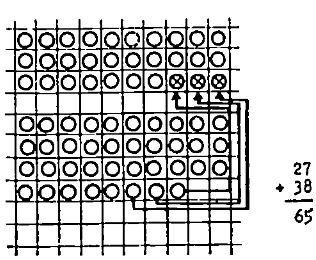
When children have mastered the combinations which total 10 they should have little difficulty recognizing sums which total more than 10. Since $7+3=10$, then $7+4$, $7+5$, $7+6$, etc. must be greater than 10. Let us show the addition of 7 discs and 5 discs on the hundred-board. The first row should show 7 discs. The second row should show 5 discs. The discs from the set of 5 should be moved one by one until we have completed the first row. We need 3 of these, therefore there should be 2 left on the second row. Thus the sum of 7 discs and 5 discs is the same as the sum of 10 discs and 2 discs. Since 10 and 2 represent 12, then the sum of 7 discs and 5 discs is 12 discs.

In adding $6+7$ (representing spools) children will visualize that they must take 4 of the 7 to put with the 6 to make a 10—therefore $6+7$ is the same as 10 and 3. Addition with the hundred-board is similar to addition performed along a number line; the hundred-board is a number line decomposed into ten sections of ten where the concept of bridging is equally prominent. To add 6 to 7 on the number line one begins at the point marked 6 and moves forward (to the right) 4 places and then 3 more arriving at 13. Thus $6+7=13$.

When showing the addition of 3 or more quantities on the hundred-board children ought to sense the expediency of filling up groups of 10 so as to state the final sum as groups of tens plus ones. Thus $8+6+9$ when combined on the hundred-board becomes first $10+4$ (2 discs have been moved from the group of 6 to the 8 group making a complete 10). Then 6 of the 9 are moved to the second row completing another 10. Since there are 3 left on the third row the sum of $8+6+9$ is 2 tens plus 3 or 23. A student might move 2 of the 9 into the top row and 4 of the 9 into the second row to obtain the same result.



The rationale of carrying is evident when one demonstrates the addition of 2 or more two-digit numbers with the aid of the hundred-board. To show the addition of 27 and 38 we set up 2 rows of tens and 7 ones and then 3 rows of tens and 8 ones. To find the composite sum we use some of the 8 ones to fill in the row containing the 7 ones. Since 3 are needed to complete the row the final result shows 2 groups of ten, and 3 groups of ten (which we had in the beginning), and 1 more group of ten, and 5 ones. The sum, clearly visible on the hundred-board is 65. There were enough ones to make another complete 10.



Two boards can be used effectively showing the tens on one board and the ones on the other. Then some of the ones can be used to make a ten. This procedure will enable children to "discover" that 2 tens and 7 plus 3 tens and 8 is the same as 6 tens and 5.

Subtraction

Subtraction is an operation which separates a given group into 2 or more subgroups and is usually taught first as a "take-away" operation, the opposite of the addition operation. If we wish to find the number of discs remaining after we have taken 3 discs from a group of 9 discs we first picture 9 discs on the hundred-board. When 3 of these are removed, one at a time, students should be encouraged to do so from right to left removing first the ninth one, then the eighth one, and then the seventh one. This backward operation will suggest a movement to the left along a number track . . . and that's the basic idea of "take-away subtraction."

Should we want to know the remainder when 5 things are taken from a group of 12 things we picture first the 12 things on the board. Children will see that the taking away operation (the 5 steps backward) will result in a remainder which is less than 10 They will first remove the 2 things on the second row and then move into the top row and take some from the group of 10. $12-5=?$

The same reasoning is applicable when we perform compound subtraction. To illustrate further let us subtract 8 from 23. First we fill in 2 complete rows and place 3 items on the third row to make 23. We start removing 8 of these one-by-one—we take off the 3 and then move into the group of 10 to get 5 more $(3+5=8)$. There will be only 15 left.

The solution of more difficult compound subtraction examples involving two digits illustrates the principle of reading and thinking from "left to right." Should we want to show the example

$$\begin{array}{r} 72 \\ -35 \\ \hline \end{array}$$

we would first show 72 items on the hundred-board. From these we are to remove 35. We will first take off 3 groups of 10 and then will take 5 ones. We take off the 2 single items and then take 3 more (enough to make 5) from one of the groups of 10. The final result is 37 and the algorism which will illustrate the thought pattern is as follows:

$$\begin{array}{r} 72 \\ -\underline{35} \end{array} \qquad \begin{array}{cccccc|r|} 10 & 10 & 10 & 10 & 10 & 10 & 10+2 \\ 10 & 10 & 10 & & & & -5 \\ \hline & & & 10 & 10 & 10 & 7 \end{array}$$

Multiplication

If the addends of an addition example are equal then the example can be solved by the operation called multiplication. Should we wish to add three groups of 2 we picture them as follows on the hundred-board:

X	X
X	X
X	X

We discover that when we regroup these items they will fill up the first 6 places. Thus 3 twos equal 6. These can also be shown with numbers after the result has been obtained—the numbered discs can be placed two on each row so that the numbers show 1 and 2, 3 and 4, and 5 and 6 on the three rows.

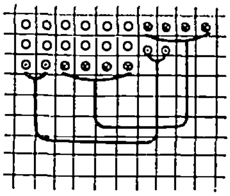
1	2
3	4
5	6

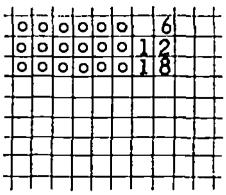
To show that 4 threes are 12 we arrange four rows of three discs and then regroup them to form sets of 10, if possible. There will be one set of 10 with 2 remaining. Thus $4\times3=12$. When numbered discs are used to show the representation of 4 threes the numbers will form the following pattern:

1	2	3
4	5	6
7	8	9
10	11	12

Children should become aware of the fact that the numbers in the last column are the "threes' facts" representing the products in the table of threes.

To show that 3 sixes are 18 we first put 6 discs on each of the first 3 rows. Now we regroup them to form sets of 10. In doing this we discover that there will be one set of 10 with 8 remaining. Thus $3\times6=18$. When the single discs are moved to fill in rows there are many ways in which this can be done—there is no particular "best" procedure. Again, when these discs are shown as numbered discs the numbers in the last column represent the products when 6 is the multiplicand. $1\times6=6$, $2\times6=12$, $3\times6=18$, etc.



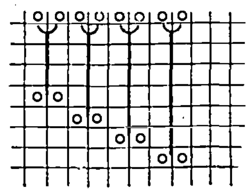


A hundred-board, with the addition of a few duplicate numbers, can be used to develop and display all of the multiplication facts and at the same time provide the opportunity for children to learn how to read tables. To do this, leave the first block of the hundred-board empty and hang all the numbers from 1 to 9 in the first row (9 will be in the place usually occupied by 10). Also, still leaving the first block empty, hang numbers from 1 to 9 in the first column. Now in the cells formed by the rows and columns hang numbers which represent the product of the two numbers which head the row and column. Thus in the cell of the row headed 4 and the column headed 7 put the number 28. Likewise in the cell of the row headed 7 and the column headed 4 put another 28. No other 28s are needed in the table. However, there are some numbers which appear more than twice as products within the table, namely, 8, 16, and 24.

	1	2	3	4	5	6	7	8	9
1									
2									
3									
4							28		
5									
6									
7				28					
8									
9									

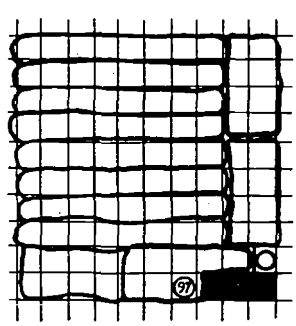
Division

Inasmuch as we have already shown ways to solve subtraction examples on the hundred-board, then division, which is also a separating or subtracting operation can also be illustrated thereon. There are two concepts of the division process, the measurement concept and the part-taking concept. We will first illustrate the measurement concept which asks one to find the number of groups when we know the size of the group (the multiplicand). An illustration of this idea is the question which asks how many groups of 2 can be formed from a group of 8. We begin with the dividend 8 and show that many things on the board. Now we remove 2 of these things and place them in a group elsewhere on the board. Then we repeat the operation with 2 more, and again 2 more until they have all been regrouped into sets of 2. We have discovered that 4 groups of 2 can be formed from a group of 8.



The example 16÷3 asks us to find the number of groups of 3 objects which can be formed from a group of 16 objects. We illustrate 16 objects on the board and then pick them off in sets of 3. We discover 5 sets of 3 objects with 1 object left over or remaining.

The example 97÷8 asks us to find the number of groups of 8 which can be formed from 97. Since the size of the sub-set is quite large and would require much tedious handling of objects we suggest the use of rubber bands, string, ribbons, etc. to encircle the group. We circle 8, then circle 8 more, etc. until all possible groups of 8 have been formed. We will discover that there are 12 groups of 8 in 97 and 1 item remaining.



The partitioning aspect of division asks us to find the size of each sub-group when the number of groups is known. For instance, 6÷3 might ask us to find the number of items in each group if 6 items are divided into 3 equal groups. We show 6 items on the board and then remove 1 at a time placing 1 in each of 3 different groups. "We put 1 here, 1 there, and 1 over there." Then we remove 1 more and 1 more and 1 more until they are all used up. Since we are trying to regroup into sets of the same size we must always put the same number in each of the sub-groups. In 6 there are 2 in each of 3 groups.

$17 \div 5$ asks us to distribute 17 things equally among 5 people. We arrange 17 items on the hundred-board and then pick them off 1 at a time putting one in each of 5 different places. Now we do the same thing again putting one in each of 5 different places, then we do it again. Now we see that there are not enough to go around again so we have learned that there will be 3 in each of the 5 groups with 2 items remaining.

After some experience with the simpler problems students will see that it is possible to remove more than 1 at a time—they could put 5 in each sub-group if they had to divide 37 by 6 and they could put as many as 10 in each sub-set if they were to divide 59 into 5 equal parts.

Fractions

The relationships inherent in both common and decimal fractions can be exhibited on the hundred-board. By the use of elastic ribbons it is easy to show that $1/2=2/4$, that $1/2=50/100$, that $25/100=1/4$, that $1/5=20/100$, etc. Also that $1/10$ is equivalent to $10/100$ and that $3/10$ is equivalent to $30/100$. If the whole board represents unity or 100% then each square represents 1 out of the 100 or 1% and is useful in showing the common fraction and per cent equivalents. The board will be useful for revealing that $1/8$ of 100 blocks represents $12\frac{1}{2}$ little blocks. Since each block represents 1 out of 100 or 1% of the whole, then $1/8$ is equivalent to $12\frac{1}{2}$%.

One column represents $1/10$ or .1 of the whole board and 6 rows or 6 columns represents $6/10$ of the board or .6 of it. Forty-three of the little blocks represent 43 of the 100 blocks or 43 hundredths of the whole. Thus were we to compare .4 with .43 we see that the .43 is larger than .4 by 3 little blocks or by .03 (3 hundredths). The board is a helpful adjunct in teaching the meaning of and the relative sizes of common and decimal fractions whose values lie between 0 and 1.

There are numerous other uses for the hundred-board which will be discovered by the teacher who uses one.

EDITOR'S NOTE. Dr. Volpel's "hundred-board" has many uses. Many people have used such a device for percentage but have not explored all the ideas he has presented. Perhaps he is right when he says it is the most valuable piece of equipment for enhancing the teaching of arithmetic. As with other devices, it is the way in which an item is used to foster discovery, understanding, and learning that makes it valuable. Teachers will see possible variations of design and construction and more values will be discovered as the use of the board becomes more common. Devices are valuable in leading children to sense the objective background for the learnings which later should become more abstract.

Kindergarten mathematics

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Miss Fitzsimons teaches kindergarten in the Lincoln Elementary School in Appleton. She also represents the kindergarten teachers on a city-wide mathematics study committee.

In our eagerness to develop mathematics concepts for today's five- and six-year-olds, are we mixing up our number ideas and merely confusing the children, or are we preparing thoughtful programs which help to develop number concepts at this particular age level? Clarity in planning and identifying activities which develop specific mathematics understandings is essential.

An effective kindergarten mathematics program has two approaches—both important. Many of our mathematics experiences develop from our everyday school living. This provides for opportune teaching. The teacher is continually alert to situations which provide the opportunities to emphasize simple number concepts. But this "catch technique," while meaningful, does not entirely satisfy the mathematical appetite of today's kindergarteners. A planned program of activities which challenges the children's abilities may also be woven into the day's living.

In developing a planned program, we may include the continued development of rational counting, the identification of numerals, the development of the cardinal concept of number, an understanding of greater than and less than (included in one-to-one correspondence) and the instantaneous recognition of small groups or sets. Materials suitable for the extension of these understandings for young children are scarce; therefore, the teacher necessarily becomes the manufacturer of many mathematical materials.

The kindergarten surprise box which frequently holds little surprise treasures such as a bright new puzzle or a soft fluffy toy bunny may also be the hidden spot for a numeral and the corresponding number of objects. When developing the concept of one, our surprise box—which is approximately 8″ square and gaily decorated with colored paper circles, squares, rectangles and triangles—contained a cut out numeral 1 pasted on a heavy 7″×7″ cardboard with a back support. With it was one small block, one plastic spoon, and one toy milk bottle. We discussed each object as we took it out of the box, the teacher commenting, "What do you call this numeral? How many blocks did we find? How many spoons? And how many toy milk bottles?"

We then directed our search about the room to discover other things of which we had only one. Typical responses were

"Look, we have one piano."

"And there is just one teacher's desk."

"We have only one round table."

"There's one punching bag."

"I have just one nose and one head." And so it continued.

Several days later the surprise box produced the numeral 2, and with it were two small blocks, two plastic spoons, and two toy milk bottles. Where else could we find two of something? Our search was again directed to ourselves and our immediate surroundings.

"I have two shoes."

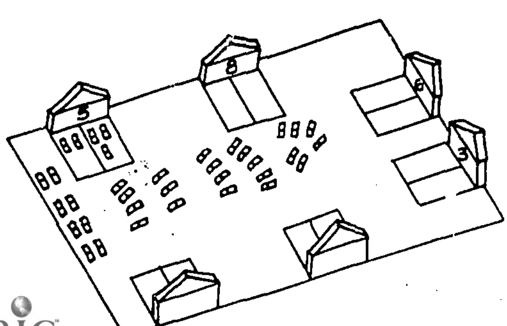
"We have two clocks—the real one up there and our play one."

"I have two eyes and two hands."

"We have two dolls in the doll house and two doll buggies."

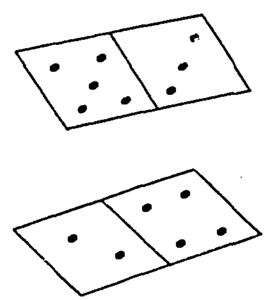
We were on our planned way. It was very natural then to find "pairs" and also introduce "duets" at singing time. Teacher judgment guided the appearance of the next numeral with its accompanying objects. In fairly like manner, we developed the cardinal concept of all numbers through ten, and also the ability to identify the numerals.

Our garage game has fascinated the children. We painted red numerals 1 to 10 on ten blocks which were approximately $6'' \times 3'' \times 1\frac{1}{2}''$. We cut and folded colored paper to represent the roofs of the garages and taped them on top of the blocks. In front of each garage we placed a paper with a heavy dividing line. A bag of miniature plastic automobiles purchased at the ten-cent store completed the materials which we needed. Using the number of miniature autos equivalent to the sum of the numerals on the garages, we were ready to play the garage game. One child plays at a time, moving the cars into the "double garage areas." In the beginning, the children arranged the cars with little regard for the dividing line, but as they continued experimenting with the game they discovered many possibilities. When running the cars into garage 6, they could put three cars in each side of the garage or five in one side and only one in the other or four in one side and two in the other side. Accordingly, the children made discoveries with each of the garages. We varied the game by changing the garages and by using only a limited number at a time. The number of autos used was necessarily changed too, so that all garages were filled and no cars left on the street at the end of each game.



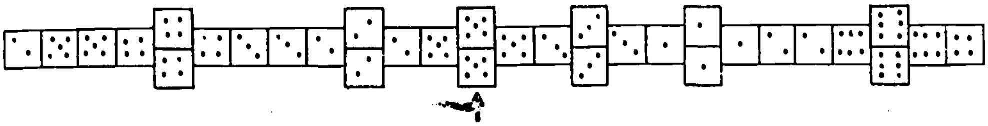
These children were identifying numerals, doing rational counting, building their own little sets or groups of cars and experiencing a clear understanding of the cardinal concept of number. The watchful teacher would frequently inquire about the car arrangements at the end of each game, thereby helping the child to develop mathematical sentences. At times it may be necessary to add a car or take away a car from some garage areas.

Our domino game proved to be enthusiastically received by the children. Square 9″ linoleum tiles were cut in half, and paper circles, arranged domino style, were pasted on them. We used various colored circles. The ones were red, the twos blue, the threes green, the fours yellow, the fives orange, and the sixes black.



With the children standing in circle formation, twenty of the children (the number of dominoes prepared) were given a domino which they held in both hands so that all the children in the circle could view them.

The teacher started the game by placing a pair of fives on the floor in the center of the room. She then called for a domino on which there is a five and something else. This is played at one side of her pair. Then she called for a five and something else to be played at the other side of her pair. As each new number appeared, the teacher called for its pair and then a matching



domino. Calling the dominoes for alternating sides of the pair of fives provided challenge and excitement for the children.

Mathematically, the children were developing an instantaneous recognition of groups or sets, as well as a clear understanding of the cardinal number.

When the children say again and again with spirited enthusiasm, "Let's play that Giant Domino game today," or "May I be the garage man today?" we may feel reasonably confident that these games are bringing a pleasant excitement and intellectual stimulation to beginning mathematics.

Whether children are counting beads on a counting board, fishing out numeral-marked fish from an artificial pond and matching them to appropriate schools of fish, scrambling and unscrambling numerals or matching children with chairs in preparation for a game, it is imperative that the teacher be acutely aware of her purpose and specific objectives in order to develop an effective and continuing kindergarten mathematics program.

EDITORIAL COMMENT.—The surprise-box idea can be expanded in many ways. For example, the box may contain a card like the following one and a set of objects for the children to place in the appropriate places on the card.

one	two	three
1	2	3

Likewise the garage-game idea may be adapted for use with boats, airplanes, farm animals, and so forth.

While making the suggested dominoes, the teacher may wish to construct some set cards for the numbers 1 through 10. These cards may then be used to provide experiences with order. For example, the cards may be handed out to the children to start the activity. The teacher asks for the child with the "1" card to place his card in the chalktray. Then the teacher asks for the child who has the card with the "next" number to place his in the chalktray, and so on. For another activity, the teacher may select a child with a number card such as the "3" card. The teacher may then ask for all children who have a number less than 3 to step forward or she may ask for all who have number greater than 3 to stand up

Our number clothesline

SYLVIA ORANS
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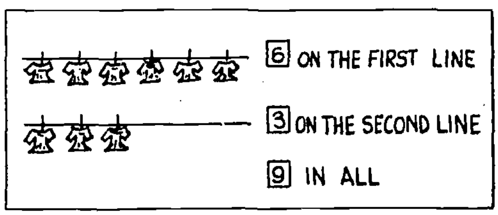
On a large, oblong piece of corrugated board I mounted two long strips of wood. From each strip I suspended a "line" (string) and hung doll clothes on it with clothespins. (The doll clothes are paper doll clothes mounted on pieces of corrugated board—when more clothes are needed, the children make them.)



Alongside one line I placed the words, "on the first line"; next to the other clothesline, the words "on the second line." A space large enough to display removable numeral cards was left between the line and the words. Cards are hung on cuphooks.

The clothesline is used for addition, subtraction, sets, and sets within sets. And, best of all, it is used by the children.

EDITORIAL COMMENT.—Clothespins or other items may be substituted for doll clothes in these activities. Experiences with equivalent sets and with "greater than" and "less than" may be provided on the clotheslines.

A picture line can be fun!

KATHERINE PATTERSON

Katherine Patterson is an assistant principal at the Laytonsville Elementary School in Montgomery County, Maryland. She has been a classroom teacher, a student-teacher sponsor, and a resource teacher in elementary mathematics.

When the young child enters school, he has many opportunities to become involved with mathematics. Among the varied classroom activities that the writer has used, the walk-on picture line is one that delights children. This activity uses movement, builds on the child's environment, and develops oral language. The pictures may be changed according to the interest of the group. The activities with the picture line are informal in nature, and they are fun! Many mathematical concepts can be introduced, such as the cardinal and ordinal use of numbers; idea of a starting point; use of a precise vocabulary such as before, after, between, how many, which one, and which place; directions on a line; counting; one-to-one correspondence; addition and subtraction.

To work with a picture line, start with a set of five pictures. As the child's concepts of number values from one to ten increase, the set may be enlarged to ten pictures. The pictures should be large cutout pictures that can be taped to the floor. A 12″ × 15″ picture is a good size.

Pictures are cut out to show the shape of the object to add to the interest level. At first, the set can include a variety of objects such as cat, ball, top, tree, and kite. After the set is assembled, the teacher can hold up one picture at a time, and the children can have a chance to identify the picture. As a child identifies a picture, he can hold the picture and stand in line with the other children side by side. Also, the class can work on spatial relationships as they place the pictures on the floor. Questions can be asked such as: "How can we place our pictures in a straight line?" "Is there the same space between each picture?"

After the pictures are taped in place on the floor, the teacher and the class can establish some rules for the games. A recall of outdoor games in which a starting place is needed will serve as a basis for deciding on the rules.

1. A starting place is needed. (We need a place to wait for directions. This place indicates that we have not taken any steps. A chalk mark or a piece of tape can be used for this purpose.)
2. Let each picture represent a step.
3. Always start on the starting line unless told otherwise.
4. Proceed from left to right unless given special directions.

Example:

Then the teacher and children play the game.

Types of questions and activities

Where will you start?

Take two steps. On which picture are you standing? Take five steps, etc.

Walk to the tree. How many steps did it take to get to the tree?

How many steps is the ball from the kite? The cat from the tree?

Which picture is two steps away from the ball?

Take two steps and one more step. On which picture are you standing? How many steps from the starting place is it?

Take four steps. Go back two steps. On which picture are you standing? How many steps from the starting place is it?

Show a pattern. →→ Explain that when arrows are used, each arrow represents one step. The arrow also shows the direction of the step. Therefore, → means one step to the right. →→ means two steps to the right. →→→ means three steps to the right. The three arrows may be arranged into different patterns such as

→ →→ or →→ →

Four may be arranged as

→→ →→, → →→→,
→→→ →, or →→→→

Show a pattern. Example:

→→→ →

Take this many steps plus one more step.

Show a pattern. Example:

→→→ →→

Take the number of steps that is one less than what the pattern shows.

Show two patterns. Example:

→→→ →→

and

→→→ →

Take the number of steps that is the smaller number of steps in the picture; the larger number.

Take one step. On which picture are you standing? Which place is the picture?

How many steps to the third picture? Which picture is it?

Walk to the last picture, first, third, etc.

Walk to the picture just before the tree, kite, etc.

Walk to the picture just after the ball.

Walk to the picture between the top and kite; the ball and the tree.

Walk to the picture just before the third picture; just after the second picture; between the first and third picture, etc.

Walk to the second picture. Take three more steps. Where are you? How many steps is it altogether from the starting place?

Walk to the picture of something that bounces.

Have children close eyes and change the order of the pictures. (This is one reason why it is a good idea to tape the pictures rather than have them painted on the floor.) Then ask the same questions as before.

Have children change the fourth picture with the last picture; the second with the fourth.

After the children have had many experiences walking the picture line and are ready for paper work, each child can make his individual picture line. The child's picture line could be made on adding-machine paper and rolled up when not in use. The same type of questions as before may be asked, and the child could point to the correct picture or could cover up the picture with a counter. After much work with the individual picture line, a follow-up activity such as the following could be provided.

→ means one step to the right; ← means one step to the left. →← means to take one step to the right and turn around and go back one step to the left.

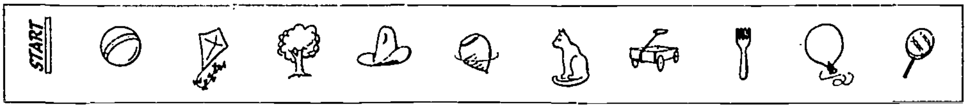
→→⇄

means go three steps to the right and at that point go one step to the left.

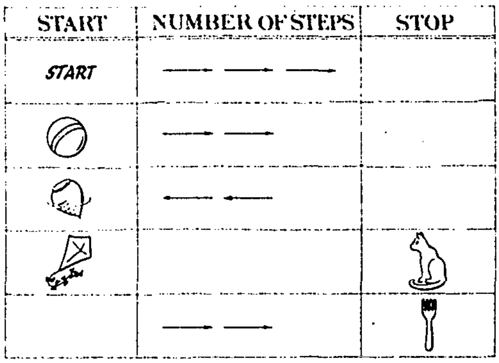
Finish the chart on the next page.

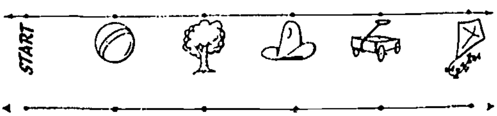
When two parts of the chart are given, the child should discover the third part for himself. A different column can be left blank each time. This provides variety.

After many experiences with a picture line, the transition to the number line is quite logical. Have a picture line made and have another line with points marked.

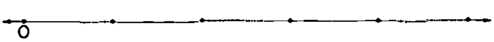


Finish the chart:





Procedure—The position on the picture line that indicated the child was waiting for direction and hadn't taken any steps was symbolized by the word "start." Therefore the symbol that could be used on the number line to indicate no steps have been taken, and the child is waiting to begin, is zero.



On the picture line, it took one step to get from "start" to the  . On the number line it takes one step to get from "0" to the next point that can be labeled "1." To get from "0" to the point after "1," it takes two steps that we can label "2," etc.

Children can see order and sequence on the number line. Likewise, they can see the reason why we start on '0" and not "1" when taking steps.

There are many more activities and games that can be done with the number line. Be creative. Try some and have fun!

Simple materials for teaching early number concepts to trainable-level mentally retarded pupils

JENNY R. ARMSTRONG and
HAROLD SCHMIDT

The project described here was carried out at the University of Wisconsin, where Jenny Armstrong is director of research and evaluation for the Special Education Instructional Materials Center. She is also a lecturer in the Department of Studies in Behavioral Disabilities.

Harold Schmidt was a graduate student during the course of the project and is currently serving in the United States Army.

As a part of a project designed to teach early number concepts to trainable-level mentally retarded pupils, a step-by-step series of materials was developed. The materials were characterized by being (1) sequential, (2) easily utilized by pupils of low-level motoric, cognitive, and verbal skills, and (3) easily constructed with inexpensive materials that are readily available to most classroom teachers.

Numeral-quantity association taught in the enactive mode

In a research study designed to compare the early number learning of trainable-level mentally retarded when taught in an enactive mode of representation (using manipulative materials) with the learning acquired through a pictorial mode of representation (using nonmanipulative materials) following the information-processing theory of Bruner (1964), it became apparent that there were very few manipulative materials available for teaching the concepts related to numeral-quantity association (Armstrong 1969). Therefore, as a part of the project a series of simple step-by-step materials was developed.

One of the major criteria of the materials was that they be manipulative in character—that is, in order to use the material the pupil must manipulate objects. Another criterion of the materials was that they be specifically designed to teach a numeral-quantity–association concept. Implicit, therefore, in the use of each material developed was manipulation and focus on a numeral-quantity–association concept.

Materials design with concern for sequential development

A series of materials structured in a very rigorous developmental sequence was then

The authors wish to acknowledge the funding of the project by the Bureau of the Handicapped, USOE, Department of Health, Education, and Welfare, under grant numbers 32-59-0500-1002 and OEG-0-8-080568-4598(032).

designed. Since the pupils for whom the materials were originally designed were at the very beginning stage of number learning (that is, several had not even learned to count by rote, let alone associate any meaningful concept of quantity with a given numeral), the first set of materials in the sequence was designed to provide a simple manipulative aid to the oral verbalization of the numerals from 1 through 5.

This first set of materials in the sequence consisted of block-chain counters (see fig. 1). Pupils were given the chains of blocks and instructed to count the blocks and verbally tell the number of blocks on each chain. This activity was repeated several times until the pupils could give verbally the numeral that expressed the number of blocks on each of the block chains. The pupils were highly motivated by this ma-

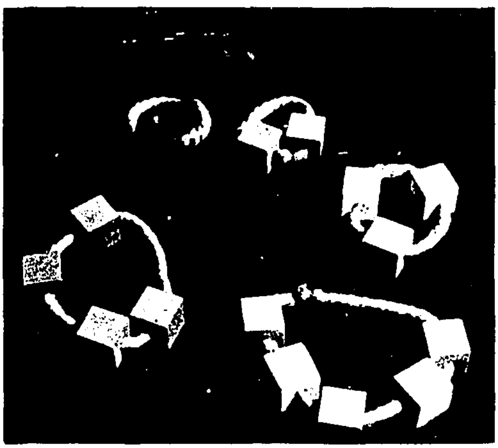


Fig. 1. Block chains

terial, and in quite a short time they were able to achieve the instructional goal.

A second set of materials that was developed for this same instructional goal was not as successful. This set of materials consisted of glass jelly jars containing varying numbers of lima beans (see fig. 2). Because of the low level of the motoric skills of this group of children, the jars were frequently dropped. This was especially hazardous as the jars were made of

Fig. 2. Bean jars

glass. Also, the lids were much too difficult to get off the jars.

The next set of materials in the sequence was designed to aid the pupils in associating the number symbols visually and tactilely to quantities of objects. This set of materials consisted of a series of plasticene molds that had insets for different numbers of candles and imprints of the associated numerals, from 0 through 5 (see fig. 3). Pupils were given one "candle counter" at a time and instructed to count the candles in the holder, to verbally give the number that told the number of candles in the holder, and then to use a finger to feel the shape of the number symbol; they were then told to look at the shape of the number symbol, verbalize it, and retrace it with a finger. The pupils enjoyed their work with this set of materials and were aided in getting one step closer to reading the number symbol. This material also

Fig. 3. Clay candle counters

provided readiness for the actual writing of the number symbol associated with a particular quantity.

The next set of materials in the sequence also allowed for both visual and tactile experiences with the number symbols as well as for quantity association with the symbols (see fig. 4). Each board had two numerals in sequence made from sandpaper to provide a tactile sensation as the pupil ran his finger over the numeral to get a feeling of the shape as well as how it would be to write the numeral. A small well in the board by each numeral provided well in accomplishing its goal, when constructing the material for future use one might wish to separate the single numerals and wells by cutting each board in half. In this way there would be only one numeral on each board (that is, 2 and 3 would not be confused with 23).

The next set of materials in the sequence was designed to provide the first experiences with writing the numerals with some guidance (see fig. 5). Each board provided a stencil for the numeral concerned and removable blocks on small pegs to illustrate the quantity. Pupils were pro-

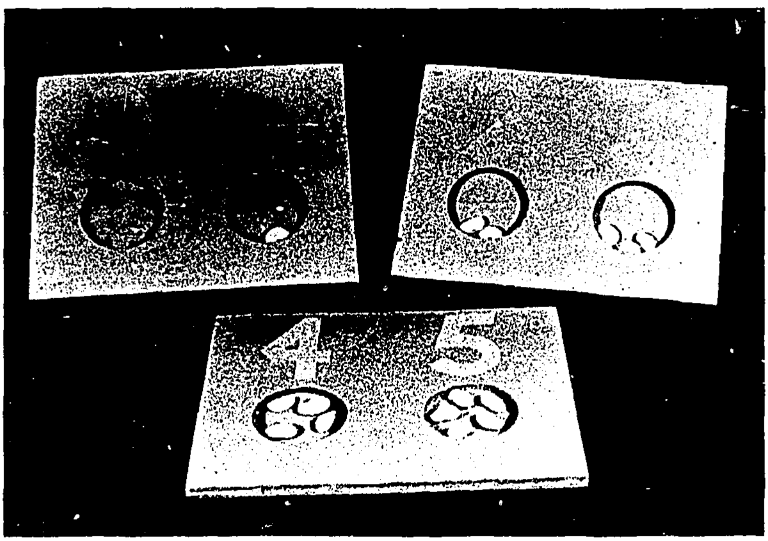


Fig. 4. Sandpaper numeral boards

a place for the pupil to place the number of beans, rocks, or other types of counters that illustrated in quantity what the numeral meant.

In using this set of materials, the pupil was instructed to work with each number individually. First he was asked to verbally identify the numeral. Next, he was asked to count out the number of beans that the numeral represented and place them in the well associated with the numeral. Then he was asked to run his finger over the numeral several times to get an idea of its shape as he repeated the number name. though this set of materials worked quite

vided with a sack of blocks, paper, and felt tip marking pens. They were instructed to place the number of blocks the numeral suggested on the board and then using the stencil portion of the board to write the numeral. After writing the numeral using the stencil, pupils were asked to write the numeral without using the stencil, using the stenciled numeral as a pattern.

The next set of materials in the sequence consisted of egg-carton counters (see fig. 6). Since the major focus was on quantity rather than on size or shape, this material served two functions. First, it served its goal as the next step in the developmental

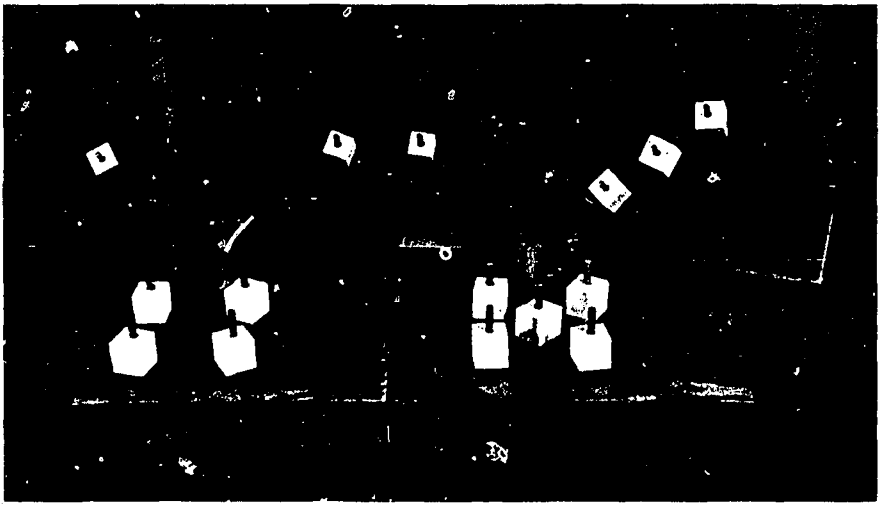


Fig. 5. Numeral stencil boards

learning sequence, and second, it served to emphasize the dominant characteristic of concern. The size of the carton did not directly relate to the number of balls the carton held (for example, the one carton and the six carton were the same size but represented quite different numbers of objects). This material was designed to take the pupils one step further in the sequence. Each pupil was provided with a can of table-tennis balls and a carton. He was instructed to fill the carton with "eggs" and then count to determine whether or not the numeral shown in the lid of the carton told the number of eggs in the carton. Pupils were asked in this case to look at a pat-

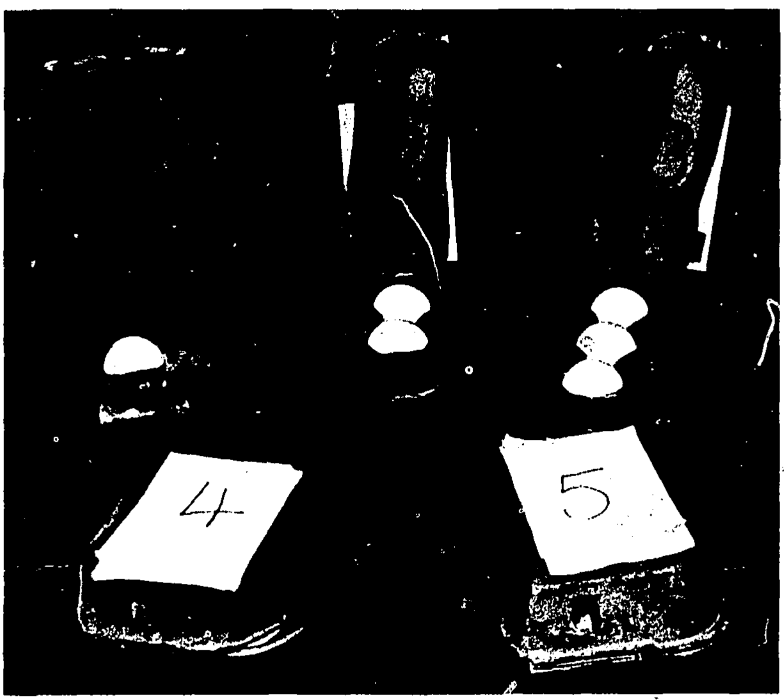


Fig. 6. Egg-carton counters

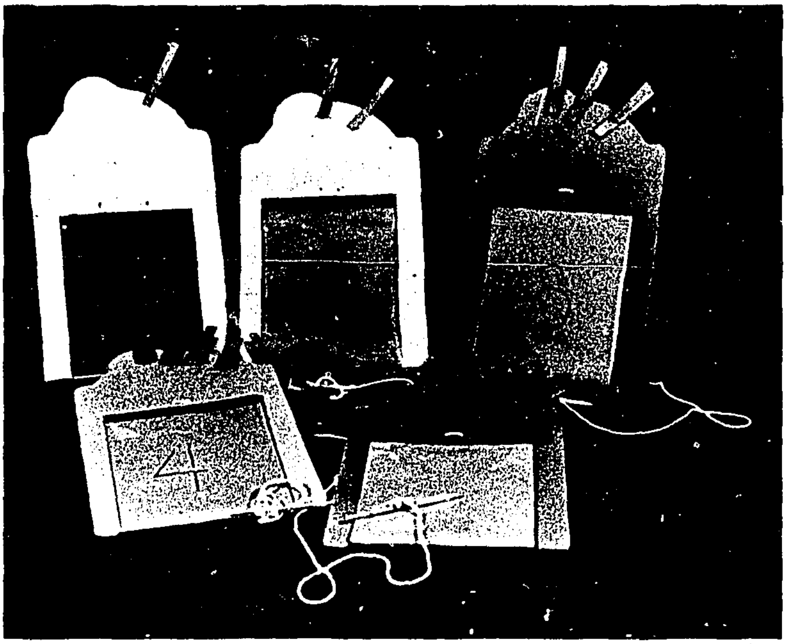


Fig. 7. Magic slate boards

tern of the numeral on the inside of the carton lid at the same time as they looked at the number of eggs (table-tennis balls) the carton would hold. Next, the pupils were instructed to close the lid of the carton and write the numeral on a tablet provided on top of the carton. After writing the numeral, they were instructed to remove from the tablet the sheet of paper on which they had written, open the carton lid and compare the numeral they wrote with the numeral pattern in the lid. This material worked quite well except for the problem of tennis-ball attrition.

The final set of materials in the sequence consisted of magic-slate tablets (see fig. 7). Each pupil was provided with a magic slate with a certain number of clothespins clipped to the top. The pupils were instructed to count to determine the number of clothespins on their slate. Then they were instructed to write on the magic slate the numeral that told the number of clothespins pinned to the slate. This material worked very well in accomplishing the desired goal. It was very flexible in its [illegible]ity for the provision of individual differences. One needed only to add or delete clothespins to provide for pupils at different levels on the numeral scale.

Summary

The purpose of this paper was to describe in detail a series of materials designed to teach numeral-quantity association to pupils at very early stages of cognitive and mathematical growth. The materials were developed to provide very small steps in the learning pattern. In this way, the learning of pupils with learning problems is most particularly facilitated. These materials can be inexpensively reproduced for classroom use and have been shown to be quite successful with at least one sample of pupils with severe learning problems.

References

Armstrong, J. R. "Teaching: An Ongoing Process of Assessing, Selecting, Developing, Generalizing, Applying, and Reassessing," *Education and Training of the Mentally Retarded* 4 (1969): 168–76.

Bruner, J. S. "The course of cognitive growth," *American Psychologist* 19 (1964): 1–15.

Yardstick number-line balance

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JOHN B. WOOD

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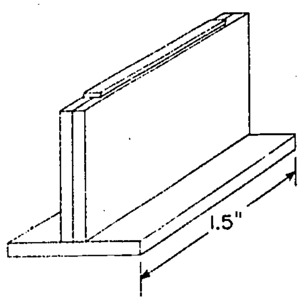
John Wood is an assistant professor in the New School of the University of North Dakota.

A balance made from a yardstick can give a visual interpretation of the balance situations expressed by equations.

Materials. One yardstick, a knife or fine-tooth saw to cut it with, glue, a piece of rubber band ¼ inch in width and about 1¼ inches long. 21 pieces of paper about ½ inch across, and from three to nine ½-inch metal washers.

Directions. Cut the yardstick at 9.5, 11, 12.5, and 14 inches. The 22-inch piece is the beam of the balance; the three 1.5-inch pieces are glued to form the fulcrum (see fig. 1); and the 9.5-inch piece is excess. Glue the small pieces of paper over the numerals (not the markings) of the beam, and relabel 25 as 0, 24 and 26 as 1, . . . , and 15 and 35 as 10. The piece of rubber band glued along the top of the fulcrum helps to keep the beam from sliding on it. If the beam does not balance right at zero, trim off the heavy end of the yardstick or stick some tape under the light end.



Fulcrum for yardstick balance

Fig. 1

Some activities. Place one or two metal washers anywhere on the balance beam: find where one more washer must be placed to make it balance; find where two more washers must be placed to make it balance. Children can use equations to record balance situations, with the equal sign representing the fulcrum. The balance is especially a good interpretation for problems like ___ + 6 = 9 and 7 + ___ = 10.

EDITOR'S NOTE. Is it possible that children might move from balancing equations as suggested by the authors to discovery of principles of the lever? As long as the beam rests freely on the fulcrum and is not fixed in position, children will experiment and find that (1) a heavier weight can be balanced by a lighter weight merely by moving one closer to the fulcrum, and (2) when the beam itself is moved, a "long" lever arm can be balanced by placing weight *only* on the "short" lever arm (the principle of the cantilever). I'm sure our authors didn't have this in mind in writing this article, but given free discovery and a movable lever arm, even six-year-olds will find the three principles of the lever. Older children can express these "discoveries" in the form of rules or formulas. Relate science and mathematics wherever necessary to the understanding of science!—CHARLOTTE W. JUNGE.

Elevator numbers

STANLEY BECKER

The principal of P.S. 191, Manhattan, a prekindergarten through third grade school in New York, Stanley Becker has taught college courses in mathematics and education and has given in-service courses for teachers.

All teachers are constantly on the lookout for games and devices that will stimulate interest in mathematics. The following description of a device called "Elevator Numbers" readily fills that need. It enables children to practice adding numbers to produce sums from 10 to 20 or more.

The idea is that children manipulate vertical strips of numbers to produce two rows of addends that must total the same stated amount when added across in either row. Each child receives a holder that allows two numbers from each column to be viewed at one time (fig. 1). He must move the strips up and down until he sees that the addends in each row total the same sum.

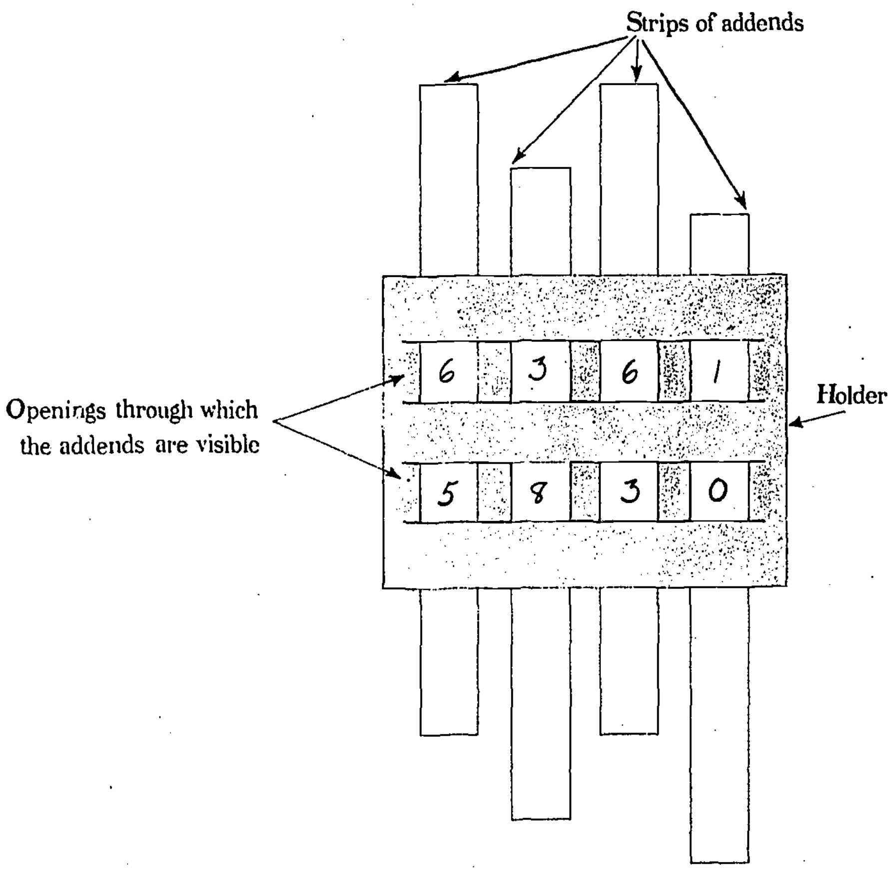


Fig. 1

Start the strips

To prepare this device, cut four or five strips, each one an inch wide. Use oaktag or some other material that is strong and flexible. Then mark each strip at one-inch intervals. Decide on the sum that you wish to appear in both rows on the card and put appropriate addends on the strips first.

In the case shown in figure 2, a sum of 16 was decided upon and the 6-3-6-1 combination (indicated by check marks) was put on the strips first. The circled numbers, 5-8-3-0, were then put on the strips by *skipping one space beneath each of the original numbers*. Skipping a space is necessary because the holder that the strips are woven through exposes only two numerals on each strip and these numerals are one space apart.

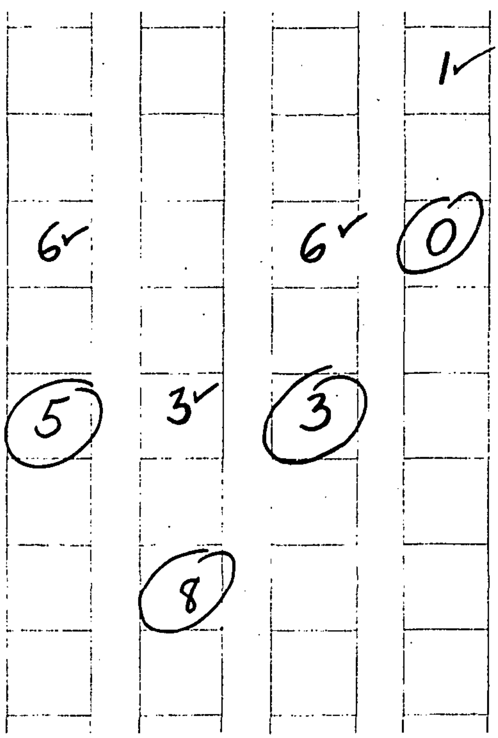


Fig. 2

Finish the strips

Now complete the strips by placing a number in each space on every strip (fig. 3). Any numbers are fine. Your card now has at least one correct solution and may have more. The order in which the children place the strips in the card is of no importance. Also, the strips can be of any convenient length. Longer strips with more numerals make the game more challenging.

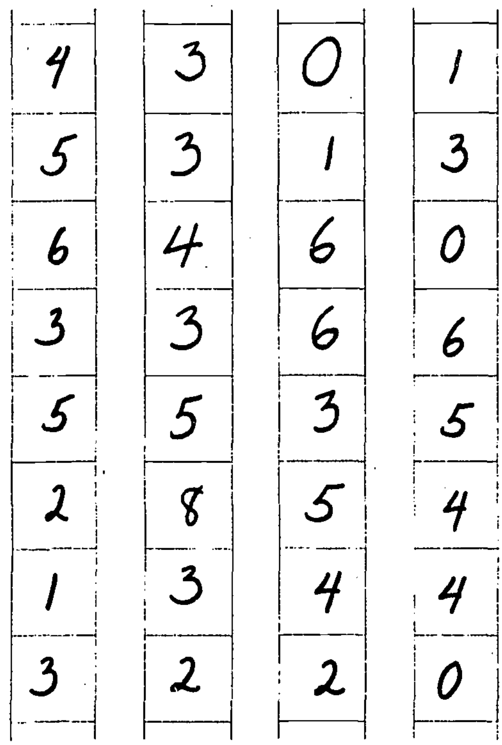


Fig. 3

Make the holder

Oaktag may be used for the holder as well. It should have four slits, one inch apart, that permit the strips to be woven through the holder in such a way that two of the numbers on each strip will be visible at one time (fig. 4). For ease of construction, place a piece of file folder or oaktag underneath this page along with a piece of carbon paper and trace the form. Then cut along the dotted lines. Using the completed holder as a model, make as many holders as desired.

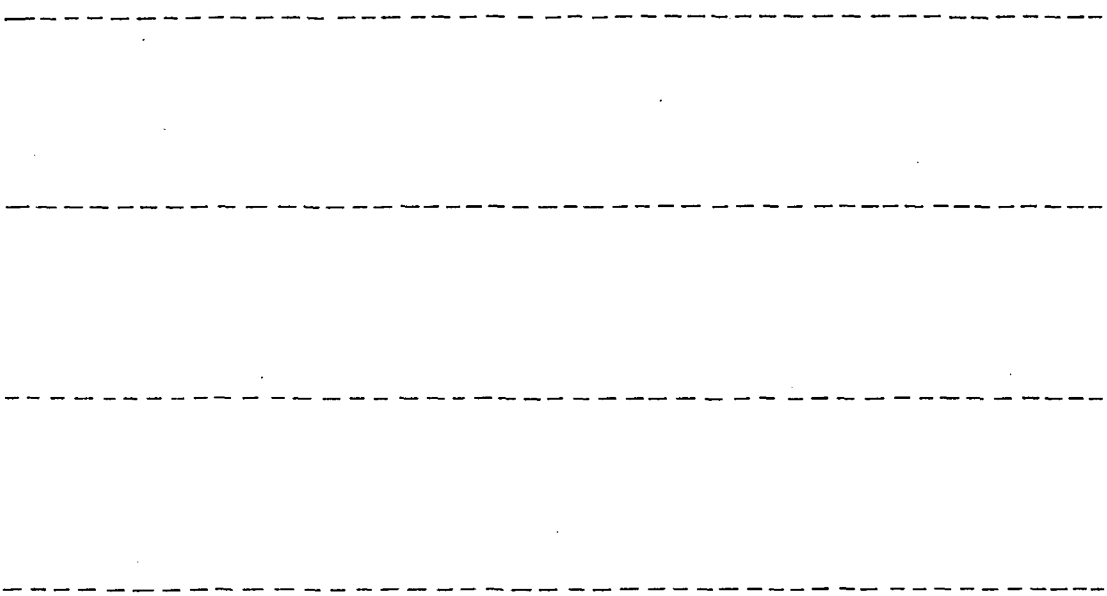


Fig. 4

Children will spend a great deal of time playing with these machines. Two second graders spent two hours finding the rows that totaled 16 and checking the strips to see if they could find any other ways to manipulate the strips so that the same sum would appear in both rows of exposed addends. They seemed to enjoy it considerably more than an equivalent amount of drill with the addition facts.

EDITORIAL COMMENT.—The students may enjoy challenging their colleagues with "elevator numbers" devices that they have made themselves. The number of strips is a variable and may be increased or decreased to satisfy the desired level of difficulty. The universe of numbers may be changed from whole numbers to integers or rational numbers in fractional or decimal form for students who enjoy a greater challenge.

Presenting multiplication of counting numbers on an array matrix

MERRY SCHRAGE

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Multiplication tends to imply an avenue of excitement and adventure for children. During my ten years as a grade three teacher, I have yet to find a year when some eager child hasn't asked the question, "When are we going to learn multiplication?" Prior to third grade the children are exposed to readiness activities in preparation for multiplication. However, the real drive to master facts and algorisms is in the third-grade mathematics sequence in Dade County, Florida.

A visual technique was developed to be used primarily during the introductory stages of multiplication. With the use of a matrix and individual cards showing the various arrays, it is possible to present and strengthen a number of basic understandings peculiar to multiplication.

The basic construction of the matrix can vary in size. I use a large one (36″ × 36″) because it is utilized and displayed for a period of time. Figure 1 shows the blank matrix chart in preparation for the multiplication facts through seven times seven.[1] A set of separate cards, which fit in the blanks of the matrix, can be easily made by using a magic marker, gummed circles, stars, or any other convenient item. The cards can be attached to the matrix with tape or other types of affixing material. Making the arrays on separate cards enables development of the matrix chart in a sequential manner according to readiness, properties, and facts. Different colors may be used for the arrays, making the commutative facts the same in color. Figure 2 shows how the array cards look.

X	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

FIG. 1.—Blank matrix

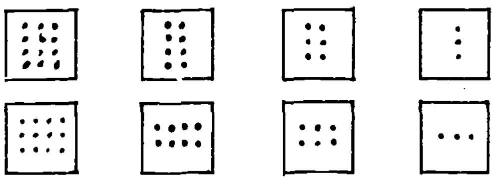

FIG. 2.—Sample array cards

[1] The matrix has not included the zero facts since the array would show only blank cards.

The primary concept that needs to be developed prior to using the matrix is the meaning and use of the array. Visual activities should be provided that enable the children to experience the manipulation and development of arrays. After activities of this type have been accomplished, the matrix chart can be introduced as a means of recording our findings in an orderly fashion.

During the initial introduction of the matrix, the children will readily recall using a matrix for the purpose of recording addition and subtraction facts. It is usually enough to indicate that we can use the matrix to find out some important ideas that the arrays show us in multiplication. Point out that the numerals across the top and along the left side of the matrix relate directly to the definition of an array. Pass out a few selected array cards to the class. Ask someone to develop a specific array on the chalkboard. "Does anyone have an array which shows the same array John has made for us on the board?" There will be two children with array cards that match the one made on the chalkboard unless the teacher has selected an array showing the product of two identical factors. Also, if some type of color scheme has been utilized in making the array cards, the two cards for the specific array will be the same color.

The next question that needs to be discussed is, "Where should these array cards be placed on the matrix?" Lead the children to a consideration of the definition of an array and how it relates to the numerals on the matrix. The children should also be encouraged to see that a quarter turn of one of the array cards produces the commutative fact.

The arrays to be developed should be approached in the manner just briefly outlined. Of course, there are many more questions that can be asked to stimulate discovery on the part of the children. The ability of the class will also be indicative of the duration that this activity will take. As the pattern of the matrix emerges through the development of the facts in terms of arrays, the matrix will lend itself to a visual understanding of the identity element of one, the commutative property and the multiplicative property of zero. Figure 3 shows an array-matrix that has been completed through the facts of five.

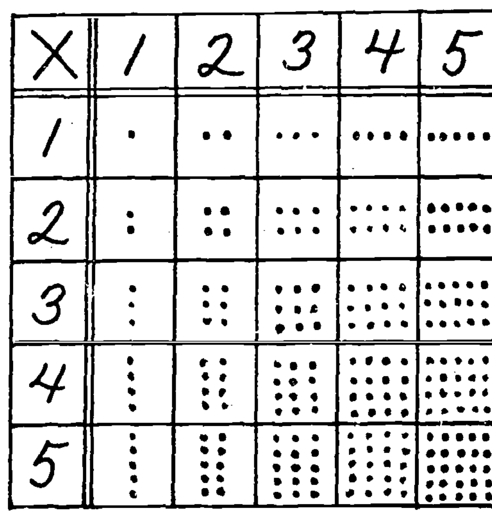

FIG. 3.—Completed array-matrix through facts of five

Through the use of the array-matrix, the children readily develop an understanding of the process of multiplication and its properties. This technique is successful with the less capable child because it provides an opportunity for manipulation and visualization of the concepts. The more capable child is greatly stimulated toward independent discovery of additional patterns and concepts.

EDITORIAL COMMENT.—Students may make their own array cards by using index cards and a hole punch. A Concentration type of game activity may be organized to provide additional experience with the multiplication fact-array relationship. One board holds the multiplication fact cards in random positions, and another board holds the array cards positioned randomly on the board. Neither the facts nor the arrays are visible until a student npts to make a match.

Multiplication mastery via the tape recorder

DALE M. SHAFER

Dale Shafer is professor of mathematics at Indiana University of Pennsylvania. This article results from his previous experience as coordinator of a K–12 mathematics program.

Most instructors who have had the responsibility of teaching children the multiplication facts will readily admit that at several intervals preceding student mastery a certain amount of drill is necessary. The word *drill* should suggest nineteenth-century pedagogy only to those individuals who do not know how this teaching technique is being imaginatively employed in elementary classrooms today.

Teachers generally view drilling the multiplication facts as time-consuming, boring, and frustrating. This is certainly the case if the teacher stands in front of the room with flash cards and methodically makes the rounds of the class. The student knows that in a class of twenty-five pupils he will be actively involved as a participant only $\frac{1}{25}$ of the time. On the one hand, if he has mastered the multiplication facts, time drags. On the other hand, if he is a slow student, he sits in class assuring himself that when his turn to participate does arrive he will surely be asked to recite one of the multiplication facts about which he is still not confident.

In this article, let us see how a tape recorder can be used to stimulate, both in students and in teachers, a more positive attitude toward mastery of the multiplication facts. The approach here suggested involves use of a reel-to-reel or cassette tape recorder, earphones, a student response sheet, and a teacher-made prerecorded tape.

Initially, the author wrote on slips of paper the 144 multiplication facts that would appear in a 12-row, 12-column multiplication table. Then one slip at a time was selected from this set so that the entries on the audio tape would be made in a random order.

The tape begins with a motivating statement encouraging the student to win the coveted "Master of Multiplication Award." This award is a large orange X placed beside his name on a chart when he has mastered the multiplication facts. The chart also provides space to record a student's progress through an entire series of prerecorded arithmetic tapes.

The tape then gives instructions about how the student is to record his answers on the student response sheet. The student response sheet for the multiplication facts contains 6 columns with 24 consecutively numbered answer spaces in each column. The problems now begin—"Number 1, five times eight [PAUSE]; number 2, nine times six [PAUSE]; . . ."—and continue through number 144.

By this time you, the reader, are probably thinking of the Herculean task involved in evaluating the response sheets. This task is greatly simplified by using a homemade correction mask constructed from half of a manila folder. Twelve different masks are used to correct a multiplication-facts response sheet.

To construct the "sevens mask" indicated in figure 1, a piece of carbon paper and a student response sheet were placed on the manila-folder material. Then a rectangular box was drawn around each area on the response sheet where a "sevens fact" was to be recorded. The carbon-paper impression was then cut out of the manila-folder material with a penknife. Above each of the rectangular holes in the mask was written the problem that corresponded to this position on the student response sheet. Finally, a straight line was used to join those boxes whose multiplication facts were related by the commutative law of multiplication.

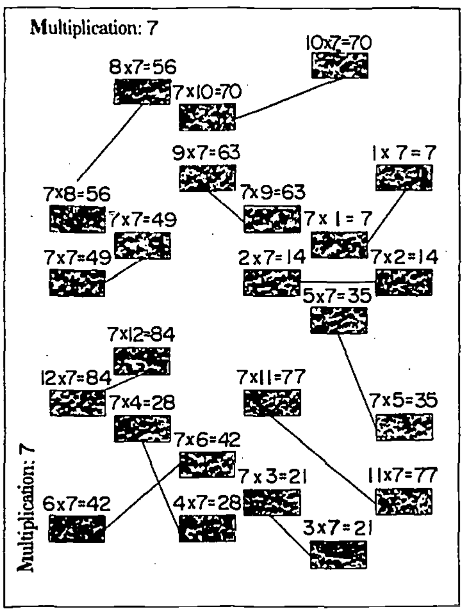

FIGURE 1

When the correction mask is laid on a student response sheet, the teacher can tell at a glance if the student has mastered or has retained mastery of the sevens table. If he has missed 7×9, it is quite easy to check whether he has also missed 9×7. The facts that need to be reinforced or retaught to a particular student are apparent through the boxes in the correction mask.

One can notice that the correction mask is labeled along the horizontal and vertical edges. This labeling provides for easy identification when the mask is being used for correction purposes and for easy location when it is stored in a file folder.

When recording a tape, a stopwatch is used to time the length of a pause between each problem. By keeping these time intervals constant throughout the tape, a rhythm of presentation is established for the student.

When the teacher experiments with this idea, he will probably want to have several tapes available on which the length of the pause interval between problems is varied to account for differing ability levels. He will also want to have available some tapes containing only a subset of the 144 multiplication facts. Whatever the case may be, he need record only one master tape. By connecting two tape recorders in series and manipulating the pause lever on either machine he can extract from the master tape whatever data he desires to record on the second tape and vary the pause interval between problems to suit his particular needs.

Therefore, by recording on tape the multiplication tables, a teacher can ascertain, whenever he desires to do so, whether a student has mastered these facts and/or whether he has continued to maintain these basic skills. When a student or a group of students place the earphones on their heads and work on the same or different tapes, the teacher can be working with the remainder of the class on some entirely different material. This approach is certainly conducive to total student participation throughout the entire teaching period.

The multiplication facts constitute only one of many areas where your instruction can be imaginatively aided by using a tape recorder with one person, a group, or the entire class. A teacher will find as he builds his library of teacher-made tapes that his approach to teaching becomes more flexible and that he has moved one step closer to individualization of instruction.

Problem solving with number-picture problem situations

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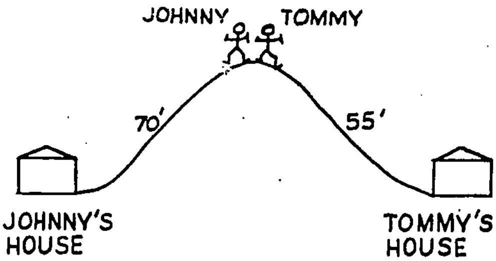
The class was inspired by the weather to develop its first picture problem situation. The teacher sketched at the chalkboard in response to children's suggestions and guided them so that basic grade-level number concepts were included. For example, the first cooperative class sketch featured a snowy hill and boys and girls with sleds. All data are not pictured; some are provided as factual information. The sketch (Fig. 1) and some of the resulting number problems were similiar to the following.

Johnny and Tommy go sledding

Johnny lived at the foot of the hill on one side. Tommy lived at the foot of the hill on the other side. They met at the top of the hill. Tommy invited Johnny to his house for hot chocolate. Johnny had to go home and ask his mother if he could go. How many feet was Johnny's trip home and back to the top of the hill? ($2 \times 70 = 140$)



Data not pictured:
18 *boys*
14 *girls*
15 *sleds*

Figure 1

How far did the 2 boys walk together when they went to Tommy's house for the hot chocolate? ($2 \times 55 = 110$)

How many feet did Johnny walk when he went home from Tommy's house? ($55 + 70 = 125$)

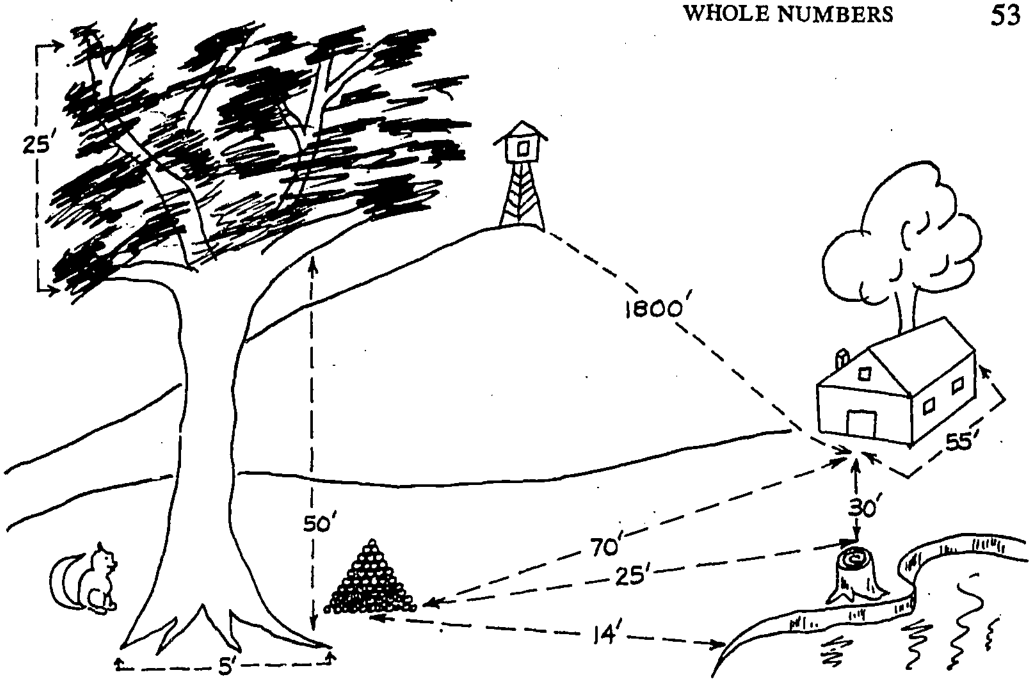
How many feet longer was the hill on Johnny's side than on Tommy's side? ($70 - 55 = 15$ longer)

It took the boys 5 minutes to pull their sleds from Tommy's house to the top of the hill and 1 minute to ride down. In 10 trips how many feet did they walk? ($10 \times 55 = 550$ for each boy or 1100 for 2 boys)

How many minutes did they spend in pulling the sled? ($10 \times 5 = 55$)

How many minutes did they ride? ($10 \times 1 = 10$)

If they had to walk 10 minutes to ride 1 minute, how far would they have to walk to ride 2 minutes? (20 minutes) 3 minutes? (30 minutes) 4 minutes? (40 minutes) 6 minutes? (60 minutes or an hour)



Data not pictured:
3 squirrels in the tree
6 fish in the lake
100 nuts in the nut pile
2 adults and 3 children in the house
5 baby birds and 1 mother bird in a nest

Figure 2

As they stood on the top of the hill, Johnny counted the children on one side and Tommy counted the children on the other side. Johnny counted 16 and Tommy counted 14. How many children did they count in all including themselves? $(14+16+2=32)$

If 18 of the total number of children on the hill were boys, how many were girls? $(32-18=14)$

If every child had on a pair of mittens, how many *pairs* of mittens were there? (32 children, 32 pairs of mittens; one-to-one correspondence between children and pairs of mittens.) How many mittens? $(32\times 2=64)$

Nine of the 15 sleds belonged to boys and the rest to girls. How many belonged to girls? $(15-9=6)$

Three children rode on a sled together and they took turns pulling the sled back up the hill. After 18 rides down the hill, how many times had each child pulled the sled back up the hill? $(18\div 3=6)$

The squirrel family

The picture problem situation about the squirrel family (Fig. 2) was first developed on the chalkboard in a similar manner with children offering suggestions on what to include, approximating distances from one place to another, and giving information not pictured to be used in problem solving. The children took turns making up and solving problems from the picture and from the information given. After the class experience the teacher duplicated the picture with some typical and appropriate problems in order to provide each child with opportunities for problem solving and to evaluate the effectiveness of the class learning experience.

Materials were distributed and the data at the top of the page was reviewed. The teacher explained that only measurements pictured by broken lines could be used in problem solving. As they completed their work the children were instructed:

1 To make up other problems about the picture.

2 To plan a picture of their own, compose problems about it, solve problems on a separate sheet of paper, and keep the solutions.
3 Exchange pictures and problems with a neighbor, solve problems, and check answers with the owner's key.

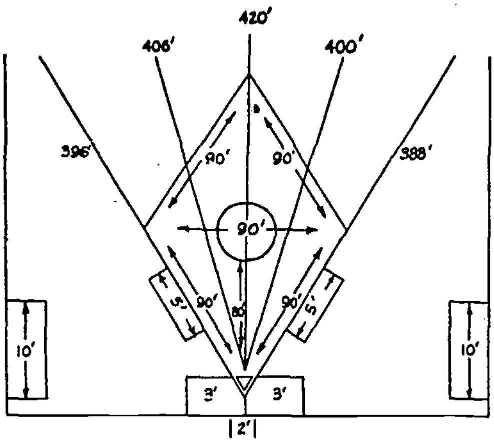
It will be noted that the level of difficulty of the problems which follow were varied to meet the needs of children with different abilities.

1 If the squirrel and a friend were going to store $\frac{1}{2}$ the nuts, how many nuts would be left in the pile? ($\frac{1}{2}$ of 100 is 50)
2 All the squirrels went to the lake to look at the fish. All the fish came up to look at the squirrels. How many squirrel eyes were looking at the fish? ($2 \times 3 = 6$) How many fish eyes were looking at the squirrels? ($2 \times 6 = 12$) How many eyes were there altogether? ($6 + 12 = 18$)
3 The whole family walked to the nut pile and back home. How many feet did the 2 adults travel? ($2 \times 70 = 140$ or 1 round trip; $2 \times 140 = 280$ or 2 round trips.) How many feet did all the children travel? ($3 \times 140 = 420$) How many feet did all 5 people travel? ($280 + 420 = 700$)
4 Twenty of the nuts are walnuts. The rest are pecans. How many pecans are in the pile? ($100 - 20 = 80$)
5 The squirrel on the ground climbed to the lowest branch of the tree and called all his friends from the highest branch to come down and help him. When all the squirrels reached the nut pile, what was the total number of feet *all* had traveled? ($25 + 50 = 75$—distance from highest branch to the ground; $50 \times 2 = 100$—distance traveled by squirrel on the ground; $75 \times 3 = 225$—distance traveled by three squirrels; $225 + 100 = 325$—distance traveled by all the squirrels.) How many yards did all the squirrels travel? ($325 \div 3 = 108\frac{1}{3}$ or 108 yards and 1 foot)
6 A squirrel went from the nut pile to the tree stump and ran $\frac{1}{5}$ of the way. How many feet did he run? ($\frac{1}{5}$ of $25 = 5$) If he jumped from the nut pile to the stump with 5′ jumps, how many jumps did he take? ($25 \div 5 = 5$)
7 Father walked from the house to the fire tower and halfway back before sitting down to rest. How many feet had he walked by then? How many yards? ($1800 + 900 = 2700$; $2700 \div 3 = 900$)
8 The squirrels were carrying the nuts to the tree behind the house. Squirrel 1 took the shortest route. Squirrel 2 took the long way. How many feet farther did squirrel 2 travel? ($70 + 30 = 100$—distance traveled by squirrel 1; $25 + 30 = 55 = 110$—distance traveled by squirrel 2; $110 - 100 = 10$; squirrel 2 traveled 10 feet farther.)
9 The little squirrel stored $\frac{1}{5}$ of the nuts. How many nuts did he store? ($\frac{1}{5}$ of $100 = 20$) How many nuts were left in the pile? ($100 - 20 = 80$ nuts)
10 The mother bird found 27 worms. She divided them evenly with the rest of the family, without cutting any. She ate the leftover worms. How many worms did she eat? ($27 \div 5 = 5$; 5 worms were eaten with 2 left over; mother bird ate 2 worms.)

The baseball game

The picture (Fig. 3) and series of problems about baseball by Daniel Chao illustrates the individual work done by members of the class.

1 One fourth of the 980 people at the ballgame were in reserved seats; the rest were in the bleachers. How many were in reserved seats? ($\frac{1}{4}$ of 980 is 245) How many were in the bleachers? ($\frac{3}{4} \times 980 = 3 \times 245 = 735$)
2 If a pitcher throws 8 balls from the pitcher's mound to the plate, how many feet in all is this distance? ($80 \times 8 = 640'$)
3 If 15 balls were used, how many were not used? ($32 - 15 = 17$)



Data not pictured:
17 players in the dugout
9 players in the field
32 balls, 15 bats for each team
980 people
15 vendors
24 hot dogs
50 cokes

Figure 3

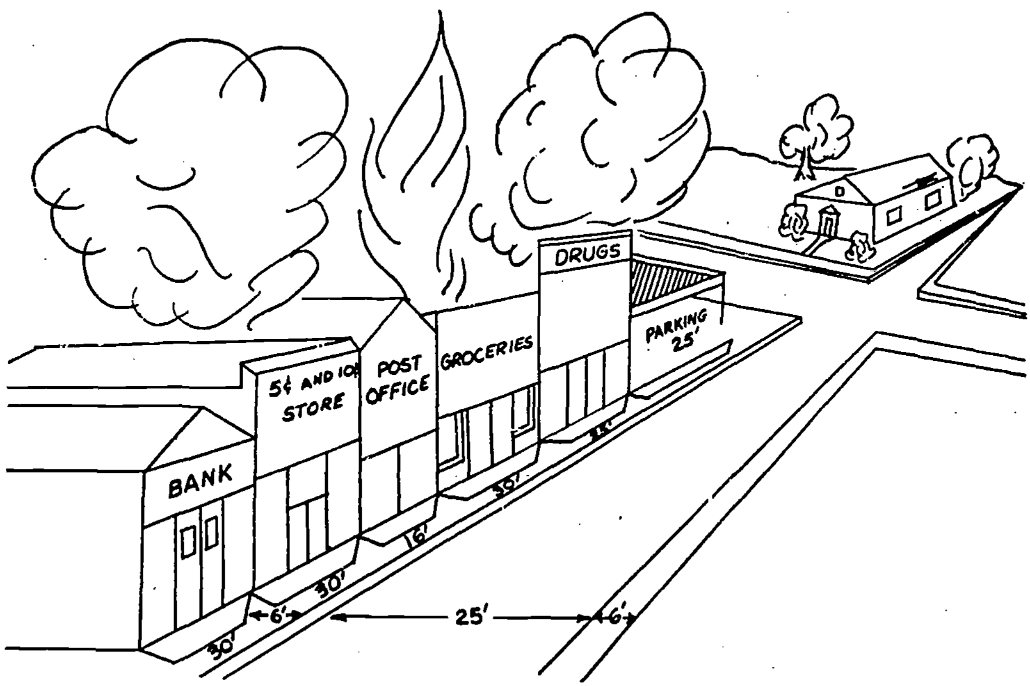
4 If there were 4 double plays, how many feet would these four players run? (180 = 1 [double play], 180 × 4 = 720 or 4 double plays.)

5 If there were 5 homers, how many feet would be covered in all? (90 × 4 × 360 or 1 homer; 360 × 5 = 1800 or 5 homers.)

The shopping center

The picture problem situations increased in difficulty with experience until the children could handle much more complicated problems and computations, such as are required for the shopping center picture (Fig. 4) and problems.

1 Mother went from the house to the grocery store and had to make a second trip to carry all the groceries. How many feet did she walk in the 2 trips? How many yards?



Data not pictured:
24 buildings in the shopping center
2 adults, 5 children, 1 dog, and 2 cats in the house
Note: The house is 5' from the sidewalk. All streets and all sidewalks are the same width. All doors are exactly in the middle of each building.

Figure 4

5	(distance from house to sidewalk)
6	(width of sidewalk in front of house)
25	(width of street)
6	(width of sidewalk beside stores)
25	(across front of parking lot)
25	(across front of drugstore)
15	(to door of grocery store)
107	
×4	
428	(4 trips or 2 round trips) (428÷3=142⅔ or 142 yards and 2 feet)

2 Susie left the house to go to the post office. She was daydreaming and reached the doors of the bank before she realized it. How many feet too far had she gone? (15+30+8=53)

3 A man painted *doubled* crosswalk lines from sidewalk to sidewalk on all sides of the intersection. How many feet did he paint? (4×25=100; 100×2=200, or 8×25=200, or 50×4=200)

4 If the dime-store owner owns ⅓ of the buildings in the shopping center how many did he own? (⅓ of 24 is 8)

5 The parking lot holds 50 cars and the charge is 25 cents an hour. If the lot is half full for 1 hour, how much will the owner collect? (25×25=625 or $6.25 for 25 cars)

6 John is going to hop from curb to curb on all four sides of the intersection. If he hops 2′ at a time, how many hops will he take? (25×4=100; 100÷2=50 or 50 hops.)

7 Jackie put on his skates at his own curb. He skated from there to the end of the bank and back to his curb. How many feet did he skate? How many yards?

25	(across street)
6	(width of sidewalk beside stores)
50	(across parking lot and drugstore)
46	(across grocery store and post office)
60	(across dime store and bank)
187	(1 trip)
×2	
374	(round trip) (374÷3×124⅔ or 124 yards and 2 feet)

Planning for a picture often involved research to find the correct statistical data, as in the case of Daniel's baseball picture. Other members of the class drew upon their imaginations to draw park scenes with fountains, park benches, painters at work, fish in the lake, flower beds, etc. Vigorous discussions with regard to the reasonableness of distances were frequent as perspective was considered. The teacher felt that the students enjoyed and profited from their experiences with class and individual number-problem pictures.

EDITORIAL COMMENT.—By involving students in the design of number-picture problem situations, the teacher may also capitalize on the interrelationship of many areas of the curriculum. Problems may be generated from the sciences, social studies, music, art, physical education, and so on. Doing the research required to validate the problem data is a most worthwhile experience.

3

Numeration

The articles in this chapter are organized in two major categories—decimal and nondecimal numeration. The significance of the base and place-value concepts is easily perceived. An understanding of our standard operational algorithms for whole numbers is heavily dependent on the student's knowledge of these concepts.

Rinker's ideas for an "eight-ring circus" should be well received if one is looking for a way to provide a wide range of activities that focus on the study of counting and place value in the primary grades. She provides a comprehensive set of directions for constructing the manipulative materials used in each of the eight rings.

Other varied ideas for working with place value are provided in the next three articles. The expanding place-value cards suggested by Ziesche should be most helpful in teaching the "numbers have many names" concept related to place value. The Placo game created by Calvo will generate many kinesthetic place-value experiences for the children. The flexibility of the counting abacus described by Cunningham allows the number of beads on each wire to be changed, thus providing multiple uses.

The role of place value in understanding the multiplication algorithm is emphasized by Haines in her illustrations with Napier's rods. She also provides directions for making and using the rods.

Several ideas for working with nondecimal numeration are given in the last three articles of this chapter. Rabinowitz provides some directions to assist selected students in constructing various types of "computers" as an enrichment activity. Teachers are encouraged by LaGanke to bring their checkers and checkerboards to school as an aid in teaching various number bases to children in the intermediate grades. Several interesting examples are provided for the purpose of illustrating her ideas. Finally, the slide rule is brought into focus by Martin in his suggestion to have junior high school mathematics students make their own slide rules for use in adding and subtracting numbers represented in various bases. Martin also provides suggestions for several other experiences that he feels will aid students in understanding the operation and construction of various types of slide rules.

Eight-ring circus: A variation in the teaching of counting and place value

ETHEL RINKER

Now teaching in the primary grades in the Balsz School District, Phoenix, Arizona, Ethel Rinker has a background of experience and professional education in New Jersey and Arizona.

This article describes a technique used by one third-grade teacher to provide enjoyable learning experiences in the study of counting and place value. The technique may be used in other grades with appropriate adjustments for the grade level and the needs of the individual classes.

"Eight-Ring Circus" begins at the very beginning of the study of counting and place value, thereby giving each child detailed instruction and experience with all materials and aids.

The various materials and aids are placed in the room in suitable sections that we call "circus rings." At least one child performs in each ring. My third graders and I have found that eight rings are desirable for us. When we feel that the time has arrived that it may be too expensive or space consuming to continue performance in a ring, as may be the case in Ring One with the use of the tongue depressors, we merely place a sign at the ring or make a mental note that this particular ring is not performing at the present time. However, the class may describe and discuss the results possible if the ring were continuing performance with the others.

The following describes the first step of performance in each ring:

1. All eyes are on the child in Ring One. He raises the first stick. When this signal is given and the stick is placed in a slot of the ones box, then all other circus rings go into action.

2. A child in Ring Two places one cardboard square in a slot of the ones box.

3. Each child in Ring Three records the numeral one in the ones place on his prepared paper.

4. A child in Ring Four brings over one bead on the wire in the ones place on the commercial place-value board. Another

Entire Scene of Eight-Ring Circus

child brings over one ring on the wire in the ones place on the handmade, pipe-cleaner place-value aid. A third child places one ring on the clip in the ones place on the handmade, paper-clip place-value aid.

5. Each child in Ring Five pushes the lever once on his small hand counter or odometer (which has been set at zero) to display the numeral one in the ones place.

6. A child in Ring Six exposes the numeral one in the ones place on the large, handmade–paper-counter odometer aid. Another child exposes the numeral one in the ones place on the small, handmade-counter place-value aid.

7. A child in Ring Seven records the numeral one in the ones place on the chalkboard, where place-value columns have been prepared.

8. A child in Ring Eight places one penny in the ones compartment of the money place-value aid.

The foregoing procedure is continued by adding one more each time and placing or recording it in the proper place. When comprehension is good and a speedup of the process is appropriate, 10, 100, or more may be added each time instead of one.

The children and I make discoveries as we go along. The relationship between a number and a numeral and the advantages of the place-value system are studied and experienced in detail.

We continue with this circus-ring technique during a large part of the school year and frequently use the aids in correlation with the work in the mathematics textbook. The place-value columns are kept on the chalkboard for a long time. The lines for the columns are drawn on the chalkboard with a special white wax pencil, so these usually do not interfere with other chalkboard use. They can be erased over many times before they wear away. The labels for the names of the periods and place values are on a strip of paper that is easily removed and returned as needed. We refer to and make use of these columns in our daily study of mathematics. Addition and subtraction and other problems are solved in the proper place-value columns. A large supply of the children's dittoed papers, like those used in Ring Three, is kept on hand for the same purpose.

The following are scenes in our classroom that show most rings at the count of 68,973. Ring One is at 1,973, with the child available for discussions.

The following materials and aids are used for the circus rings:

1. Ring One

Approximately 2,000 sticks (tongue depressors)

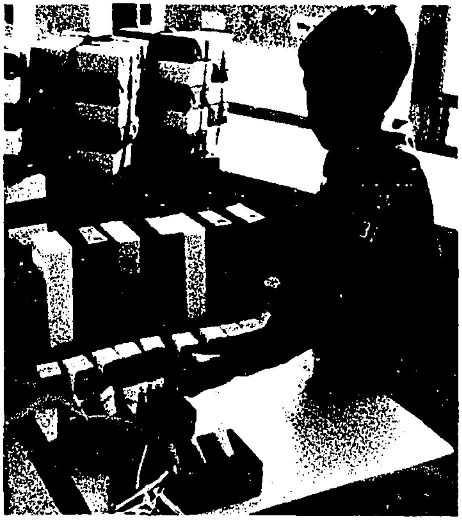
A box with nine slots for supporting

single sticks

A box with nine holes for supporting bundles of sticks, ten in each bundle

Rubber bands for bundling the sticks

2. Ring Two



Approximately 2,000 cardboard squares (the weight of tagboard), 1⅞-by-1⅞ inches

A ones box with nine slots for supporting single cardboard squares

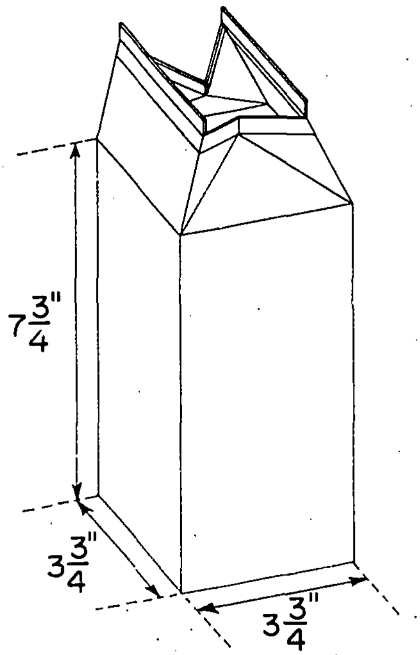


Fig. 1

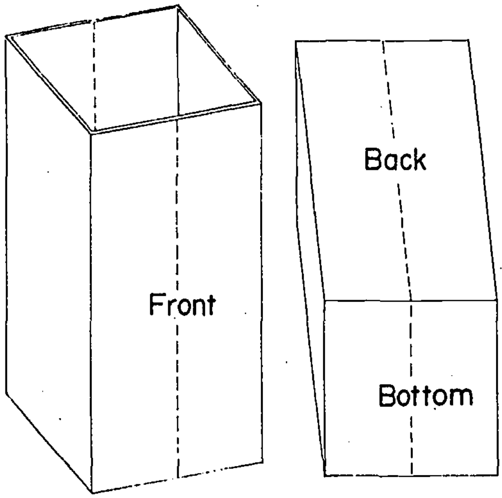


Fig. 2

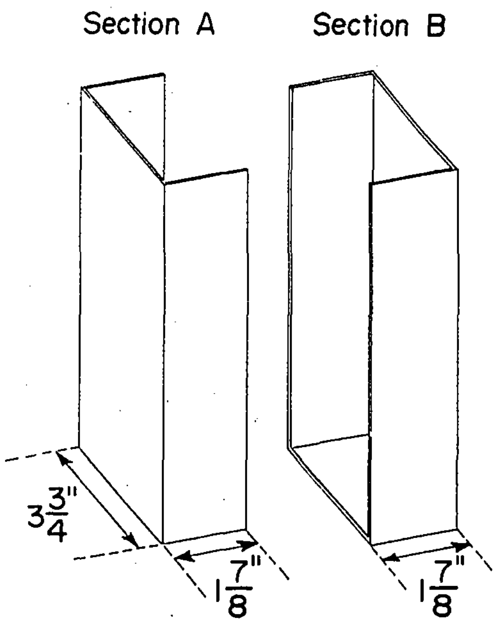


Fig. 3

A tens box with nine slots suitable for supporting packs of cardboard squares, ten squares in each pack

Rubber bands for packing the cardboard squares

Half-gallon milk cartons to be transformed into oblong boxes, approximately 7¾-by-3¾-by-1⅞ inches

When ten of the tens packs are accumulated, they are packed together and secured

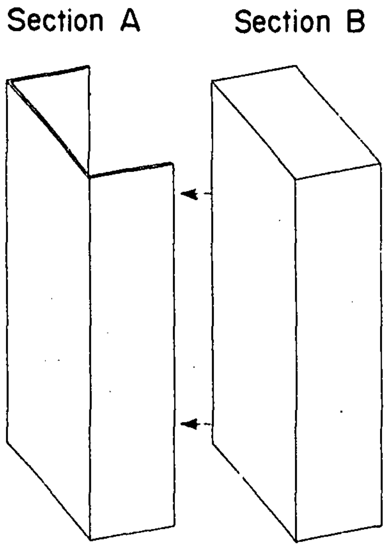


Fig. 4

with a rubber band and placed in a third place, the hundreds place. The hundreds packs stand on the table without additional support.

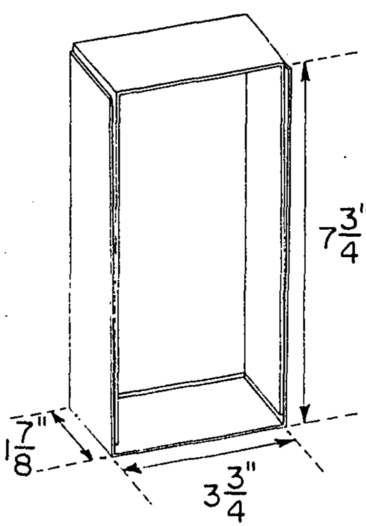


Fig. 5

When ten of the hundreds packs are accumulated, they are arranged in two rows in the oblong box, as shown in figure 6.

The oblong box containing the 1,000 cardboard squares is placed in the thousands place on the table. The first oblong box is left open so that the contents can be seen at all times. Boxes representing 1,000 can be closed by fitting the two sections *A* and *B* of the divided milk cartons [illegible] each other so that all sides are closed. The boxes may or may not be filled. The closed boxes are tied and covered, if desired, with foil or paper to look more attractive. At this stage the child may be capable of visualizing 1,000 cardboard squares inside each closed box, but he may refer at the same time to the box that is open, which contains the 1,000 cardboard squares.

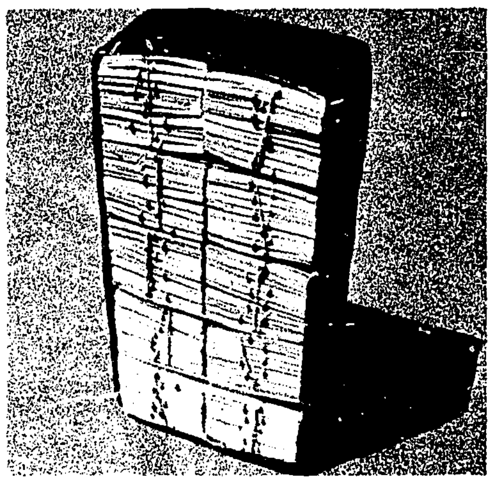


Fig. 6. Oblong box containing 1,000 cardboard squares

When the ten-thousands place is needed, ten of the oblong thousands boxes are tied together to represent one 10,000, as shown in figure 7. We can go on like this, as far as we wish to go.

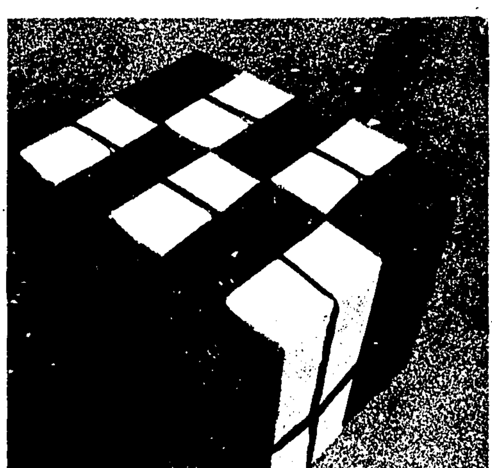


Fig. 7

3. Ring Three

   Dittoed papers showing ruled columns for place values and names of the periods, similar to what is shown on the chalkboard for ring seven

4. Ring Four

   Commercial place-value board to billions

   A handmade, pipe-cleaner place-value aid made with various colored pipe cleaners bent in a U shape, with nine rings on each pipe cleaner and each pipe cleaner glued to a base as shown in figure 8

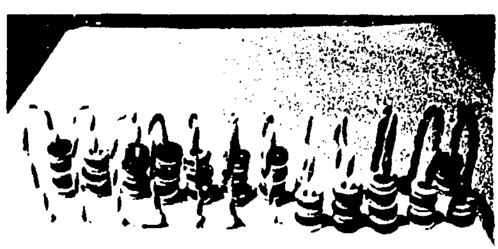


Fig. 8.

A handmade, paper-clip place-value aid made with giant paper clips, each pulled apart to form a horizontal base and a vertical pole, so that the desired number of bent paper clips can be glued in a row on a strip of cardboard as shown in figure 9

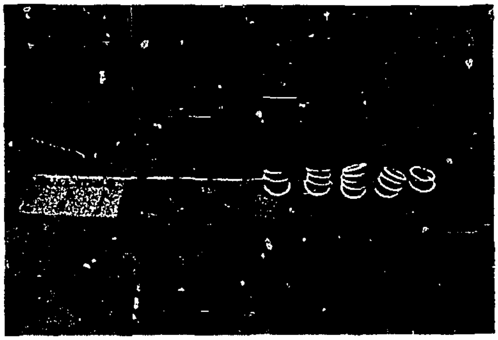


Fig. 9.

Rings—from curtains or tin cans or made from pipe cleaners, nine for each place—to be used with the above paper-clip place-value aid

5. Ring Five
Small, commercial hand counters and

Fig. 10

discarded odometers as shown in figure 10

6. Ring Six
Large, handmade, paper counter-odometer aid to hang from a chart rack as shown in figures 11 and 12

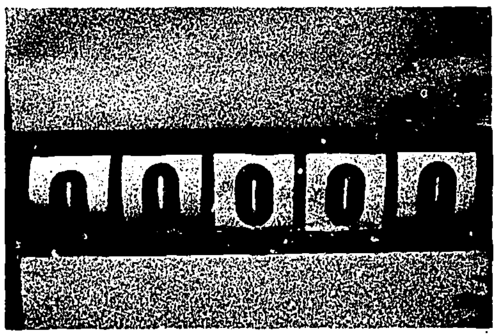


Fig. 11

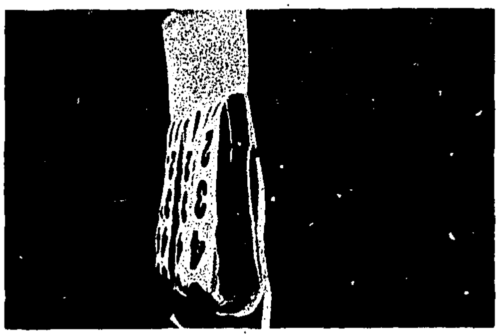
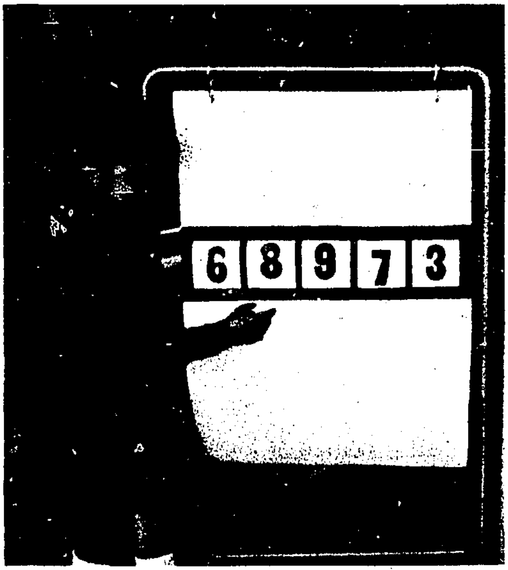


Fig. 12



Small, handmade counter–place-value aid as shown in figure 13

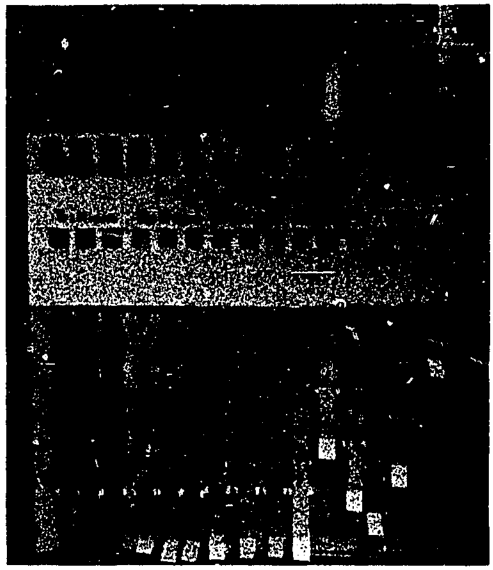
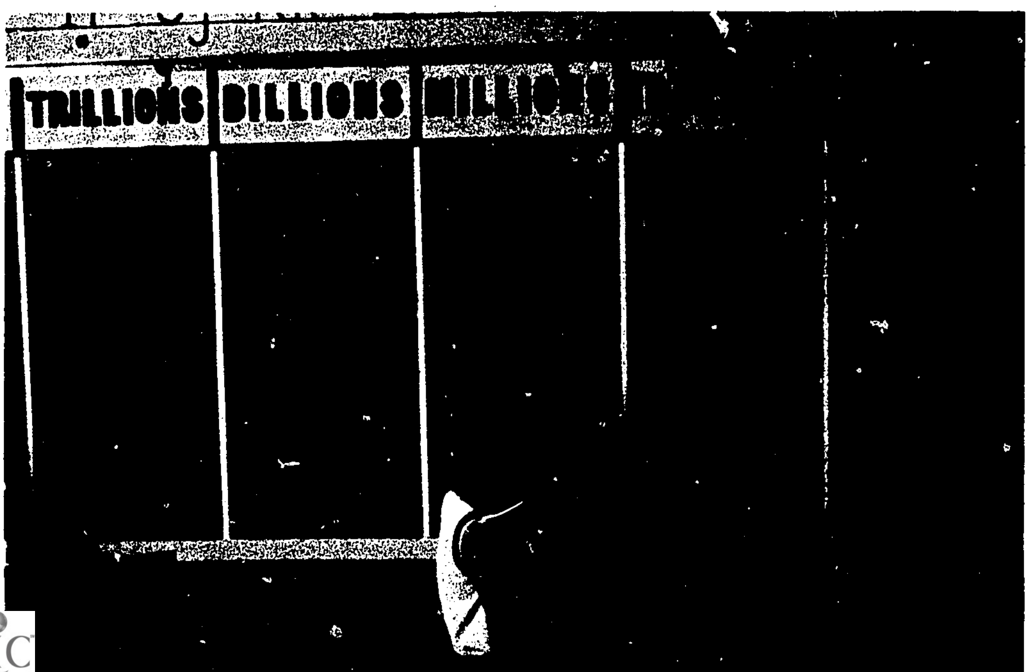


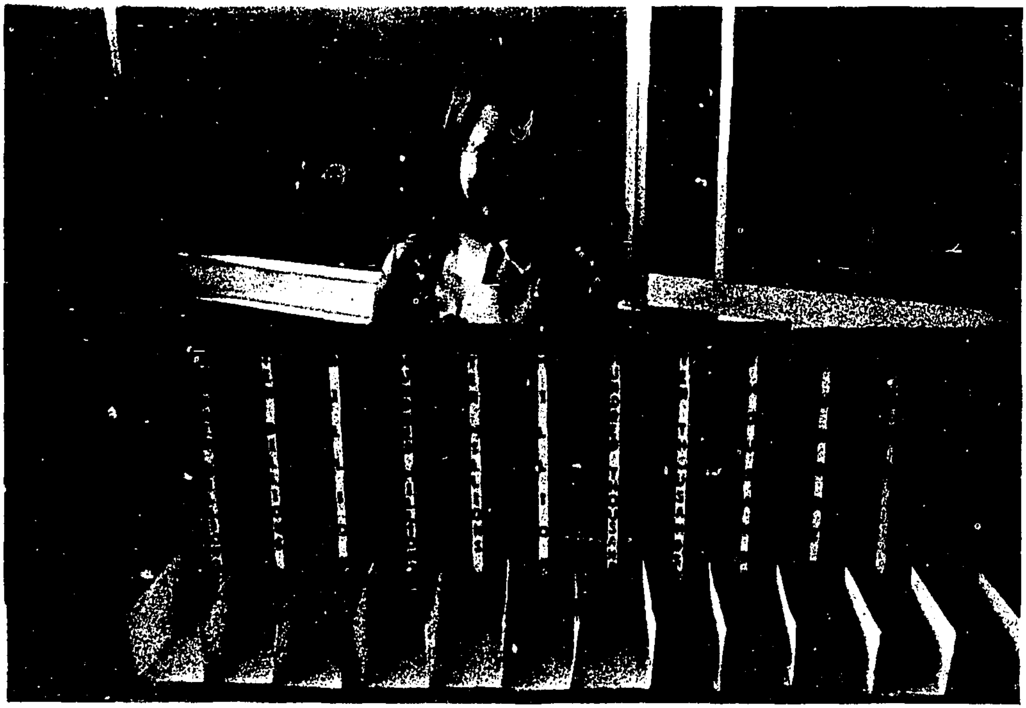
Fig. 13

7. Ring Seven

Place-value columns drawn on the chalkboard with white wax pencil

Strip of paper labeled with the names of the periods to go across the top of the columns on the chalkboard





8. Ring Eight

   Handmade, money place-value aid made from quart milk cartons stapled side by side after the flap and one side have been cut away from each carton

   Real or play money:

   (1) Pennies for the ones place
   (2) Dimes for the tens place
   (3) One-dollar bills for the hundreds place
   (4) Ten-dollar bills for the thousands place
   (5) One-hundred-dollar bills for the ten-thousands place
   (6) One-thousand-dollar bills for the hundred-thousands place
   (7) Ten-thousand-dollar bills for the millions place
   (8) And so on

Figure 14 shows a play bill that can be made from green butcher paper.

DOLLARS			1	0	0	0
PENNIES	1	0	0	0	0	0

Fig. 14

Through the years, my teaching of counting and place value has evolved to the present method, Eight-Ring Circus. It has proved to be a considerable improvement over the comparatively vague old ways. We began with just a few circus rings, and as equipment was accumulated more rings were added. The whole idea grew from the apparent needs of the children for more detailed experience for understanding in this learning situation.

Eight-Ring Circus has been truly an enjoyable experience in learning through the use of concrete teaching materials.

Understanding place value

SHIRLEY S. ZIESCHE
Gulf Gate School, Sarasota, Florida

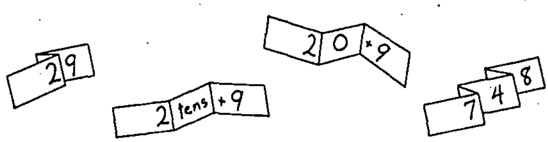


FIGURE 1

Have your pupils ever had difficulty understanding place value? Perhaps my discovery could help you, too.

A few years back I was trying to explain to my second graders how the numeral 29 meant 20 + 9, 2 tens and 9 ones. The brighter ones in the class could rename most numerals accurately but, I felt, without really understanding the meaning, even though we had used various concrete materials. I took a plain strip of paper and folded it accordion fashion twice (see fig. 1). I wrote a "2" close to the fold on the left and the "9" on the right. Within the fold I put the word "tens" and a "+" sign so that when the paper was pulled open, the children saw 2 tens + 9. I made a duplicate, only this time I put a "0" instead of the word "tens." This one opened to read 20 + 9.

I faced the class again and asked them to rename 29. As the answer was given, I expanded the paper showing 2 tens + 9. Then I asked how much 2 tens would be. At the response, "20" I showed the other paper saying, "Then 2 tens + 9 is the same as 20 + 9, 20 + 9 names the same number as 29."

The reaction of sudden understanding that I saw reflected on the children's faces led me to pursue my discovery. I made cardboard expansion cards with pockets and sets of number symbols from 1 to 9 on smaller cards to fit the pockets. (The zero was placed on the expansion card itself.) Some expansion cards had "tens +"; others, "0 +" on the inside. Then I gave them to the children to experiment with renaming of numbers. Later I made expansion cards to include hundreds and thousands places. These were used to help the pupils understand that regrouping does not change the value of the numeral; e.g., the numeral 9483 could be expanded to 9 thousands + 4 hundreds + 8 tens + 3 ones; or it could be partially expanded to

94 hundreds + 8 tens + 3 ones; or to 948 tens + 3 ones (see fig. 2).

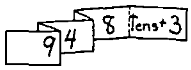


FIGURE 2

For my fourth grade, I made a set of cards with hundreds, tens, and ones that were fitted into larger pockets of millions, thousands, and units (see fig. 3).

I found that most of my pupils mastered place-value concepts much more quickly with these expansion cards. Even seventh and eighth graders found expanded notation much easier after experimenting with these expansion cards.

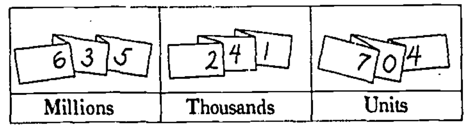


FIGURE 3

Try making a set for your class. Or better still, have each child make his own set. Then, relax and have fun with your mathematics lesson.

EDITORIAL COMMENT.—Another version of the expanding place-value card is shown below.

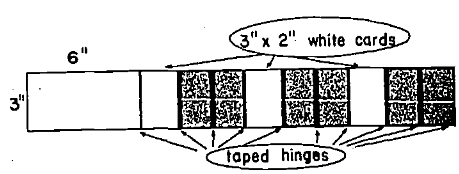
295=	2	hundreds	9	tens	5	ones

In order to provide flexibility, the numeral card and the digit cards are laminated or covered with clear contact paper so that the numeral and digits may then be written on the cards with a grease pencil. Thus the expanding place-value card may be used over and over again with different numerals.

Materials needed:

1 card 6" × 3" (white—laminated or covered with clear contact paper)
3 cards 2" × 3" (white—laminated or covered with clear contact paper)
6 cards 2" × 3" (yellow—assembled in pairs with taped hinge)
1 piece clear contact paper 6" × 3"
3 pieces clear contact paper 2" × 3"
tape
marking pen

The cards are assembled as shown by making tape hinges on the back at each joint to facilitate folding.



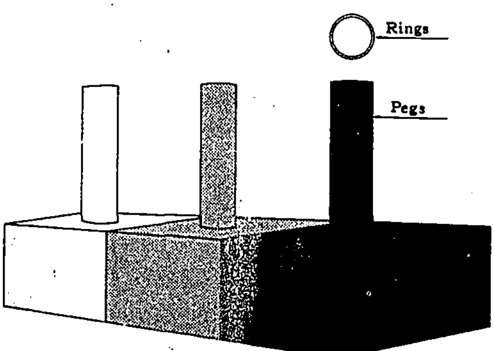
Placo—a number-place game

ROBERT C. CALVO *Woodland Hills, California*

This game emphasizes fun *and* the learning of positional and place-value aspects of our number system. *Placó* (pronounced play-so) is a number game that boys and girls can play to good advantage in the classroom. The children blindfold each other, one at a time, and then attempt to fit the rings on the highest place-value pole (see diagram). One value of the game is the manipulative aspect. The children enjoy handling the rings, fitting them on the dowels, and then computing to find the amount they "win."

While Placo is chiefly a place-value game, in actual classroom practice it contributes to other operations in arithmetic as well. It provides stimulating practice in computation and enables children to have fun while they play the game. It can be used with success in Grades 2 through 6 and can be refined to challenge even the brightest students. It may be played by as few as two children or as many as the whole class.

Three colored dowels are glued on a base, as shown. These dowels are stationary. The dowel to the left is highest in value, the one to the right is least in value. They may be painted blue, red, and white in that order. They are labeled 100's, 10's, and 1's. Thus 100's (in a whole-number game) are blue, 10's are red, and 1's white.



PLACO

One player is blindfolded and given some rings (9–18) of which he may put not more than nine on any one dowel. This blindfolded player then tries to place the rings on the dowels. The other players (split in teams) watch in glee. The object of the game is to get the largest score by putting the rings on the highest-value dowel. After all rings have been placed, the blindfold is removed, and the whole class or a small group computes the amount. The other players then take their turn. Players can play with a goal of 10,000 or with a time limit, the highest score winning. In playing decimal-fraction Placo, the blue dowel represents ones; the red dowel, tenths; and the white dowel, hundredths.

Placo cards may be developed to use with the game. These cards are used as follows: After each turn, a player selects one of the cards and follows the directions given. They say, "Double your score" or "Cut your score in half" or "Subtract three tens," etc. This serves to provide variety in the game. These cards can be developed to reinforce specific skills needed by the class or to provide the class with needed drill. Cards might give directions to add, subtract, multiply, or divide. In addition to dramatizing and creating added interest, they serve to equalize scores of fast and slow learners without penalizing the fast ones.

A word here about the use of colored dowels. While some experts say that place-value instruction should not depend on color, we can't rule it out completely at early levels of learning. In place value, it is important that the child knows that "this ring is in the place to the right" or "that peg is in the units column," but that doesn't mean color can't be brought in to augment these instructions. It merely means color should not be used as a sole means of identifying number-place. If color can help the children gain a concept or can make a game more attractive, then its use is defensible.

Other uses can be found for this game. One possibility is to use it with sight-saving classes by cutting niches in the base to identify the corresponding number-value. Computation could be done mentally. Another potential use would be to roll special dice and compute the score in whatever number base the dice show when rolled. The dice could be made so only certain number bases would result.

Materials list for Placo

1. Piece of wood 12 inches long, 4 inches high, and 6 inches wide

2. Doweling (also known as "closet poling"), $1\frac{3}{8}$ inches in diameter, three pieces of 7 inches each

3. Drapery "cafe curtain" rings, $1\frac{1}{2}$ inches inside diameter, 12 to 20

4. Cards and tacks

5. Sandpaper

6. Paint—white, red, and blue high gloss

7. Carpentry tools, including drill and saw

8. For variation noted, 3-by-5 cards

9. Large handkerchief or clean rag for blinder

10. Oak tag for power chart

Building the game

1. Drill three holes equally distant from each other in the 12-by-6-by-4 inch piece of wood, making these holes 1 inch deep and $1\frac{7}{16}$ inches in diameter.

2. Sand one end of each 7-inch piece of doweling smooth. Put some white glue on the bottom inch of the dowels and the inside of the drilled holes. Slip the dowels in and allow to dry overnight.

3. Purchase cafe curtain rings of the size specified in the materials list.

4. Paint the game three different colors, as mentioned previously, making the dowels and the base of matching colors.

5. You may want to put two small nails about 3 inches from the top of each dowel (exact placement would depend on the thickness of the curtain rings). In this way, players could put only 9 rings on each peg—a 10th would have nothing to keep it from falling off. This style of counting uses only 9 units. When 1 more is added, it becomes the number on the left.

Making a counting abacus*

GEORGE C. CUNNINGHAM

Walton High School, Walton, West Virginia

Use of the counting abacus (Fig. 1) may begin in the first grade and continue

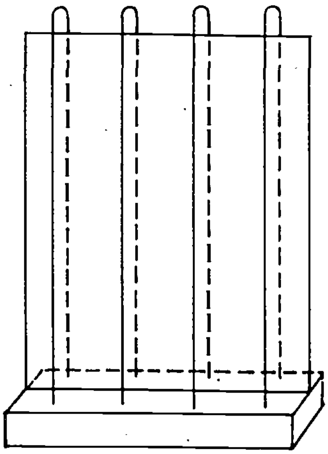


FIGURE 1

through high school level in general math courses. It is a very effective tool in showing the action of addition, subtraction, and regrouping. By changing the number of beads used, a teacher may show these same actions in other base systems.

The cost of constructing the counting abacus will be less than $2.00. You will need the following list of materials:

- Board 6″ x 8″ x 3″ (less than $.50)
- Electric drill and $\frac{7}{64}$″ bit or hammer and a nail about $\frac{7}{64}$″ diameter
- 4 all-metal coat hangers
- 2 pairs of pliers
- 3 shirt cardboards
- $\frac{3}{4}$″ diameter wooden beads (less than $1.50 for 100)

The board must first have a cut $\frac{1}{2}$ inch deep down its center length (Fig. 2).

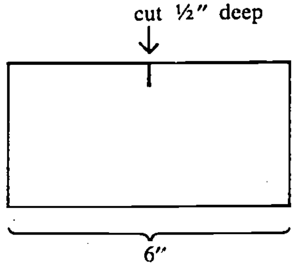


FIGURE 2

Ask the lumber company to make this cut for you if you do not have a circular saw. Most companies will do this for you

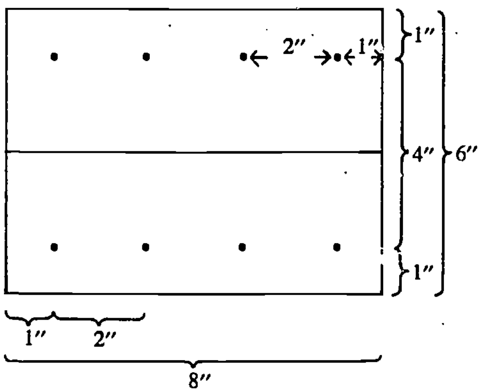


FIGURE 3

free. Draw a line 1 inch from each long edge. Then, beginning at one end, measure 1 inch along this line and mark a point. Measure 2 inches from this point and

* Developed at modern math workshop, West Virginia State College: Miss Lynch, instructor.

put another point. Do this until you have 4 marks. Repeat the same process on the other edge (Fig. 3). With the electric drill, drill holes $\frac{5}{8}$ to $\frac{7}{8}$ inches deep at each point (Fig. 4). The hammer

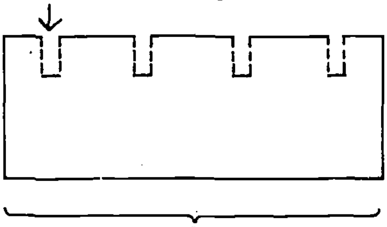


FIGURE 4

and nail may be substituted to make these holes if you do not have an electric drill. For a more attractive board you may sand it or sand and paint it. Put it aside.

The coat hangers must first be separated at the neck. To do this take a pair of pliers and twist the neck of the coat hangers in a clockwise motion until it separates. Using both pairs of pliers, bend the coat hangers into a straight line. Then break off each end so that you have a wire about 32 inches long. With both hands grasp the coat hanger in the middle and bend it into a semicircle with a diameter of 4 inches (Fig. 5). The beads are now ready to be put on the wire.

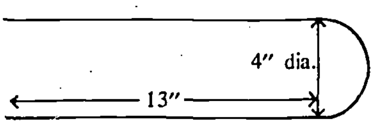


FIGURE 5

The use of 18 beads on the first three places and 9 beads on the last place is recommended. This will allow you to show how to regroup after borrowing and carrying.

The third part in making the counting abacus is making the cardboards. Most cardboards are 8" wide but must be cut to $10\frac{1}{2}$" high. (Cut 1 cardboard 8 x 10.) Draw horizontal lines on the larger cards. On the first line, behind each place value, write the name of that place value. For base five write ones, fives, twenty-fives, 125's. On the next line above, write the place value in exponential form: 5^0, $5'$, etc. On the next line write the place value in expanded notation: 1, 5, 5 x 5, etc. Use the two 8-x-$10\frac{1}{2}$ pieces for this, with the smooth surfaces showing. Tape the tops of these two cards together. This gives them added stability. Use both sides and each end to put 4 more base systems on the 8-x-10 card. This card is to be inserted between the two longer cards for ease of storage. When in use it can be placed in front of the other cards to show the base system being discussed.

Now you must put all of these different parts together. Place the beads on each wire, insert the wires into the holes drilled in the boards, put the cardboards in the slot, and the counting abacus is ready to use.

If a board cannot be obtained, a temporary base may be made from a block of styrofoam. Use a knife or razor blade to cut the slot, and use only one cardboard. The wires may be punched into the styrofoam without any problem. The teacher can use this type of counting abacus to show the class; but it cannot be used very long by the children, because the holes will soon become enlarged and the wires will come out.

A simplified "open-end" version can be made with 4 straight sections of coat-hanger wire. Prepare the board as before,

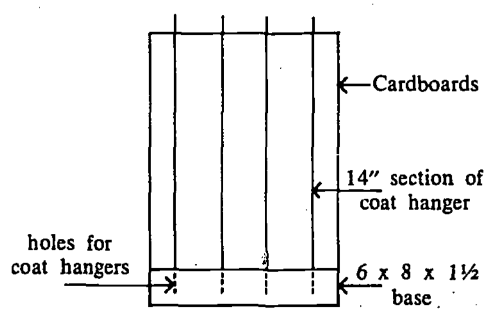


FIGURE 6

but mark and drill the holes on one side of the board only. Cut a 14" section from the bottom of the coat hangers. Insert the wires and cards into the board (Fig. 6). The beads should be kept in a small box or bag to prevent their being spilled and lost.

Concepts to enhance the study of multiplication

MARGARET HAINES
Fairview Elementary School, Elizabethtown, Pennsylvania

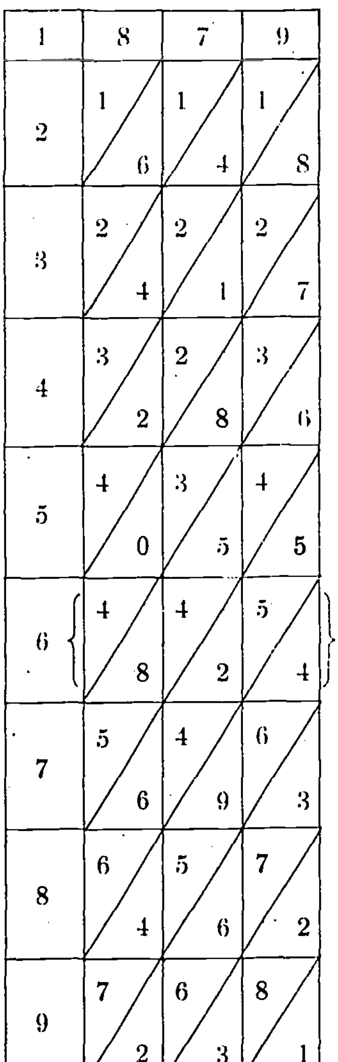
The use of a system called Napier's Rods will make multiplication an interesting study for both fast and slow learners. These were invented about 1617 by a Scottish mathematician named John Napier for performing multiplications. These can be made and used for other number bases, but for the present let's consider only base ten or our decimal system.

For best results and proper manipulation these rods should be cut from heavy cardboard or thin plywood about nine

1	2	3	4	5	6	7	8	9
2	/4	/6	/8	1/0	1/2	1/4	1/6	1/8
3	/6	/9	1/2	1/5	1/8	2/1	2/4	2/7
4	/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

Figure 1

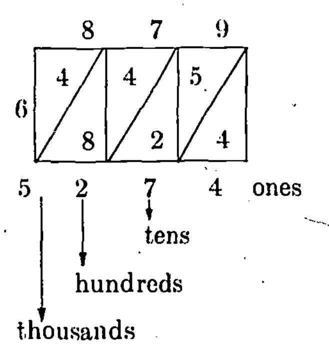
inches long and one inch wide. Four strips are needed if one set of answers is placed on each side of the strip. However, in our fifth-grade room we like to show all the answers at once in order to make more difficult problems so we used eight strips. All one-inch blocks should be divided diagonally to take care of two place answers. We used a woodburner pen to make our numbers show more distinctly. The answers on the rods represent multiples from 1 through 9. When all the rods are finished a coat of shellac will preserve the printed matter. (See Figure 1.)



The strips are movable and can be arranged to form any multiplicand. The strip representing the multiplier is placed at the left-hand side. The product is found by *arranging* the *answers* found on the Napier's rods in a *certain order*—an order that corresponds to the order of the digits in the multiplicand. Please note that these preceding sentences prove to be very valuable in teaching the meaning of multiplication terms that often seem elusive to many pupils. Pupils have to master these terms in order to use the rods—something they need not do in ordinary written multiplication.

Suppose the example to be multiplied is 879×6. Arrange rods 8, 7, and 9 to form the multiplicand. Place the rods in the order which corresponds with the order of the digits in the multiplicand. Place the strip representing the multiplier at the left of the three strips you have arranged. (See Figure 2.)

To find the answer, find the multiplier, 6, at the far left. Start at the right of this row and read the answers following the diagonal lines. The answer for the ones place (6×9) is 4; the answers for the tens place (6×7) is 5 to carry plus 2 or 7; the answers for the hundreds place (6×8) is 4 to carry plus 8; then a thousands place is needed for 1 to carry plus 4 or 5. The answer, therefore, is 5274.



Check: 879
×6
5274

gure 2

The example may require multiplication by two numbers, such as

$$\begin{array}{r} 879 \\ \underline{\times 36} \end{array}$$

In this case the answer for multiplication by 6 is found and recorded as indicated. Then the row across from 3, the tens part of the multiplier is found. It must be remembered that multiplication is by 30; therefore, 7 becomes 7 tens or 70. Other digits in the answer are 26370.

$$\begin{array}{rl} 5274 & (879\times 6) \\ \underline{26370} & (879\times 30) \\ 31644 & \text{(product)} \end{array}$$

Check:

$$\begin{array}{r} 879 \\ \underline{\times 36} \\ 5274 \\ \underline{26370} \\ 31644 \end{array}$$

My pupils check these without my reminding them for they feel that just once one must be wrong.

Halving and doubling

Another method of multiplication to erase the need for memorization of all multiplication tables (save the twos) has been devised. In fact, there is sufficient evidence to establish that it was known to the ancient Egyptians as early as 2000 B.C. The principle is simple enough that it may be introduced in the fifth grade and will serve as an introduction to more difficult mathematics since it introduces doubling and halving of numbers.

Suppose one has two sets of numbers to be multiplied and suddenly forgets all multiplication facts except the two's. If one number is halved, it matters not which one, and the other doubled until the halving reaches a quotient of "1" the product will be the same as that of the original number. There is only one small item to remember—cross out all even numbers which are opposite even numbers in the halved column. While halving numbers discard all remainders.

Halving	*Doubling*
27	34
13	68
~~6~~	~~136~~
3	272
1	544
	918

Example: $27\times 34 = ?$ Six is even so omit from both columns *before* adding remaining figures in doubling column.

Now add and check using the regular method:

$$\begin{array}{r} 27 \\ \underline{\times 34} \\ 108 \\ \underline{81} \\ 918 \end{array}$$

Children need not be coaxed to check this type of multiplication and I found my brighter pupils loved it.

EDITORIAL COMMENT.—It is also possible for children to make their own sets of Napier's rods by using tongue depressors. The actual construction of these rods by elementary school children in grades 4 through 6 may accomplish several objectives. For some children this activity may provide another vehicle to review the multiplication facts, may be a stimulating enrichment experience, or may be an interesting link with history. For others, the rods may simply be a curiosity to be explored and used in performing the multiplication algorithm.

Building "computers" for nondecimal number systems

FREDERICK R. RABINOWITZ
Mathematics Consultant, Philadelphia, Pennsylvania

A rash of comic-strip entries provided an unexpected form of motivation for a Philadelphia sixth-grade class. Mrs. Katharine McFadden's class at the John H. Webster School used them as a springboard for studying various systems of numeration. The class had been introduced to the binary system early in the year, had achieved very well, and had enjoyed the hours working with it. But when the cartoons on "new mathematics" appeared later in the year, and interest again became high, the pupils sought additional outlets for their interest in numeration systems.

First, in connection with their work with the binary system the children constructed a binary "computer," which consisted of a black box, dry-cell batteries, switches, and the necessary wiring for lighting five bulbs, which represented the ones, twos, fours, eights, and sixteens places.

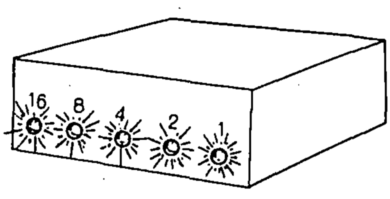


FIGURE 1

A particular base-two number could be named, and then an operator would choose the correct combination of switches to light that number on the "computer." Of course, it did not "compute," but it did serve as an effective teaching aid in base-two numeration.

The boys made individual computers of varying degrees of sophistication. Somehow, the girls felt that this was a "manly" manifestation and so did not engage in this type of construction. With outside consultant help, Mrs. McFadden set up means for the girls, and the boys, to construct less complicated "computers" and adapt them for different bases.

Wire coat hangers were used for the binary system. Ten strips of manila oak tag paper, each 2″ × 2½″, were cut. The numeral 0 was inked on five of them with a felt-tip marker and the numeral 1 inked on the remaining five. A "one" and a "zero" were stapled together around the base of the hanger and thus operated as one unit when they were rotated. The size of the hanger and the size of the cards allowed for five places from the right in the binary system. Consequently, the greatest number that could be computed was 11111_{two}, or 31 in our base ten.

A diagrammatic example of the use of this binary "counter" is shown in Figure 2.

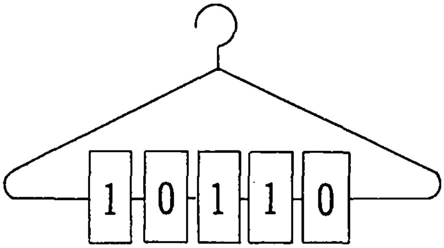


FIGURE 2

$10110_{two} = 22_{ten}$

This device worked well for the binary system, but what about the ternary system? Here, again, no obstacle was met, since a coat hanger and the same-sized tag strips were used. Now, however, one more numeral was needed in the base-three system; so three strips were stapled around the hanger, providing the three numerals necessary for the ternary system, thus:

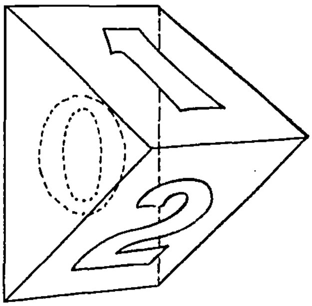


FIGURE 3

Admittedly, these computers were a bit more difficult to manipulate than the ones designed for base two; yet they were effective. The greatest base-three number possible on this computer was 22222_{three}, or 242 in base ten.

Four pieces of oak tag were used for each place in base four, as in Figure 4. It was possible to turn the "dials" here to }. This meant that 33333_{four} = en.

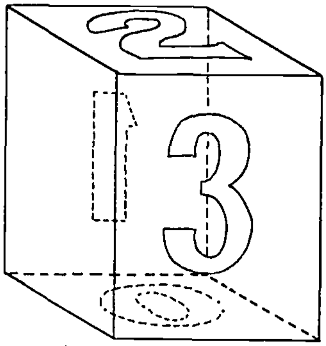


FIGURE 4

This still did not satisfy the more adventurous minds, so they tackled base six in a different way. Their computer was modeled after an odometer in an automobile. Six cylinders were made to revolve around a horizontal axis consisting of a dowel stick suspended on two wire hanger supports (Fig. 5).

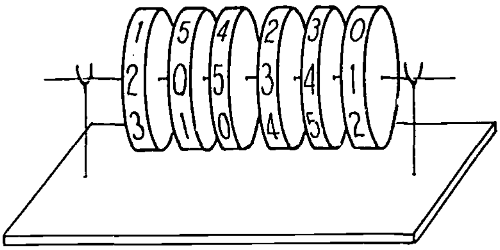


FIGURE 5

A strip of paper was pasted over the lateral face of each cylinder. On each strip had been written the six digits necessary for base six—0, 1, 2, 3, 4, 5. A pupil would merely turn the dial to select the proper base number. In Figure 5, the center row of digits would produce $205341_{six} = 16765_{ten}$.

Not every sixth-grade class in the author's scope of responsibility could deal with number bases in this manner. These activities are for enrichment and must be identified as such from the outset.

However, a great deal of mathematical learning as well as enjoyment can result from these activities. The children reaped both of these harvests.

Let's use our checkers and checkerboards to teach number bases

LUCILE LAGANKE *Wilmington College, Wilmington, Ohio*

In this day of teaching modern mathematics, one notices the tremendous numbers of manipulative materials being manufactured for the sole purpose of teaching number concepts. However, the teacher might find interesting math materials at home.

Why shouldn't the teacher and his pupils bring their checkers and checkerboards to school, to use in teaching the idea of various number bases to fifth and sixth graders.

A checkerboard has sixty-four squares and twenty-four checkers. The pupil, by drawing four rows of eight squares, the size of those on his checkerboard, on an appropriate size of paper, can change the checkerboard into an arithmetic tool.

The squared piece of paper can be placed at the top of the checkerboard. The first row should be marked from right to left with the number base being used, each number having the proper exponent above the base number, to indicate its value. The second row should have the numerical equivalent of each base number, written in base ten. Then, two rows could be used for manipulating the checkers.

A pupil could place checkers on each space of the fourth row, and then push the checkers into the proper spaces in the third row, to build numbers. This is really using the checkerboard as an abacus.

Figure 1 is a sample of what a paper guide for base two or the binary system would look like.

To express the base-ten number 21, in base two a pupil would push the checkers into the proper squares in the third row.

If checkerboards would take up too

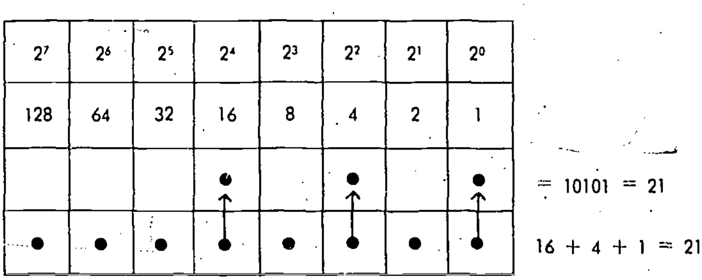


FIGURE 1

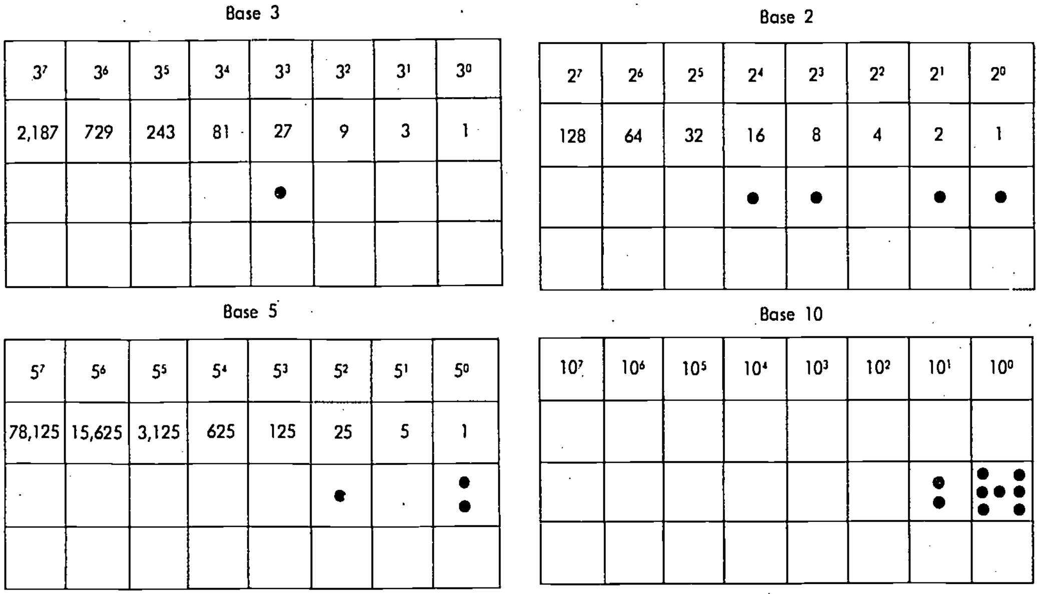


FIGURE 2

much space in the classroom, the teacher could make hectograph checkerboard sheets, and as each number base was studied, the pupil could work out his own number bases and write the correct numbers in the proper squares. (See Fig. 2.) Cardboard or paper disks could be used in lieu of checkers.

These number boards could be used to build numbers, to add numbers, to subtract numbers, and to compare how a certain number appears in different number bases. Drills might be given to see how quickly the children have learned to perceive the component parts of a number.

Adding several numbers in a different number base should be fun, for as soon as a pupil obtained the maximum number of checkers permissible in each square in the number base being studied, he would then discard the excess number of checkers, and move one checker into the next column to the left. He would really be "carrying." (See Fig. 3.)

It would probably be wise for the child to start with base ten, and then learn to

The Number 27 in Four Bases

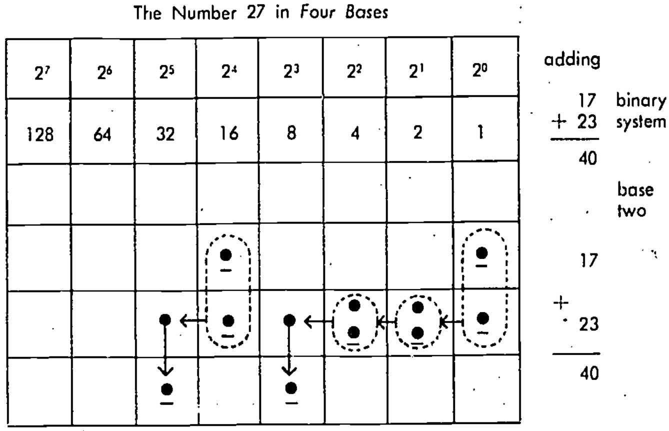
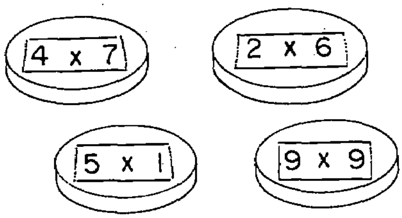


FIGURE 3

work in base two, base three, and base five. The number of symbols or checkers that could be used in each number system could be found out by the "discovery" method. Other bases might be presented in the junior high grades.

There are many possibilities in teaching various aspects of number bases, by using a checkerboard and checkers. Probably all the members of the family will join in with the children in playing checkers, a math game for the twentieth century.

EDITORIAL COMMENT.—The checkerboard may be used in a variety of ways as a teaching aid in mathematics. For example, to provide a "practice experience" with basic multiplication facts, the teacher may affix a piece of masking tape on top of each checker and write a multiplication fact on the tape.



Two students may then be permitted to play a regular game of checkers with the following modifications to the rules: Before a player can move a checker, he must correctly state the product for that checker (the other player checks to see that he gives the correct product); before a player can make jumps, he must give the correct product for every checker he jumps; before a player can accept a crown, he must give the correct product for at least one of the checkers available for use as a crown.

Another variation for a practice activity with checkers is to affix a piece of tape to each black square on the checkerboard as well as to each checker. Then write a numeral on each piece of tape to provide for the desired practice. For example, if addition of tens is the desired practice experience, the teacher writes two-digit numerals such as 10, 20, 30, and so forth on the pieces of tape. Then as two students play a game of checkers, they are required to give the correct sums generated from the two addends—one on the checker being moved and the other on the checkerboard square where he lands.

Discovering the mathematics of a slide rule

J. GREGORY MARTIN, JR. *Ithaca, New York*

Mr. Martin has been a junior high school teacher. He is now in his second year of study on a doctoral program in mathematics education at Cornell University.

Like mathematics teachers in most junior and senior high schools, I find myself looking for ways to refresh my own enthusiasm as well as new ways to channel the enthusiasm of my students. One method I find to be effective is to introduce a topic that is not in the text, and at the same time is a topic I have not presented before.

Thus, a few years ago, when several students asked about learning to use a slide rule, a couple of my fellow math teachers jumped at the chance to capitalize on the student's enthusiasm. In fact, for the past three years, the school bookstore has maintained a large inventory of slide rules. The pros and cons of incorporating the use of a slide rule into any math class, and in particular into the seventh and eighth grades, were discussed, quite naturally, among us teachers.

Initially I could see no justification for slide-rule drill in any math class; however, one of our teachers, a man who had come to us from several years of engineering, altered my thinking. To him, practice with the slide rule, like practice with any implement, gives to the user the skill and confidence necessary to render the tool useful. Thus, a little time given each year will bring this needed confidence and skill to the student for the time when he may need to make use of it in physics and chemistry.

While I conceded the point, I could not bring myself to tolerate the sight of the "things" in any class of mine that went by the name of mathematics. But my prejudices not withstanding, we soon had an infectious rash of two-dollar K.E. slide rules growing to epidemic proportions. The symptoms of the disease became manifest in my own classroom. I inoculated all my students against the slide-rule disease by warning that I intended to confiscate any commercial slide rule that I might see in my class.

Of course, it would have been a mistake not to utilize the force of this infectious and wide-spreading enthusiasm. Thus, one day I announced that we would take up the topic of why a slide rule works. Several students wanted to know if this were now the signal to rush forth to the book store—the quarantine period seemed over. Yes, we were now going to have slide rules, but not commercial ones; instead, each student would make his own. This is very easily done with cardboard and scissors. (See Fig. 1.)

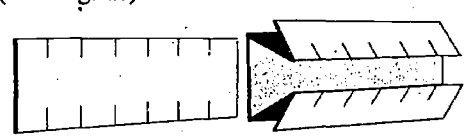


FIGURE 1

What follows is an outline of ideas which can be successfully presented to junior high school students, or which could be profitably used in conjunction with topics on exponents at any level.

By numbering the marks of the sticks

with consecutive integers as shown in Figure 2, we are able to obtain answers to ad-

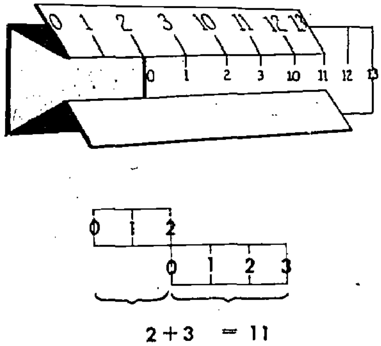


FIGURE 2

dition combinations. In the figure, we are adding

$$(2 + 3 = 11)_{\text{base four.}}$$

The slide rule in Figure 2 is also set to add the combinations

$$2 + 0,\ 2 + 1,\ 2 + 10, \text{ etc.}$$

It should be clear what is happening in the illustration. The outside stick contributes 2, the inside stick contributes 3, and the answer is read back on the outside stick. Using slide rules marked in different bases is of great interest to students; however, it is probably best to use base-ten numerals first. Later, if it is appropriate, the left end of the stick could begin with negative numbers.

If other teachers in a building have already devoted time to slide rules, and even if they have not, the students will very soon tire of just adding. They will be impatient to multiply. But the teacher should, in my opinion, counter any student suggestions to multiply with a statement that the slide rule can do nothing but add and subtract. It will probably already have occurred to someone that a subtraction problem takes place on the slide rule for each addition problem. Thus, in Figure 2, the slide rule also gives

$$(11 - 3 = 2)_{\text{base four}}$$

or it could be regarded as

$$2 - 0,\ 3 - 1,\ 10 - 2, \text{ etc.}$$

Slide rules are generally not constructed to supply answers to addition combinations. This is because the combinations are well known, or very easy to determine. However, the student-constructed slide rule can be of aid when addition is done in a numeration system with a negative base, for then it is not quite so easy to determine sums. (An article entitled "Using a negative base for number notation" by Chauncey H. Wells, Jr., appeared in THE MATHEMATICS TEACHER, February 1963.)

After the students have spent some time constructing and using slide rules that add in different bases, they will be ready to consider multiplication. At this point I display a commercial demonstration slide rule. (I always expect some student to tell me I am breaking my own rule about commercial slide rules, but this response has never come, apparently the students are too absorbed in learning.) Using the large slide rule, I demonstrate a couple of multiplications. "See, it works just like your adding slide rules—but because the numbers are arranged in a strange manner, the answer is the product instead of the sum."

Students and teachers will derive maximum satisfaction if the class is challenged to find the way in which the numbers are placed on the slide rule. Of course, the students should be warned that they will not be able to find a scheme which will locate every number. I tell my junior high school students that it would be a very smart twelfth grader who could devise a system for placing every number and, in fact, most twelfth graders probably could not place any number.

In some years, I have allowed the challenge to drag on all year, while I periodically gave hints. Discovering the scheme is not an easy task, and any student who can independently find the solution is indeed perceptive. For the teacher's part, he will most likely find that several hints will be needed before the students are able to discover the method.

Before the students can find the method for placing a few numbers, or before they will be able to understand an explanation of this, they must have some experience with exponents. When such examples as

$$(3^2)(3^5) = 3^{2+5}$$

are familiar to the students, it can be explained that we are using addition to find a product, a slide rule will only add, but a slide rule will find a product. Even this example will not allow every student to discover the method for finding products. Of course, the student must know that $n^0 = 1$, with n not zero since 0^n is undefined.

Once the method for locating numbers is known, it is a good exercise for the students to construct a slide rule like the one shown in Figure 3. The numbers on the upper

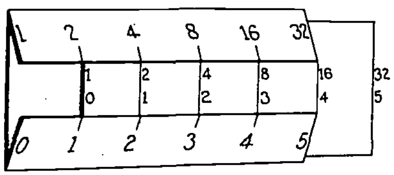


FIGURE 3

halves of the sticks are just powers of 2. Thus, on the top scales we have the slide rule set for the multiplication,

$$2 \times 4 = 8,$$

or

$$(2 \times 2,\ 2 \times 8,\ 2 \times 16,\ 2 \times 32).$$

On the lower halves of the slides we have the addition,

$$1 + 2 = 3.$$

Thus,

$$2^1 + 2^2 = 2^3$$

is for the slide rule just the same as

$$1 + 2 = 3.$$

After the pattern of numbering the slide rule is known, the student will be quick to follow up with numberings other than those that use 2 as the base.

A natural extension, and a most profitable one, is the use of the slide rule with negative exponents. Here, concrete meaning and lasting understanding can come to the student in a very easy and natural way. Another obvious extension would be fractional exponents. This additional way of viewing the concept is very helpful and meaningful, though this may not be as inherently clear as the previous notions mentioned.

A cardboard slide rule is not a practical slide rule, and it would be a mistake to assume that the student is acquiring any degree of skill in manipulation. However, using a homemade slide rule allows the student the opportunity for mathematical understanding, and it gives him a chance to "see" why it works.

4

Integers

The system of integers is beginning to play an increasingly important role in the mathematics curriculum of the intermediate grades. As elementary school teachers, we shall be seeking more and more ways to make the set of integers and their operations meaningful to our students. The four brief articles in this section are a start in that direction.

Mauthe describes a game in which upward motions on a ladder correspond to positive integers and downward motions correspond to negative integers. The composition of movements corresponds to addition. Imagine the glee of students as they reach the top of the ladder or as their opponents drop off the ladder.

Frank's article provides a model for addition and subtraction of integers. A "take away" interpretation of subtraction is given for positive *and* negative integers. Frank relates several concepts—integers, number lines, and operations—while incorporating much fun in her shuffleboard game.

Very often our students come away with the impression that number lines are either horizontal or vertical and that positive is always up or to the right. Milne describes a number-line model that not only preserves the ordering of the integers and their readability but also shows students that the number line may have many orientations.

The sign rules for multiplication of integers seem to baffle some of our students. Some can follow the abstract arguments and are willing to accept the rules; other students need more tangible illustrations. Pratt's flip cards offer students an interesting way to visualize and correlate these rules.

Climb the ladder

ALBERT H. MAUTHE

Albert Mauthe is chairman of the department of mathematics at A. D. Eisenhower High School at Norristown, Pennsylvania. He is on leave from his position this year to participate in the Experienced Teacher Fellowship Program in Mathematics for Prospective Supervisors at Teachers College, Columbia University.

Here is a new game. It is called "Climb the Ladder." The rules of the game are very easy to learn, and you will have lots of fun playing Climb the Ladder.

To play the game, you will have to make a ladder and a dial-spinner. (These will be shown later.)

As many players who wish to play the game together may do so. The object of the game is to go above the $^{+}12$ step on the ladder. Anyone who falls below the $^{-}12$ step on the ladder has fallen off the ladder, and he is out of the game. Thus a player can win the game in either of two ways:

1. if he is the first person to go above the $^{+}12$ step on the ladder, he wins the game; or

2. if he is the last person remaining on the ladder after all the other players have fallen off the ladder (by falling below the $^{-}12$ step), he wins the game.

Everyone starts by putting his marker at "Zero", the "0" step. We will call the numbers above 0 "positive numbers," and we will call the numbers below 0 "negative numbers." We will not call 0 either "positive" or "negative"; 0 will be called "zero."

The first person spins the spinner. If the spinner stops on a $^{+}1$, $^{+}2$, $^{+}3$, or $^{+}4$ space, the player moves his marker *up* the ladder the number of steps indicated (1, 2, 3, or 4). We will call the numbers $^{+}1$, $^{+}2$, $^{+}3$, and $^{+}4$ on the spinner-dial "positive numbers."

If the spinner stops on a $^{-}1$, $^{-}2$, $^{-}3$, or $^{-}4$ space, the player moves his marker *down* the ladder the number of steps indicated (1, 2, 3, or 4). We will call the numbers $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$ on the spinner-dial "negative numbers."

If the spinner stops on the 0 space, the player does not move up or down the ladder. We will call this number "zero."

Each player takes one turn at a time until he has fallen off the ladder or some player has been declared the winner.

Let us watch Betty, Bob, and Dave play the game. We will watch each player's progress up the ladder on the demonstration ladder. At the same time, we will keep a record of each player's steps at the right side of this page.

	BETTY	
Betty begins. She starts at Zero, 0.	Start	0
She spins a $^{+}3$. Thus, she goes 3 steps *up* the ladder.	Add	$^{+}3$
She is now on step $^{+}3$.	Step	$^{+}3$

Bob and Dave each take their first turns. Bob lands on $^{-}2$, Dave on $^{-}4$. The second turns for Betty, Bob, and Dave look like this:

	BETTY	
Betty was at $^{+}3$.	Start	$^{+}3$
She spins a $^{+}1$. Thus, she goes 1 step *up* the ladder.	Add	$^{+}1$
She is now on step $^{+}4$.	Step	$^{+}4$

	BOB	
Bob was at $^{-}2$.	Start	$^{-}2$
He spins a $^{+}3$. Thus, he goes 3 steps *up* the ladder.	Add	$^{+}3$
He is now on step $^{+}1$.	Step	$^{+}1$

	DAVE	
Dave was at $^-4$.	Start	$^-4$
He spins a $^-2$. Thus, he goes 2 steps *down* the ladder.	Add	$^-2$
He is now on step $^-6$.	Step	$^-6$

	BOB	
Bob spins a $^-2$ on his third turn. He was at $^+1$.	Start	$^+1$
Thus, he goes 2 steps *down* the ladder.	Add	$^-2$
He is now at $^-1$.	Step	$^-1$

	BOB	
Bob spins a 0 on his fourth turn. He was at $^-1$.	Start	$^-1$
Thus, he does not move either *up* or *down* the ladder.	Add	0
He is still at $^-1$.	Step	$^-1$

The game continues until one player passes $^+12$ or all but one player passes $^-12$.

Make your own game

Materials needed.—We'll want to play this game in class. But before we begin playing, we'll want to make our own game to keep. To make the game, each person in the room will make the game from these materials:

1. A square piece of heavy cardboard, about 15 centimeters on a side (15 centimeters is about 5¾ inches).
2. A piece of heavy paper or cardboard, approximately 8 centimeters by 28 centimeters (that is, about 3 inches by 11 inches).
3. A paper clip.
4. A thumbtack.
5. Several small pieces of cardboard or other objects to serve as markers.
6. A ruler, preferably one with centimeter markings.
7. A protractor.
8. Compasses (to draw a circle).

The dial-spinner.—On the heavy cardboard, draw a circle, with center in the middle of the cardboard, with a radius of 5 centimeters (about 2 inches).

Study the spinner-dial shown in Figure 1. Note that there are nine different sectors, four for the numbers $^+1$, $^+2$, $^+3$, and $^+4$, four for the numbers $^-1$, $^-2$, $^-3$, and $^-4$, and one for 0.

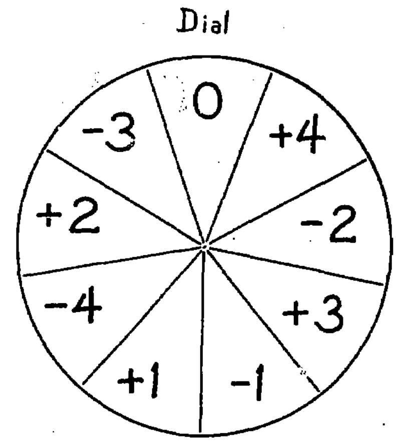


FIGURE 1

Do you remember that there are 360 degrees in one complete turn around the center of a circle? How many degrees must there be in each of the angles around the circle if we use 9 sectors that have equal angles?

Draw a radius from the center of the circle you have drawn to any point on the circle. Measuring with your protractor, draw another radius so that the two radii make an angle of 40 degrees at the center of the circle.

Continue making 40-degree angles until you have nine equally spaced radii, and nine sectors of the same size around the center of the circle. Label these sectors as shown in Figure 1.

Take a paper clip and bend the last loop out so that it looks like the one shown in Figure 2.

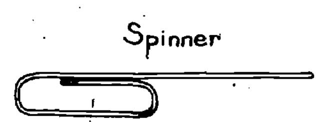


FIGURE 2

Place a thumbtack through the oval still remaining in the paper clip, and place the thumbtack through the center of the circle you have drawn. Your spinner-dial should look like the one shown in Figure 3.

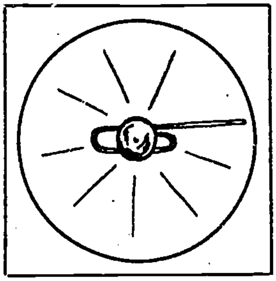


FIG. 3. —Thumbtack through oval of paper clip.

Make sure the paper clip spins freely around the dial.

The Ladder.—On a heavy piece of paper (or cardboard), draw a straight line segment 24 centimeters long (or 9 inches, if you do not have a ruler marked in centimeters). Make a mark at the bottom of the line segment. Then mark off each centimeter (use ⅜ inch, if you used a 9-inch line segment), until you get to the top of the line segment.

Label each step from $^-12$ to $^+12$, as was done with the ladder in Figure 4.

Let's play!

Although any number of players can play together, let's divide into groups of about three players each and try playing our new game. Each group will use one ladder; each player will use his own marker.

With each turn, be sure to move your marker up or down the ladder. And be sure to keep score, as we did for Betty, Bob, and Dave.

If your game finishes before the others, play another game. Keep score for this game also. If time runs out before anyone has won the game, the player highest on the ladder will be the winner.

Have fun!

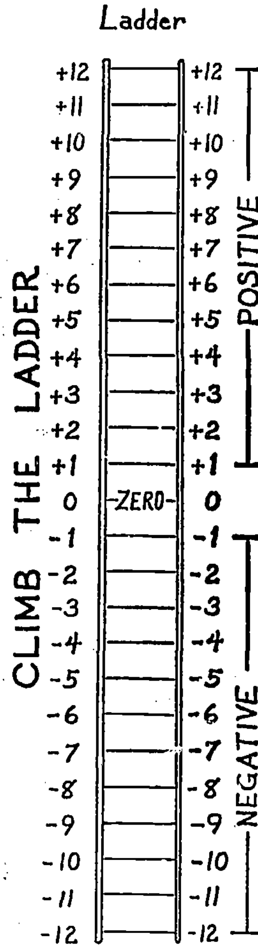


FIGURE 4

EDITORIAL COMMENT.—You may wish to have the student write a number sentence to describe each of his moves on the ladder. This will provide the connecting link between the manipulations and movements on the ladder and the symbolic representation of these movements. Many times the student fails to realize that these game experiences have any relationship with the symbolic language of mathematics.

Play shuffleboard with negative numbers

CHARLOTTE FRANK
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"Shuffleboard, anyone?" Can you imagine how a student-filled math classroom would respond to that invitation? Compare that with the reaction to a teacher announcing, "Students, today we are going to learn the four fundamental operations of the set of integers." By the time the teacher had finished saying the word "fundamental" his pupils would have tuned out both him and his lesson. Negative Shuffleboard can be used to introduce the four basic operations of negative integers to the elementary school youngster in a fun setting.

For the few who have never played, regular Shuffleboard is a game in which players use long-handled cues to shove wooden disks into scoring beds of a diagram. Each player tries to push his own disk into one of the scoring areas or to knock his opponent's disk out of a scoring area and hopefully into a penalty area.

By adding positive and negative signs, the playing field (Fig. 1) can easily be transformed into Negative Shuffleboard (Fig. 2).

Other adjustments necessary to fit our new game to the facilities of a classsroom are the following:

1. Target diagrams are drawn on paper sheeting large enough to cover the student's table.

2. Red and black checker-size disks replace the larger disks normally used.

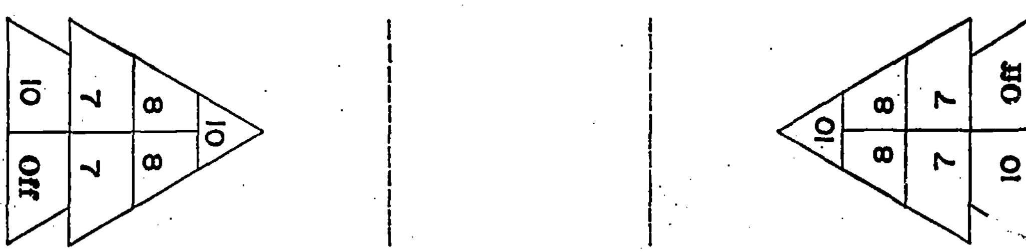


FIGURE 1

FIGURE 2

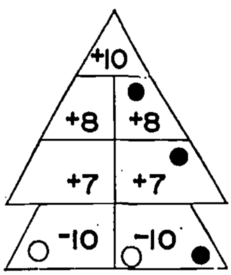
3. A pencil, pen, or ruler is used as a propelling instrument.

4. The ratio of the playing field to the disk is increased in order to allow for more scoring activity.

5. A number line is used to keep a running score.

When Anne and Harley played the game, their mathematical computations developed this way:

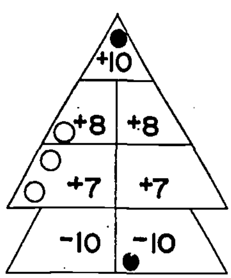
ROUND 1



Anne (○ disks) $^{-}10 + {}^{-}10 = {}^{-}20.$
or $2 \times {}^{-}10 = {}^{-}20.$

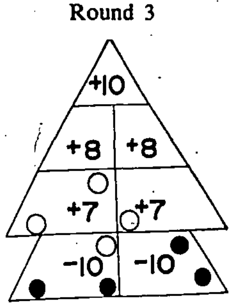
Harley (● disks) $^{+}8 + {}^{+}7 + {}^{-}10 = {}^{+}5.$

Round 2



Anne (○ disks) $^{+}7 + {}^{+}8 + {}^{+}7 = {}^{+}22.$
Harley (● disks) $^{+}10 + {}^{-}10 = 0.$

Round 3



Anne (○ disks) $(3 \times {}^{+}7) + {}^{-}10 = 11.$
Harley (● disks) $4 \times {}^{-}10 = {}^{-}40.$

This game continues until a previously agreed upon winning score is reached.

Some of the computations at another table, where Matthew and Sidney were seated, were as follows:

Situation 1.—Matthew had $^{+}7$, $^{+}8$, $^{+}10$ and one disk out of limits, giving him a total of $^{+}25$. When Sidney's turn came he successfully knocked Matthew's disk out of the $^{+}8$ box and into the $^{-}10$ box. Matthew's $^{+}8$ was taken away; written mathematically, it was $-{}^{+}8$. This number sentence describes the last play:

$$^{+}8 - {}^{+}8 = 0.$$

Situation 2.—Sidney successfully knocked his own disk out of the $^{-}10$ penalty box. Translated into our language of numbers, Sidney had a $^{-}10$. When he took away the $^{-}10$, it gave him a 0 for the box. This number sentence is written:

$$^{-}10 - {}^{-}10 = 0.$$

Situation 3.—After Round 4 Matthew had 30 points more than Sidney had. With three disks still to shoot, Sidney told the group watching that he could win if Matthew were unlucky enough to end the round with a $^{-}30$. That would be written by the following number sentence:

$$^{-}30 \div 3 = {}^{-}10.$$

The target diagrams vary from table to table to meet the needs of the individual

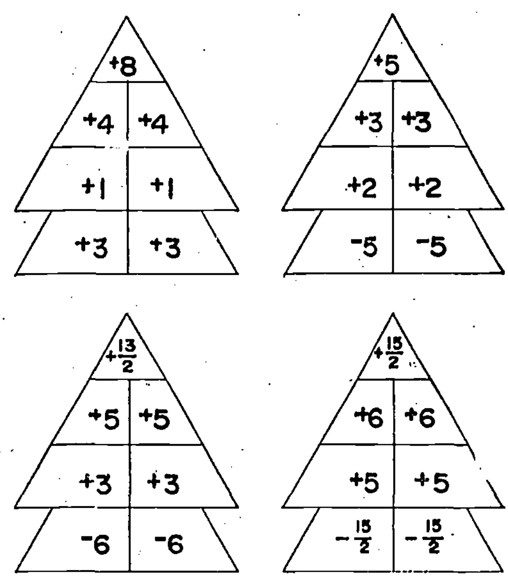


FIGURE 3

children. Some suggestions for a few possible changes to match the learning styles of your students are indicated in Figure 3.

Editorial Comment.—Many variations of this activity are possible. A beanbag game format could be used in a manner similar to that suggested for the shuffleboard activities. The game arrangement will actively involve the children and provide an interesting format for skill development. If the students can be encouraged to record their scores on a number line and write equations to represent the operations, they will gain additional insight into the operations with directed numbers.

Number line: Versatility

ESTHER MILNE
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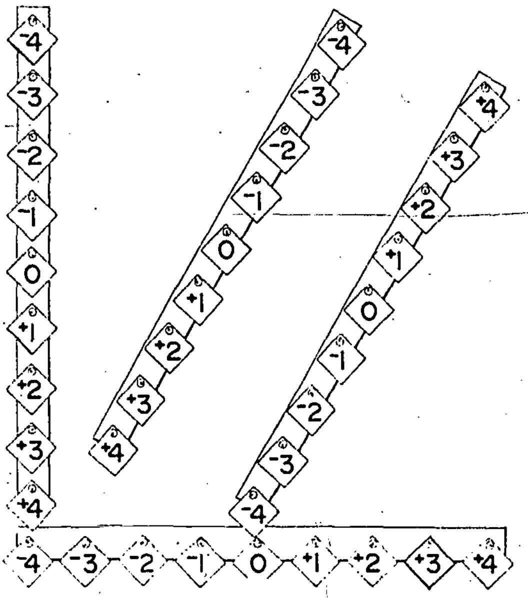
How can you show a number line at innumerable positions? How can you, at the same time, show that the positive integers may be on either side of the origin? Is there a simple device you can construct to illustrate all possibilities?

Such was the problem confronting a in-service group in our school district. While writing a unit on intuitive ideas for an introduction to integers, the group wanted to indicate all possible varieties of a number line. The use of a stick about a meter long, a dozen or so thumb tacks, and a like number of small tags, each with a hole punched off-center (like key tags) provided the solution. The tags, marked with names for positive integers, zero, and negative integers in order of value, are tacked on the stick so that they rotate freely. As the stick is turned to any position the numerals appear right side up.



EDITORIAL COMMENT.—The benefits to be derived from this teaching aid could be increased by having each student make his own model. A variation of this device could be made from clothesline with the numeral cards tied to the clothesline at intervals with string.

A teaching aid for signed numbers

EDNA M. PRATT

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Each year the time comes when it is necessary to illustrate the multiplication of signed numbers. There are many approaches to helping the student develop insight and skill. Some of the more common are multiplying two negative numbers using the number line,[1]* an inductive approach,[2] the modern mathematical proof approach,[3] the statement approach,[4] or the model approach,[5] but some young mathematicians still have a hard time understanding these ideas. The following is a very effective device which may be used to help students discover and then reinforce the operation of multiplication of signed numbers.

Each student is given a pack of twenty 3-inch-by-5-inch plain filing cards which have been cut in half so that they become two packs of 3-inch-by-2½-inch cards. On the first card of the first pack, the student proceeds to draw a picture of a road and a small car at the top left of the card. On the second card, the same road is illustrated but the car is advanced one-eighth inch to the right; on the third card the car is advanced another one-eighth inch to the right. The process is carried on until the last card in the pack shows the car at the right hand side of the card or at the end of the road. Thus Pack No. 1 shows the car on the road going forward, step by step.

A similar procedure is followed with the second pack of cards, except that the first card shows the car to the extreme right and each succeeding card in turn is drawn so *that the car is one-eighth inch to the left.* The last card shows the car to the extreme left. Pack No. 2, therefore, shows the car going backward, step by step.

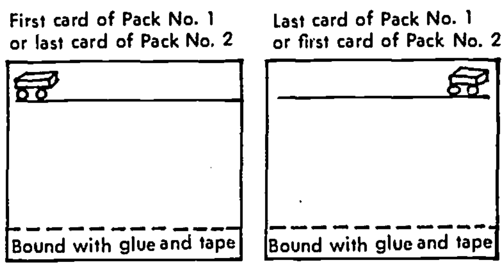


FIGURE 1

Figure 1 shows the first card of Pack No. 1 or the last card of Pack No. 2, and the last card of Pack No. 1 or the first card of Pack No. 2.

Each pack of cards is bound together at the bottom with glue and tape so that it is convenient to spread the top of the pack with the thumb and see each card in turn.

The cards are used to illustrate the multiplication of signed numbers in the following manner:

*Footnote references are given at the end of the article.

1. To illustrate (+)(+) = (+), Pack No. 1 is used. The cards (one by one) show the car going along the road in a forward direction (+). If the cards are flipped forward (+), the result is that the car appears to go in a forward direction (+).

2. Again using Pack No. 1, (+)(−) = (−) may be illustrated. The cards show the car going along the road in a forward direction (+). If the cards are flipped backward (−), the result is that the car appears to go in a backward direction (−).

3. To illustrate (−)(+) = (−), Pack No. 2 is used. The cards (one by one) show the car going in a backward direction (−). If the cards are flipped forward (+), the result is that the car appears to go backward (−).

4. Again using Pack No. 2, (−)(−) = (+) may be illustrated. The cards show the car going along the road in a backward direction (−). If the cards are flipped backward (−), the result is that the car appears to go in a forward direction (+). By regulating the spacing of the car and the rate of flipping of the cards, the quantity values may be demonstrated.

After carrying out the above process, each student is given another set of cards and allowed to demonstrate the multiplication of signed numbers in his own way. Some of the motions illustrated are as follows:

1. A boy swinging upward (+) and downward (−).
2. Piling up rocks (+) and removing rocks (−).
3. Notes on a musical scale: up (+) and down (−).
4. Climbing a ladder (+) and descending a ladder (−).
5. Ship sailing forward (+) and sailing backward (−).
6. Airplane flying eastward (+) and westward (−).
7. Kite going up (+) and a kite being pulled in (−).
8. Clock going in a clockwise direction (+) and in a counterclockwise direction (−).
9. Pail being filled with sand (+) and emptied (−).
10. MONMOUTH REGIONAL HIGH spelled forward (+) and backward (−).

There seemed to be no end to the imagination of the students. This approach reinforced the traditional approaches and in addition was enjoyed.

References

1. Adam Pawlowski, "Multiplication of Integers," THE ARITHMETIC TEACHER, XII (January 1965), 64.
2. William G. Annett, "Products of Signed Numbers, an Inductive Approach," THE MATHEMATICS TEACHER, LIV (March 1961), 169-70.
3. John Rechzek, "The Product of Two Negative Numbers," *New Jersey Mathematics Teacher*, XIV (October 1961), 26-27.
4. Joseph P. Marra, "Minus Times a Minus," *New Jersey Mathematics Teacher*, XVIII (March 1961), 10.
5. National Council of Teachers of Mathematics, Eighteenth Yearbook, *Multisensory Aids in the Teaching of Mathematics* (New York: Columbia University, 1945), pp. 139-45 and 376-77. (EDITOR'S NOTE.—Teachers using this device may find it necessary to establish a forward and backward direction on the card. Not all students will understand readily that as the car moves to the left this is called "backwards" and is represented as −1, −2, −3, etc.—CHARLOTTE W. JUNGE.)

Rational Numbers

5

Exploration of the system of rational numbers begins early in the elementary school program. Children in the primary grades work with "parts of wholes" and "parts of sets." The fraction symbol is introduced early. In later grades students learn to work with different names for the same fractional number. The notations include fractions, ratios, decimals, and percents. Whole number operations are reinterpreted in the system of rational numbers, and computational algorithms for the different notations are formulated.

The title of Jacobs's article echoes the theme of this collection—if children are given the opportunity to manipulate concrete materials, they will more likely be able to reason with the abstractions these materials represent. Jacobs describes how the familiar egg carton can be used to generate equivalent fractions and to find products of rational numbers.

The game of Bingo, as developed by Cook, provides a medium for working with equivalent fractions. The game is more than simple practice in recognizing equivalent fractions, since the children manipulate sets of objects to discover a rule for converting a fraction to one of its equivalent forms.

The article by Drizigacker and the one by Rode describe aids that are useful in showing the relative sizes of rational numbers. In Drizigacker's game, the emphasis is on putting rational numbers in order by using properties of the fraction symbol. Rode's activity involves matching fraction pieces to fraction symbols and deciding whether the piece is too small or too large to "make a whole."

Hauck provides a three-dimensional model for visualizing percentages and their relationship to common fractions and decimals. He shows how to use the model to explain how percents function in problem situations. The short article that follows describes a two-dimensional percentage board suggested by Bailey.

Lyvers provides directions for constructing a fraction circle kit that is used to make rational number operations more meaningful to children. The fraction strips described by Jacobson can be used for similar purposes. Note that these strips may also be used with children other than the special education groups suggested by the title.

Division of rational numbers seems to be one of the most difficult operations to make understandable to children. Many students do not always comprehend rationalizations that involve transformations of complex fractions. McMeen suggests using a set of fraction wheels that have been carefully constructed to demonstrate certain key division problems. Perceiving the division relationship in a manipulative setting will open the door for greater comprehension at a more abstract level.

The last article in the section describes a game used to review skills in computing with fractions. Zytkowski's game board has the added advantage of showing students that substituting different names for a given rational number in a problem does not change the final answer to that problem.

If the hands can do it the head can follow

ISRAEL JACOBS

A teacher in the mathematics department at Anning S. Prall Junior High School in Staten Island, New York, Israel Jacobs reports here on a mathematics laboratory project with seventh-grade students.

Maureen looked down at the egg cartons. Tentatively she separated the singles, then she took the three sets of four out of the group and put them aside. Hesitatingly she continued the process of separating the cartons until the whole set of egg cartons was apart. She pursed her lips and stared down at the disassembled cartons. A pencil and paper lay nearby to record comments. She reread the activity card:

Materials: One set of nested egg cartons (one twelve, two sixes, three fours, six twos, and twelve singles).
Report any pattern you find.
Comment: What do you think of all this?

While Maureen was absorbed in her activity, the other three girls in the group kept up a running commentary:

"These hold only six eggs."

"And two of these make one whole."

"What about these little ones?"

"How many are there of them?"

"One, two . . . how many of them have you got? Eleven . . . no, twelve of them."

"Twelve makes one whole, too!"

Maureen sucked the eraser of her pencil a moment and then wrote: "They all fit into each other." She paused and then added, "It has to do with fractions."

The pencil and paper were put aside. The four girls started to rearrange the pieces, fitting them into various groupings, nesting them into each other to make one whole.

Maureen and her group were involved in an experimental project at Intermediate School 27 (Anning S. Prall) on Staten Island, N.Y. It was a mathematics laboratory program that started on May 21, 1971 and continued every Friday thereafter until the end of June. Some one hundred twenty children in the seventh grade were involved. Within each class, the children were divided into teams of four or five of their own choosing. The teams were seated around double desks that had been turned so that the youngsters could face each other. Except for two pilot programs, none of these children had ever participated in such an activity before.

A mathematics laboratory approach is not new. In England, the Nuffield Mathematics Project was set up in 1964. The Madison Project promotes an individualized laboratory approach. The ARITHMETIC TEACHER has included reports of a number of mathematics laboratory projects. The purpose of a mathematics laboratory is to involve the hands of the youngsters—an approach that will, we hope, lead to the understanding of concepts.

The program at Prall was not funded. All games were constructed by the author with the help of the youngsters. Many of the activities were culled from the mathematics laboratory course for teachers conducted by Professor Judith Jacobs at Richmond College, Staten Island, N.Y. Suggestions for other activities came from the pages of the ARITHMETIC TEACHER and from various textbooks.

In the mathematics laboratory, the role of the teacher was to hand out prepared activity cards that provided instructions for the pupils to follow and to give limited assistance if it was needed. At first the teacher's function was to walk around, smile, and to each anguished question answer, "I don't know. What does the card say?" If a group was completely stalled, then the attitude of the teacher changed. A leading question from the teacher usually precipitated meaningful action.

Maureen and her group were playing with egg cartons that the children had collected and brought in. Twenty groups were involved in this activity. A majority of the groups did not relate this game to fractions immediately or discover any kind of relationships among the various pieces beyond discovering the parts of twelve. One group turned the cartons into a tossing game. They used the 12-section piece as a receptacle and tossed the singles at it. The winner was the one who scored the highest number of "ins." Other groups used the cartons as erector sets to build various shapes.

The second activity with egg cartons included specific questions:

How does the smallest carton relate to the largest carton?

How does the next smallest carton relate to the largest carton?

What other relationships do you find?

The response to this activity was positive. Answers to specific questions were well within the students' experiences. It was on this activity that a group of advanced students came up with some interesting results.

Two teams in the advanced class had moved their desks together and formed a group of eight. Their work area was covered with egg cartons, bottle caps, and sheets of scribbled paper. A debate was in progress. Sue placed bottle caps in the egg receptacles, counted them, made notations on a sheet of paper, and handed the paper over to Glenn. Seven heads bent over as they checked it. They looked up and Glenn said, "It works!"

"What works?" I asked.

"Our computer. A computer to multiply fractions!" said Sue.

"How does it operate?"

A single 12 egg carton was set out and Sue explained. "Suppose you want to show a fraction of 5/6 on the computer. Place a bottle cap in ten of the openings. . . ."

I held up a hand. "Let's backtrack just a little bit. Ten? Why ten?"

She sorted through the sheets on the desk and pulled out a diagram. (See figure 1.)

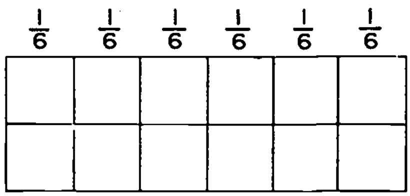


Fig. 1

"Each double up and down row. . ."

"Vertical."

"Each vertical row is 1/6 of the egg carton. If we want 5/6," she explained, "we place a large bottle cap in ten of the holes. Now our computer reads 5/6."

"But," I objected, "you have ten caps in the computer. How does that come out 5/6?"

"Look," said Glenn, pointing to another diagram. (See figure 2.)

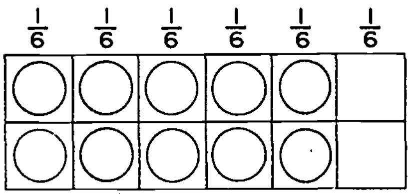


Fig. 2

"This computer is a 12-hole computer. So we double the 5/6. That's why this computer shows ten bottle caps in a 12-hole egg carton. Suppose we want to multiply 1/2 times 5/6. Now we place small bottle caps over the large bottle caps, but only on half of them like this." (See figure 3.)

"Hmm. How would you show 1/7 on your computer?"

"Easy!" they chorused.

A 2-section egg carton was added to

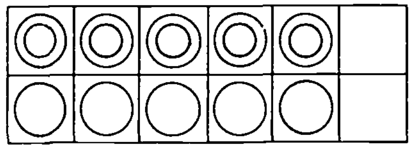


Fig. 3

the original set and figure 4 was the resulting diagram.

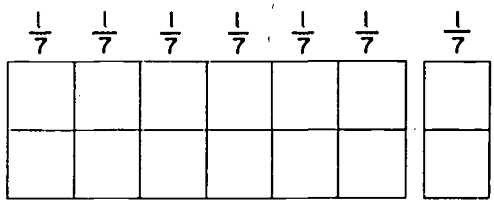


Fig. 4

"What would 1/2 of 3/7 be?" I asked.

"Let me show it now," said Sue, "I found it first."

"Go right ahead," said Glenn, "but I figured out the 14-hole computer to start with."

Sue went on to explain. "Since all the holes in the egg cartons total fourteen we call this a 14-hole computer. Now for 3/7, all I have to do is put six large caps in the openings, because each vertical row has to be filled completely." She went on, "For 1/2 of 3/7, all we do is put three small caps over the large ones, and the result—3/14!" (See figure 5.)

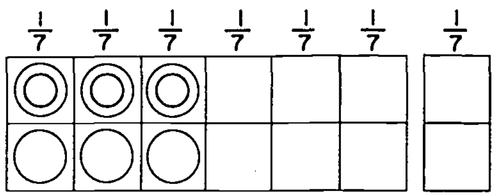


Fig. 5

"That's all very nice," I conceded. "The egg cartons are conveniently set up in sets of pairs. Suppose you want your computer to show 1/3 of 3/7?" With that I walked away.

In a little while, Sue and Glenn signaled frantically. The egg cartons had been arranged as in the diagram in figure 6.

"Now," explained Glenn triumphantly, "we have a 21-hole computer."

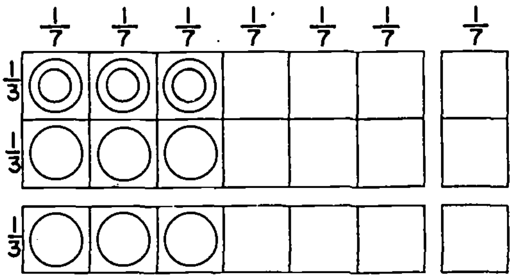


Fig. 6

"Why 21?"

"Because," explained Sue, "if you want 1/3 of 3/7 we need a "whole" into which both 3 and 7 can go evenly. And that is 21."

Glenn interposed. "In this 21-hole computer, 3/7 means nine large caps in nine holes. To take 1/3 of 3/7, we put three small caps over the three large ones, and the computer now reads 3/21."

"Very good! Write it up." I walked away.

It is possible that either Sue or Glenn may have come across fraction diagrams before and applied this knowledge to the egg cartons. Whether this exercise in fraction multiplication was a discovery or just reinforcement, it was a worthwhile experience for the whole group.

From the front of the room, left, a boy was screaming, "I won! I won!"

He was in a group of four playing the spinner fraction game. Two of this same group had been active in constructing the game. Round cardboards, eight and one-half inches in diameter and in various colors, had been cut up into pie slices of 1/2, 1/4, 1/8, 1/16, and 1/32. (See figure 7.) In cutting up these slices, the youngsters quickly learned that it took two 1/32-pie slices to make one 1/16-pie slice.

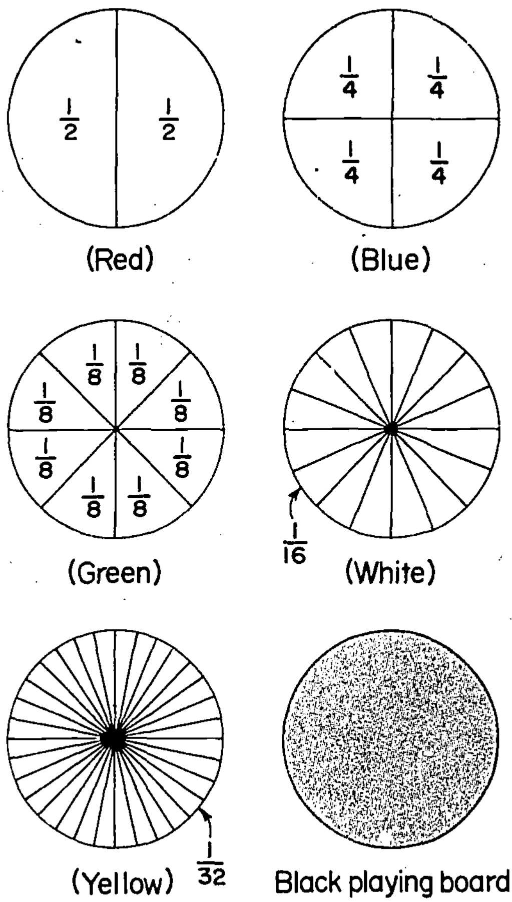
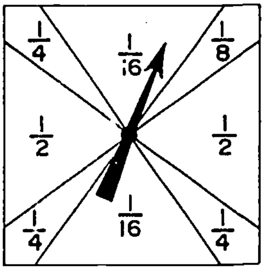


Fig. 7

The spinner was made in the school shop. The sections on the spinner board



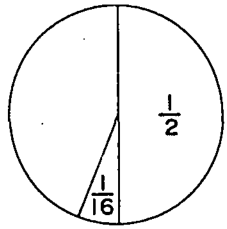
Spinner

Fig. 8

itself are labeled 1/2, 1/4, 1/8, 1/16, as shown in figure 8.

The wooden arrow, pegged loosely to the board, is spun by each player in turn. Each player has to select from the general pile of pie slices a piece corresponding to the fraction the arrow on the spinner points to, and he places the piece on his playing board (the black one). To win, all fractions on the playing board must total up to one whole. If the spinner selects a fraction that does not fit on his board, then the player must return an equal amount to the general pile.

Figure 9 shows the playing boards of four players (A, B, C, and D) in progress. On his next turn, C gets 1/4 on the spinner. Since this cannot be used on C's board, he has to return a 1/4 slice to the general pile so that C's board then looks like figure 10.



C's Board after he lost the "1/4" slice

Fig. 10

After he lost the 1/4-pie slice, C had just two slices left, 1/2 and 1/16.

Suppose C had had no 1/4-slice on his board. He would have had to change his 1/2 to two 1/4s and give up one of them. Such changes were checked carefully by each player. The students learned to add and subtract these fractions without the use of pencil and paper.

Spinner fractions was the most popular game, and all the youngsters were anxious to play with it. It was the spinner that intrigued them. At one time when the spinner was out of action for a while, cards with fractions marked on them were substituted. The cards were not as much fun for the students as the spinner.

The second activity with the spinner fraction game added something new. It was not enough to tabulate the number of wins and the ways in which a game could be won. Now the goal was to determine which fraction on the board came up most frequently. The students were to try 100, 500, and 1000 spins at a time.

One group in each class worked on this activity. All of the groups that were involved concluded that the "winningest" fraction was 1/2. One boy simply wrote "The more we spin the more it's 1/2." No one group came up with any specific reason why this was so, although one of the boys did comment that he could make a better shaped arrow!

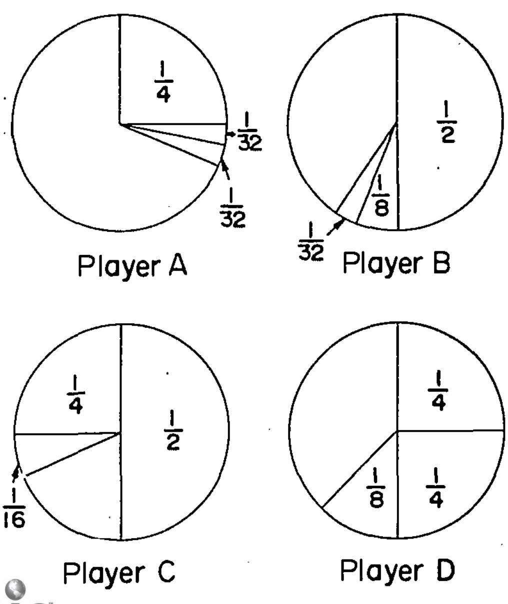


Fig. 9

"Does the shape of the arrow have anything to do with 1/2 coming up most often?" I asked.

He did not know.

Another activity card read as follows:

SCALE MEASUREMENT

Materials: road map, string, 12-inch ruler,

Use string to follow along the main roads between selected cities.

How many miles are there between the selected cities?

Interest in this game varied. Some groups were absorbed all period. They were the ones who had worked out the meaning of the scale at the bottom corner of the map. Other groups, those who could not figure out the meaning of the mileage scale, tired of this game very quickly and went to something else. Still others just sat and gossiped until the scale idea was explained.

Another activity card gave the following instructions:

MEASUREMENT IN INCHES, FEET. AND YARDS

Materials: piece of string. 12-inch rule.

Measure as many objects in this room as possible.

Record results in feet and yards. Compare results with other groups for accuracy.

The group that showed the most interest in this activity (and had the most fun) was in a class of low achievers. They had a wonderful time measuring each other's heights, including the height of the teacher. They measured desks, chairs, the length of the floor, and even the length of a girl's blond hair. They did discover that different measurers came up with different results for the same object measured. Errors were quickly recognized. The boy who translated 72 inches into 5 feet was corrected by everyone in his group. Generally this activity evoked such enthusiasm as to involve everyone in the group.

From the youngsters' point of view, the mathematics laboratory was a huge success. There was always laughter, and at s, the joyous noise level was so high that the place did not sound like a classroom at all. No one was cracking the obvious learning whip. There were no tests to take, no homework to worry about, no notes to copy. It was a period of freedom. On Fridays the students sought the author out on the stairways, in the hallways, and in the lunchroom; or they stopped by his room to ask only one question, "Are we having math lab today?"

Two obvious questions can be raised. What did this experiment prove? Is a mathematics laboratory a worthwhile program of learning on an extended basis?

The experiment was implemented toward the end of the school year, and it was of short duration. Generally the activities were haphazard; the pupils could try one game or another, visit, or gossip. Learning was casual, self-oriented, and untested; but it was there, whether we call it discovery or reinforcement. Whenever an activity card was deliberately vague and allowed the youngsters a choice of direction, the usual response at first was a dead stop. As the program progressed, the children caught on to the new idea: they were to proceed on their own.

The strict regimen in mathematics is all too familiar. It does not reach all of the youngsters. The phrase *under achiever* was coined for that group of students who find mathematics too difficult to comprehend. A mathematics laboratory could be programmed once a week and it could be coordinated with the week's lessons. It should make more use of manipulative activities than pencil and paper activities. In our experimental mathematics laboratory program, the youngsters who physically worked with inches, feet, millimeters, and centimeters learned about measurement more quickly (and had a great deal of fun with it) than those who learned with pencil and paper exercises and table memorization. On an extended basis, learning with the hands can be fun, and it can become more meaningful to the student.

The role of the teacher has to be more

than to just stand aside and hope that Johnny will make a great discovery or get started on his own. Usually Johnny does neither. It is in a laboratory setting, with proper guidance, that Johnny can learn to get started on his own and perhaps go in a direction of his own choosing and still have fun doing it. *But he must be placed in such a position again and again* before he learns that he is allowed to wander off on his own and that this wandering has value.

During the final Friday program, some of the youngsters brought in wood dice they had made in shop.

"Why can't we use these in the games?"

"You can. How true are they?"

"They're pretty good. I made them."

"How would you test them out?"

"Use them, I guess."

"Would you like to try it out and bring in a report?"

"How many times would I have to roll them?"

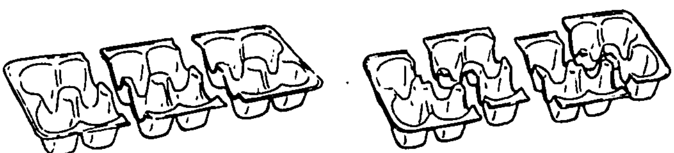
"100, 500, maybe 1,000 times."

"Boy! It'll take all summer!"

"Not as long as that. Is it worth a try?"

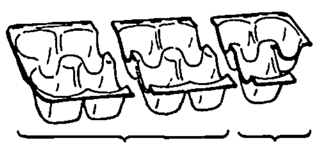
"O.K., 1,000 times!"

EDITORIAL COMMENT.—It should be apparent that the egg carton pieces may be used to identify equivalent fractions. Therefore, we can also use the egg cartons for adding and subtracting rational numbers. For example, suppose we wish to add 2/3 and 1/4. First we try to find a number of pieces from which we can easily make both thirds and fourths (that is, we find a common denominator). This turns out to be twelve.



thirds fourths

To find the sum we consider:



$$\begin{aligned} & 2/3 + 1/4 \\ = \; & 8/12 + 3/12 \\ = \; & 11/12 \end{aligned}$$

Fraction bingo

NANCY COOK
The Evergreen School, Seattle, Washington

To almost all fourth graders, the concept of *one-half* is meaningful and well understood. Many fourth graders also know that *two-fourths* names the same fractional number, but *why* it does is frequently not understood. To help my class of fourth, fifth, and sixth graders to better understand and to discover for themselves how to find other names for a fractional number, I made the game that they named "Fraction Bingo."

Playing cards were made of lightweight poster board, using many bright colors. These cards consisted of three rows of common fractions with three fractions in a row, the center space being a free space (see fig. 1). Various names were used for the more frequently used fractional numbers, such as 1/2, 2/4, and 3/6. Fractions that could also be written as decimal fractions, such as 5/10, 25/100, and 2/10, were also used.

The game is played like Bingo. The "caller" picks a card from the "caller's pile," and anyone who has on his card some name for that numeral covers it with a disc. "Three-in-a-row" wins.

PLAYING CARD

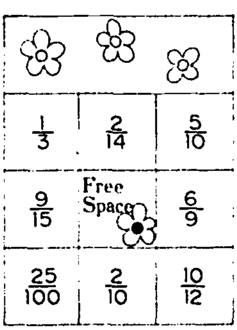


FIG. 1. Playing Card.

Originally the playing cards were five by five, with a free space in the center; but to search a card of twenty-four fractions for a particular value proved discouraging to the novice as it took too long and the game moved too slowly. When the game was changed to the three-by-three cards described above, even the beginner could manage the search among eight fractions. The game now moved quickly and the students did not lose interest.

There are many ways you could start to play this game. I chose to use as "callers" the simple fractions and made small cards with these names and illustrations to be used as the "caller's pile." (See fig. 2.)

"CALLERS"

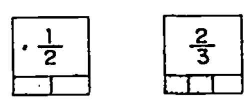


FIG. 2. "Callers."

When I was explaining the game to the class, someone asked, "What do you mean 'another name' for the same fractional number?" I put a pile of plastic discs on the table and, taking two discs, said that these two discs could be considered to be a whole, a unit—say, one whole candy bar (an easy concept for the pupils since many candy bars now come as doubles, two small candy bars in one big wrapper). I then asked how to divide the candy bar into two equally sized (congruent) pieces. They all knew how it could be done. What do we call one of the two congruent pieces? One-half, of course; and we reviewed the fraction 1/2 and discussed what the denominator and numerator represent. Then, taking three discs. I called the three discs my whole candy bar. How can it be divided into two congruent pieces? The class agreed that one disc could be broken into two congruent pieces. I explained that I didn't want to ruin (break, cut, or tear) any of the discs and asked if there were another way? They agreed that there wasn't, but someone remarked that we could still write down another name for 1/2: 1½/3. I then took four discs and called the four my candy bar. Everyone agreed that two of the four discs would represent one-half of the new candy bar. I asked what one-fourth would be. One disc. What would two-fourths be? Two discs. Hadn't we just given another name for two of the four discs? Yes, one-half. And they agreed that one-half and two-fourths were two different names for the same value. Now they were ready to play Fraction Bingo.

We started the game. I called one-fourth, and the search was on. Everyone was manipulating discs in an effort to find other names for one-fourth, but not everyone was using the discs in the same way. One child took four discs, lined them in a row, and separated one somewhat from the other three. He said the one disc was one-fourth of the whole. In another row directly under the first row he put four more discs, making eight discs. He now had two discs separated from the other six and said the two discs represented two-eighths of the whole. He continued in this fashion, each time checking his card for the new name he found. Another child took four discs, separating one a little from the others. He first let each disc stand for one (leading to one-fourth). Without taking any more discs he let each of the four discs stand for two (leading to two-eighths), and he continued in this way, checking his card for the new names.

By the end of the first day two students had discovered that multiplying the numerator and denominator by the same counting number led to another name for the original rational number. But not everybody was that quick. By the next day a few students had their "favorite cards" memorized and they spent their extra time helping friends. On the third day one boy said to me, "I know that eight twenty-fourths is another name for one-third. I figured it out in bed last night!"

By the fourth day everyone had some system other than discs alone to find other names for the rational numbers, systems that they themselves had worked out, systems that were meaningful to them. Many had discovered "the rule" of multiplying the numerator and denominator by the same counting number, and we spent some time finding out why that rule worked. Many already knew why it worked just from the work done in discovering it, and the rest had laid the groundwork necessary for complete understanding of it.

EDITORIAL COMMENT.—The bingo type of game can be used in many areas of the mathematics program, especially where drill and practice are required. Care must be taken, however, that individual games do not become too long and drawn out.

The cards can be used to review basic facts. Sums and products, for example, can be written on the spaces. The caller would give the two addends or factors, and the players would cover the appropriate sums or products. For variation, the caller could name the sum with the addition problem written in the space on the card.

The bingo cards may also be used for identifying numerals in other bases or in ancient numeration systems. Pictures of geometric shapes may be drawn on the card with the caller giving the name of the figure.

FRIO, or FRactions In Order

ROWENA DRIZIGACKER, *Scottsdale, Arizona*

FRIO is a game that will help students understand the relative value of common fractions. It can be played and enjoyed by third graders as well as the upper-grade students and even by their parents. It can be made as simple or as difficult as desired.

For a learning deck you will need twenty-six cards. On these write the numerals that represent the halves, thirds, fourths, fifths, eighths, and tenths. With this small deck two players may play. Later more fractions may be added, and more players may play at once. Also, when the game is being learned, each player is given five cards. Later each may be given up to ten cards, depending on the number playing and the number of cards being used in the deck.

The cards are placed face up in front of the player in the order in which they are dealt to him. Each player in turn draws from the pile that is left face down, or he may take the last card to be discarded. If this new card will help him to get his cards in order from low value to high, he exchanges it for one of his; if not, he discards it.

The object of the game is to be the first one to arrange his cards in order beginning with the smallest value. When a player has done this, he yells, "Frio!"

Now for a sample hand. Suppose a player is dealt the following cards:

$$\boxed{\frac{1}{5}} \quad \boxed{\frac{2}{3}} \quad \boxed{\frac{7}{8}} \quad \boxed{\frac{1}{10}} \quad \boxed{\frac{3}{4}}$$

On his first turn he draws $\boxed{\frac{2}{8}}$ and replaces the $\boxed{\frac{2}{3}}$ with it.

When his next turn comes, there is a $\boxed{\frac{1}{8}}$ on the discard pile, but he decides to draw. Next he draws $\boxed{\frac{2}{4}}$ and replaces the $\boxed{\frac{7}{8}}$ with it.

$$\boxed{\frac{1}{5}} \quad \boxed{\frac{2}{8}} \quad \boxed{\frac{2}{4}} \quad \boxed{\frac{1}{10}} \quad \boxed{\frac{3}{4}}$$

Now all he needs is a card to replace the $\boxed{\frac{1}{10}}$ that is between $\boxed{\frac{2}{4}}$ and $\boxed{\frac{3}{4}}$. As soon as he has done this, he has a FRIO and wins the game. If he draws a card that will not help, he discards it.

Score may be kept in various ways. A record may be kept of the number of wins. (This is the simplest way and is suggested for beginners.) A win may count 50 points, and play may continue until one player reaches 500 or to any desired goal. Bonus points may be given if all the fractions are in "lowest terms" (basic fractions) or if all have the same denominator.

Students and teachers will think of many variations, and these will add to the value of the game. It is surprising how much this game can contribute to the students' understanding of fractions.

To prepare the children to play FRIO, it is helpful to practice arranging the fractions in order. Each pupil should make his own list and perhaps refer to it while he is learning. Such a list would include

$\frac{1}{10}, \frac{1}{8}, \frac{1}{5}$ or $\frac{2}{10}, \frac{1}{4}$ or $\frac{2}{8}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}$ or $\frac{4}{10}, \frac{1}{2}$ or $\frac{2}{4}$ or $\frac{4}{8}, \frac{5}{10}, \frac{6}{10}$ or $\frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{6}{8}$ or $\frac{3}{4}, \frac{8}{10}$ or $\frac{4}{5}, \frac{7}{8}$, and $\frac{9}{10}$.

According to the background and ability of the students, different methods of determining the above order may be used. Fractions may be represented as parts of circles or pictured on a number line and then compared to establish relative size. Common denominators may be found or, for older players, decimal fraction equivalents may be considered. The game is a good incentive for learning the decimal equivalents.

EDITORIAL COMMENT.—Many card games may be adapted to mathematical drill and practice. Rummy is easily converted to drill on basic facts. Ten or more sets of four cards are prepared with different names for the same number on each of the four cards (for example: $12, 7 + 5, 8 + 4, 9 + 3$). To play, students must match corresponding number names.

Old Maid can be adapted to drill with fraction equivalences. Pairs of cards naming equivalent fractions are used. One "monster" card is inserted in the deck. Students play out the game by matching pairs of equivalent fractions.

"Crazy eights" can be converted to "crazy ones" for fraction recognition. Suits can be made from fraction denominators. In the play of the cards, the suit can be changed by playing a "crazy one" card.

Make a whole—a game using simple fractions

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The game called "Make a Whole" helps develop the concept of fractional numbers by using concrete examples; it gives reinforcing experience with the use of fractions and their equivalents. It is suggested for use at third-grade level and above, depending on the achievement of the students.

The material required for the game is pictured in figure 1 and described more completely in a later section. The sections are made in four different colors, equally distributed; that is, of 8 sections representing $\frac{1}{2}$, 2 could be red, 2 blue, 2 green, and 2 yellow. The area for the "Whole" is also prepared in four different colors. The faces of the die are marked with a star and the following fractions: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$.

Rules of play

The object of the game is to use the fractional sections to construct a multicolored disk, the "Whole," in the black mat frame. The Whole may be constructed with any of the differently colored sections so long as they fit together to be equivalent to a whole.

Two to eight children may play the game in individual competition (in which case the color of the area for the Whole makes no difference), or they may play on color teams as determined by the color of that area.

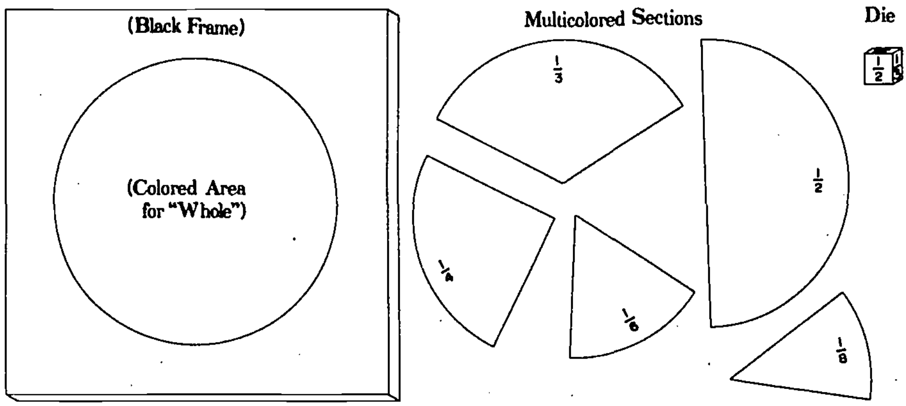


Fig. 1

The fractional sections are placed in a box, from which they are to be selected by the children as they roll the die, in clockwise order.

To determine who goes first, each child rolls the die seen in figure 1. The child rolling the star or the fraction with the greatest value goes first.

At each turn the child must decide whether he can use one of the sections represented by the fraction showing on top of the die.

If the player cannot use the section, he passes.

If he decides to take the section and can use it, he places the section in his mat. (If the section taken completes the Whole for that player, then the game is over and that player has won. If it does not, the roll of the die passes to the next player.)

If he decides to take the section and cannot use it, he loses his next turn.

If he passes up a section he could have used, he loses his next turn.

If the star is rolled, the player may choose any section he can use.

The winner is the first to construct a multicolored Whole.

The game may be played in various other ways, according to rules designed by the teacher. For example, a child might be asked to construct five Wholes with no more than one of a kind of fractional section (½, ⅓, etc.) to go into each. For advanced students, the numeral identification may be omitted from the sections.

The materials may also be used by the teacher to show equivalent fractions in classroom demonstrations and by the individual student in working with equivalences at his desk.

Construction of game materials

A complete list of pieces follows, the color of the mat disks and sections being equally divided among whatever four colors are used.

Mats, 8
½ sections, 8
⅓ sections, 12
¼ sections, 16
⅙ sections, 24
⅛ sections, 32
Die, 1

The material used for the black mat frame and the fractional sections is inexpensive tagboard (poster board) in different colors. Because the Wholes constructed by the students are to be multicolored and because the fractional sections come in equal numbers of the different colors, the colors give no clues to value.

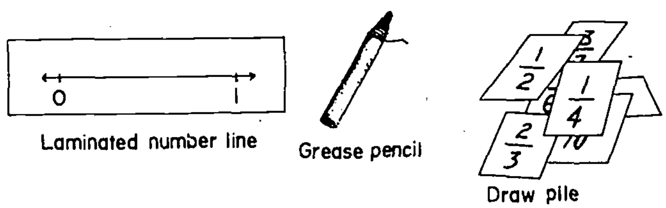
The mats were made first by cutting 8 eight-inch squares from black tagboard. Disks six inches in diameter were cut from the center of these squares, and the frames were then glued to other eight-inch squares, 2 of each of the four colors chosen.

Five disks in each of the four colors were then constructed, with a diameter of six inches. A compass was used for accuracy. The disks were then divided, with the aid of a protractor, into sections corresponding to ½, ⅓, ¼, ⅙, and ⅛. The central-angle measurements for the five sections are 180°, 120°, 90°, 60°, and 45°, respectively. Depending on the level of the children, the sections may or may not be marked with the fractions to which they correspond, as seen in figure 1.

The die was made from a cube cut at a lumber yard so that each edge had a measurement of three-fourths of an inch. The five fractions were painted, one to a face on each of five faces of the cube. A star was placed on the sixth face. The cube was sprayed with a clear varnish to prevent smudging.

EDITOR'S NOTE. Here is a device that can be used for independent learning activity. I dare say the game could be varied by the construction of rectangular sections and mats, so that children would view fractional parts in a broader spectrum. CHARLOTTE W. JUNGE.

EDITORIAL COMMENT.—The game may be extended to include shapes other than circles. The game may also be played on a number line with the draw pile being pieces of paper with fraction names. A student draws a piece from the pile and marks on his number line a segment representing that fraction.



Concrete Materials for Teaching Percentage

ELDON HAUCK
Seattle, Washington

MANY PROBLEMS AND DIFFICULTIES in elementary school arithmetic can be resolved through: (1) better instructional methods, (2) more instructional materials, (3) better instructional materials, (4) concrete materials for each basic phase of the subject, and most important (5) the *know how* in using materials in the classroom. Concrete materials for teaching and learning should have: (1) the property of being handled in such a way as to conduct a learning situation to the handler or observer, (2) the inherent structure to present to the learner a sequential pattern leading to the abstract, and (3) a specific relation and application to the subject being treated which, at the same time, gives an immediate experience into reality. Concrete materials will enable the child to participate in discovery and develop organization and insight.

Leading into Percentage

Understanding of problems and ability to solve problems using common and decimal fractions provide the pupil with a background which will facilitate his understanding of percentage. A thorough review and reteaching of these fundamentals is desirable prior to introducing percentage. Stories about the invention of per cent and its use will arouse interest. Newspaper advertisements showing per cents may be brought to school and displayed and discussed, a store or bank may be visited, and pupils may talk with their parents and others about the use of percentage. These activities should pave the way to a readiness for a study of percentage. All pupils who bring things to the classroom should have an opportunity to present and discuss them with the class.

Our goal is an understanding of per cent together with an ability to solve problems. The factors which lead to an understanding of the thought processes of solutions are:

(1) Changing fractions to decimals and per cent.
(2) Changing decimals to fractions and per cent.
(3) Changing per cent to decimals and fractions.
(4) Per cent means "hundredths."
(5) Of means times (×).
(6) Per cent is a simpler way of working with fractions.
(7) To find a per cent of something is the same as finding a part.

There are two chief difficulties encountered in the teaching of per cent. These are: (1) *the inability to understand the relationship of per cent to fractions and decimals* and (2) that 100% means the whole of *anything*. A third difficulty is encountered in that, if one whole thing equals 100% then two whole things (like objects) equal 200%, etc.

An understanding of the above listed factors and a doing away with the difficulties encountered may be achieved wholly or in part with the use of the "Percentage Box" as illustrated on page ten.

Figure 1 shows the entire box as being equal to 1 or 100% or 1.00; Figure 4 (section A) is equal to $\frac{1}{2}$ or 50% or .5 and figure 2 (section B) is equal to $\frac{1}{4}$ or 25% or .25.

A combination of A and B equals $\frac{3}{4}$ or 75% or .75.

An introductory procedure of explaining the box and showing the above; with discussion and questions would show a definite relationship between fractions, per cent and decimals.

Remove section C from the percentage box (hide sections A and B) and point out that you have something in your hand, section C (figure 3) and that you hold all of this something. Ask what per cent it equals (it is important to keep the inside section printing hidden from the class). In most cases they will answer, 25%. Repeat that you have the whole thing and nothing less. They will soon deduce that you have 100%. Then show them the printing you have been hiding from them. Let them decide what each part and combinations of two or more parts equal in per cent, decimals or fractions, e.g., C2 equals what per cent? What fraction, what decimal? C1 and C2 equal what per cent? What decimal?, what fraction?, etc.

Now bring the other sections, A and B, out of hiding; place all together again and go over the procedure. Point out to them (their questions will call for answers) that all of anything is equal to 100%, but a part of anything can also be equal to 100% of that part.

THE PERCENTAGE BOX

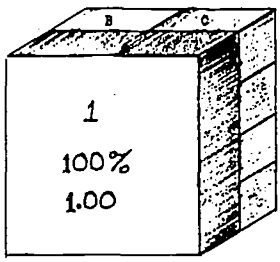


FIG. 1

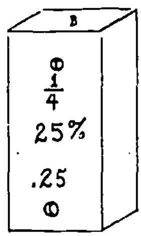


FIG. 2

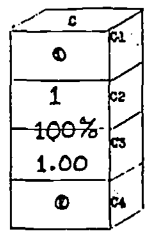


FIG. 3

The box in entirety is a cube. Its dimensions are flexible. It is held together by dowels in section A fitting into the holes of sections B and C.

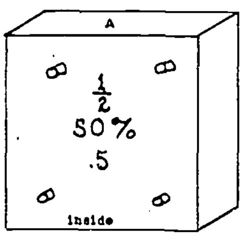


FIG. 4

Section C is held together in the same manner: C1 having the dowel and C4 having the hole with C2 and C3 each having a dowel on one side and a hole on the opposite side.

Then separate section C from the box and separate its parts. Holding C1 in your hand, state that you have a whole thing in your hand and ask them what per cent it equals. They'll answer, 100%. Place C2 beside C1 and say, "If C1 equals 100%, what do both of these, C1 and C2, equal?" If they do not come up with the correct 200%, repeat the process. They will! Then add C3 and C4 to the group for an eventual 400%. Work back through the processes and let the students demonstrate and ask questions about the percentage box.

The second and third difficulties mentioned above have now been surmounted. Repetition of the procedure will help the great majority of pupil's through these obstacles.

To change fractions to decimals and per cent show the whole box and ask, "How many hundredths in one whole thing?" "To find how many hundredths in one-half a whole thing, what would you do?" "If you wanted to find how many hundredths are in one-fourth of a whole thing what would you do?" "Per cent is just another name for hundredths. If you know how many hundredths are in something, can you tell what per cent it equals?" etc.

To change decimals to fractions and per cent, "You know how many hundredths are in a whole thing, how would you find what part of a whole thing fifty-hundredths equal?" "Five-tenths?" "How many hundredths do five-tenths equal?" "How would you find what part of a whole thing twenty-five hundredths equals?" "Fifty per cent?" "Twenty-five per cent?" etc.

To change per cent to decimals and fractions, "What is the difference between 100% and 1.00, 25% and .25, 50% and .5?" "What would you have to do to the per cent to make it like the decimals?" "You know how to change per cent to decimals, can you find the fractions they equal?" "These are called equivalents."

The above procedures have brought about the fact that per cent is just another way of saying hundredths (accompanied by understanding).

Bring forth the percentage box once more. "Let us assume that this box is equal to one hundred cows." "If the whole thing is equal to one hundred cows, how many cows would one-half of it equal? 50%? 25%? $\frac{1}{4}$? 75%? $\frac{3}{4}$?" etc.

To find a per cent of a number raise these questions: "Fifty per cent of one hundred cows equals how many cows?" "25% of 100 cows equals how many cows? One hunded per cent of 100 cows equals? Two hundred per cent of 100 cows equals?" etc.

Now show them the process of working the problem out on paper by changing the per cent to hundredths (decimal) or a common fraction and multiplying. And later when the above has become known to the majority, give them the following step:

To find what per cent one number is of another—"If the whole box equals 100 cows and one-half the box equals fifty cows, how would you find what per cent fifty cows is of one hundred cows?" "Are fifty cows a part (fraction) of 100 cows?" "What part?" "This part equals what per cent?"

"Twenty-five cows equals what per cent 100 cows?," etc., etc. If 100 cows equal one hundred per cent, what per cent would 200 cows equal?" "Two hundred cows are equal to what per cent of 100 cows?" etc.

Now show the problem solving processes: "find the part, change this fraction to a decimal, change the decimal to a per cent." And later:

To find a number when its per cent of the whole is known:

"The whole box is equal to one hundred cows. Twenty-five cows are 25% of how many cows? Seventy-five cows are 75% of how many cows?" etc. "Now let's forget about the box and the hundred cows! Ten cows are twenty-five per cent of how many cows? Twenty-five cows are fifty per cent of how many? Fifty cows are 100% of how many cows?" etc.

The cows need not be used entirely, but the cow is a good example because a cow, when divided, is no longer a cow but a beef.

And now the problem working process can be brought in: finding the answer by either first determining 1 per cent and then 100 per cent, or by dividing the stated number by its per cent expressed as a decimal.

The above exercises should allow each student the knowledge that he can work the problems in his head and therefore most certainly can do them on paper. The practice involved will now be more of a part of his makeup than if the methods had been taught to him by *rote!* His understanding of all fractions should be more complete and his work habits in percentage should be of the desirable nature.

Money may also be used to illustrate further the above points. A silver dollar, 50 cents, 25 cents, dime, nickel, and a penny. The only drawback is the inability to divide the silver dollar. In its favor rests the children's familiarity with money. Once they get the idea from the percentage box, money can be used to good advantage. Whenever a difficulty arises, refer to the percentage box again and again.

Conclusion

Much has been said about and many materials supplied for learning certain parts of arithmetic such as the number system. fractions, and measures but percentages has been neglected. There is need for an interchange of ideas on teaching methods that teachers have found beneficial. These ideas should properly come from teachers experienced in working with children. Better teaching will help the individual child growing up with a feeling of courage, accomplishment, and understanding. Lack of understanding is the basis of fear and the average adult today has a fear of mathematics brought about chiefly through school experiences in this subject. The average adult holds the subject as one of extreme importance in getting along in life and berates his child, hoping the child will learn better than the parent.

Good learning of arithmetic depends upon good guidance and teaching. A good teacher needs not only how to deal with children but also he needs a good background in the materials he is teaching. In many cases this calls for in-service education of teachers in the selection and use of instructional methods and materials. Some new policies in requirements in the teacher training colleges are also desired.

A PERCENTAGE BOARD

Miss Inez Bailey of Rawlins, Wyoming likes to use a large square divided into 100 small squares so that each small square represents 1% and the large square 100% or a whole. When a per cent is mentioned, the children count as many squares as are necessary to represent the given per cent. They then find what part this is of the large square. This teaches them per cent and fractional equivalents in a meaningful way. When a part of a per cent is introduced, they again study their percentage square and see that only a part of one of the small squares is meant. This helps children to visualize and understand the difference between $\frac{3}{4}\%$, for example, and $\frac{3}{4}$ or 75%. It is a simple but very helpful device.

Such a percentage board can be made of materials such as oilcloth, muslin, slated blackboard cloth, or any suitable material.

A Fraction Circle

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OF ALL THE TOPICS presented in the 7th and 8th grade mathematics courses, those dealing with fractions, decimals, and percentage seem to be the most troublesome and the least understood by the student. Often meaningless rules have been committed to memory by the student, but little comprehension has taken place. Occasionally there are students in a class who may add denominators, or insist that 1/4 times 1/2 does not give the same result as 1/2 times 1/4, or become confused that 1/4 divided by 1/8 is 2. Some students do not know or are not sure that 1/3 is larger than 1/4. These are just a few examples of problems common to many classrooms. I believe that many of these problems have their roots in one or more of the following:

1. The basic idea of number and the nature of our decimal system has never been thoroughly understood.
2. The student has been a victim of the ". . . invert the 2nd term and multiply . . . do as I say" type of training.
3. The student has not had the opportunity to discover for himself what a fraction really is and some of the relationships which exist among fractions.

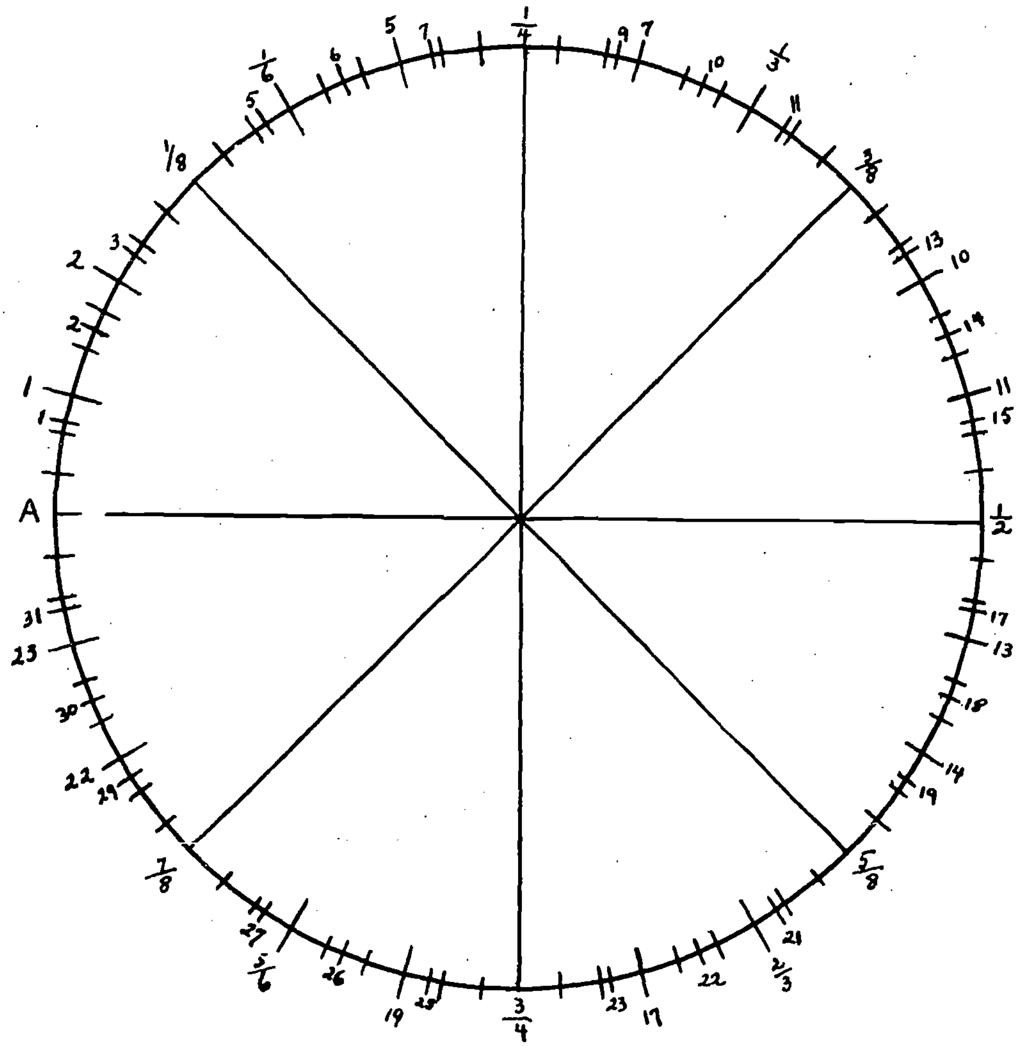
With this third idea in mind, I have made up a device which I hope will enable students to appreciate and understand this business of fractions to a greater degree.

A circle is drawn on a piece of paper or cardboard—a 6″ circle seems to work very nicely. The circle is then divided into 24 equal parts. I found that by dividing each of these parts into thirds would increase the versatility for certain combinations of fractions. Since I wanted to use the circle for halves, quarters, thirds, sixths, and eighths, it was necessary to also divide the circle into 32 equal parts to handle the multiplication of fourths and eighths. The halves, quarters, thirds, sixths, and eighths are labeled with the proper symbol. The 24ths are represented by small whole numbers (numerators), as are the 32nds, but with a different color. The 72nds do not need to be labeled. A good protractor is needed in order to do an accurate job, preferably one which is graduated in half-degrees.

Next, five 6″ circles are cut from colored 6-ply cardboard or construction paper. It is convenient to have a different color for each circle to facilitate recognition. Then each circle is cut up into its respective unit fraction parts. This should produce 23 parts which are labeled with the proper fraction symbol on both sides. It is also necessary to put on each part 1/2, 1/3, and 1/4 marks on both sides. This can easily be accomplished by placing the pieces on the completed fraction circle and marking them accordingly. This completes the construction of the kit.

Arbitrarily, I chose the "9 o'clock" position on the fraction circle as the starting point for the operations of addition, multiplication, and division. The fraction pieces are placed on the circle in a clockwise direction beginning at the radius OA. Suppose we want to add 1/3 and 1/4. The proper fraction pieces are selected and placed on the circle. The two parts added together reach the 14 mark on the circle which is 14/24 or 7/12. More than two fractions with unlike denominators can be added in the same manner, such as 1/3, 1/4, and 1/6.

Subtraction is performed by locating



the minuend on the circle and laying the fraction piece(s) representing the subtrahend in a *counterclockwise* direction from the radius of the minuend. For example, if we wish to subtract 3/8 from 5/6, place three of the 1/8 pieces on the circle in a counterclockwise direction beginning at the 5/6 mark. The result we read off the circle as 11/24.

Division can be accomplished by placing the number of fraction pieces equal to the divisor on the circle in a clockwise direction until the fraction represented by the dividend is reached or *passed*. The number of pieces needed to reach the dividend (it may be a mixed number) is the quotient. Perhaps the problem is to divide 5/6 by 1/3. This requires placing all three 1/3 pieces on the circle. We notice that it takes all of two of the thirds and part of the third piece. We see that 5/6 which is our dividend coincides with the 1/2 mark on the third piece so that our answer is 2 1/2.

Multiplication is a bit different in that the fraction pieces representing either of the two fractions to be multiplied can be placed on the circle, but sometimes one works much better than the other. In the case of 1/6 times 3/4, a piece representing 1/6 can be placed on the circle and then locating the fraction on the circle which coincides with the 3/4 mark on the piece will give us the answer. Also, placing three 1/4 pieces on the circle and noting that half of the first piece represents 1/6 of the three 1/4 pieces, the fraction coinciding with this mark must also give us the correct solution.

With a little experimentation and imagination, it is possible to handle many combinations in each of the four operations. One of the pleasant features is that most answers can be read off directly, and even when adding and subtracting fractions with unlike denominators, there is no need for finding a common denominator. At most, one may occasionally have to reduce to simplest form.

Provisions for the use of fifths may be added if desired. This could probably best be accomplished by making up another fraction circle divided into fifths, tenths, 20ths and 40ths instead of including them on the circle described above.

It would be ideal for each student in the class to make his own kit, either in part or entirely. All 24 pieces can easily be made from colored 6-ply cardboard which is durable, easily cut, and is readily available at a low cost. The parts can be kept in individual envelopes which would fit in a notebook.

The device described here may be more elaborate than necessary for students who are just beginning the study of fractions. The use of halves, quarters, and eighths may be sufficient. For those who have not yet learned to use compasses and protractor, it may be necessary for the teacher to make up mimeograph or ditto copies of the fraction circle and let the student fill in the fraction symbols. Also the teacher may have to prepare the cardboard or construction paper with the circles and parts already drawn on them, but the student can cut them out. For the more advanced student, the entire kit can probably be made with only a little assistance from the teacher. Just having the student computing the number of degrees required for each of the parts of the circle will prove to be a valuable learning experience for him.

I believe that this project would provide the student with the much needed "discovery" type of learning. The student is able to manipulate fractions physically rather than struggle with meaningless abstract symbols. He can readily "see" that 1/5 is larger than 1/8, that 1/6 is half of 1/3, and that 7/8 divided by 1/6 means how many sixths are contained in 7/8? It must be emphasized that how much a student "discovers" is dependent upon how skillful the teacher is in asking the kind of questions which will foster discovery. Even though this is not intended to be a toy, the "game element" can be used to advantage to stimulate interest.

Fun with fractions for special education

RUTH S. JACOBSON

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A device to aid the learning of addition and subtraction of fractional numbers named by like and unlike fractions has been developed in a class for children with special learning disabilities at Boston University School of Education. It is a manipulative device that provides immediate feedback on the correctness of the child's answer to a problem.

This method may be used as preparation or reinforcement for addition and subtraction of fractional numbers when different denominators are involved. Although the example in this article involves halves, fourths, and eighths, the device may also be presented in other fractional values, such as fifths and tenths. This device appears enjoyable to use and, most important, gives the child a method of attack when working with the often difficult concepts of addition and subtraction of fractional numbers.

The aid consists of an eleven-by-eight-inch acetate projectual and corresponding fractional parts (fig. 1). Acetate is used because (1) it can be used on the overhead projector by the instructor and (2) the student can see through the material to the line diagram underneath. The fractional parts are made from a second projectual of another color. This second acetate is cut into pieces that represent fractions. Thus the child's aid has two components, the acetate and the matching pieces, the latter being kept in an envelope.

The author is grateful to John J. Callahan, Boston State College, for assistance in the preparation of this article.

The procedure for the understanding of addition is as follows:

1. Place the acetate projectual (fig. 1) on the desk.
2. Gather the pieces that represent the fractional numbers to be added, for example, $\frac{1}{2} + \frac{2}{8}$ (fig. 2).
3. Join the pieces and place them on the portion of the acetate marked "1 whole."
4. Slide the pieces down until the right end meets a line—for this example, $\frac{3}{4}$ (fig. 3).
5. Look above the line to the name of the point. This represents the sum of the two fractional numbers.

At this time in the lesson some children might discover that $\frac{1}{2} + \frac{2}{8}$ also equals $\frac{6}{8}$ or $\frac{12}{16}$. This would be a suitable time to discuss the fact that $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{12}{16}$ name the same number. When our class discovered that many seemingly different results are sometimes found by using this procedure, we developed a rule stating that although all answers are correct, the best result is the solution on the line closest to the line labeled "1 whole," the simplest form of the fractional number.

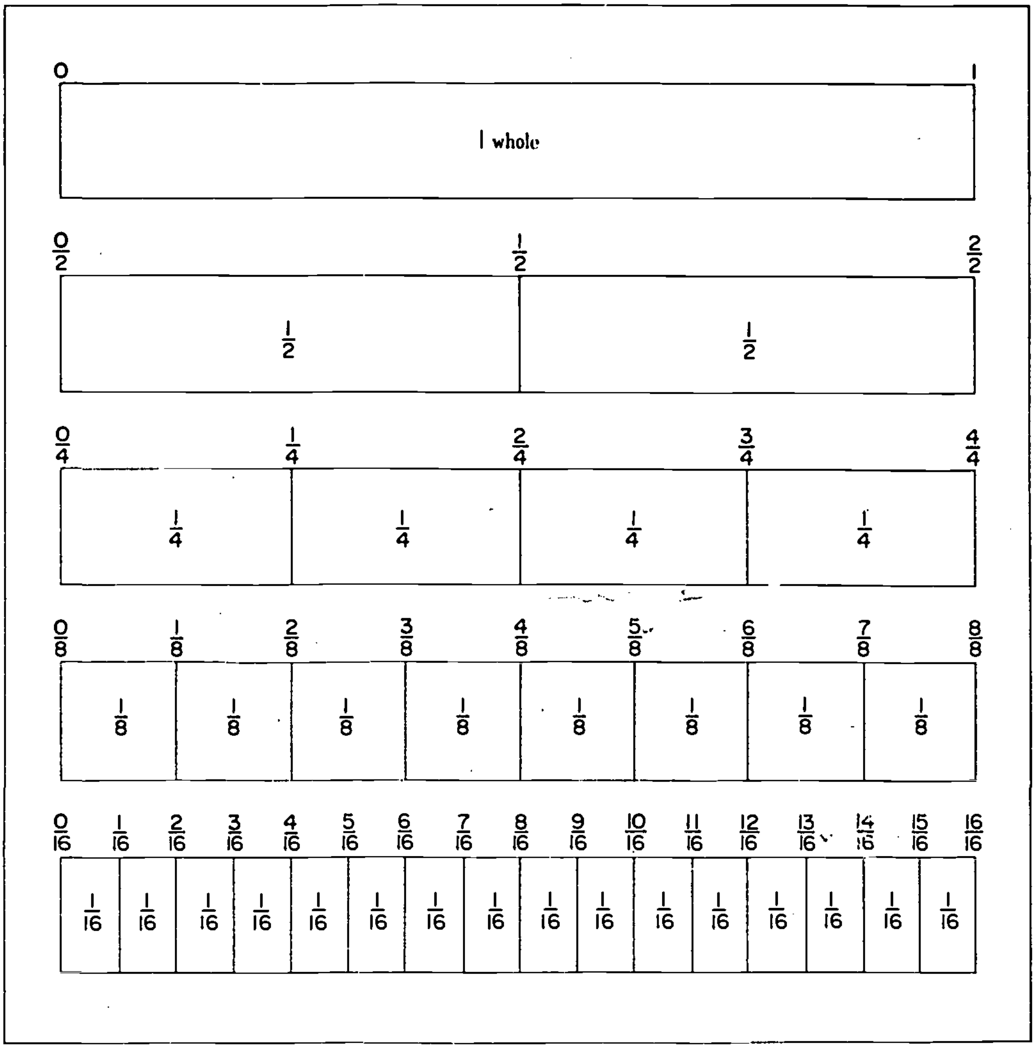

Fig. 1

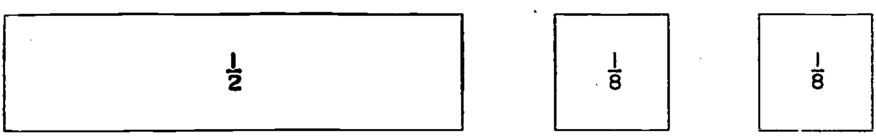

Fig. 2

For subtraction the first and second steps are the same, but in step three the form is different.

In the example $\frac{1}{2} - \frac{1}{8}$, the piece that represents $\frac{1}{8}$ is placed on top of and on the right hand side of the piece that represents $\frac{1}{2}$. The child starts at the top moves the pieces down until the left of the $\frac{1}{8}$ piece is aligned with a point named by a fractional numeral, for example, $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$ (fig. 4).

Many problems with fractional numbers less than one were pursued in our class and the answers were derived with the help of this aid. Further, after using this device for a short period of time many of the children could solve similar problems with paper and pencil only.

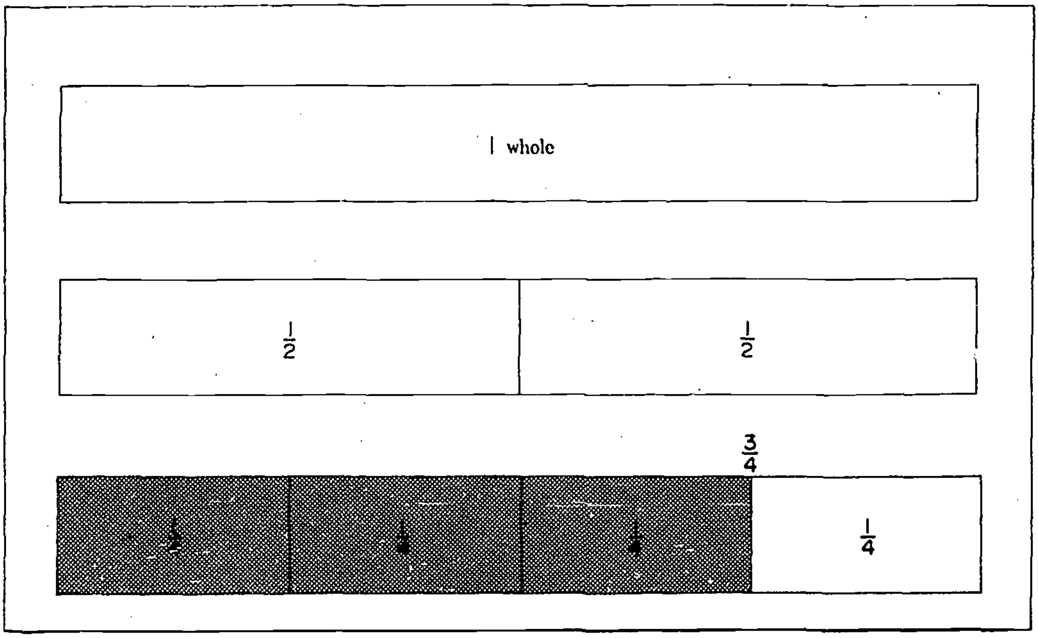

Fig. 3

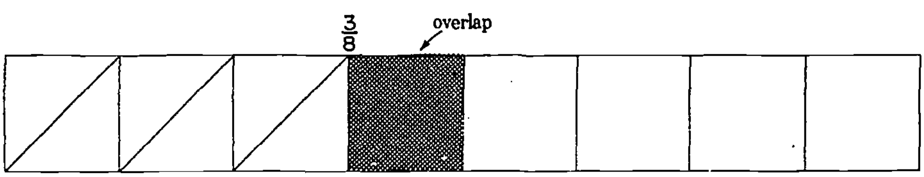

Fig. 4

Editorial Comment.—Similar devices may be used for whole-number operations. Pieces are made to represent the whole numbers from one to ten. For example, a ten-inch strip can represent ten units. The pieces may be used to find basic sums as well as to show the structure of the "teen" numbers, as in 12 = 10 + 2.

This type of aid may also be used for decimal fractions and percents. Making the unit strip ten inches long allows for easy division into tenths. If an inch ruler with 1/10-inch subdivisions is available, it is easy to subdivide the unit into hundredths. Or, in keeping with trends toward metrication, make the unit strip one decimeter long. Tenths and hundredths will then correspond to the centimeter and millimeter marks on the ruler.

Division by a fraction—a new method

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Editor's Note

Presented at the annual meeting of the National Council of Teachers of Mathematics at Chicago, Illinois, April, 1961.

* * *

Great progress has been made in the past decade in making the various operations of arithmetic intelligible for the learner. However, a few arithmetic operations continue to challenge our best efforts in this direction. Of these more challenging operations, I can think of none more difficult to rationalize than the operation of division by a fraction in which we "invert the divisor and multiply." My experiences with hundreds of college education majors have led me to observe that very few of these students come to college understanding the rationale of the operation. This is understandable when we consider the mechanical way in which the operation has been presented in most elementary schools.

In the past decade, marked by renewed interest in the mathematical rationale of arithmetic, we have seen some arithmetic texts attempting to present division by a fraction intelligibly. On the other hand, many authors continue to present the operation largely in a mechanical fashion and justify their treatment on the grounds that (1) the operation is too difficult for pupils to comprehend fully other than in the simplest case of division by a unit fraction, (2) an understanding of the operation is relatively unimportant, and (3) the operation is seldom encountered by most people in everyday life, and hence it should be accorded only minor attention. These authors usually continue to clarify their position by stating that the really important things with regard to the operation are for the student to understand (4) when to divide, (5) how to judge whether the obtained quotient is reasonable, and (6) how to verify in simple cases that the "invert and multiply" process really works.

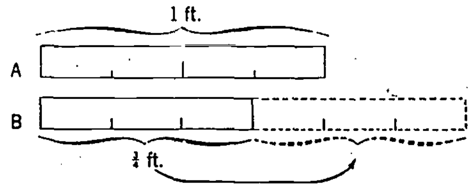
While agreeing at least partially with the latter statements, I wish to take exception to the first three. It is my position that the operation is not too difficult for most pupils to understand if presented properly and that an understanding of it is very important is rounding out a clear understanding of the four fundamental operations and their interrelationships with respect to whole numbers and fractions. While continuing to stress the social aim of arithmetic instruction, contemporary mathematical programs have been placing increased emphasis upon sound mathematical development. In keeping with this trend it is important that we leave no stone unturned in our efforts to find improved ways of rationalizing division by a fraction even though it be true that the operation is used infrequently by many people in everyday life.

New method provides rationale

For the past decade I have been searching for a better way to rationalize division by a fraction. I now believe that I have discovered such a method which I encourage the reader to try. Not only does my method lead to a clearer understanding of the operation, but it promotes a deeper understanding of the role of a reciprocal or inverse ratio. When in the past I have measured the effectiveness of my college instruction using conventional procedures, I have usually found a substantial portion of my students still unable to supply an intelligent explanation of why we multiply the reciprocal of the divisor by the dividend in dividing by a fraction. I cherish the hope that usage of my method will greatly alleviate this situation. Recent experiences with my college students and two classes of sixth graders have been very encouraging in support of my method.

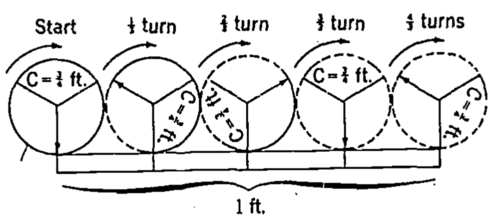
My method provides no easy road to understanding for those students incapable of good thinking, and I am convinced that a demonstration of the method by the teacher is not alone sufficient. Since the method follows the pattern of many scientific experiments, it should be presented in a laboratory approach through which the students make discoveries using their own laboratory materials and record their findings as they progress from one learning sequence to another. Demonstration of principles and generalizations in front of the class using a demonstration set of materials should follow but not precede individual experimentation. When approached in this way, I am convinced that a majority of sixth graders or teachers college students are capable of achieving a mature understanding of the division operation.

My method employs number wheels to illustrate the well-known concept of "measurement division" and enables the student to see easily how many times the divisor is contained in the dividend. Originally I used sticks of various lengths to illustrate "measurement division" and found them quite effective. However I found that students using the sticks were prone to make a certain type of mistake. For example, in measuring a stick one foot long (A) with a stick three quarters of a foot long (B) they were apt to state that the three-quarter foot stick was contained one and a quarter times in the one-foot stick, not clearly perceiving that the remaining one-quarter foot was one-third of another measure with the divisor.



$$A \div B = 1 \div \tfrac{3}{4} = 1\tfrac{1}{3} = \tfrac{4}{3}$$

However, when I shifted from sticks to number wheels to portray the role of various divisors, this difficulty vanished. This is due primarily to the ease of determining the number of turns the number wheels make as they are rolled along the number line by means of the markings on the number wheels. The diagram below illustrates how easy this is. Familiarity with the

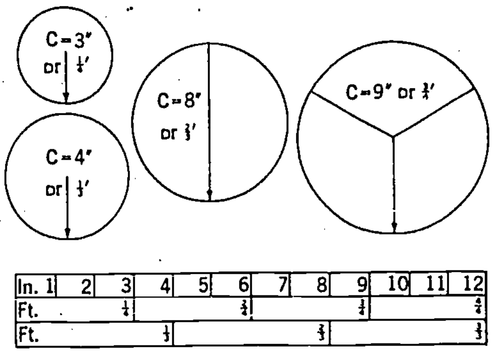


many uses of wheels in our modern world also assists students to grasp their significance quickly in this new division setting.

Construction and use of pupil materials

Prepare a stencil, and mimeograph on long paper four circles and a paper ruler for each pupil as shown on the following page.

The drawings in this article are reduced in scale, but the pupil materials should be



full size to prevent confusion. To get the twelve-inch ruler on a stencil it will be necessary to draw it lengthwise on the stencil. To draw the four circles on the stencil having circumferences of 3 in., 4 in., 8 in., and 9 in., set your compass to radii of 23/48 in., 31/48 in., 1 13/48 in., and 1 21/48 in. respectively. Constructing circles having the correct circumferences requires some skill and the teacher is advised to first experiment with a compass and make the circles before cutting the stencil. Since any error in the radius produces about three times as much error in the circumference, the construction radii above are given to the nearest forty-eighth of an inch. A mechanical drawing ruler usually includes this scale.

After the pupils have been given mimeographed sheets showing the four circles and ruler they can cut them out with their scissors. The circles should be pasted on cardboard cut to the same shape to make them rigid. When a common pin is inserted through the center of the number wheel, the wheel may then be rotated about its axle. In cutting out the circles, the pupils should be instructed to make them a trifle large, for after they have been tested on the ruler for exactness they can be trimmed down to the correct size. Have the pupils print the inch measure on one side and the foot measure on the other side of their number wheels.

Now that the individual materials have been constructed, the pupils are ready to begin experimenting with them. It would be a good idea for the teacher to mimeograph a guide sheet as indicated below. The guide sheet will not only direct the pupils from one learning sequence to another, but it will provide appropriate places for them to record their findings and guide them in making generalizations. The teacher should illustrate briefly with her large demonstration materials how to roll the number wheels along the appropriate scale on the number ruler and count how many times the wheels turn in traveling a given distance. She should lead her pupils to observe that a number wheel dramatizes the role of the divisor, the number ruler indicates the dividend, and the number of turns of a number wheel indicates the quotient.

The tables in the Guide Sheet below provide experiences with both whole numbers (inch measure) and fractions (foot measure). This procedure not only helps to relate the two measures, but it gives the pupils more confidence in working with fractions since they can see how the same principle works with whole numbers. The numerals in parentheses indicate the answers pupils should obtain using their experimental materials. Pupils should be instructed to first insert the missing quotients in each table and then to complete the generalizations which follow. Pupils should not be forced to use their materials if they are sure they can operate on a mental level without them.

Pupil guide and data sheet (with answers)

See Table 1 on p. 125. The wheel in Experiment 1-*B* had (3) times as large a circumference as the wheel in Experiment 1-*A*. Hence the nine-inch wheel turned around only (1/3) as many times as the three-inch wheel in traveling twelve inches.

The divisor in Experiment 1-*D* was (3) times as large as the divisor in Experiment 1-*C*. Hence the divisor in 1-*D* was contained in one foot only (1/3) as many times as the divisor in 1-*C*.

Suppose that Bicycle B has wheels three

Table 1

Experiment	1-*A*	1-*B*	1-*C*	1-*D*
Dividend (length of ruler)	12 in.	12 in.	1 ft.	1 ft.
Divisor (wheel of circumference)	3 in.	9 in.	1/4 ft.	3/4 ft.
Quotient (turns of wheel)	(4)	(1 1/3)	(4)	(1 1/3)
Quotient written as an improper fraction	(4/1)	(4/3)	(4/1)	(4/3)

times the circumference of those of Bicycle A. If both bicycles travel the same distance, the wheels of B will turn around only (1/3) as many times as those of A.

We know that $1 \div 1/4 = (4)$, and since 3/4 is (3) times as much as 1/4, therefore $1 \div 3/4 = (1/3)$ of $4 = 1\ 1/3$ or (4/3).

See Table 2. The wheel in Experiment 2-*B* had (2) times as large a circumference as the wheel in Experiment 2-*A*. Hence the eight-inch wheel turned around only (1/2) as many times as the four-inch wheel in traveling twelve inches.

The divisor in Experiment 2-*D* was (2) times as large as the divisor in Experiment 2-*C*. Hence the divisor in 2-*D* was contained in one foot only (1/2) as many times as the divisor in 2-*C*.

Suppose that American Car B has wheels two times the circumference of those of Foreign Car A. If both cars travel the same distance, the wheels of B will turn around only (1/2) as many times as those in A.

We know that $1 \div 1/3 = (3)$, and since 2/3 is (2) times as much as 1/3, therefore $1 \div 2/3 = (1/2)$ of $3 = 1\ 1/2$ or (3/2).

Study the quotients of 1-*C*, 1-*D*, 2-*C*, and 2-*D* when expressed as improper fractions. Now look at the divisors. Could you predict how many times the fractional divisor is contained in one whole by writing the divisor upside down for the quotient? This is called inverting the divisor. We call 4/3 the reciprocal of 3/4.

$1 \div 4/5 = (5/4) \qquad 1 \div 5/6 = (6/5)$

$1 \div X/Y = (Y/X)$

The reciprocal of the divisor tells how many times a fraction is contained in (1) whole. The truth of this generalization may be verified by proving the foregoing divisions by multiplication:

$(5/4) \times 4/5 = 1 \qquad (6/5)$ of the 5/6ths $= 1$

There are (Y/X) of the *X*/*Y*'s in 1.

The pupils should now be led to extend their thinking to distances which are multiples of 1 foot. Table 3 on the following page illustrates one such approach, and the teacher may wish to construct other tables of a similar nature. The dividend may be portrayed by laying several number rulers end to end. An equally good procedure is simply to measure again and again on a single ruler according to the size of the dividend. Many pupils by this time probably will be able to omit this experimental stage and proceed immediately to the level of abstract thinking involved in completing the table.

Since the divisor in Experiment 3-*A* is contained (3/2) times in one whole, in Experiment 3-*B* we found that the same divisor is contained in two wholes, (2) times as much or $(2) \times (3/2)$ times, and in Experiment 3-*C* we found that the same divisor is contained in five wholes, (5) times as much or $(5) \times (3/2)$ times.

Table 2

Experiment	2-*A*	2-*B*	2-*C*	2-*D*
Dividend (length of ruler)	12 in.	12 in.	1 ft.	1 ft.
Divisor (wheel of circumference)	4 in.	8 in.	1/3 ft.	2/3 ft.
Quotient (turns of wheel)	(3)	(1 1/2)	(3)	(1 1/2)
Quotient written as an improper fraction	(3/1)	(3/2)	(3/1)	(3/2)

Table 3

Experiment	3-*A*	3-*B*	3-*C*	3-*D*	3-*E*
Dividend (length of ruler)	1 ft.	2 ft.	5 ft.	1/3 ft.	4/3 ft.
Divisor (wheel of circumference)	2/3 ft.	2/3 ft.	2/3 ft.	2/3 ft.	2/3 ft.
Quotient (turns of wheel)	(3/2)	(2)×3/2	(5)×3/2	(1/3)×3/2	(4/3)×3/2

If $1 \div 2/3 = 3/2$,
then $1/3 \div 2/3 = \overline{(1/3)} \times 3/2$
and $4/3 \div 2/3 = \overline{(4/3)} \times 3/2$,
$M \div Y/Z = \overline{(M)} \times \overline{(Z/Y)}$,
$M/N \div Y/Z = \overline{(M/N)} \times \overline{(Z/Y)}$.

To divide any number by a fraction *invert* the divisor and multiply it by the number.

Construction and use of demonstration materials

In order that the demonstration materials be clearly visible in front of the class, they should be much larger and made of more durable materials. Multiplying the dimensions of the pupil materials by three provides a convenient size and at the same time results in a change from foot measure to yard measure, a valuable variation. Anyone possessing minimum woodworking skills can easily groove a board 36 in. long to guide the wheels as they are turned. Inch measure may be printed on the top of the board, thirds of a yard on one edge, and fourths of a yard on the other edge. When working with fractional parts of a yard, the appropriate edge can be turned so as to face the class. A minimum of four number wheels should be constructed of plywood, masonite, or plastic. The four number wheels should have circumferences of 9 in., 12 in., 24 in., and 27 in. Their construction radii to the nearest forty-eighth of an inch are respectively 1 21/48 in., 1 44/48 in., 3 39/48 in., and 4 14/48 in. Print the circumference in inches on one side and yards on the opposite side. Holes should be bored through the centers of the wheels so that a nail or other object suitable for turning the wheels may be inserted. The number wheels and number board may be used in exactly the same way as the pupil materials, keeping in mind that they are in yard measure instead of foot measure. If desired, still larger wheels whose circumferences are multiples of one yard might be constructed. This would permit the teacher to relate division by a fraction to division by a whole number. Thus, for example, it would be easy for the pupils to see that a wheel having a circumference of two yards will turn only one-half time in traveling one yard. This type of measurement division often has little meaning for pupils in spite of its simplicity.

Summary

Division by a fraction has long been regarded as one of the most difficult algorisms, if not the most difficult, to rationalize with pupils. Preliminary testing of my experimental approach to the operation indicates that encouraging numbers of pupils are capable of developing a mature understanding of the operation. The unique feature of the method is the employment of number wheels of various sizes to portray the role of the divisor. Using their easily constructed materials pupils first discover that a unit fraction, $1/Z$, is contained Z times in one whole. Comparing this quotient with the one obtained when dividing by a nonunit fraction, Y/Z, they deduce that since Y/Z is Y times as large as $1/Z$ it will be contained only $1/Y$ as many times as $1/Z$. Hence they generalize that $1 \div Y/Z = Z/Y$. And finally, the pupils are led to discover that when the dividend is a multiple or fraction of one whole, the quotient will be that number times the reciprocal of the divisor.

A game with fraction numbers

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The game described below is a unique way to provide practice in the addition, subtraction, multiplication, and division of fractions. The game can also be used to teach ratios as well as the sequence of numbers.

Use a 2-by-2 ft. piece of oaktag or cardboard to make a playing board. Divide it into spaces as shown below, then write the fractions as figure 1 shows.

A	Start	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{5}{10}$	$\frac{6}{12}$
B	$\frac{2}{3}$	$\frac{4}{6}$	$\frac{6}{9}$	$\frac{8}{12}$	$\frac{10}{15}$	$\frac{12}{18}$
C	$\frac{3}{4}$	$\frac{6}{8}$	$\frac{9}{12}$	$\frac{12}{16}$	$\frac{15}{20}$	$\frac{18}{24}$
D	$\frac{4}{5}$	$\frac{8}{10}$	$\frac{12}{15}$	$\frac{16}{20}$	$\frac{20}{25}$	$\frac{24}{30}$
E	$\frac{5}{6}$	$\frac{10}{12}$	$\frac{15}{18}$	$\frac{20}{24}$	$\frac{25}{30}$	$\frac{30}{36}$
F	★ Winner	$\frac{6}{7}$	$\frac{12}{14}$	$\frac{18}{21}$	$\frac{24}{28}$	$\frac{30}{35}$

FIGURE 1

On some small pieces of tagboard $3\frac{1}{2}$ in. by 8 in. write the following symbols and numerals: $-\frac{1}{2}$; $+\frac{1}{3}$; $\times\frac{1}{4}$; $\div\frac{1}{5}$; $+\frac{1}{6}$; $\times\frac{1}{7}$; $-\frac{1}{8}$; $\times\frac{1}{9}$; $+1\frac{1}{2}$; $\times2\frac{2}{3}$; $\div3\frac{3}{4}$; $+4\frac{1}{5}$; $\times5\frac{5}{6}$; $-6\frac{6}{7}$; $+7\frac{7}{8}$; and $\times8\frac{8}{9}$.

$\boxed{-\frac{1}{2}}$ $\boxed{+\frac{1}{3}}$

$\boxed{\times\frac{1}{4}}$ $\boxed{\div\frac{1}{5}}$

FIGURE 2

In figure 2, examples are shown of how the front side of the small pieces of tagboard would be labeled.

On the back of each small piece of tagboard write the answers for the problem that is on the front. There will only be six possible answers, since there are only six rows: *A, B, C, D, E, F;* the rest of the fractions in the row are equivalent to the first fraction in that row. Figure 3 shows how the back side of $-\frac{1}{2}$, $+\frac{1}{3}$, $\times\frac{1}{4}$, $\div\frac{1}{5}$ would look respectively.

Two to six students can easily play this game, or the class may be divided into groups.

$-\frac{1}{2}$	$+\frac{1}{3}$	$\times\frac{1}{4}$	$\div\frac{1}{5}$
A 0	A $\frac{5}{6}$	A $\frac{1}{8}$	A $2\frac{1}{2}$
B $\frac{1}{6}$	B 1	B $\frac{1}{6}$	B $3\frac{1}{3}$
C $\frac{1}{4}$	C $1\frac{1}{12}$	C $\frac{3}{16}$	C $3\frac{3}{4}$
D $\frac{3}{10}$	D $1\frac{2}{15}$	D $\frac{1}{5}$	D 5
E $\frac{1}{3}$	E $1\frac{1}{6}$	E $\frac{5}{24}$	E $4\frac{1}{6}$
F $\frac{5}{14}$	F $1\frac{4}{21}$	F $\frac{3}{14}$	F $4\frac{2}{7}$

FIGURE 3

The rules of the game are as follows:

Each player will begin with the first numeral on the large tagboard. which is the fraction ½.

He will, without looking, pick a small piece of tagboard with a symbol and fraction on it.

He will then perform the mathematical operation in order to arrive at the correct answer.

If the player gets the answer correct, he moves one space. If he is wrong, he remains where he is. (The answer can be verified by checking the back of the card used by the player in performing the mathematical operation.)

The first student to reach "winner" is the "mathematical champ" or "whiz kid."

All answers must be reduced to their lowest terms. Improper fractions must be changed to mixed numbers.

Each player is given only one chance per card, then it is the next player's turn.

The fraction on the small piece of tagboard or cardboard should always be considered to represent the second numeral in the problem; thus, it will either be the second addend, the minuend, the multiplier, or the divisor.

Example:

To begin the game, player A, without looking, picks a card. The card states × ¼. Since all players begin with the first square on the large oaktag board (½), the player must multiply ½ × ¼.

Player A's computations:

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}.$$

Player B checks the back of the card being used to see if the answer is correct. The answer is correct, so player A moves to the next space, 2/4. It is now player B's turn. Player B misses his problem. Therefore, he must remain at the same place until it is his turn again.

If there are no more than two players, it will be player A's turn again. Only this time instead of being at ½, he has now moved to 2/4. He draws another card; the card states − ½. Player A must subtract − ½ from 2/4.

Player A's computations:

$$\frac{2}{4} - \frac{1}{2} = \frac{2}{4} - \frac{2}{4} = 0.$$

Player B checks his answer by looking at the back side of − ½ to find the answer. He sees that the answer is correct, so player A moves to the next space, 3/6. It is now player B's turn again. Player B should always get his turn last, since player A was first.

If the game is used by the teacher in row competition a few times, the pupils should be led to see the relationship between numerals. The student should be asked what numeral would follow the sequence in each row; then this can be expanded to lead the pupils to see the relationship between the numerator and the denominator of each fraction.

6

Geometry and Measurement

Geometry in the elementary school is treated informally and intuitively. The teaching of geometry centers more on the use of manipulative aids than any other topic in elementary school mathematics. A child is introduced to space concepts by the physical objects for which geometric ideas are the abstractions.

The NCTM publication *Readings in Geometry from the Arithmetic Teacher* contains a number of articles that describe manipulative aids for teaching geometry. We have not attempted to duplicate the materials contained in that collection.

The lead article in this section describes an activity with the geoboard. Wells uses a nine-pin board, but it is possible to construct boards with a greater number of pins. The article by Fisher on pegboards in the last chapter of this book also describes activities suitable for the geoboard.

Bruni describes two methods for justifying the sum of the measures of the angles of a triangle. The first is the well-known experiment in which the "three corners" are rearranged to suggest a straight angle. The second uses an intuitive notion of limits. Backman and Smith suggest models for investigating the classification of triangles by properties of their sides and by properties of their angles.

The outdoors is the laboratory in which Woodby suggests activities with the angle mirror. This is a device for establishing right angles over large distances. It is an easy step from the angle mirror to kaleidoscopes and to investigations of other reflective properties of mirrors.

Students usually have little difficulty accepting the product relationship for finding the area of rectangular regions. Colter describes a manipulative aid that helps students rationalize the area formula for a circular region. The aid relates the circle area to formulas for the area of a rectangular region and the circumference of a circle.

Hall's article on the Pythagorean theorem does more than provide students with a visualization of the relationship of the squares of the sides of a right triangle—it also explores relationships between dissimilar figures with equal areas.

Children are usually enthusiastic about constructing models of geometric solids, especially when they use the models for holiday decorations or contemporary mobiles. Wahl provides us with a set of patterns as well as directions for the construction of a tetrahedron, a hexahedron, an octahedron, a decahedron, and an icosahedron.

Using material originally designed for crafting artificial flowers from a viscous plastic solution, Wahl dips wire frames for various solids and comes out with some very creative surfaces. One of the surfaces generated by this technique is the Möbius strip.

Kaprocki shares a method for wrapping a number line around the face of a clock, thus giving children a better idea of the hour scale on a clock. One could go another step and fashion a similar number line for the minute scale.

The last article shows how to integrate many facets of geometry and measurement into the construction of a very ancient device—the sundial. Wahl has simplified the construction so that elementary school children can make a reasonable sundial that actually works.

Creating mathematics with a geoboard

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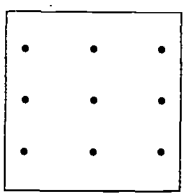
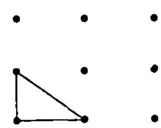
A nine-pin geoboard is easily constructed from 3/8-inch plywood and 5/8-inch panel pins, the pins being about two inches apart. A large quantity of colored elastic bands is also needed.



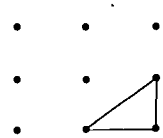
A piece of wood, some pins and some elastic bands! What is this contraption for? What shall we do with it? The geoboard is essentially a piece of material for a child to work creatively with; it gives rise to many activities that lead to discussions of an open-ended nature. The nine-pin board has the advantage of being so simple that the child believes that any mathematics that may arise can soon be sorted out. Results are quickly and easily recorded on squared or dotted paper; this provides a definite advantage to those who find written communication of their mathematics more difficult that the mathematics itself.

To illustrate the mathematical flexibility of the geoboard, this article will discuss some lessons experienced with a group of children. One board and a pile of elastic bands between two children is ideal—a partner is there to encourage, correct, and argue with. Initially the children just put the bands on the board, and the girls soon needed more bands of particular colors to continue the patterns they were making (possible work on symmetry here?). During this free activity stage, the teacher is watching for situations that particularly interest the children and that are suitable for development and extension.

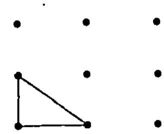
Some children became interested in the smallest triangles.



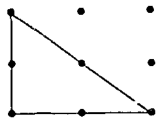
How many triangles the same as this one can you make on the board? A problem arose immediately. Is



the same as

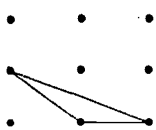


Some children said yes; others, no. Some wanted to include this one.



What do we mean by "the same"? After much argument most of the children seemed happy with counting the first two only as the same, agreeing with Michael who said, "If I cut one out of cardboard I can fit it on top of the other." Arguments of this type challenge mathematical interpretation. Definitions, classification, and communication are all involved. Ideas of sets and equivalence relations are behind the thinking, and the teacher should be aware of this although the children may not be. The children might not have classified  and  as "the same," and the problem would then have developed differently; but once a field of discourse has been agreed upon, mathematics will develop.

How many of the smallest triangles are there then? All the children except one reached for elastic bands and the board and set out to make all of them. They soon found recording was essential as the boards became overcrowded. James, on the other hand, thought for a moment and came over and said, "There are sixteen." His explanation was "Four fit into a small square, and there are four small squares." When asked to find how many of these



there are, he went away and began putting bands on a board.

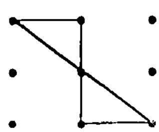
The discussion moved on to larger triangles. How many different triangles (in Michael's sense) can be made on a nine-pin board? This can be played as a game between two children, alternately making and recording triangles different from previous ones. An extension to the different quadrilaterals is more challenging although both games should result in a draw. Some of the sixteen different quadrilaterals are not easy to find. Moving the elastic bands on the board illustrates how figures are related dynamically, a different kind of activity from drawing quadrilaterals on squared paper.

I list some other starting points on the classification of shapes on a nine-pin board which a teacher could draw upon if the opportunity arose.

1. How many different squares can be made on the board? How many of each type?
2. Consider the four-sided shapes. Which ones have an axis of symmetry? How many can you give a familiar name to?
3. Look at the triangles you can make. How many are isosceles? How many are right-angled? How many of each type are there on the board?
4. Make shapes with more than four sides. What is the largest number of sides you can get?
5. How many straight line segments of different length can be made?
6. What shapes cannot be made on the board?

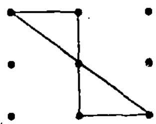
Although the teacher should be aware of these possibilities, I consider it preferable to follow up the children's own suggestions. It is essential to *listen* to the children and to allow them to agree on a method of classifying, which may not be the teacher's own method.

In discussing point 4 one of the girls produced

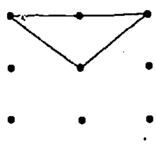


and said it had six sides. The class soon divided itself into those who said it had four sides and those who said it had six.

The argument was settled for this class by one boy who said, "If you think that



has six sides, then,

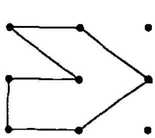


has four."

On larger geoboards there are so many of the various shapes that it is not practical to go through all these questions again, but if the question is more definite (e.g. make an isosceles trapezium or a right-angled triangle that is not isosceles) the activity could be worthwhile. The children are able to create the shapes themselves without the difficulties of ruler and pencil. Discussion of various points arising can follow with the whole class (if necessary) very quickly. Teacher: "Make a shape like mine on your board." Useful ideas are developing as the child moves the band to the correct position before the discussion of the particular point even begins.

In this article emphasis has been upon classification and counting, but there are many other mathematical situations that may develop from a geoboard. Chapter 8 of *Notes on Mathematics in Primary Schools*, Association of Teachers of Mathematics, Cambridge University Press, is an invaluable source of ideas and problems for both children and teachers.

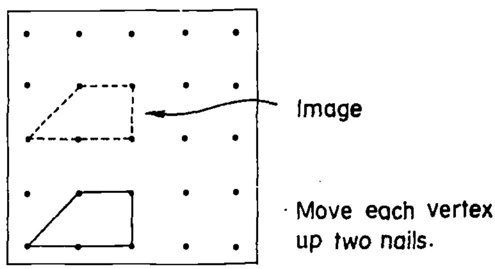
Let us end with a problem for the teacher. For a nine-pin geoboard the largest number of sides a simple closed polygon can have is seven. One solution:



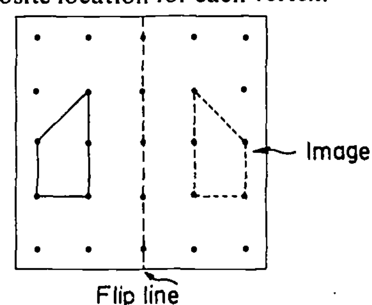
What about a sixteen-pin board, twenty-five pin board, n^2-pin board?

EDITORIAL COMMENT.—The geoboard may be used to display the three motions of transformation geometry—translations (or slides), reflections (or flips), and rotations (or turns). Rubber bands of two colors could be used, one color to represent the original figure and the second color for the image.

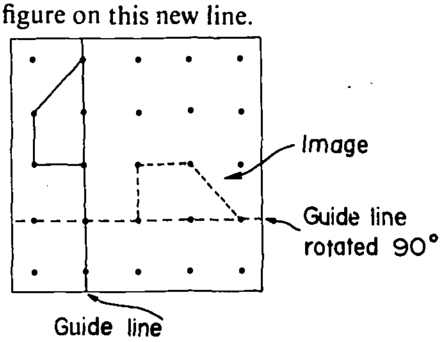
Translation
Move each vertex up two pins.



Reflection
Identify a flip line and find the opposite location for each vertex.



Rotation
Rotate a guide line and then build the figure on this new line.



A "limited" approach to the sum of the angles of a triangle

JAMES V. BRUNI

An assistant professor of education at Herbert H. Lehman College, City University of New York, Dr. Bruni teaches mathematics education courses for elementary school teachers.

One of the exciting discoveries children can make about triangles is that *the sum of the measures of the angles (in degrees) of any triangle always equals 180.* Teachers often lead them to "discover" this phenomenon by having them make all sorts of triangles, use a protractor to measure each of the angles, record the measure of each angle for individual triangles, and look for a pattern or rule. This method gives the child the opportunity to practice using the protractor for measuring angles.

Another approach frequently used is to have students take a piece of cardboard cut into any triangular shape and ask them to perform the following experiment:

Make a black border around the triangular shape. This will serve as a model for a triangle. Label the insides of the angles 1, 2, and 3 respectively, as in figure 1. Now tear or cut the model into three parts and rearrange the pieces as in figure 2. You have now moved the angles so that they form one large angle. That large angle is a straight angle and contains 180 degrees. Since combining the angles is like finding the sum of their measures, this procedure shows that the sum of the number of degrees in the three angles is 180.

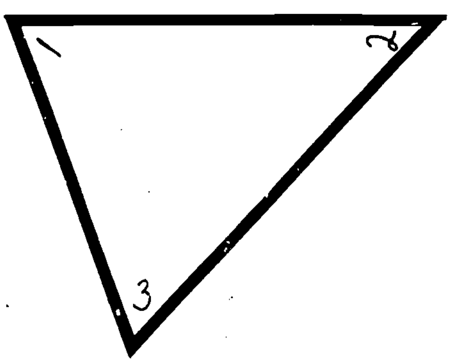


Fig. 1

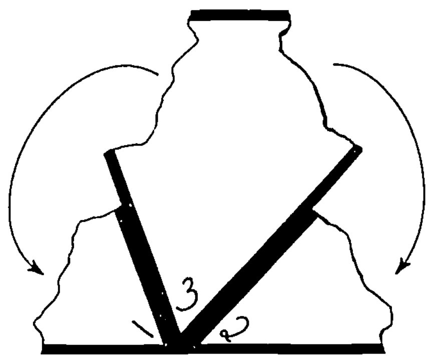


Fig. 2

The following approach to the discovery of the sum-of-the-angles property of triangles is less well known.[1] It is a very useful one because it is an intuitive demonstration that employs the concept of a mathematical limit as the student examines this property of the angles of a triangle.

You will need to construct a very simple model using a small piece of wood or heavy cardboard, two nails or brass fasteners, and a rubber band. Hammer two nails

1. This approach is introduced in Emma Castelnuovo's *La vìa della matemàtica—la geometrìa* (Florence: La Nuòva Italia, 1970).

almost entirely into the wood at points A and B as illustrated in figure 3. Stretch a large elastic band around the nails, forming a figure that looks like a line segment. At about the midpoint of that "line segment," pull upward on part of the elastic band so that you form vertex C and, consequently, the model of a triangle (fig. 4). As you pull at point C you make different isosceles triangles. Ask students to examine the angles of these triangles.

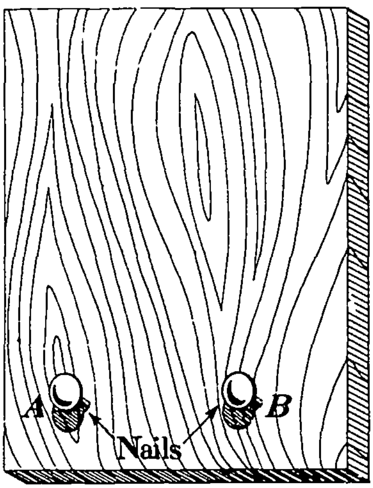


Fig. 3

At one point the sides are all the same length, forming an equilateral triangle. All the angles of an equilateral triangle are congruent and each contains 60 degrees. The sum of the measures of the three angles in degrees is 180.

Next, pull upward at C and form other triangles. Ask students, "How is the size of $\angle C$ changing as I pull upward?" and "How are the sizes of $\angle A$ and $\angle B$ changing?" As you pull the elastic upward at C, $\angle C$ gets smaller and smaller while $\angle A$ and $\angle B$ get larger and larger (fig. 5). Ask students to imagine the elastic stretching farther and farther toward an extreme case (before which the band would break!), with $\angle C$ getting closer and closer to 0 degrees in measure and $\angle A$ and $\angle B$ looking more and more like right angles. Again the sum of the measures of the angles in degrees approaches 180 $(0 + 90 + 90)$.

Now slowly release the elastic band downward and have students examine the triangles formed. They should note what happens to the angles now: $\angle C$ gets larger and larger while both $\angle A$ and $\angle B$ get smaller. $\angle C$ increases in size until it looks almost like a straight angle of 180 degrees (fig. 6); meanwhile, $\angle A$ and $\angle B$ seem to be getting closer and closer to 0 degrees in measure. Once more the sum of the measures of the angles of the triangles formed seems to be fixed at 180 $(180 + 0 + 0)$.

Although the preceding experience *does not prove* that the sum of the measures of the angles of a triangle is always 180 (in degrees), it suggests that this conclusion is reasonable. Actually, only isosceles tri-

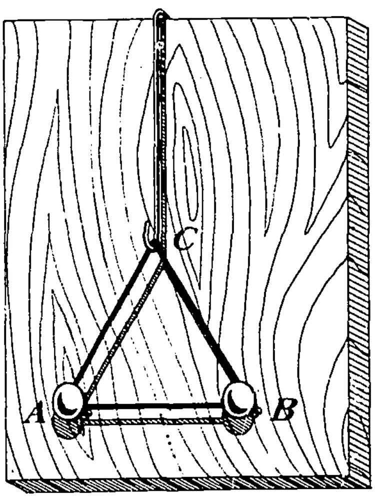


Fig. 4

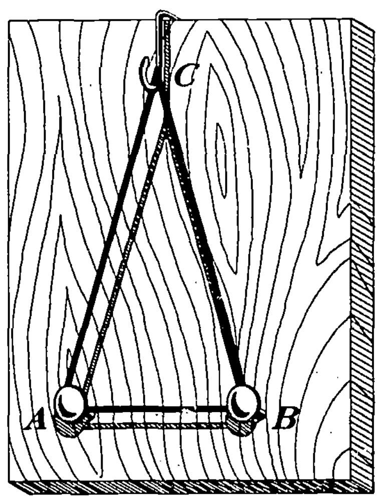


Fig. 5

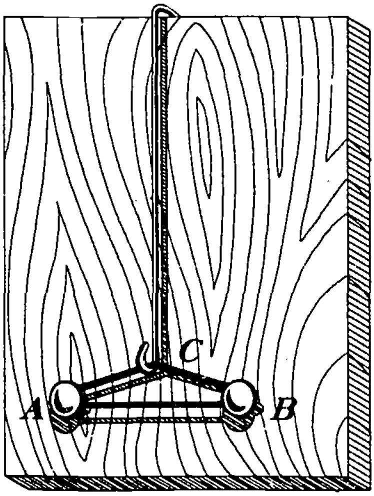


Fig. 6

angles are formed, and the student uses his intuition to discover what is happening to the angles as the model is transformed.

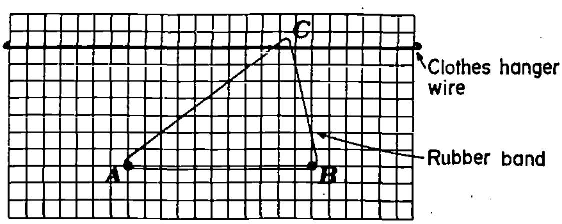
This experience is a valuable way of taking another look at the sum-of-the-angles property of triangles because it involves thinking in terms of *limits,* a fundamental concept in mathematics. As the elastic band is stretched upward the base angles are increasing in size. They each *approach* 90 degrees in measure while the third angle, or vertex angle, *approaches* zero degrees. The extreme case is never reached, since no triangle would exist in that extreme case (you could not have one angle of zero degrees in a triangle). Yet the important point is that the base angles get as close to 90 degrees as you want to make them while the vertex angle gets as close to zero degrees as you wish.

Similarly, in releasing the band downward, you can make the base angles as close to zero degrees as you wish while the vertex angle increases toward 180 degrees. Again, no triangle would exist in the extreme case, since you would end up with a line segment. Yet you can get as close to the extreme case as you wish.

This intuitive demonstration of the special property of the sum of the angles of a triangle can help students develop an early foundation for an understanding of the concept of a mathematical limit.

EDITORIAL COMMENT.—A similar board may be used to explore the effect of changes in the altitude of a triangle on the area of the triangular region. On the board shown in figure 3 of the article, rule off a square-grid system. As the vertex C (in fig. 4) is drawn upward, count the number of square regions enclosed by the figure. If you establish a given position of C as the starting position, you can then investigate areas for altitudes that are given multiples of the original altitude (such as twice as high, half as high, and so on).

A second board may be constructed by starting with the board shown in figure 3 and ruling off the square-grid system. Now fasten a piece of clothes hanger parallel to the line determined by vertices A and B. Slip a rubber band over the wire before fastening. Now vertex C can move along the wire. What happens to the area of triangular region ABC as vertex C is moved along the wire?



Activities with easy-to-make triangle models

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Strips of colored tagboard and brass fasteners can be used to make a set of models for demonstrating a variety of properties of triangles. The models are inexpensive, durable, and easy to store, and they can be made by the classroom teacher for use in classroom demonstrations. They also are easily assembled by pupils for use in mathematics-laboratory activities.

Construction

Specifications and directions for a set of six models illustrating the two classification systems for triangles (*sides*—scalene, isosceles, equilateral—and *angles*—acute, right, obtuse) are given below.

Each model is made from three half-inch-wide strips of colored tagboard (poster board) and three short brass fasteners. Use a paper cutter to cut the strips from sheets of colored tagboard. Holes for the brass fasteners are punched half an inch from each end of each strip. (A ticket punch makes a hole of acceptable size.) After assembling the strips, trim the corners to make them pointed. (See fig. 1.)

Listed below are lengths for the tagboard strips which have been found satisfactory for each of the six models.

Side Classification	Side 1	Side 2	Side 3
1. Scalene (no two sides congruent)	9″	8″	7″
2. Isosceles (two sides congruent)	9″	9″	7″
3. Equilateral (three sides congruent)	9″	9″	9″

Angle Classification	Side 1	Side 2	Side 3
1. Acute (three acute angles)	9″	8″	7″

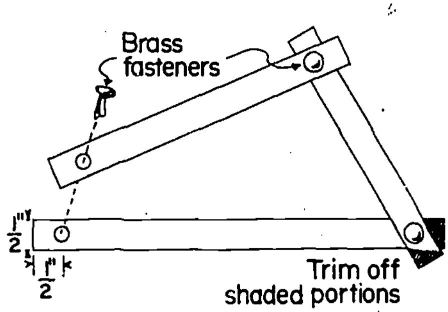

Fig. 1

2. Right (one right angle)	11″	9″	7″
3. Obtuse (one obtuse angle)	10″	7″	6″

Suggested activities

1. *Detecting differences between models*

Students are given the set of three models illustrating the side-classification system (or angle classification). Assuming that the students have not as yet been introduced to the classification system, they are asked to state ways in which the three models differ from each other.

2. *Sorting the models by type*

The models are gathered together. Students are asked to select a model, identify its type, and list its characteristics. (For sorting activities, all models should be made from the same color of tagboard. Otherwise, there is a strong possibility that students may learn to identify the models by color rather than by the desired characteristics.)

3. *Constructing the models*

Assuming that the students have learned the classification systems, they are given a set of prepunched tagboard strips, brass fasteners, a ruler, and a protractor and are asked to assemble models for each of the six basic types of triangles.

Students may also be asked to construct models for certain classification combinations. For example, they might be asked to construct an isosceles right triangle (possible) or an obtuse equilateral triangle (impossible).

4. *Designing new models*

Students are asked to create a set of models that illustrate the basic types of triangles using different strip lengths than those given above.

5. *Developing perimeter formulas*

The models for the side-classification set are constructed so that only strips of the same length are the same color. Students may then use the color cues to develop the three formulas for the perimeter of a triangle: (*a*) scalene, $p = a + b + c$; (*b*) isosceles, $p = 2a + b$; and (*c*) equilateral, $p = 3a$.

EDITORIAL COMMENT.—The model construction suggested in the article is easily extended to polygons of more than three sides. These models, however, will not be rigid. Students may be asked to "stabilize" the models, thus introducing the engineering function of diagonals and exploring the characteristics of triangle rigidity.

The angle mirror outdoors

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Measurement activities outside the classroom have a special appeal to children if they use some optical device. A simple and useful optical instrument can be made by fastening two small mirrors so that their faces form a dihedral angle of 45°. The instrument is sometimes called an "optical square" because it can be used to establish perpendicular lines.

How to make the angle mirror

Two mirrors, some small pieces of wood, and some glue are all that is needed. The size of the mirrors is unimportant. However, since tiny ones work just as well as large ones, and a small instrument is easier to handle, mirrors about one inch square are suggested. One way to mount them is to glue a small rectangular block to the back of each mirror and then glue the two blocks to a flat piece of wood. First glue one block so that the attached mirror is vertical to the base, then glue the other mirror so that the angle between the planes of the two mirrors is 45°. The reflecting surfaces should face each other. See figure 1.

The simplest way to obtain a 45° angle for fixing the position of the mirrors is to fold the square corner of a sheet of paper. Of course, a draftsman's 45° triangle could be used if one is available. Another way to mount the mirrors is to cut a wedge-shaped piece of wood with a miter saw, and then glue the mirrors directly to the faces of the wedge.

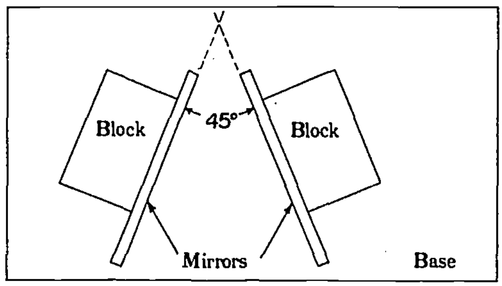


FIG. 1. Top view of the angle mirror

How to use the angle mirror

Basically, the angle mirror just described has only one capability—it can be used to establish lines that form a right angle. The lines are lines of sight between objects. In figure 2, the observer at O looks in the direction of an object A. He then brings the angle mirror into the position shown, and while still looking toward A sees object B by double reflection in the mirror. Thus B appears to be in line with A. When this situation occurs, angle AMB is twice the size of the angle at P formed by the mirrors. Since this angle between the mirrors is

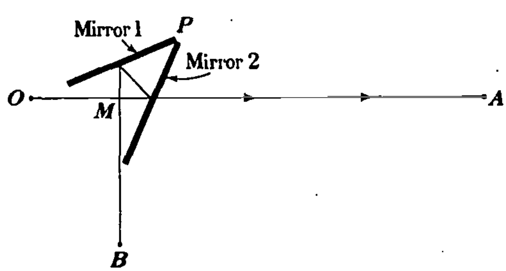


FIG. 2. Sighting with the angle mirror

45°, angle AMB is 90°. (For an analysis of why this is true, see the article by Hart that will be published in one of the fall issues of the journal.)

Since the angle mirror is used out-of-doors, the distances are large compared with the dimensions of the mirror. So the mirror (held close to the observer's eye), is considered a single point, M. A plumb bob on a string fastened to the angle mirror can be used to locate a point on the ground directly below point M. It takes some practice for the observer to see the correct image and to determine when the image of the image of B is lined up with A. It helps if the objects A and B are people who can move as directed, so the observer can visualize the lines involved. In learning how to use the angle mirror, the observer should move slowly while looking toward A and notice the movement of the image of B.

The hardest part is to discover what you are trying to line up. Remember that you are trying to determine two lines at right angles to each other, and the mirror will be the vertex of the right angle.

After making your angle mirror, go outside with two friends and find two lines that are perpendicular to each other, on the tennis court or playground. Stand at the corner so that the angle mirror is above the vertex of the right angle and face toward A. Have one friend stand at A facing you.

Without using the angle mirror, you see friend A. Now have your other friend stand at B, facing you, but you keep looking straight at A. So you cannot see friend B. But now hold the angle mirror up in your line of sight between you and A, and rotate it until, by looking in mirror 2, you see the reflection of B's reflection in mirror 1. Now your eye, the image of the image of friend B, and friend A are all lined up, and the angle AMB is a right angle.

Now try constructing a rectangle. Begin by fixing two points P and Q and stretching a string between these two points to one side of the rectangle.

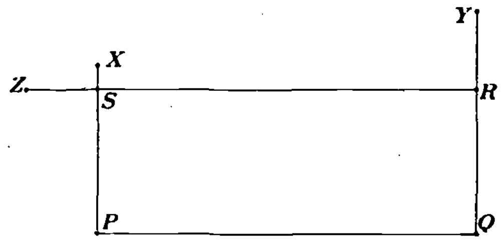


FIG. 3. Constructing a rectangle

The observer then stands at P and sights toward a friend standing at Q. In the mirror, the observer sees the image of another friend at X. He directs friend X to move until X is lined up in the mirror with friend Q. Friend X then marks his position and stretches a string from P to X so that angle QPX is a right angle. The observer then moves to point Q and in the same manner directs a person Y to move so angle PQY is a right angle. Stretch a string from Q to Y. Now several alternatives are possible for completing the rectangle. For example, the observer could pick any point R on the ray QY and locate Z so that QRZ is a right angle, and stretch a string from R to Z. S is the intersection of PX and RZ.

Notice that no distances were measured in constructing the rectangle $PQRS$. Only three right angles were established, and the fourth angle turns out to be a right angle, since the sum of the angles of a quadrilateral is equal to four right angles. Another method would be to mark off the same distance on the rays PX and QY to establish points S and R. In the special case when the distance chosen is the distance PQ, the rectangle would be a square.

Another basic construction with the angle mirror begins with a line (again a stretched string between A and B) and a given point P on that line, as shown in figure 4. The observer at P positions a person at X so that APX is a right angle. He can check the accuracy of the mirror by checking the angle BPX. If BPX is not also a right angle, as judged by the image in the mirror, then there is an error in the construction of the mirror. This error can be overcome by finding the two points X and

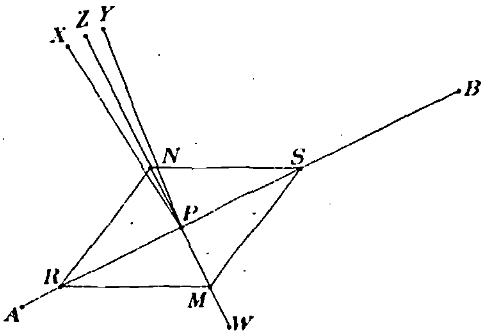


Fig. 4. Establishing perpendicular lines

Y equidistant from P that apparently determine right angles APX and BPY. Then mark point Z midway between X and Y. Stretch a string from Z to W so that the string passes through P, and ZW, will be perpendicular to AB. From this basic construction a rhombus can be made by marking M and N so that $PM = PN$. If $PN = PR = PS$, $MRNS$ will be a square. (If the diagonals of a quadrilateral bisect each other and are perpendicular, the quadrilateral is a rhombus. If, in addition, the diagonals are equal, the figure is a square.)

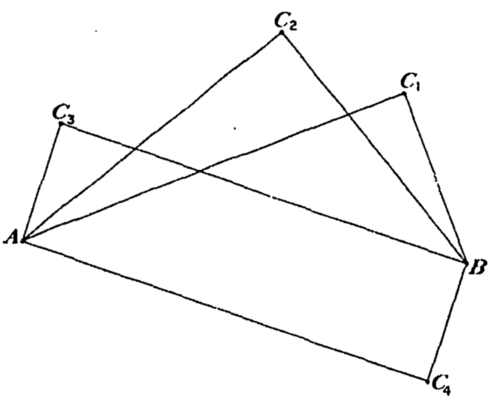


Fig. 5. Locus of the vertex of a right angle

A special application of the angle mirror allows children to discover for themselves that the locus of the vertex of the right angle of a triangle with hypotenuse fixed is a circle. Let two fixed points A and B determine the hypotenuse. See figure 5 above. Stakes are set at A and B, and a student with an angle mirror is asked to locate point C so that ACB is a right angle. He marks his point C and stands there. Another student finds another point C_2 such that AC_2B is a right angle. Now every student is asked to use his angle mirror and locate himself anywhere he chooses so that he is at the vertex of a right angle, with rays through A and B. Students see at once that the locus is a semicircle by observing where other students are standing. Usually someone will try out the other side of line AB. Another way to dramatize the locus is to have a student, one who is skillful with the angle mirror, walk wherever he can and still keep the stakes lined up in the angle mirror. This technique gives a spectacular result if the ground is covered by freshly fallen snow.

When students have mastered the use of the angle mirror, they can use it in a variety of applications. For example, they can realign (or establish) the baseball diamond or the football field. Outdoor construction with stretched strings for lines is particularly appealing. Children who construct a rectangle or a rhombus on the playground, given only a ball of twine and an angle mirror, experience something quite different from a similar task done at their desks using ruler, compass, pencil, and paper.

Editorial Comment.—Students are fascinated by the reflective properties of mirrors. You might consider constructing kaleidoscopes or studying parabolic and spherical mirrors.

The instrument described in this article is designed to construct right angles. Students may wish to investigate other methods of making right angles, such as the Egyptian knotted --pes or paper-folding techniques.

Adapting the area of a circle to the area of a rectangle

MARY T. COLTER

Presently employed by the Department of Defense on the island of Taiwan, Mary Colter is teaching military and civilian dependents. Her previous experience includes teaching Navajo Indians in a New Mexico high school.

One of the most difficult problems for a student studying the areas of regions bounded by geometric figures, regardless of whether he is in the fifth or ninth level in mathematics, is the area of a circular region. The student may memorize the formula, know where to plug in the values, consistently arrive at the correct answer, but never comprehend and master completely the problem. The teacher assumes Johnnie knows what the area of a circular region means by hearing him repeat some "fed facts."

The areas of regions bounded by squares, rectangles, triangles, or trapezoids are relatively easy for the student to visualize. Often teachers show by blocks, cutouts, three-dimensional objects, and so on, how to find the area of a region, especially if it can be expressed as a certain number of square units in the base row multiplied by a number of rows, or "base times height." The area of a circle is not as easy to illustrate, and by the time the class has reached the study of circular regions, they may be tired of deriving formulas. Sometimes the teacher, unsure of himself, approaches this topic as a memorization lesson.

Working with the fifth to the ninth levels, an easy way to teach the area of a circular region is to make the circular region "fit" a rectangular region. This also provides an excellent way to estimate the area of a circular region, thus allowing the student to see if his answer is reasonable. The only materials needed for the class are construction paper, scissors, paste, and a ruler.

The study of areas logically begins with rectangles, squares, parallelograms, trapezoids, and triangles. After careful development of the formula for the area of a rectangle, when the time comes for the study of circles and circular regions, the class may ask whether the area of a circular region can be adapted to the area of a rectangular region.

The study of circles begins with experiments to see how the circumference of a circle compares with the diameter. It may take a couple of class periods and half a hundred circular objects, or each student may do about five experiments at home and report the results in class. The measurements are tabulated, and the class looks for a pattern. Experiments continue until it is evident that in every instance the circumference of a circle is approximately three times the diameter of the circle. The ratio of the circumference to the diameter is identified as pi, and this leads to the formula for the circumference of a circle, $c = \pi d$.

The circle has been defined as the set of points equidistant from a given point called the center of the circle. The distance from the center to any point on the circle is identified as the radius, and a few measurements of diameters and radii confirm the fact that $d = 2r$. The class is now ready for experiments with the areas of circular regions.

Circles of many different radii are drawn on construction paper, and the many circular regions are cut out. The center of each circle should be clearly labeled. Each student is given one of these circular regions to work with. Thus each student can proceed through the following steps:

1. Draw a diameter. Mark the two radii on the diameter, and mark one-half the circumference ($c/2$) on each semicircle as shown in figure 1.

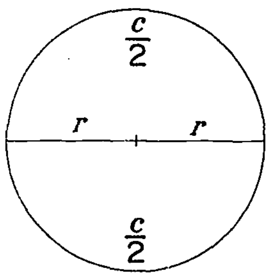


Fig. 1

2. Cut along the diameter. You now have two semicircular regions. Put these two semicircular pieces together so the edges coincide.
3. Beginning at the center each time, cut along many radii, coming very, very close to the edge of the circular region on each radius but not cutting the edge. This results in a string of pie-shaped wedges as shown in figure 2. (Generally speaking—the size of the circle will affect this—each semicircular region should have about ten wedges.)

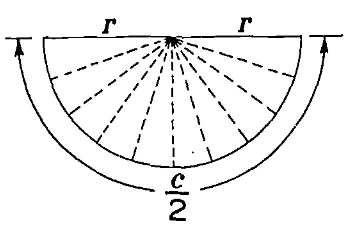


Fig. 2

4. Paste one string of pie-shaped wedges on a different piece of colored paper in such a way as to suggest a rectangular region as shown in figure 3.

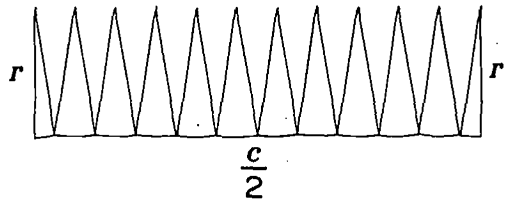


Fig. 3

5. Note that the length of this rectangular region is one-half the circumference of the circle ($c/2$) and the height is the radius of the circle (r). Half the rectangular region is covered with the semicircular region.
6. Paste the second string of pie-shaped wedges on the first as shown in figure 4 so that the rectangular region is covered and the wedges are not overlapping.

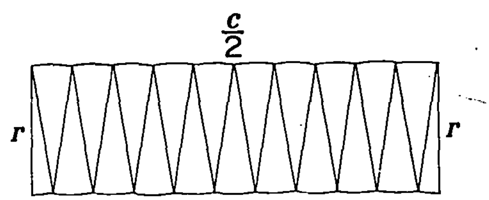


Fig. 4

7. Note that the circular region has now been adapted to a rectangular region, and it is possible to estimate the area of the circle. Since the area of a rectangular region is "length times width," the area of the circular region is one-half the circumference times the radius.

After practice in estimating areas of circular regions, the class is ready for the development of the formula for the area of a circular region.

Through their earlier experiences with measuring the circumferences and diameters of many circular objects, students learned that $c = \pi d$, or $c = 2\pi r$. The circular region was cut into two parts, thus each part included one-half the circumference, or $c/2$. Using the formula $\frac{1}{2} \times c = \frac{1}{2} \times (2\pi r)$, or in simpler form, $c/2 = \pi r$, the length and the width of the rectangular region are given in terms of the radius of the circle. (See fig. 5.) With a little help and direction, the students are able to see that the area of the rectangular region ($l \times w$) in terms of the circle is $(\pi r) \times r$. Thus the formula for the area of a circular region is $A = \pi r^2$.

Practice in finding the areas of circular regions follows the development of the formula. In each problem, students are encouraged to estimate the area using the idea $c/2 \times r$ before using the formula itself in calculating the area of the circular region.

It is amazing how much confidence students will develop in a problem that was once dreaded—how to find the area of a circular region. Students are also intrigued when they find that the area of a circular region can be made to fit a rectangular region.

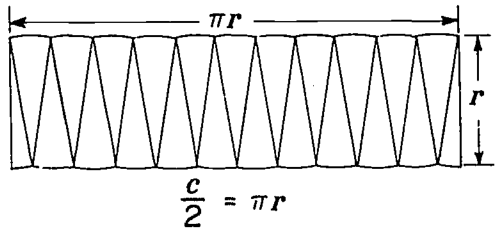
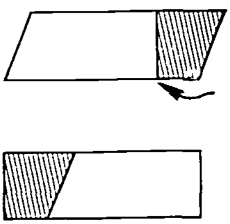


Fig. 5

EDITORIAL COMMENT.—The technique of partitioning a region and rearranging the pieces to form another figure may be used to generate area formulas for other figures.

A parallelogram region can often be rearranged to form a rectangular region.

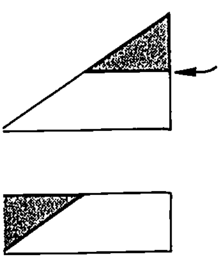


Cut along an altitude.

Rearrange to form a rectangular region of equal area with base and altitude equal to that of the parallelogram.

A triangular region can also be converted to a rectangular region.

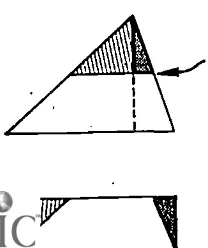
Right triangle



Cut at half-way point of the altitude.

Rearrange to form a rectangular region of equal area. The base of the triangle is also the base of the rectangle. The rectangle is half as high as the triangle.

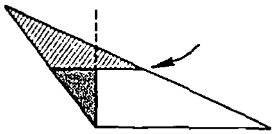
Acute triangle



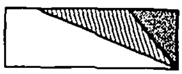
Cut at half-way point of altitude.
Cut top piece along altitide.

Rearrange to form a rectangular region of equal area. The base of the triangle is also the base of the rectangle. The rectangle is half as high as the triangle.

Obtuse triangle



Cut at half-way point of altitude.
Cut along altitude.



Rearrange to form a rectangular region of equal area. The base of the triangle is also the base of the rectangle. The rectangle is half as high as the triangle.

A Pythagorean puzzle

GARY D. HALL *North Judson San-Pierre Schools, North Judson, Indiana*

This puzzle was constructed as a project for a graduate course. It has been used in teaching sixth-grade mathematics.

Purpose

The purpose of this project is to teach the Pythagorean theorem to children who have no background in plane geometry. The ideas of squared numbers and the concept of a 3-4-5 right triangle are also introduced.

The child is taught these related concepts through the use of a brightly colored manipulative puzzle that guides him to form relationships involving area.

Materials

1 right triangle with sides 6 inches, 8 inches, and 10 inches.

1 square (side 6 inches) and 1 parallelogram (sides 6 inches and 10 inches; an angle equal to the smaller acute angle of the triangle described above). Each of these figures has an area of 36 square inches. They should be painted the same color.

1 square (side 8 inches) and 1 parallelogram (sides 8 inches and 10 inches; an angle equal to the larger acute angle of the triangle described above). Each of these figures has an area of 64 square inches. They should be painted the same color.

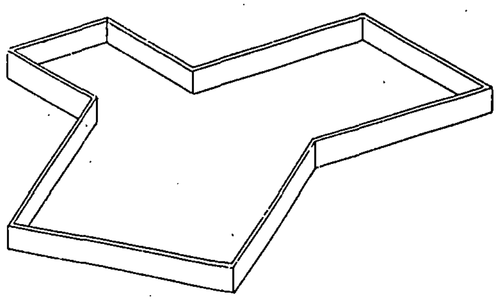


Fig. 1

1 square (side 10 inches) with an area of 100 square inches.

50 squares (side 2 inches), each with an area of 4 square inches; 9 should be painted to match the 6-inch square; 16 should be painted to match the 8-inch square; and 25 should be painted to match the 10-inch square.

1 frame like that shown in figure 1.

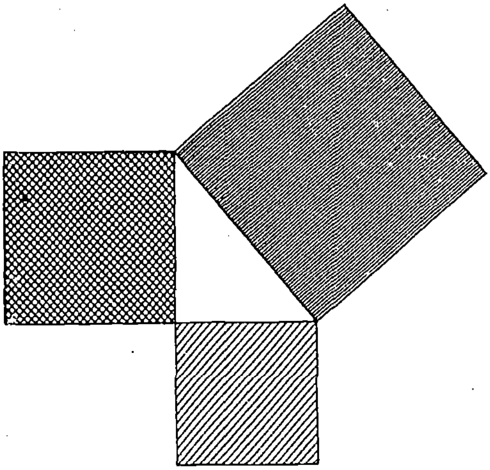


Fig. 2

Procedure

Empty the puzzle frame and give the child the basic puzzle pieces (the triangle and the three large squares). He should construct a figure such as that shown in figure 2.

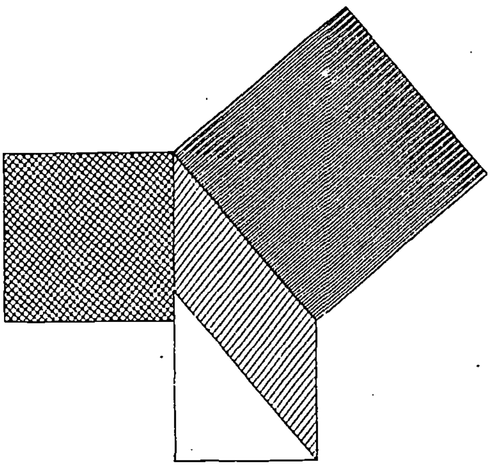


Fig. 3

Empty the puzzle again and give the child the pieces, substituting the smaller parallelogram for the smallest of the three squares. (Note: These two pieces are painted the same color to facilitate the conclusion that they are the same area.) The child should then conclude that the small parallelogram and the small square are the same size by constructing figure 3.

After emptying the puzzle the third time, give the child the largest square, the smallest square, and the triangle and substitute the larger parallelogram for the medium square. (These two are also painted the same color and are the same size.) The child should construct figure 4.

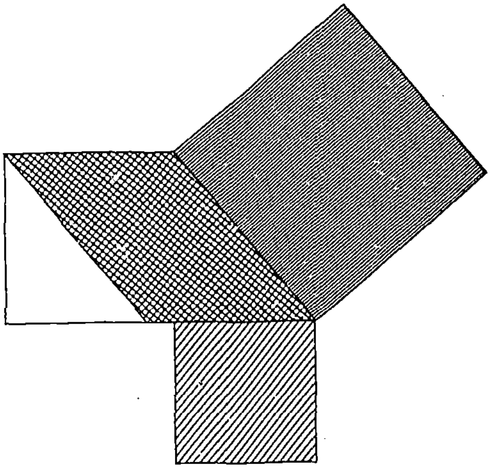


Fig. 4

The fourth construction (fig. 5) is

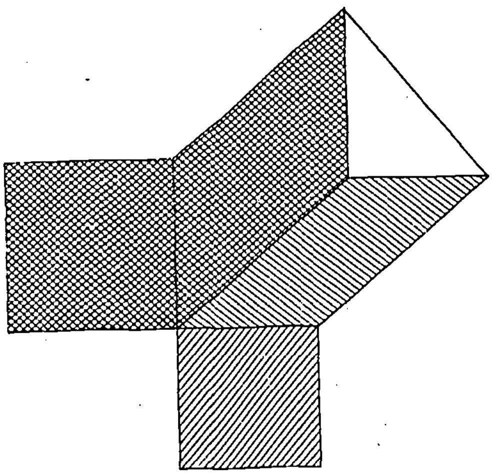


Fig. 5

formed by emptying the puzzle frame and giving the child the two parallelograms, the smallest square and the medium square, and the triangle. After making figure 4, the child should conclude that the two parallelograms are the same size as the largest square, hence the two smaller squares are also the same size as the largest square.

The next series of constructions is designed to reinforce the idea of the Pythagorean theorem and to introduce the idea of squared numbers and the concept of a 3-4-5 right triangle.

Empty the puzzle and put in the fifty small squares. The nine squares painted to match the six-inch square go in the small compartment, the sixteen squares painted to match the eight-inch square go in the middle-sized compartment, and the twenty-five squares painted to match the ten-inch square go in the largest compartment. Also, put the triangle in the middle. Have the child take the squares from the large compartment and fill up the two smaller compartments. Then have him take the squares from the two smaller compartments and construct various designs, such as those

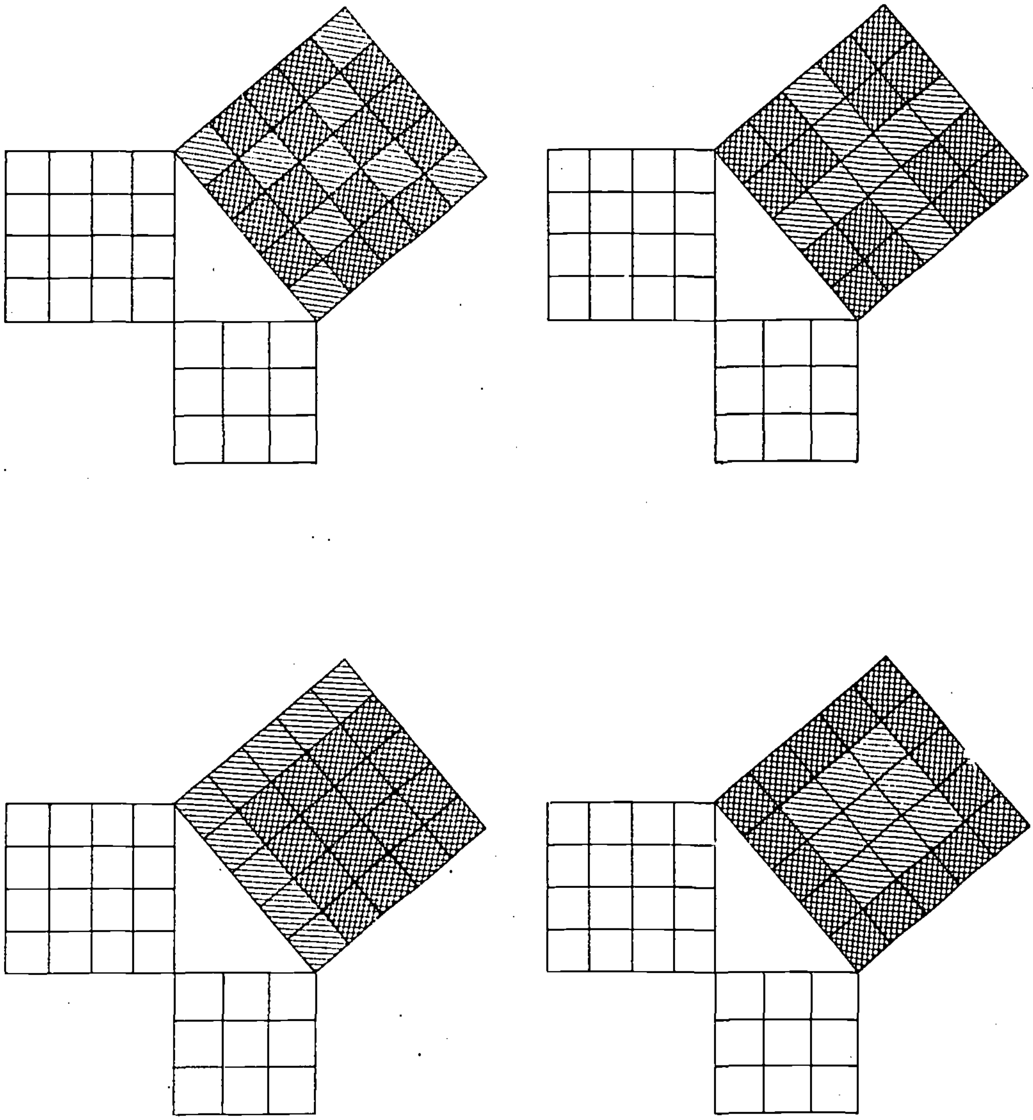


Fig. 6

shown in figure 6, in the large compartment.

Use of the puzzle would not necessarily require a rigid lecture type of presentation such as has been outlined. If students are merely allowed access to the puzzle, many of them will make interesting discoveries in their free time.

Easy-to-paste solids

M. STOESSEL WAHL *Danbury State College, Danbury, Connecticut*

Geometry has gained an accepted place in the mathematics curriculum of the elementary school. Elementary teachers have found that their pupils enjoy using compasses to construct designs and mathematical figures. Some teachers, who have tried the construction of three-dimensional solids, have found the pasting technique difficult for young fingers when conventional patterns were used. Here is a set of patterns designed to make the pasting problem easy enough so that third-grade students can succeed with the simpler solids. Teachers have been most enthusiastic about their pupils' delight and success in using them.

The construction of these five solids—tetrahedron, octahedron, decahedron, icosahedron, and hexahedron—is similar, so general directions will be given. The basic pattern guides for each solid are included.

The materials needed include a compass, a straightedge or ruler, pencil, scissors, and an instrument, such as a stylus or table knife, for scoring the lines. The oak tag used in file folders is a satisfactory construction material. Construction paper is more colorful, but does not hold up as well. A good glue that dries quickly is important (Elmer's, Uhu, or rubber cement).

The directions given here are quite detailed because younger children need very specific instructions. The construction of triangular grids with a compass should be practiced first because accuracy of the grid construction is important.

Construction of the triangular grid

1 Place the oak tag paper on a newspaper padding to prevent the compass point from slipping.
2 Using a ruler, carefully draw a straight line across the middle of the paper.
3 Mark a distinct point near the middle of the line.
4 Set the compass to any convenient radius. An inch and a half is a good radius for beginners. The same radius will be used throughout the construction.
5 Place the point of the compass on the marked point and swing a complete circle.
6 At the point where the circle intersects the straight line, place the point of the compass and draw another circle. Continue in the same manner across the line.
7 Continue drawing circles by placing the point of the compass where two arcs intersect. The paper should be covered with intersecting circles.
8 Draw straight lines between the points of intersection of the circles so equilateral triangles are formed. (It should

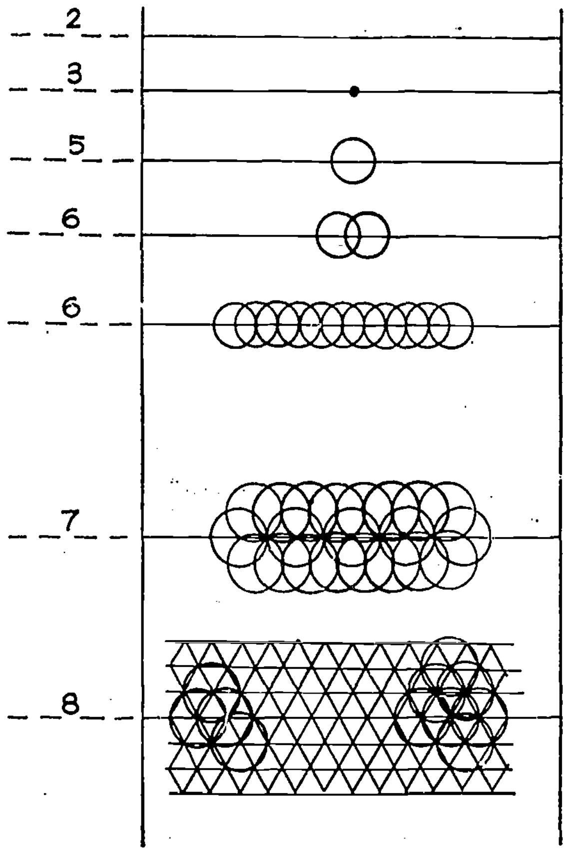


Figure 1. Triangular grid

look like a piece of graph paper formed by triangles instead of squares. See Figure 1.) This is the basic triangular grid for four of the solid patterns.

Directions for transferring pattern, and scoring, cutting, and folding

1 Transfer the numerals and letters from the pattern guides in this article to the constructed grid. Check to see that no numeral or letter is omitted. (All pasting edges are supplied in the pattern.)
2 Using a ruler and a stylus, or semi-pointed instrument, score each line in the pattern. This ensures the necessary sharp folds.
3 Cut out the pattern. Also cut wherever bold lines occur.
4 Carefully fold on all lines. (Just one line not folded can spoil the solid.)
5 Optional. Fold the solid experimentally to get the feel of its shape. Some solids, especially the tetrahedron, can be temporarily secured without glueing by tucking in the Y flaps (Fig. 2).

Directions for pasting and finishing the solid

1 Pasting is done so the numerals appear on the inside of the solid.
2 For patterns containing X labels, start by pasting the triangle labeled X5

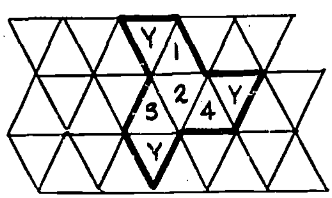


Figure 2. Tetrahedron

directly over the triangle labeled 5, the triangle X8 over the triangle labeled 8, etc. Paste so the triangles fit together carefully and hold long enough for the glue to take a firm hold.

3 A string should be added before the last flap is pasted. It should be attached to a small button or object large enough not to pull out.
4 After all X flaps are pasted, the Y flaps are pasted to the outside of the solid. (A more experienced person could tuck the Y triangles inside, but this is too difficult for the younger children.)
5 The solid can now be admired!

The delight in the child's eyes will more than compensate for the slightly grubby results of the first efforts. It is wise to work with the simplest solid, the tetrahedron, first. After the child has sensed the importance of accuracy in his grid construction, and felt the need for care in scoring and patience in pasting, then it is time to try the octahedron and the decahedron (Figs. 3 and 4).

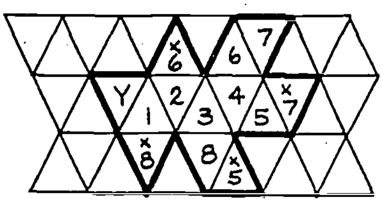


Figure 3. Octahedron

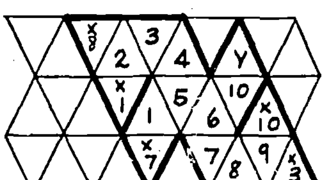


Figure 4. Decahedron

The icosahedron is more difficult. The teacher may prefer to ditto this pattern on oak tag for the first experience with this twenty-faced solid (Fig. 5). This may be accomplished by hand-feeding the oak tag into the ditto machine. Later, the students

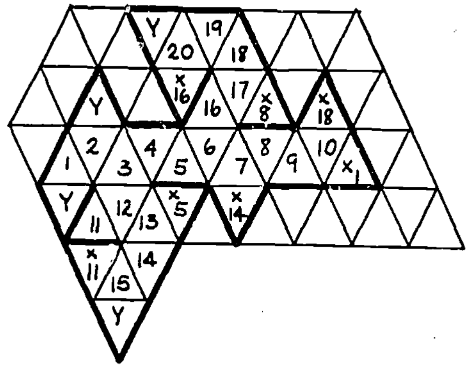


Figure 5. Icosahedron

who wish can construct their own icosahedrons, as they did the previous solids. It is interesting to note that often the slower children are more successful than others because they have developed the patience that brighter children seldom take time for. It is a good experience for the slower learner to have the brighter child look to him in admiration!

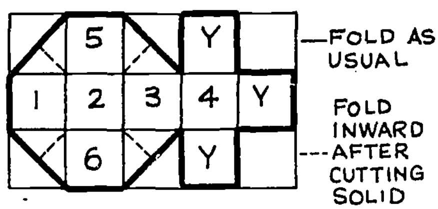


Figure 6. Hexahedron

For the cube, or hexahedron, a square grid is needed (Fig. 6). On the elementary level it is preferable to use graph paper for this pattern. This pattern, too, differs from the conventional pattern so that pasting is made easier. In this pattern some lines are folded in and some folded out, so a little more care is needed in scoring and folding.

Some teachers like to motivate their pupils by displaying a solid made by the teacher as evidence of the kind of work the pupil may be capable of doing in a more advanced grade. Figure 7 is a pattern for a sixty-faced solid for the teacher to construct. It is based on the triangular grid construction and is also easy to paste.

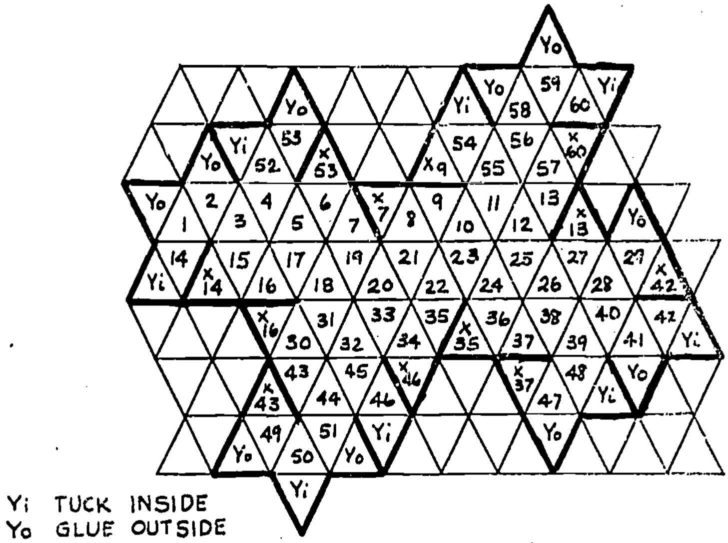


Figure 7. Sixty-faced solid

Decoration

Some pupils cover the solids with glitter, or spray-paint them. Some find ingenious and inexpensive ways of decorating their solids. The art or home economics departments may cooperate on these projects. The solids are beautiful when covered with velvet and decorated with sequins, but quite difficult to make.

At times, these solids are used as Christmas tree decorations, but many states have new laws forbidding flammable tree decorations. The children can also make them into attractive mobiles by using a bit of wire (cut coat hangers) and black string.

Most of these solids were known to Plato and Archimedes, ancient Greek mathematicians who lived in the third and fourth centuries B.C. Mobiles were popularized by the American artist Alexander Calder in our own century. It is interesting that an elementary school child can enjoy success in making a mobile that combines the contributions of geniuses who lived more than twenty-three centuries apart.

EDITORIAL COMMENT.—A number of different materials may be used to construct geometric solids. Patterns may be traced onto acetate sheets (leave out the glue flaps). Use plastic model cement to form the edges. These figures are very effective for placement on the stage of an overhead projector. Students may be asked to identify the three-dimensional figure from its two-dimensional projection. Or you may ask the students to place a cube on the projector so that its projection has a hexagonal outline.

A permanent-soap-bubble geometry

M. STOESSEL WAHL
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For years the study of soap-bubble geometry has been a source of surprise and fascination to individual students. However, it has not been a satisfactory technique for classroom study: just as the teacher holds up a good specimen for the class to admire, the bubble bursts!

Fortunately, because a new art medium is available, the students and their teacher can now make and study permanent specimens. This new medium comes under various trade names and can be purchased in hobby shops that sell the new viscous fluid for making fancy glasslike flowers of various colors. The wire they sell for making the flowers is suitable for our geometric models, but any thin wire can be used.

The materials needed are wire, a wire cutter (old scissors might do), and at least one bottle of the purchased liquid. Long-nosed pliers help make neater models, but bending the wires with one's fingers will suffice.

Students usually begin by fashioning wire tetrahedra or hexahedra about 1 1/4 inches on a side, with convenient handles about five inches long (see fig. 1). The finished wire model is completely immersed in the liquid and then held up to the air for a few moments so the thin film can dry. Contrary to expectations, the film does not coat the outside of the model. Surface tensions within the liquid force it to form special patterns inside the polyhedra. When dry, it can be passed from student to student for examination and discussion.

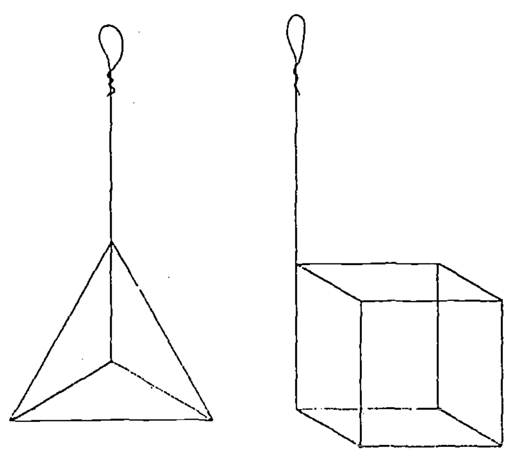


Fig. 1

This paper was presented in a workshop at the NCTM meeting in Boston, 13 November

For the inquiring student who wishes to find out why the surface tension forced the liquid into such surprising patterns, there is an excellent paperback book available. It describes lectures on soap bubbles given in London in 1889 to a juvenile audience by Sir Charles Vernon Boys.[1] For the student whose interest or curiosity leads him further, various pyramid and prism models lead to interesting discoveries. By twisting a wire in spiral fashion around a cylinder and attaching each end to a wire handle before immersion, a helix is formed that is beautiful as well as mathematical (fig. 2).

One of the most fascinating forms is fortunately the easiest to make. Here are the directions:

1. Bend a 1/4-inch hook at one end of a 15-inch strand of wire.
2. Make two twists of wire around a cylindrical pill bottle of about 1 1/4 inches in diameter.
3. Fasten the hook from the end of the wire onto the remaining length of wire handle. Pull it taut and fasten well by twisting the hook.
4. Pull the bottle out from the wire encircling it.
5. Gently pull the two loops apart so there is about half an inch between them. (See figure 3.)
6. Immerse the model in the liquid.
7. Pull it out and hold it up to dry.
8. Examine it closely. How many sides does it have? Could a flea traverse the inside and outside without walking over an edge?

Immersion of the wire model produces a Möbius strip.

How far the student can go with this technique is limited only by creativity and interest. Soap-bubble geometry is an open-ended study correlating mathematics and science. Luckily, it is inexpensive enough for classroom laboratory use. It can also be easily utilized as a mathematics and science station in the newly popular open classroom.

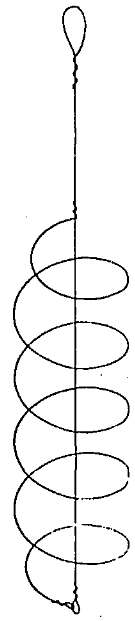


Fig. 2

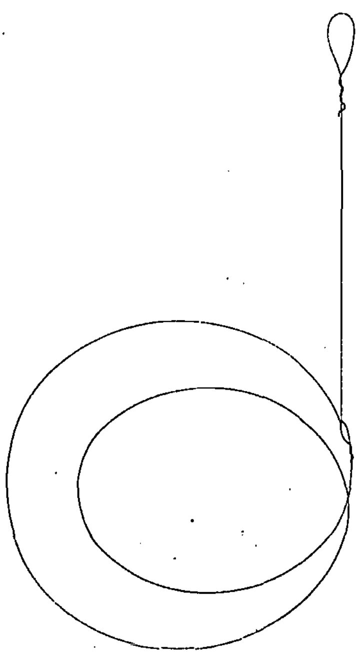


Fig. 3

1. C. V. Boys, *Soap Bubbles and the Forces Which Mold Them*, Science Study Series. Garden City, N.Y.: Doubleday & Co., Anchor Books, 1959.

Cleo's clock

CLEO KAPROCKI

Forest City Elementary School, Forest City, Florida

In my teaching I have developed and successfully used the idea of associating the number line and the clock. Since the children had had experience with the number line, my idea was to relate this experience to the telling of time.

A six-foot folding carpenter's rule—the type with metal joints—was used as the base for the number line. Starting at one end of the rule, and at each joint thereafter, a 2½-inch diameter tag was fastened to the rule. Each tag had one numeral on it, and the tags were ordered from one through twelve (1, 2, 3, . . . 12). To avoid confusion, the rule was painted white, thus eliminating the original numerals on the rule from any consideration.

The circular clockface, which was cut from ¼-inch plywood, was 28 inches in diameter. A handle and stand were cut as part of the face to facilitate the clock's use (see figure 1). The clock had a backstand so it would be freestanding (see figure 2).

Magnets were attached 2 inches from the edge and 6 inches apart. The magnets were positioned where the hour marks on a clock would ordinarily be found (see figure 1). The face of the clock was painted white, and removable felt eyes, mouth, and mustache were added as motivational factors for first graders.

To use the clock, a student (or teacher) places the end of the rule so that the tag labelled 1 is on the first magnet to the right of top center. The joint with the tag marked 2 will land on the next magnet as you move clockwise. By bending the rule at the "2" joint, the tag marked 3 will land on the next magnet. Continuing this procedure, the result will be the "number-line rule" bent around the face of the clock (see figure 3).

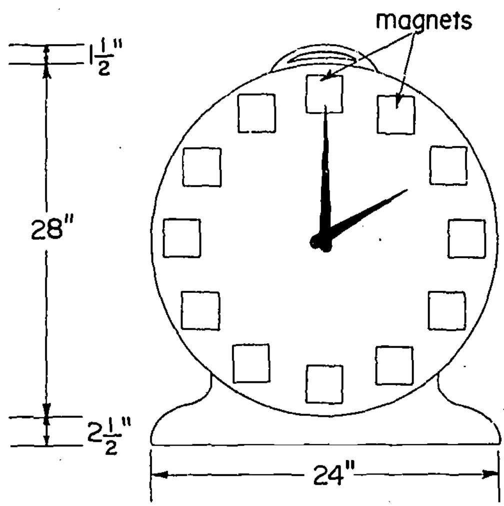


Fig. 1

The clock is versatile because the number line can be removed, and individual numbers can be given to students to place

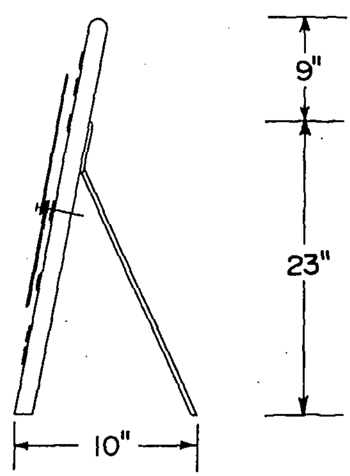


Fig. 2

in the appropriate places. Since the clock is large enough for the students to see across the room, children can set the hands of the clock at a particular time and ask other students to read the indicated time. The clock can be used to test telling time by removing all, or some, of the numbers.

This clock was found to be a motivational device that attracted the children's attention and held their interest. Its use proved to be a problem-solving method for students who found telling time difficult. Because of its versatility, this clock can have other uses in the classroom.

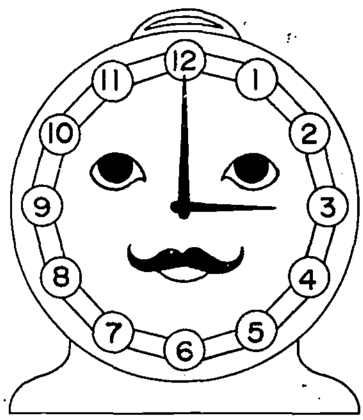
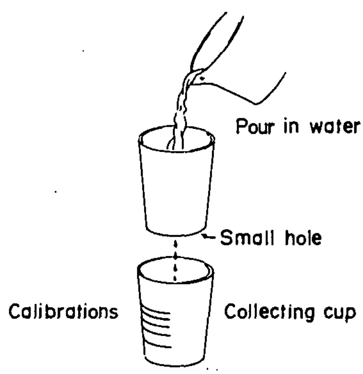


Fig. 3

EDITORIAL COMMENT.—Kaprocki suggests relating a number line to the hour function of a clock. One may also use a number line for the minute sequence. Students can make the two number lines simultaneously, thus noting the factor-of-five relationship between markings on the two scales.

Time telling may also be investigated through the use of a sundial as featured in the next article. You may also want to construct a "water clock" using paper cups. The collecting cup may be calibrated by drawing marks on the cup to show the water level after 30 seconds, 60 seconds, 90 seconds, and so on.



"We made it and it works!"* the classroom construction of sundials

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M. Stoessel Wahl teaches courses in mathematics to teachers of elementary school at Western Connecticut State College. Formerly, she taught at Lincoln School, Teachers College, Columbia University, and at the State University of Iowa.

In the early days, each household had its own sundial where time was read from the slowly moving shadow traced on the dial face by the movement of the sun across the skies. "Dialling"[1] was a subject included in the school curriculum. Later, the standardization of time and the general use of accurate clocks rendered sundials obsolete. Although there has been renewed interest in these time-telling garden ornaments, many of today's children have never seen them function.

It is fun to make your own sundial, but most available instructions are too mathematical to be used by the elementary pupil. Fortunately, an old book on sundials,[2] printed in 1902, contains an elementary set of instructions formulated by H. R. Mitchell, Esq., of Philadelphia. Mr. Mitchell's rules have been simplified by the present author; for the past thirty years her students (and many of their pupils) have had the pleasure of making their own dials.

This set of instructions, written in careful detail for elementary pupils, can be easily adapted to the junior high school level. For instance, where elementary pupils are directed to draw perpendiculars and parallels, the junior high student can be asked to construct them. Simple overhead transparencies can aid the pupils in following the directions.

For the basic construction each pupil needs an oaktag file folder, a compass, a protractor and ruler, a sharp pencil, and a pair of scissors. A plastic right triangle is useful. Other equipment can later be used to make a more permanent dial.

Construction of the gnomon

The gnomon is the part of the dial that casts the shadow of the sun. Its name comes from a word meaning "one who knows," for the shadow tells the correct sun time. To make a gnomon, one needs to know the latitude of the place where the dial will be used. One also needs to know the longitude of the location to correct the readings to clock time. Both readings should be made as precisely as possible.

1. Write your latitude and longitude as read from the map.
2. Place your file folder so the folded edge is nearest you. With a pencil

* These materials were presented at a workshop, "We Made It and It Works," at the forty-eighth annual meeting of the NCTM in Philadelphia, April 1967. Forty teachers from Florida to Canada constructed sundials, each one custom-made for his own latitude.

1. *Dialling* is the proper spelling as it was used in all the old mathematical treatises and texts.

2. Alice Morse Earle, *Sundials and Roses of Yesterday* (New York: The Macmillan Co., 1902).

mark a point on the fold, one inch in from the left edge. Label it *A*.

3. With *A* as a vertex use your protractor to lay off an angle equal to your latitude.
4. From *A* measure three inches along this line. Mark the point and label it *C*.
5. Draw a lightly dotted line, making *C* the vertex of a right angle. Extend this line till it intersects the fold. Label the intersection *B*. (See fig. 1.)

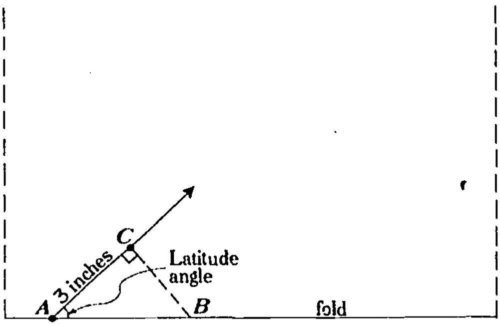


FIGURE 1

6. The length of the hypotenuse of your gnomon *AB* will be important in the construction of the face of the dial. Set your compass to exactly equal *AB* as a radius and put your compass aside.
7. If you wish to make a fancy gnomon, draw a curved line between points *A* and *B* where you now have a dotted line. Leave *AB* and *AC* unchanged.
8. On line *AC* mark off a point *E* so *AE* equals ¾ of an inch. On line *AC* mark off a point *F* so *CF* equals ¾ of an inch.
9. Draw pasting flaps about one inch wide. (See fig. 2.)
10. Cut out your gnomon, carefully cutting through both thicknesses of cardboard so your gnomon will have a double thickness.
11. Score along line *EF* on both thicknesses to make folding easier. Fold

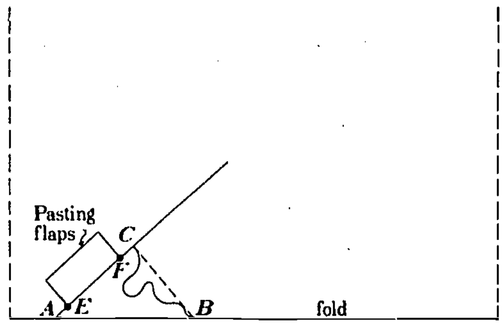


FIGURE 2

back on these lines. Your gnomon should stand upright. Later it will be inserted in the face of the dial.

Construction of the dial face

1. Near the center of your folder mark a point *O* for the center of the dial.
2. With *O* as a center, and *AB* the length of the hypotenuse of the gnomon as radius, draw a circle.
3. With *O* as center, and the length *AC* (three inches) as a radius, draw an inner circle.
4. Draw a diameter through *O* extending through both circles.
5. Place a protractor with its center on the point *O* on the diameter. Lightly mark 15 degree angles in the upper half of the circle. Using a straightedge, mark points where these 15 degree angles intersect the inner and the outer circle.

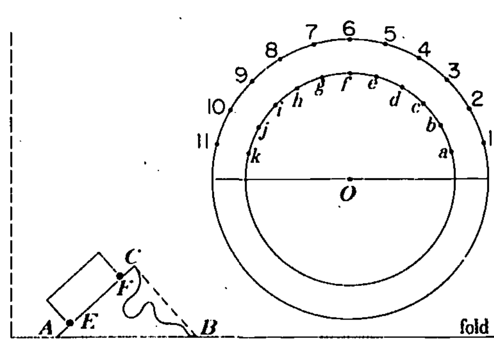


FIGURE 3

6. Label these points on the inner circle with letters *a* through *k* counterclockwise. Label the points on the outer

circle with numerals 1 through 11 counterclockwise. (See fig. 3.)

Note: In steps 7 and 8 draw lines only in the region between the two circles as in figures 4*a* and 4*b*.

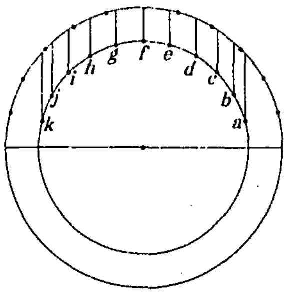


FIGURE 4a

7. Through point *a* draw a line perpendicular to the diameter line. In like manner draw perpendiculars to the diameter from points *b* through *k*.

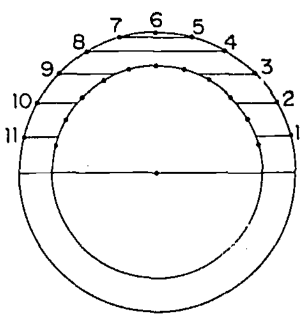


FIGURE 4b

8. Draw lines parallel to the diameter by connecting lines between points 1 and 11, 2 and 10, etc.
9. Find the point where the perpendicular through point *a* and the parallel through point 1 intersect. Call attention to this important point by drawing a small circle around the intersections noted by the ordered pairs, (*a*,1), (*b*,2), (*c*,3), (*d*,4), (e,5), (*f*,6), (*g*,7), (*h*,8), (*i*,9), (*j*,10), and (*k*,11). You will see that these points form a smooth curve. All the construction work completed to this point was done to locate these points accurately. (See fig. 5.)

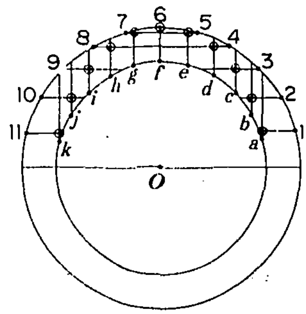


FIGURE 5

Marking the hour lines

1. Carefully draw lines connecting each circled intersection point with the center of the dial *O*.
2. Extend the two lower lines into the lower half of the dial. (See fig. 6*a*.)

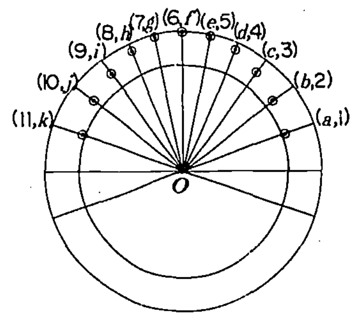


FIGURE 6a

3. The diameter is also an hour line for six in the morning and six in the evening.
4. The hour line perpendicular to the diameter is the noon or twelve o'clock line. The hours can now be marked on the dial face. (See fig. 6*b*.)

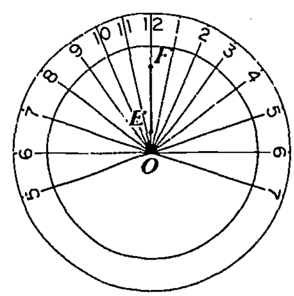


FIGURE 6b

5. On the noon line measure in ¾ inch from O and label it E. On the noon line measure in ¾ inch from the inner circle and label the point F.
6. Using scissors or a razor blade cut a sharp line through line segment EF.
7. Into the slash EF insert your gnomon so the A of the gnomon coincides with the dial center O and the Es and the Fs of the gnomon match those of the dial face. The gnomon will be perpendicular to the dial face.
8. Cut out your sundial by cutting through both thicknesses of folder.
9. The mathematical portion of the dial is now completed. It can be placed in the sun with the gnomon pointed true north (not magnetic north) to the location of the north star.

Making the dial more attractive

Once the hour lines have been established mathematically for your locality, the dial can be enlarged, made smaller, or mounted on fancier shapes such as an octagon. An art teacher or a shop teacher can help the students make permanent dials of metal or wood. Some mathematics teachers without this additional help must rely on their own ingenuity. It is possible to sew Roman numerals and hour lines on the cardboard with heavy carpet thread. The dial face can then be covered with foil (kitchen foil or fancy florists' foil) and gently pressed with the fingers so the numerals and hour lines stand out nicely. The resulting dial is one that the pupil can take home with pride.

Reading the sundial

With the gnomon pointed true north, the shadow cast by the sun reads true sun time. Hence, it corresponds to our artificial clock time only four times a year. Unless the pupil is made aware of that fact he might be disappointed. Questions raised by the difference between sun time and standard time can lead to a study of important scientific facts. The science teacher can be helpful in showing how latitude and longitude affect these differences. Using data from the world almanac one can construct a graph (or use an analemma) to find how many minutes the shadow is "sun fast" or "sun slow" that day. A student desiring greater accuracy can make a longitudinal correction as well. Since almanac tables are given for the edge of time zones, each degree of longitude variation changes the shadow by four minutes. If this change is incorporated in the making of the original graph the student will be able to read his sundial time to a surprising closeness to our clock time.

Conclusion

The time used for making a sundial in the classroom can be justified from a mathematical standpoint. The student's skills in using angles, parallels, perpendiculars, and ordered pairs are put to a practical use. The student learns to follow directions, and the subject matter can easily be correlated with science, art, and shop, or can be used by the social studies teacher in the study of man's measurement of time. The best justification for the mathematical effort lies in the eyes of the student who announces triumphantly, "I made it and it works!"

7

Multipurpose Aids

The articles that follow differ in focus somewhat from those presented in earlier sections. Some of these articles do not describe manipulative aids in the strict sense of the word. We include them because they illustrate how one manipulative aid can be used in teaching widely divergent concepts or because they describe how one kind of construction or technique can be applied in a wide variety of situations.

The first article describes a truly versatile manipulative aid—the pegboard. Anyone who has access to masonite pegboard and a sack of golf tees can equip the class with sufficient pegboards for everyone. Fisher describes how this aid can be used in teaching perimeter, area, angle measurements in a polygon, coordinate geometry, place value, properties of operations, number theory, and ratio and proportion. One can hardly ask for more!

Golden's article on games in mathematics is important for two reasons. First, it shows that games can be useful teaching aids. Second, it points up the fact that all teaching aids are not necessarily made by the teacher or by commercial distributors. Some of the most successful ones are those created by your own students.

Too often we complain that not enough is done to show how mathematics is related to other areas of human endeavor. Beougher's article gives some very helpful suggestions for relating mathematics and science. Even more pertinent to the theme of this collection, Beougher shows how the models we use and construct for science teaching have a role to play in teaching mathematics. Included among his devices are a gravity model, a solar-system model, and instruments for measuring sun elevation and for calculating heights of tall objects.

Ma[illegible]lative devices have their role in the earlier stages of concept development, but how does one make the transition from the concrete to the more abstract or symbolic modes? Williams suggests that classroom charts developed by teachers and students can help. The charts can be summaries of things experienced firsthand.

The bulletin board can be more than something that lies flat against the wall. It can have three dimensions and come to life. Children can manipulate the components of the board as they develop the illustrated concepts. Hall gives us some hints for making bulletin boards that are both attractive and useful teaching aids.

Finally, there is the dream of using all our manipulative aids in an appropriate setting and in the proper place in the curriculum. The mathematics laboratory provides one medium for integrating many of the physical activities suggested by the use of manipulative aids. Davidson and Fair describe how their school transformed a small room into a place in which students could carry out a math-lab program. Their success and enthusiasm should give each of us the incentive to try out the many varied ideas in this book and to share with others our own successes.

The peg board—a useful aid in teaching mathematics

ALAN A. FISHER *Mt. Tabor School, Portland, Oregon*

A peg board* is an excellent arithmetic teaching aid. It will assist the teacher in presenting new concepts and help the students visualize these ideas in a concrete form. It is useful at any elementary level and is helpful in making algebra and geometry understandable. (Fig. 1)

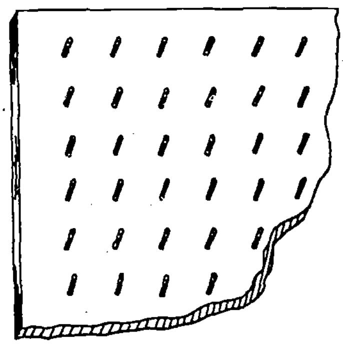


Figure 1

How to use it

Perimeter and area

Colored rubber bands slipped over the pegs make an attractive teaching aid in presenting areas and perimeters of rectangles, squares, parallelograms and triangles. An arrangement like Figure 2 can lead to discussions, such as: "What a square looks like," "Can any smaller squares be made by using more rubber bands and the same four pegs?" (Fig. 3) "How many triangles can be formed?" "What a right triangle is and how to determine its area." The perimeter may be measured with a dressmaker's cloth tape and the area determined in square inches. Bisecting angles look similar to Figure 4. The distances *AB*, *BC*, *CD*, and *AD* are

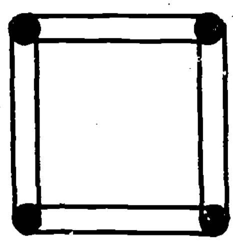


Figure 2

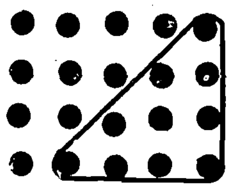


Figure 3

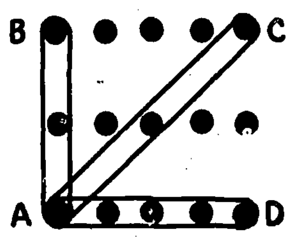


Figure 4

* The peg board may be made by gluing wood dowels cut into one-inch lengths in the holes of the Masonite peg board, leaving two-inch spaces between pegs. The material can be purchased at any building-supply store or lumberyard. The size of the board may vary, but a 2 by 3 foot size with 144 pegs has proved satisfactory for me. Most 100 boards have 10 rows with 10 pegs each but many times a larger number of pegs are needed, and those pegs not in use can be separated by rubber bands. A white background sets off any display which is arranged on the pegs, so that the display can be seen from any part of the classroom.

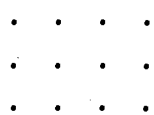
equal, which shows that the figure is a square. It may also be observed that the diagonals bisect the angles.

Place value and hundred board

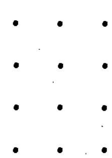
The board can be used in developing the idea of grouping by tens and place value. One-inch wooden beads or blocks with holes large enough to slip over the pegs will help with the explanation. Start with the basic idea of how many blocks can be put on the first row, giving the idea of ten. If two rows are completed, we have twenty, etc. This material can also be used to represent parts of one hundred, to present per cents greater than one hundred, and also amounts less than 1 per cent, if washers are slipped over one peg.

Commutative law for multiplication

The commutative law for multiplication can be demonstrated by placing the blocks over the pegs in this design.



After this is done, ask the students if the answer will be the same by multiplying three times four or rotating the board a quarter turn and showing four times three.



The law of compensation, or the idea of inverse variation, can be demonstrated

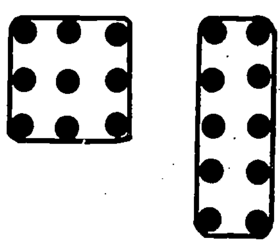
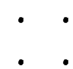


Figure 5

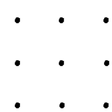
very effectively, too. Start the discussion by using the chalkboard to present $6\times4=24$, $(6\div2)\times(2\times4)=24$, or $(2\times6)\times(4\div2)=24$. Using any of the combinations, the answer will be 24. Then demonstrate on the peg board with the aid of rubber bands. One may present the problem, "How could we change the dimensions of a fence and still have the same area?" (Fig. 5)

Squares of numbers

To represent squares of numbers, place beads on the pegs to form squares, four beads thus



will represent 2^2; nine beads thus



will represent 3^2. This can continue, being limited only by the size of the board, possibly 10^2 or 12^2.

Equal ratios

Equal ratios are made more interesting by using beads to show all of one ratio and only half of the proportionate ratio on the adjoining pegs. Students can complete the ratio. See Figure 6.

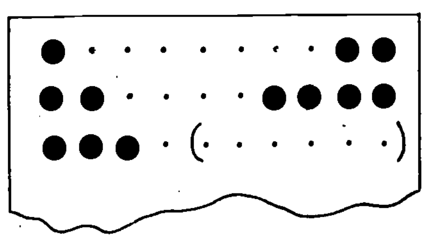


Figure 6

Co-ordinate axes

In the selection of quadrants in teaching graphing with co-ordinate axes, it is helpful to start with a number line locating the positive and negative values on the hori-

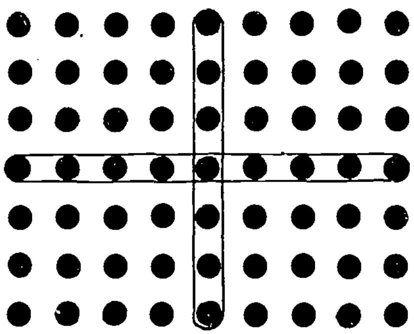


Figure 7

zontal axis. This fixes in the student's mind the idea as to which direction to count for the value of x, or the first of the ordered pair. Then, work with the vertical axis may be begun in order to give the students practice in placing the value of the second of the ordered pair, or y coordinate. When graphing, the position of the rubber bands can be adjusted to suit the demonstration. (Fig. 7)

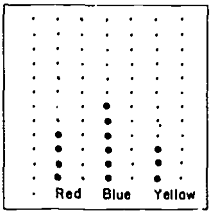
Degrees in a polygon

The formula 180° $(n-2)$ is used in determining the total number of degrees of the interior angles in any polygon and can be made clear by starting with the concept of 180° in a triangle. Students will enjoy forming as many triangles as possible by stretching rubber bands from *one point* to all other points or pegs in the polygon's perimeter. Interest will be high when they discover that the number of triangles formed is always two less than the number of sides of the polygon.

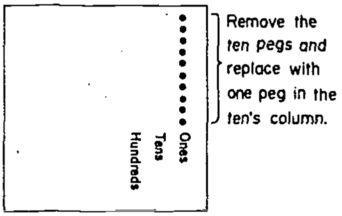
As teachers use the peg board, they undoubtedly will see many other uses for this device in representing mathematical ideas.

EDITORIAL COMMENT.—Teachers may also use the pegboard to create statistical graphs, to explore the distributive property, or to function as an abacus.

To illustrate a simple example of graphing, consider asking each member of your class to identify his favorite color—red, blue, or yellow. On a pegboard identify three columns, one for each color. As each student calls out his color, place a peg in the proper column. The completed board produces a bar graph showing the number of students selecting a given color.



To use the pegboard as an abacus, identify columns with place values. Then establish the rule that each time a column contains ten pegs, the student should replace them with a single peg in the column to the left.



Fostering enthusiasm through child-created games

SARAH R. GOLDEN

Sarah Golden is a classroom teacher of a combination first- and second-grade class at Halecrest School in Chula Vista City School District, Chula Vista, California. Mathematics is a favorite subject for her and for her class.

How can the young child develop enthusiasm for mathematics, maintain needed skills, gain mathematical insight, and work at his individual level in a relaxed classroom climate of success?

One of the possible ways developed as a result of a recent visit by Donald Cohen, Madison Project Resident Coordinator in New York City, to my combination first-second class. He introduced an original Guess the Rules board game involving movement of imaginary traffic. Following this experience, one boy tried to make up his own Guess the Rules game while the other children played with commercial and teacher-made games. I noted that in playing some of the games, variations of the original rules were used spontaneously by small groups to increase total scores possible. For example, the chalkboard scores for Ring Toss ran into the thousands, although the indicated score for each toss was only 5 or 15. "Oh," said Jeff, sitting on the floor with one foot extended, "When he rings my foot, it counts for one hundred. When he rings Robert's foot, that's five hundred." Excitement ran high.

The next day we discussed the possibility of each child creating his own board game. The following criteria were established:

1. The game must be fun to play.
2. The game must have rules.
3. The game should enable children to play together and become friends.
4. The children must learn from the game.

Mathematics, science, reading, and spelling were suggested by the children as the content areas. The mathematics games might include, they said, go ahead and go back, counting, addition, subtraction, and multiplication. Dismissal time arrived all too quickly.

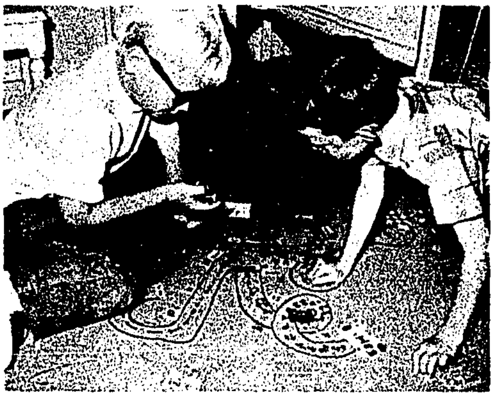
Traci (Grade 2)
Object: Get to finish line.
Rules: Use spinner or die to indicate number of moves. Correct response, stay on that box. Incorrect response, back to previous box.[1]

FINISH
$1219 + 0 = \square$
$970 - 1 = \square$
$468 + 1 = \square$
$\square + 1 = 10$
$10 = \square + \triangle$
$\square = 8 - 10$
$\square + \square = 12$
$\square = 8 + 12$
$15 = \triangle + \square$
$\square = 2 - 15$
$16 = \square + \square$
$\square = 3 - 8$
$19 = \triangle + \square$
$\square = 10 - 20$
$18 = \square + \square$
$\square = 15 - 30$
$\square = 10 - 30$
$30 = \square + \square$
$100 = \square + \square$
START

FIG. 1. Number Game

1. It was surprising that Traci deliberately planned to have negative numbers as responses in her equations.

The following morning one girl submitted a sketch of her game which used dice with three golf tees as pawns. Her explanations were quickly grasped by the class. As she proceeded to make a more durable copy of her game, children used the following vocabulary to describe the situation: Traci's game, Traci's pattern, Traci's copy, Traci's original, Traci's guide, Traci's directions, Traci's instructions.



Eager to start on his own, each child then developed a game on a sheet of chipboard about 12 by 22 inches. Dice, spinners, or number cubes were used to indicate direction and number of moves. In some cases, answers on cards were to be matched with the corresponding problem in the game. The games showed a wide variation in difficulty, ranging from simple counting to the use of negative numbers. Danny asked, "What if you landed on 3 and your card said, 'Go back 6'? Where would you be? Off of the board?" He went on, "I guess that would be a negative number, and you'd have to get 'Go ahead 6' to get back to where you were on 3." Danny thought about this and incorporated negative numbers in his Bang Bang Chitty Chitty game by using a starting line with negative numbers behind it leading to the Repair Shop. The twenty-seven games included such names as Try and Guess; The Happy Game; Go Ahead, Go Back; Treasure Island; The Whacky Racer; Streets and Numbers; The Counting Game; and Bang Bang Chitty Chitty.

In making games that involved rounded shapes, the children encountered the problem of dividing the rounded portions into congruent regions. They then learned how to use simple protractors to construct such regions. Several lessons on curves followed, including simple, complex, open, and closed curves.

Upon completion of the games the children were given repeated opportunities to explain the object and rules of their games and to play them with their classmates. Scores were tallied and comparisons such as "I beat him by 35 points" were made.

Deborah's Streets and Numbers game led to a discussion of odd and even numbers. We took a walk in the neighborhood to find out how the houses were numbered.

For two players. Robert (Grade 1)

Object: Get Snoopy from the yard to his dog house.

Rules: Use spinner to show number of boxes to advance. Correct answer allows player to remain there until next spin. Wrong answer makes player go back to his preceding position.[2]

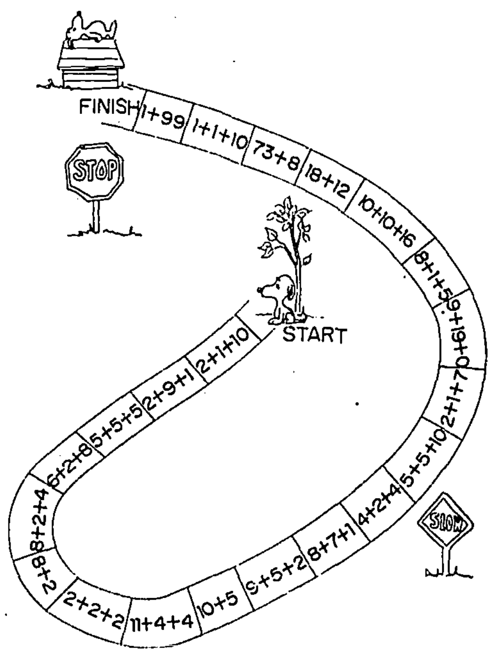
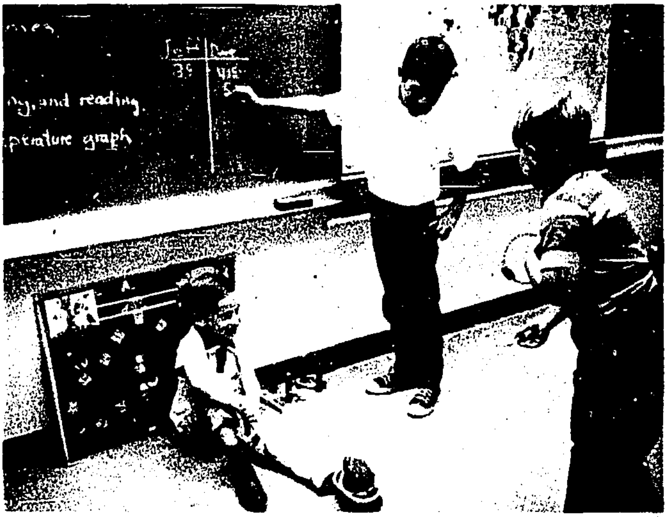


FIG. 2. Snoopy Game

2. Robert, a first grader, was interested in using three addends.



Each child made a map of his street and the houses on it and numbered the houses. We invited a member of the Chula Vista City Engineering Staff to come to school to explain how numbers were assigned to new homes in the city. Deborah revised her game to reflect her new learnings in this area—odd numbers on one side of the street, even numbers on the opposite side.

For one or two players. Kathy (Grade 1)

Object: Put the greatest number of cards where they belong in the counting ladder.

Rules: Players take turns picking top card from a shuffled pack. Cards show numerals 6, 12, 14, etc. Player puts the card on his side of the ladder. If incorrect, player loses turn. Each player counts his correct responses at end of game to determine winner.

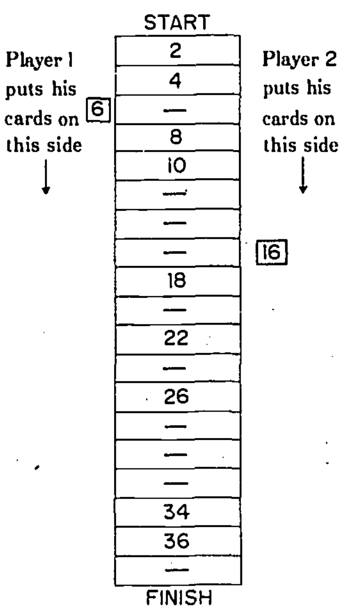


FIG. 3. Counting Game

The Bang Bang Chitty Chitty board game was devised by four boys as an outcome of their interest in mathematics, model cars, races, and the story of *Chitty Chitty Bang Bang*. Speed limits were discussed. The boys consulted *Sports Encyclopedia*, almanacs, and other books for information on speed records and reasonableness of posted speed signs. Of course, the intricate speedway had to be made wide enough to accommodate the toy racing cars used. Much experience in measuring resulted.

Summary

Child-made games foster enthusiasm and contribute to learning in several ways. Children learn through a pleasurable medium of their own creation. The varying degrees of complexity in the games are commensurate with a child's mathematical concepts and interests. In devising the games, the children must try out their ideas and pursue them to a reasonable outcome. Language development is advanced as the children are highly motivated to explain the games clearly to others. There is an

Leslie (Grade 1)
Object: Get to home.
Rules: Use die to indicate number of boxes to advance. Then follow directions in box.

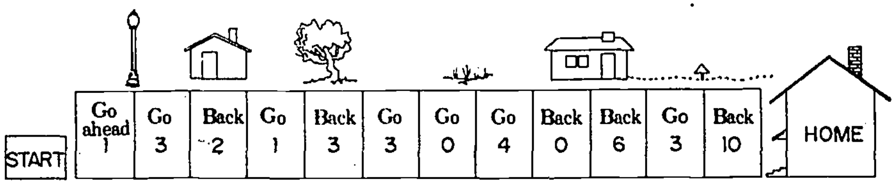


FIG. 4. Happy Game

exchange of information between players, especially when more than one correct response is possible. The teacher gains information concerning individuals and their levels of thinking. For example, it was surprising to note some children's use of difficult combinations and negative numbers and their free use of commutative and as-

For two players. Danny (Grade 2), Chris (Grade 1), Robert (Grade 1), Bobby (Grade 2)
Object: Get to finish line.
Rules: From a stack of cards, each player takes a card, which tells him what to do.[3]

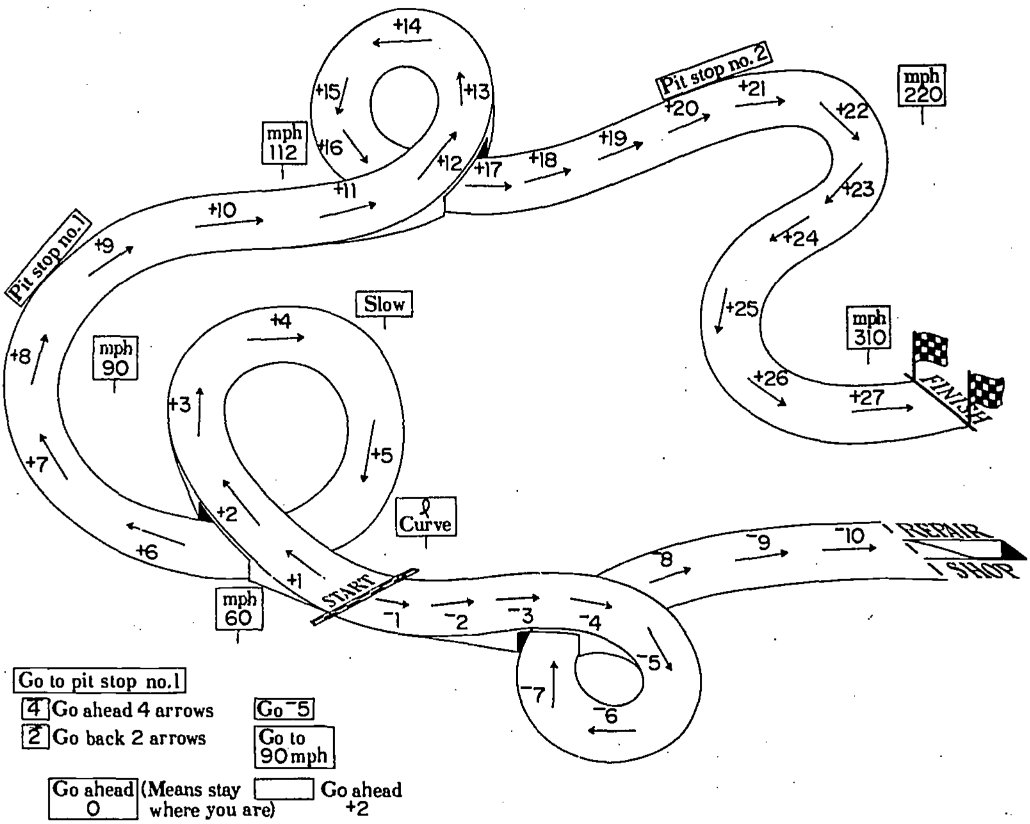


FIG. 5. Bang Bang Chitty Chitty

[3] Numbers back of starting line were negative numbers and led back to the repair shop with its grease etc.

sociative properties of addition in computing total scores. The games provide teachable moments for the teacher to introduce pertinent material in greater depth.

Bobby (Grade 2)

Object: Get to treasure at end.

Rules: Use spinner to indicate number of moves. If answer is correct, stay until next spin. If answer is not correct, go back to preceding place.[4]

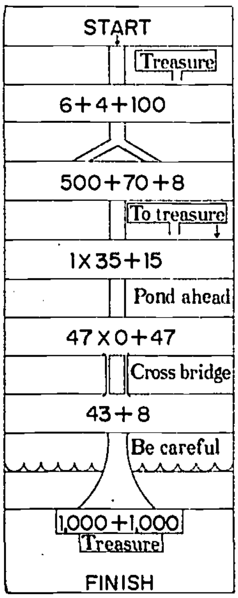


FIG. 6. Treasure Island Game

4. Each child has a marker of a different color or of a different kind.

Chris (Grade 1)

This game is played similarly to the Snoopy Game (fig. 2). This first grader included multiplication and subtraction in his game.

Object: Get to finish line.

Rules: Use number cube to indicate number of boxes for move. Correct answer allows player to remain there until next spin. Wrong answer makes player go back to his preceding position. If player lands on "Go back," he must go back the number of places indicated in the box.

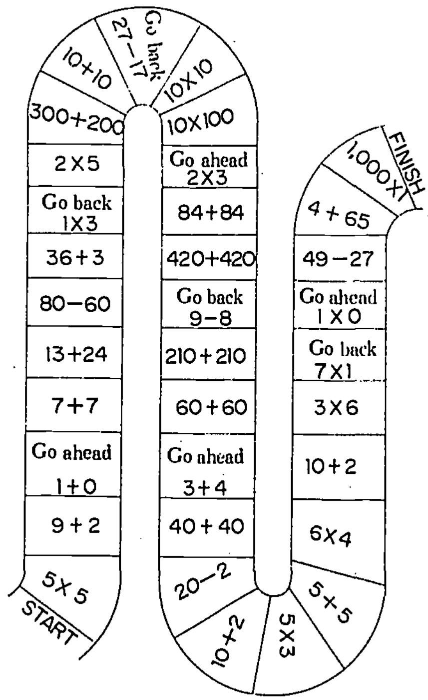


FIG. 7. Try and Guess Game

Blast-off mathematics

ELTON E. BEOUGHER

Elton Beougher is an assistant professor of mathematics at Fort Hays Kansas State College, Hays, Kansas. He teaches courses in mathematics and mathematics education for elementary and secondary teachers.

A more appropriate and informative title for this discussion might be "Astronomy, Space Travel, and Arithmetic." An attempt will be made here to support the claim that the study of arithmetic and of topics from space science can be coordinated in the elementary school program. Further, it is claimed that this coordination can enhance both the mathematics and science areas and that each can promote learning in the other. The author hopes to provide some ideas that teachers might use to give variety to their lessons.

A number of specific gains can be made through this endeavor. A few that will be illustrated are:

1. Practice using large numbers
2. Practice in estimation and rounding of numbers
3. Provision of numerous examples of the use of ratios
4. Development of skill in measuring and using angles
5. Provision of examples of scale models and development of skill in building such models
6. Provision of examples of many geometric figures, both plane and solid
7. Practice in use of the function concept, that is, relationships between varying quantities
8. Development of a better understanding of our system of time measurement
9. Drill in the fundamental arithmetic operations
10. Provision of many examples for graphing
11. (Trite as it may sound, but still valid) preparation of students to be intelligent citizens of the space age

The reader will note that the advantages claimed generally deal with mathematics understandings and skills. This reflects the fact that the author's main interest is in mathematics education.

At this point the reader may say, "Stop! Put the countdown on HOLD. These claims could be made for many areas of science or life, and such advantages could accrue in many other areas." This is correct. However, there are distinct advantages in favor of the approach through astronomy and space travel.

One advantage is that students already have words in their vocabulary to prepare them for this study: g-force, weightlessness, LM, satellite, zero g, orbital path, apogee and perigee, solar orbit, extraterrestrial, and so on. (This may not be true of the teacher's vocabulary!)

A second advantage is that a great deal of motivation is already present for this study. These are truly children of the space age with whom we deal. Space travel is as real and acceptable to them as a drive to the next town. If the reader doubts the motivation possibilities, he should take a look at the Saturday morning and after-school shows on television which cater to children. Science fiction and space travel have long been the strong suit of these television hours. Consider as added evidence the attraction of children to space toys and games.

A third, major advantage is that there are a multitude of aids, booklets, and films available on the subjects of astronomy and space travel, many of which are free in class quantities. At the end of this article is a short, annotated list of such resource materials and sources of aids.

Specific topics

In order to add credence to the previous claims, some specific examples of how this study might be accomplished in the elementary school are in order. A more extensive listing of topics and projects is appended at the end of the article. Each topic might be one day's lesson, or, if enough interest developed, could be expanded into a unit.

1. *Connection between astronomy and time measurement.* Few people realize the strong connection between our position in space and our measurement of time. A historical look at time measurement might be revealing in this aspect. It is recorded fact that ancient peoples told time by observance of heavenly bodies—the sun, moon, and stars. Our present-day units of time—years, months, days, hours, minutes, and seconds—all developed out of these early methods. To show the arbitrariness of our time system and its structure, a study of the length of "days" and "years," and possibly "months" (if they had moons), of other planets would be very revealing. For example, Mercury's "day" is about 59 Earth days and its "year" is about 88 Earth days. In view of these facts, a lively discussion could ensue from the question, "If we lived on Mercury, how could we measure time?"

Pupils of early and middle elementary grades would be fascinated by the calculation of their ages in Mercury "years" or Jupiter "years." They would be surprised to find they were less than one Jupiter "year" old. (Jupiter's "year" is about 12 Earth years.)

2. *Study of astronomical navigation.* There are two facets of this—earthbound navigation and navigation in space. A teacher and his students could study the relationship between earthbound navigation and the angle of elevation of the sun at a given spot on the earth. The measuring of this angle could be accomplished by a simple instrument whose construction is discussed later. The study of the connection between this angle and the ideas of longitude and latitude could then lead to an interesting study of navigation.

Navigation in space is seen to be a much different and more complex task. This results from at least three facts. No reference system such as latitude and longitude on earth exists in space. On earth, points on land and in the ocean stay fixed. The poles and equator do not move relative to the earth. In space, no such fixed points exist. Any reference system is constantly moving, since all bodies in space are in constant motion. Secondly, two numbers suffice to fix a position on the earth (latitude and longitude), but in space three or more numbers are needed. Thus space navigation necessarily would be based in a three-dimensional geometry (or maybe four-dimensional, if time needs to be considered) rather than in a two-dimensional one. This is a fascinating study in itself. A third inherent difficulty of space navigation is that much more planning and plotting of the course must be done before the flight begins. This was evident in the recent moon flights. Only minor corrections can be made after such a trip is in progress. Essentially, the flight must be completely preplanned down to the most minute detail.

A study of navigation with your students could lead to fruitful results for later work in algebra, since there is a strong connection between navigation and graphing and graphing plays a vital role in algebra.

3. *Measuring distance in space.* The distances betwen natural objects in space are so great that ordinary units of measure and notation for numbers become unwieldy. For example, the distance to the star nearest to us (besides the sun) is approximately 25,800,000,000,000 miles. Generally, astronomers and other scientists write such large distances in scientific notation. For example, the number mentioned would be written as 2.58×10^{13} miles in scientific notation. Another example would be the

mean distance of the planet Pluto from the sun, 3.67×10^9 miles. This would be 3,670,000,000 miles in usual notation. (Note the relationship between the power of 10 and the number of places that the decimal is moved in changing from one notation to the other.) Astronomers also use another unit of measurement in naming such large distances. This is the light-year. One light-year is the distance that light travels in one year at the fantastic rate of 186,000 miles per second (approximately). To provide an idea of the enormity of this, there are 86,400 seconds in one day. In each of those seconds light would travel 186,000 miles. Think now of the 365¼ days per year to get an idea of the size of a light-year. To place it in the familiar framework, this is approximately 6,000,000,000,000 miles. In this system, the closest star is 4.3 light-years away. (This means it takes 4.3 years for light to reach us from that star.) As mentioned before, this is 25,800,000,000,000, or 2.58×10^{13}, miles.

Another fascinating study is that of methods of measuring distances in space. There are a number of sources on this subject of "indirect" measurement, and many of these are within the grasp of elementary school students. Later we will discuss the applications of some of these methods which your students could make.

4. *Variation of gravity on the planets.* A fruitful study of ratios could accrue from this topic. Pupils would be highly motivated to find what their weight would be on other planets and the moon. They would be surprised to learn that if they were fairly good high jumpers, on the moon they could probably jump over their family's car with ease. Furthermore, they could probably pick up their own mother and carry her around, assuming she weighed about 100 pounds. This is because the moon's gravity is $\frac{1}{6}$ the earth's, and objects would weigh $\frac{1}{6}$ their earth weight on the moon. Your students could find the weight of familiar objects and multiply by $\frac{1}{6}$ to find moon weight so as to test their "moon strength."

However, your students would be astonished to learn that on Jupiter they might not even be able to lift a medium-sized dog, since the gravity of Jupiter is about 2½ times that of Earth. The gravity of the sun is about 28 times that of the earth. In such a situation, one of your students could not even lift his own arithmetic book! The possibilities for practice in multiplication and in the use of ratios are almost limitless.

5. *Shapes of orbits.* Most of your students would be aware from diagrams on television and in the newspaper that orbiting objects, whether man-made satellites or planets, do not travel in perfect circles. The path is an ellipse. You (or your students) can easily construct an ellipse by using two pins stuck into a board, a pencil, and a piece of string with its ends tied together to form a closed loop (see fig. 1).

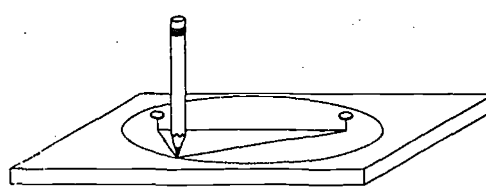


Fig. 1

The pencil point is placed within the closed loop and then the loop is pulled taut by the point. A closed figure is then drawn on the board by moving the pencil around, keeping the string taut. Viewed from above, the result is as in figure 2. The closed figure drawn is the ellipse.

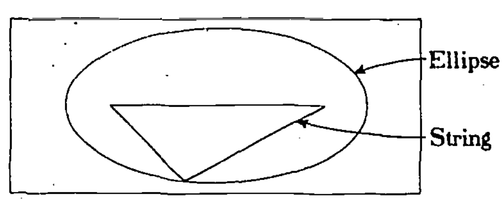


Fig. 2

The two points where the pins are placed are called foci (singular, focus). When a planet orbits the sun, the sun is at one of these foci. In general, when a small body orbits a much larger one under gravitational influence, the large body is at a focus of the ellipse. There are precise mathematical laws governing this motion which would be out of reach of most students' minds, even at the upper levels of elementary school. However, the motion paths are simple enough to understand.

The other topics on the list at the end of the article offer many possibilities for study. With a little bit of thought and preparation and some reading on the part of the teacher, each topic could be developed into an interesting lesson for students. What should be kept in mind is that the purpose of the lessons is to use the subject matter of astronomy and space travel to illuminate and expand arithmetic lessons and to motivate students in arithmetic. The sources listed at the end of the article would prove valuable in this light. Some of these would be useful strictly for their content of facts. Others list aids that may serve to make the teacher's task more appealing both to him and to his students.

Models and instruments

One possibility for an interesting activity is the construction of simple measuring instruments and devices for demonstrating various concepts. One such device works nicely to illustrate number 18 on the list of topics. The device is made using a spool, two weights (large fishing weights from the sporting-goods store will do nicely—experiment to find best sizes), and a piece of string about 3½ feet long. Assemble these materials as in figure 3. Weight 1 is whirled around in a circular path by grasping the spool. Weight 2 will hold weight 1 in a path of a certain radius depending on the relative weights of 1 and 2. Weight 2 acts as the force of gravity does. If we increase the "force" of gravity by increasing the weight of 2, then we note that weight 1 "orbits" closer to the spool. Experiments will show that it will also orbit faster in this new position. Planets in orbit around the sun behave in this way. The planets close to the sun—Mercury, Venus, Earth, and

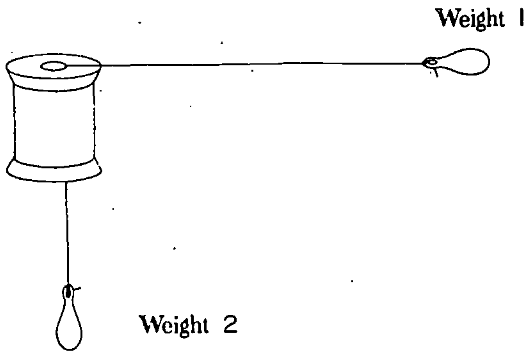


Fig. 3

Mars—move faster in their orbits than the ones farther out—Jupiter, Saturn, Uranus, Neptune, and Pluto. This is because gravity is stronger close to the sun than it is farther from the sun.

In addition to the mental practice to be gained from these studies are the skills to be learned in planning and carrying out projects involving mathematics and space science. Some such projects are suggested in the list at the conclusion of the article. This is only a small sample of the possibilities.

There is much to be gained from the building of scale models. The topics of ratio, multiplication, division, and scales are just a few concepts that are involved. Also, pride in the completed project can contribute much to its value as an educational tool. Depending on the age of the child, the teacher might have to offer more or less guidance in the construction.

Building of a scale model of the solar system

A word of caution should be offered here. It would be impossible to construct a scale model of the solar system all on one scale, that is, with the size of planets scaled the same as the interplanetary distances, and have it fit in a classroom. If this were attempted, in order for Mercury to be large enough to see (say ⅛ inch in diameter) the model of Pluto would have to be about 25 miles away from the model of the sun. (This gives some idea of astronomical distances). So unless you want your "scale" model of the solar system to be 50 miles across, you'd better use two separate scales! A convenient choice might be ⅛ inch = 1,000 miles for the size of planets and 1 inch = 100,000,000 miles for the size of the solar system. This would make the solar system model about 6 feet across. A model of the sun could not be made on the same scale as the planets and have it fit in your model solar system (it would be necessary to make it 9 feet across). Thus it would be best just to indicate its position at the center of the solar system and not have a scale model of it in your solar system.

Projects involving graphing

Projects that could certainly involve graphing are numbers 3, 5, and 8 on list B. An easily made instrument could be used for measuring the angles in numbers 3 and 8. The instrument is constructed by using a nail, a piece of string, a small weight, and a piece of scrap wood about a foot square (see fig. 4). The nail is placed at the center of the arc on which the degree scale is indicated (i.e., at the center of the circle of which the arc is a part). The weight and string are attached to the nail and are used as a plumb line. In use, the instrument is held so that the string passes through the 90-degree mark on the scale.

To measure the angle of elevation of the sun, the device is held plumb so that the nail casts a shadow down across the scale, and the angle is read from where the shadow intersects the scale. The angle measured in figure 4 is approximately 35 degrees. In carrying out number 3 of list B, it would be important to measure the angle at the same time every day (preferably at noon) and to face the instrument in the same direction. A record of these measurements over a length of time would contribute to an understanding by students of the relationship between the changing angle of the sun's rays striking the earth and the change of seasons. A point of caution that should be stressed to your students is that they should **never look directly at the sun** in carrying out any of these measurements.

One source indicated that nine-year-old students carried out project 6 with use of a kit available for less than two dollars. The telescope was powerful enough for one to see many details of the moon's surface and possibly to observe four of Jupiter's moons. They could be seen with the naked eye were they not so close to Jupiter as to be blotted out by its bright light.

A number of the sources listed under C describe how to construct sundials of various types. This would be instructive again in relation to our system of time measurement as well as valuable as a project to be carefully completed.

Indirect measurement and triangulation

To get a feel for the methods of indirect measurement used by astronomers, students could carry out "earthbound" measurements indirectly. Either trigonometry or ratio and proportion could be applied. Use of the latter would involve measuring the length of the shadow cast by the sun of an object of known height (a student) and, at the same time, measuring the shadow of an object of unknown height (a tree). These two are then compared as follows:

$$\frac{\text{boy's height}}{\text{boy's shadow length}} = \frac{\text{tree's height}}{\text{tree's shadow length}}$$

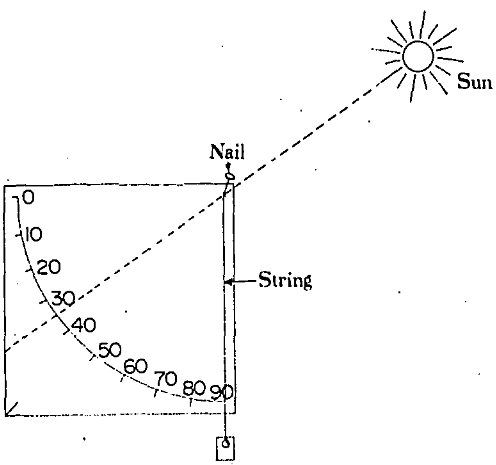


Fig. 4

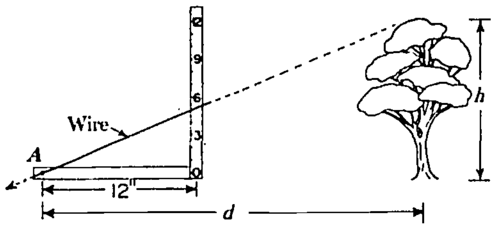


Fig. 5

The use of ratio is obvious. This is somewhat simpler than the method astronomers use, but the basic principle is the same—"triangulation."

An easily constructed instrument for finding measurements indirectly is based on the principles of triangulation. Materials needed are two pieces of scrap lumber about two inches wide and one foot long. These are fastened at right angles and a one-foot scale is drawn on one as in figure 5. A sighting wire is attached at *A* by use of a small bolt or screw so that the wire can be rotated around *A*. To use the instrument, one sights along the wire at the top of the object whose height, *h*, is to be measured (fig. 5). Care must be taken so that the lower edge of the horizontal board is kept horizontal (a plumb line along the vertical board would facilitate this). The distance to the object, *d*, is then measured and a ratio is set up as follows:

$$\frac{\text{reading on scale}}{12} = \frac{h}{d}$$

Again the use of ratios is obvious. Pupils in the later grades would have had experience with ratios, proportion, and similar triangles and could easily comprehend the mathematics of the instrument.

Conclusion

The versatile teacher can make many uses of the instruments described previously. Furthermore, the topics, projects, and sources given in the appended lists would provide a wealth of material for the same teacher.

A. Some suggested topics for study
1. Connection between astronomy and time measurement
2. Navigation and astronomy
3. Units of measure and methods of measuring distance in space
4. Variation of gravity on planets
5. Relative size of planets
6. Relative distance of planets from the sun
7. Length of "years" of planets
8. Shapes of orbits of natural and artificial satellites
9. Number of moons of planets as related to size of planet
10. Weightlessness and orbiting
11. g-forces
12. Shapes of spaceships and reasons for these shapes
13. Size of rockets as compared to man
14. Size of payloads compared to size of rockets
15. How rockets work
16. Decrease in pull of gravity as distance from earth increases
17. Why planets are spherical
18. Speed of orbiting object as related to distance from object it is orbiting

B. Some suggested projects
1. Construct scale models of the sun and the planets.
2. Construct a scale model of the solar system.
3. Keep a record of change of the angle of elevation of the sun over time and study the use of such a record by ancients to determine seasons.
4. Observe and time an eclipse of the moon.
5. Keep a record of the change in the length of days with the seasons.
6. Build a simple telescope and calculate its power.
7. Observe the moon's craters and compare their size with the size of land masses on the earth.
8. Determine the relationship between the angle of the tilt of the earth's axis with the plane of its orbit and the seasons.
9. Construct a sundial.
10. Measure distances and heights indirectly by triangulation.
11. Taking the conditions of amount of light, temperature, gravity, and so on, on a given planet, determine the physical characteristics of beings that could live there. Perhaps even write a story about their lives.

C. Sources of content, ideas, and aids

Anthony, James K. "Events That Led to the Discovery of Pluto." *School Science and Mathematics* 53 (January 1953): 316–18.

Arey, Charles. *Science Experiences for Elementary Schools*. New York: Bureau of Publications, Teachers College, Columbia University, 1961.

Bernardo, James V. "The Space Age: Its Impact on Education." *School Science and Mathematics* 63 (January 1963): 5–19.

A short history of the space age and NASA. Discusses the role of NASA in elementary and secondary education and details the aids and services available to educators through NASA.

Blough, Glenn O., and Marjorie H. Campbell. *Making and Using Classroom Science Materials.* New York: Dryden Press, 1954.

The Book of Popular Science. New York: Grolier.

Deason, Hillary J., and Ruth N. Fey. *Science Book List for Children.* Washington, D.C.: American Association for the Advancement of Science, 1961.

Gleason, Walter P. "Real Astronomy." *School Science and Mathematics* 54 (January 1954): 31–38.

Discusses construction and use of some easily made instruments that demonstrate astronomical concepts and principles: (1) instruments for measuring the altitude of the sun, (2) sizes for a scale model of the solar system, (3) a sundial, and (4) a model to demonstrate how gravity keeps a satellite in orbit.

Johnson, LeRoy D. "A Physical Science Demonstration: The Planets." *Sci[illegible] Science and Mathematics* 53 (October 19[illegible]): 569.

Jones, Louise M. "A Trip to the Moon." *School Science and Mathematics* 60 (June 1951): 486–87.

Discusses an excellent and promising interdisciplinary idea: Have your class plan a trip to the moon—planning food, clothing, jobs they would take in transit, and so on.

Kambly, Paul, and Winifred Iadley. "Elementary School Science Library for 1961," *School Science and Mathematics* 62 (June 1962): 419–38. "Elementary School Science Library for 1962," *School Science and Mathematics* 63 (May 1963): 387–414. "Elementary School Science Library for 1963," *School Science and Mathematics* 64 (June 1964): 467–82.

Good selection of astronomy, mathematics, and general science books and books on rockets, jets, and space travel. Each book is annotated and the grades 1–6 are given for which the material would be most appropriate.

Mallison, George C., and Jacqueline V. Buck. *A Bibliography of Reference Books for Elementary Science.* Washington, D.C.: National Education Association, National Science Teachers Association, 1960.

National Aerospace Education Council. *Aerospace Bibliography.* Washington, D.C.: Government Printing Office, 1970.

———. *Free and Inexpensive Pictures, Pamphlets and Packets for Air/Space Education.* Washington, D.C.: National Aerospace Education Council.

A list of many free and inexpensive materials, including sources from which these may be obtained by teachers.

Nelson, Leslie W., and George C. Lorbeer. *Science Activities for Elementary Children.* Dubuque, Iowa: Wm. C. Brown Co., 1955.

Norris, Theodore R. "An Inverse Square Relationship in Science." ARITHMETIC TEACHER 15 (December 1968): 707–12.

Report of a sixth grader's experience in experimenting and drawing conclusions based on data and observations.

Pettorf, H. Ronald. "Fundamental Ideas of Space Travel." *School Science and Mathematics* 63 (May 1963): 405–10.

Background information for the teacher (to catch you up with your students!).

Ray, William J. "Just for Fun: From Arc to Time and Time to Arc." ARITHMETIC TEACHER 14 (December 1967): 671–73.

Reuter, Kathleen. "Sixth Graders Compose Space Problems." ARITHMETIC TEACHER 11 (March 1964): 201–4.

Report of fruitful results of discussions in class which followed some of the first U.S. orbital flights. A host of student-developed problems based on recorded facts about the flights are given.

Ruchlis, Hy, and Jack Engelhardt. *The Story of Mathematics: Geometry for the Young Scientist.* Irvington-on-Hudson, N.Y.: Harvey House, 1958.

Excellent resource for teacher. Practical problems in many areas, including space travel.

Smith, Bernice. "Sunpaths That Lead to Understanding." ARITHMETIC TEACHER 14 (December 1967): 674–77.

Deals with the problem of helping children understand the summer and winter solstices and the equinoxes.

"Space and Oceanography Bibliography." *Instructor* 79 (January 1970): 68.

A fairly extensive bibliography of audiovisual materials, pamphlets, and books suitable for elementary school students.

EDITORIAL COMMENT.—Have you thought about correlating your studies of ancient numeration systems with social studies (ancient history)? Or perhaps you might consider relating mathematical language to the study of other languages. Have you realized that when you teach latitude and longitude, you are teaching a mathematical coordinate system for a sphere?

The Function of Charts in the Arithmetic Program

Catherine M. Williams
Ohio State University, Columbus

Many teachers do an effective job of using concrete manipulative materials to help children with initial stages of the formation of arithmetic concepts. All too frequently, however, the teachers shift so rapidly from these concrete, visual experiences to abstract verbal symbols that children are unable to breach the gap. Children need intermediary helps paced to their emerging insights. Charts are one type of visual aid which can be developed to help children along the road from need for concrete materials to ability to operate with abstractions.

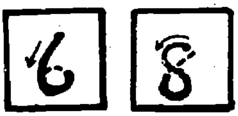
Usually children should participate in the development of the chart. This participation fosters the habit of organizing one's own ideas and of resourcefully working out one's own self-helps. Then, too, participation in developing a chart serves as insurance against introduction of material in chart form before the children are ready for the organized presentation. Because the pupils understand a chart which they have helped to work out, it becomes the meaningful reference which they need temporarily.

One of the common mistakes in the teaching of arithmetic is that charts (this includes all types of tables) which appear in arithmetic texts are introduced before experience has made them meaningful. Not even the most simple of organizational charts should be introduced until after concepts have been developed through rich first-hand experiences. This seems obvious when we remember that the sense, the meaning of any logically organized form is a development process which evolves as experience provides new insights. Since the meaning of any logical organization is an outgrowth of one's own experiences, children should not be denied opportunities to build up their own organization. After children have worked out their organization, they have basic understanding and are ready critically to study the mathematician's mature organization. Only then will they be able to appreciate his refined terms and short-cuts and/or his inclusion of information which they may have overlooked or lacked.

In this article it will be possible to explore only a limited number of situations wherein the teacher-pupil developed chart can be an important aid to the learning of arithmetic. Often chart-making situations can profitably include all members of a class group. However, when learning is paced to maturation, teachers will find that some charts ought to be worked out with sub-groups to meet their readiness needs.

Developing Charts

Teachers have discovered that beginners master forming letter symbols more readily when the manuscript alphabet is exhibited for their reference. Comparable guidance for the formation of figure symbols is seldom provided. However, as children learn the number symbols, teacher and pupils together might well work out a helping chart. For such a chart black crayon might be used for making the symbols. Red crayon might well be used for arrows which would indicate the path the pencil should take. Perhaps a dotted line could be used to indicate the part to be made last, for example.



Sometimes charts should not be completed all at once but should evolve cumulatively as new experiences contribute further pertinent information. A simple chart of this type might bear the title, *Signs We Should Know*. Signs of operation, the equal signs and the dollar sign are among those which would have a place on this chart.

When children reach that stage of maturity where familiarity with the terms for designating the numbers involved in each of the four fundamental processes is desirable, a chart might be developed. This chart would include a simple example of each process, accompanied by the proper term for designating each number. This chart would become a useful reference until such time as children are thoroughly familiar with all of the terms.

The three charts already mentioned typify the sort of information which is usually introduced with care, but which becomes unnecessarily complicated for children because they are not provided with helps to tide them over until the information is firmly established. The child for whom the coordination required to form the figures is a struggle should not be denied adequate reminder concerning the path the pencil should take. Children who are uncertain of which operation is indicated or those for whom memory fails and terms are a matter of guesswork cannot develop the sort of confidence which engenders further success. We teachers are short-sighted, indeed, if we fail to develop with children reference materials so that they can confidently and resourcefully undertake their tasks. Given such helps along with encouragement to become independent of these aids, children will delight in reporting, "I don't have to look anymore," or "I'm going to try this time and not look even once."

Occasionally charts should be developed simultaneously with the initial learning experience. A chart, *Rectangles*, might be developed when children are learning to recognize and identify the rectangle. In this case actual objects in the classroom, the door, the windows, and the like are being studied and, if recognized as belonging, are listed. The children might next bring in cut-outs from periodicals and mount them on another chart, *More Rectangles*, thereby extending the number of objects recognizable as rectangular. After children have had similar experience with square, circle and other common geometric figures, teacher and pupils together might well summarize the information in a chart, *Shapes Have Names*. This chart would include the picture symbols accompanied by appropriate word symbols.

Another type of chart which should closely follow experience is illustrated by a picture chart of liquid measure. After young children have used water to fill and empty into one another of the standard measures, along with ordinary containers such as milk bottles, a simple picture chart *Liquid Measure* would serve to summarize the information so recently discovered and verified through the experience. The chart would serve also as a reference until such time as a reminder becomes no longer essential. When children are more mature, the pictures would be unnecessary. However, participation in working out the organization of tables of measure, following initial experience, can enrich the learning at all levels. A chart regarding the measurement of time for example should not be developed all at once. It should be an outgrowth of experiences which, perhaps, had their origin in the discovery that seven days equal a week. When week, month and year have meaning, their relationship might be worked out into a simple chart. Information gathered over a few school years is necessary before a complete table of time measures can be meaningfully developed. The advantage of pupil participation in organizing their own table-type chart obviously is not that they will develop a unique chart. The advantages in their participation are that the learners think through what belongs and work out where it belongs in the scheme. When this type of chart seems complete to the learners, they should be encouraged to compare their results with standard tables found in textbooks designed for their level of maturity. Children who have this sort of experience have been spared rote memorization of another's organization of information, *they have been guided to organizing their own information into a table.* This latter procedure affords a very different quality of experience for the learner.

Let us turn attention to developing insight and facility with various ways of grouping whole numbers. If purposeful learning experiences do not occur with frequency and variety, the teacher should supplement them with contrived experiences. Experiences should be planned to utilize a variety of concrete materials for children to think with, to experiment with, to use for verification which leads to formulation of generalizations about groups of numbers. When children demonstrate understanding as they work with concrete materials, they are ready to organize information in simple chart form. An example is a chart, *About 8*, made to show the addition facts about eight. For an easy-to-make chart of this type dark background satisfactorily sets off white, half-inch, self-sticking discs (Dennison's Pres-a-ply Labels). Not only does this chart summarize the various combinations which total eight but also it is useful in establishing the relationship between addition and subtraction.

Charts may well be developed to build understanding of a process, for example, division. A chart of this sort might grow out of a situation where children need to find how many five-cent items can be purchased with thirty-five cents. Children should be encouraged each to figure out his own way of arriving at the solution. One child may perform successive subtractions. Another may fence off dot pictures in this fashion ... / .. .. / .... Another may use multiplication as an aid. When solutions have been arrived at, the teacher might ask each child to demonstrate the means by which he arrived at seven. The various ways should be studied and, perhaps, become a part of a chart *We Learn To Divide*. The various methods of arriving at solutions can help children to important discoveries. Experimentation will show that the new method, division, is shortest and fastest. At this point the teacher has an obligation to help children to realization of the fact that people lived for many years before anyone discovered this faster, shorter method called division. Children will appreciate mathematics if constantly they are helped to see that number is an ingenious system worked out by man as his insights developed. After several experiences with the new process, division, children should be ready to work out what additional information they think belongs on the chart, *We Learn To Divide*. Producing such a chart challenges children to self-reliant thinking and, because it builds both confidence and insight, serves as impetus for using the short-cut, division.

A group of charts which will aid in developing insight into the process of division consists of several helping charts. *Helps for Dividing by 7* would be but one of the series. Others would be developed by substituting various ones of the digits for the seven. To make the seven chart, mark off the paper into seven vertical spaces and ten horizontal spaces. The top rows from left to right would read

1	2	3	4	5	6	7
8	9	10	11	12	13	14.

Use of the chart is facilitated if the figures which are multiples of seven are shown in color. Such a chart helps to build understanding especially when remainders are involved. Confronted by the example, $7\overline{)45}$, children will recognize 42 as the product nearest 45. They will readily arrive at the quotient, six, with three remainder. Later when, children need to transfer remainders into fractional parts, this chart enables them *to see* that the three is three toward the next seven or three-sevenths.

When children are learning to understand large numbers both whole and those involving decimal fractions, they need a great deal of guided experience designed to build insight into value dependent upon position. When children experience a large number the teacher has both opportunity and obligation to help them to some realization of what such a number really signifies. A case in point occurred in a fourth grade engaged in a study of prehistoric animals. The earth was found to be about two billion years old. The teacher inquired if anyone could write two billion in numbers and a volunteer did so correctly. Then she suggested that the group might work together to find out how much two billion is. They worked out a chart *This Is Two Billion*, by starting with ten, $10 \times 10 = 100$ and so on. In this manner the teacher helped to extend the tens concept and to build understanding of position as well as to show children a good way of attempting to grasp the meaning of a large number.

Another type of chart, one showing the composition of a number, is also helpful in developing understanding. For example, the number 630,734 is partly composed of:

6 hundred thousands	600,000
3 ten thousands	30,000
0 thousands	0,000.

Breaking down the number and making a total for verification provides insight. The finished chart may be referred to in checking subsequent performances in reading and writing numbers.

A very different kind of chart is one worked out to indicate ways numbers are written and read for special business usage. This sort of chart would be worked out only after children had developed considerable facility with reading numbers. For gaining background information for making a chart to indicate correct reading of numbers in business relationships, children might be asked to investigate how telephone numbers, street numbers, license plate numbers, dates and the like are read. Experiences will differ widely. Some children will be ready to cite the difference in the way 1492 is read if it is a date, a telephone number, or, perhaps an amount of money. Other children will need time to collect data. When they have collected pertinent information, the teacher will guide their activities so that pupils become familiar with the variations in each category. For example, at least four types of telephone numbers are essential. Typical of the four are these: 5081, 6000, 7700, 2-2209. After each, the correct reading should be indicated.

Production of a time line can become another rich chart-making experience. If pupils work out a time line of their own, they discover relationships between fixed dates. With the complete chart for ready reference the scope and sequence of time become significant.

The direct teaching chart is another important type of chart. A simple illustration of this type is one titled *Things in Pairs*. For such a chart children will clip from periodicals pictures of items belonging in pairs and before mounting, sort the pictures into those in which the pair is two separate items and those in which the "twoness" is apparent but the two are *fastened together* as are a pair of eyeglasses. The term pair takes on meaning and the concept is extended through the activity. Other concepts such as that of instruments of measurement can be extended through the process of making simple picture charts.

Another illustration of the direct teaching chart is one designed to show square measure. On wrapping paper or building paper of adequate size, children can measure off a square yard, the square feet in a square yard, and the square inches in a square foot, all in actual size. Relationships are apparent when they see the real thing instead of only the scale representation which appears in textbooks.

Not all charts will be teacher-pupil made. At times teachers will want to make up their own demonstration charts. The concept of average, for example, can be clarified by use of a few charts designed for the purpose. Such charts are effective teaching tools if used to stimulate thoughtful examination, followed by development of tentative generalizations which are experimentally tested.

Usually we teachers overlook numerous opportunities for pointing out the role number plays in other areas. Children will gain new understanding and appreciation if introduced to a chart showing fractions in music. They will be interested to discover that relationships between notes, and between the notes and rests in a measure are actually fractional relationships.

The few charts which it has been possible to discuss indicate that charts can be functional aids to the teaching of arithmetic. Each teacher will recognize many instances wherein the teacher-pupil developed chart will meet the needs of her particular group.

What is to be done with charts once their initial purpose is served? If placed in a flip-chart rack or in an oversize "book," the charts will serve as reference so long as they are needed. Also, they will serve as valuable review material and as milestones of progress. Since participation in the making of the charts is so important a part of the learning experience, it is obvious that most of the charts should be discarded at the end of the school year. Because they should be discarded it is neither necessary nor desirable that they be made of expensive, long-lasting material. Newsprint size 24″×36″, both plain and ruled, is inexpensive paper which proves satisfactory for most of the charts mentioned. Because it is large enough for material to be made visible from every point in the classroom, it is often preferable to the more expensive poster boards, usually available only in smaller size. Crayons and brush pens are both satisfactory for writing. Both are easy to write with and both provide easy-to-read markings.

EDITORIAL COMMENTS.—Classes that use the laboratory approach may find student-made charts an excellent way to summarize laboratory activities. Teachers may find that "activity cards" or laboratory directions can easily be presented in chart format.

Using charts in the classroom may save many classroom hours often wasted writing material on chalkboards. Once prepared, the charts may be saved for future classes. Laminating them will add many years to their life expectancy.

Bulletin boards for elementary school arithmetic

DONALD E. HALL
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The bulletin board, an often-neglected type of visual education in the field of elementary arithmetic, is a powerful means for getting ideas across, an excellent attention-getter and interest-stimulator.

The following suggestions are given for the construction and maintenance of bulletin boards in the areas encompassed by the field of number. The manner in which these materials and children's specialties are displayed indicates the type of teaching done and the concern for the use of one more effective avenue of communication in arithmetic education.

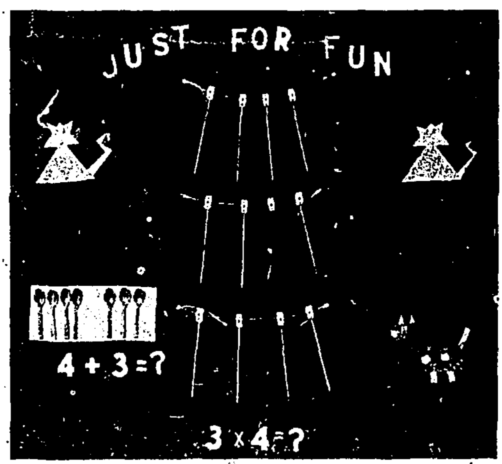


Figure 1

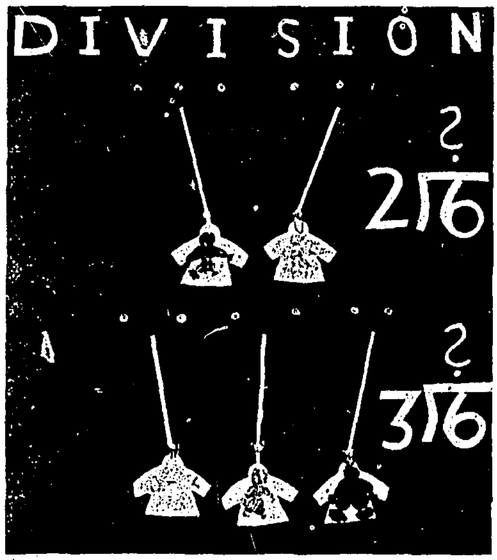


Figure 2

1 To sharpen a particular point that you wish to emphasize, provide a bulletin board *title;* often two-toned letters in definite contrast to background color are effective.
2 It is not always necessary to place titles in the center of the board. Lettering on a slant is permissible, especially in the upper grades.
3 Attention is drawn to the board by the wording of the title. Sometimes questions such as "How many sets?" or "Can you figure this out in the binary system?" serve this purpose very well.

4 Experiment with three-dimensional materials. Lettering made from rope, sandpaper, etc., often enhances the ideas. Wire or staple "things" to the bulletin board. Cones, cubes, and cylinders, as well as pictures, may be used to illustrate effectively a concept which, without the dimensional approach, might not be understood.

5 Watch carefully the colors chosen for mounting pictures and other background material. If the material or object is predominantly warm in color, it should be counterbalanced with a cool color. Try parts of silver and gold gift boxes for mounts and frames.

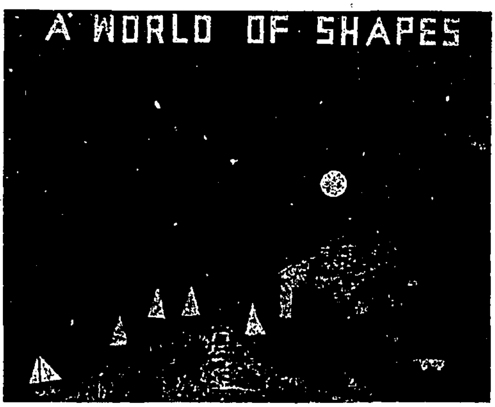


Figure 3

6 Definite emotional appeal is provided when there are distinct contrasts of light against dark. Dark mounts tend to advance and light ones to recede.

7 The following props are usually available to all teachers: mailing-tubes, wire coat hangers, glass blocks, ping-pong balls, whitewashed stones, seeds, corrugated paper, shoe boxes, colored cellophane, yarn, rope, twine, sand, cotton, cork, plastic, felt, tiles, gummed paper, balsa wood, straws, dowels, paper milk-bottle caps, metal bottle caps, etc.

Usually, completed bulletin boards serve their purpose in two or three weeks. The exception is the type of display to which additions are made around a central theme or understanding daily or weekly. In a display where Steps One, Two, and Three are used, the final step is sometimes left open-ended for the children to discover.

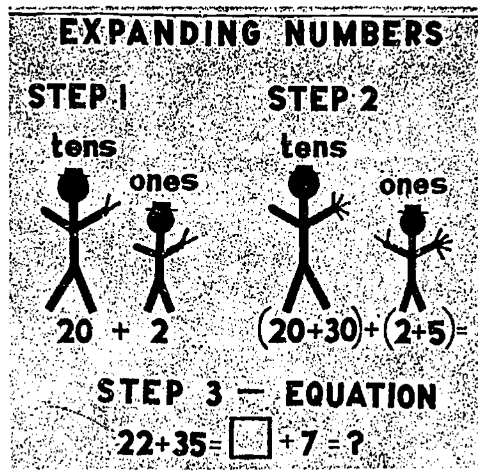


Figure 4

The type of bulletin board probably used least of all is the *Review Board*, yet it is an excellent way of summarizing a particular subject or area in arithmetic.

A crowded, monotonous display is actually worse than no display at all. This is also true of bulletin boards put up hurriedly with the use of one thumbtack or pin per picture, or pictures poorly cut and mounted. An uncluttered board is to be desired, for it is indeed true that too much material crowded onto a given space gives the viewer visual indigestion. Generally speaking, stapled papers and pictures lack dignity and good taste. Common or plain pins are sometimes less offensive than thumbtacks; masking tape is more easily controlled than transparent tape.

Through the use of convenient materials and simple procedures, any elementary classroom may be revitalized in the area of arithmetic. It is hoped that these suggestions will be used as a springboard for the investigation of vast, new possibilities in teaching arithmetic to today's children.

A mathematics laboratory—from dream to reality

PATRICIA S. DAVIDSON and
ARLENE W. FAIR

Patricia Davidson is coordinator of mathematics for the Newton Public Schools of Newton, Massachusetts.
Arlene Fair is an elementary teacher who serves as a mathematics coordinator at the Oak Hill School in Newton.

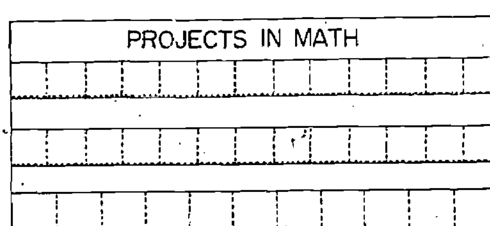
Just a year ago last September, the mathematics laboratory program at the Oak Hill Elementary School was still a dream. A few plans had been made during the previous spring for an initial purchase of materials, and the first job in the fall was to find "the place." It was decided to convert a small storage room on the basement floor into a mathematics laboratory. By the end of the year, the dream had become a reality—the laboratory approach to learning mathematics had become a vital part of the mathematics program in the whole school.

Since visitors to the lab ask so many of the same questions—"How did you organize the room?" "Just how did you begin?" "What is the purpose of a math lab?" "How do you organize the children?" "Are the classroom teachers involved?" "How does the lab program relate to the class work?" "What materials are needed?"—we should like to share some of our experiences with those of you who may wish to develop a laboratory program.

Getting started

First of all, we were going to have to wait several months to have the room painted (Does this sound familiar?)—so with the help of the children, we cleaned, washed, and made the room colorful and attractive in other ways. Some sixth-grade boys, who enjoy this type of work and are particularly adept at it, took great pride in repairing and painting—in bright colors—some discarded tables and chairs, which were then decorated by the art teacher. (We have been asked several times where to obtain such delightful furniture!)

Along one of the long walls we hung a piece of fabric on which we had sewn pockets to hold individual problem cards and booklets.



Such a rack can be made any height or length. To be most serviceable, the background should be of a dark color with bright colored pockets adding to the decor. The pockets can be made in different sizes to accommodate various-sized cards, booklets, or project sheets, and they can be

All photographs used in this article were taken by William Winston.

labeled if they are to be used for the same thing all the time. We never have labeled ours because we are constantly evaluating the written materials that we are developing, and with the help of the children we are always adding new ideas.

The other long wall had shelving from floor to ceiling which we used for storing the concrete materials. We organized and labeled the shelves in categories so that each item could be found easily and the children could learn to put away their own things. Since our particular shelves had six major sections, we used six main headings:[1] *Blocks* (with subheadings such as Attribute Blocks. Geo-Blocks, Poleidoblocs, Logical Blocks, Discovery Blocks, . . .); *Geometric Materials* (with subheadings such as Models, Mirror Cards, Geoboards, . . .); *Measuring Devices* (which included drawing tools and all kinds of balances); *Numerical Games* (with subheadings such as Calculators, Chips, Abaci, Kalah, Chi-Wah-Ree, Tuf, Imout, Heads Up, Card Games, Slide Rules, Cuisenaire Rods, . . .); *Strategy Games* (with subheadings such as Tower Puzzle, Qubic, Twixt, Jumpin, . . .); and *Shapes and Tiles* (with subheadings such as Tangram Puzzles, Pattern Blocks,[2] Mosaic Shapes, Linjo, Three-D-Dominoes, . . .).

In addition, the room had some book display racks to hold other kinds of student instructional materials and cupboards that contained labeled boxes of miscellaneous items like paper, pencils, oak tag, construction paper, scissors, string, rubber bands, needles, thread, buttons, tongue depressors, egg cartons, beans, sticks, and so forth.

A large sheet of brown paper helped to cover one cracked wall while serving also as a useful backing for hanging students' work—for example, various geometric designs, curve stitching, and graphing projects. Also early in the fall, some students made coat-hanger mobiles with colorful three-dimensional models cut from contact paper to hang from the ceiling.

We placed a bulletin board on the fourth wall with pictures from magazines headed with the caption "Guess How Many." Underneath was a small table with jars of peas, beans, macaroni, rice, etc., for the "Peas and Particles" unit. A few study carrels made by students out of Tri-Wall[3] cardboard and a portable blackboard completed that side of the room.

We started out with one small patch of rug so that children could work on the floor, and since that time parents have donated enough scraps to "cover" the floor patchwork style. Pupils enjoy working with materials on the floor. The rugs help to cut down the noise level and also the amount of furniture necessary.

Oak Hill Mathematics Laboratory

The picture was taken in October after about a month's effort to transform a dirty, dingy room into a warm environment that "looks like" a Math Lab.

Several teachers who have visited the lab have used many of these ideas to con-

1. For a detailed listing and ordering information for most of the materials mentioned here, see P. Davidson, "An Annotated Bibliography of Suggested Manipulative Devices," THE ARITHMETIC TEACHER, October 1968. These six categories were a consolidation of the fifteen used in this article.

2. Not available at the time of the bibliography, Pattern Blocks were developed recently by Elementary Science Study. The materials and Teacher's Guide are available from Webster Division, McGraw Hill.

3. For information on Cardboard Carpentry write to Education Development Center, Inc., 55 Chapel Street, Newton, Massachusetts 02160.

vert a corner of their own classrooms into a math lab area.

Working with children

Although we were able to have some small groups of children working with materials during September—as they helped us get the room ready—the lab really opened in October.

During the month of October, our aim was to have every student come to the lab for at least one 1/2–1 hour session. In order to make it possible for each teacher to come to the lab along with her students, we worked with whole classes, which often meant that there were too many children for the small room. But the orientation for a teacher with her own students is very necessary. In these first sessions, we tried to set a tone—

1. that you can learn math not only with paper and pencil, but also through the use of manipulative materials.

2. that the math lab approach involves active investigating, exploring, hypothesizing, looking for patterns, and "doing" rather than being told or shown.

3. that mathematics is many things; that there are often many right answers to a problem; and that usually you can check your hunches yourself by means of the materials.

4. that although much of the work will seem like fun and games, all of the lab experiences can be related to specific math concepts, to problem-solving techniques, or to modes of mathematical thinking.

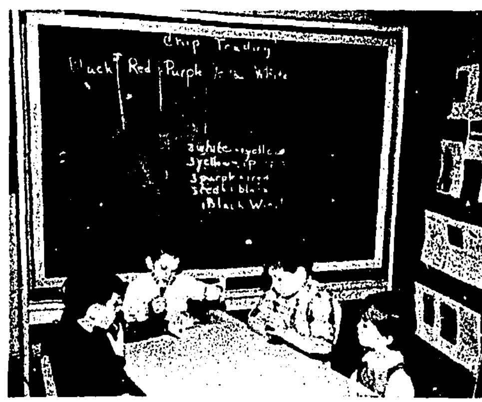
5. that at the beginning, the lab teacher will choose what activity you embark on, but once you have pursued enough of the materials to know what some of the possibilities are, you will be given some choice.

6. that care of the materials is the responsibility of each student; that the loss of one piece may mean that the entire set of materials is unuseable; and that you are responsible for putting the materials you used in their proper place.

7. that often projects will be started or materials introduced in the lab that will be followed up in your classroom or at home.

Training of teachers

Simultaneously, the classroom teachers were trained. We tried to do different things with each class during that first month, in order to get most of the materials in use. Also within each class, groups of children worked with several different things. For example, in one third-grade class, a little group worked with Pattern Blocks; another little group with balances; another little group with geoboards; while a fourth group did chip trading activities,[4]



Chip trading for understanding of place value

two students worked with Tangram Puzzles, and two explored Geo-Blocks. When each class left the lab after their initial session, the students took one or two things —for example, chips and geoboards—back with them. The teacher was responsible for reading manuals and getting help from us to be able to continue the work. Those children who had used these particular materials also helped the other children get started.

There were also workshops for all or

4. See P. Davidson, "Chip Trading—a Strategy for Teaching Place Value," *Professional Growth for Teachers*. Third Quarter Issue 1968–69, and "Exchanges with Chips Teach 'Borrowing' and 'Carrying,'" *Professional Growth for Teachers*, First Quarter Issue, 1969–70. Croft Educational Services, Inc., 100 Garfield Avenue, New London, Connecticut 06320.

some of the teachers on Tuesday or Thursday afternoons throughout the year. (We are fortunate in Newton in that elementary teachers have released time every Tuesday and Thursday afternoon.) Also several

Measuring with spring scales

members of the faculty attended citywide math workshops, and needless to say, there were many informal discussions in the teachers' room. Within the building, as well as citywide, there was a constant sharing of ideas and of written sheets or math lab cards.

Expanding the program

After that first month, the lab really "exploded" to permeate the whole school. As much as possible, a lab schedule was built around the math periods of the entire school, so that each class including kindergarten and special classes could have 1/2–1 hour each week. A copy of the schedule was given to each teacher who was responsible for sending her children to the lab on time.

How many children came at once? Occasionally, the children still came by classes with the teacher so that a unit with such materials as geoboards, rods, balances, or various measuring devices could be introduced by the math lab teacher (Mrs. Fair, who had some time released from other teaching duties) with the help of the classroom teacher, and then taken back to the classroom.

More often a teacher would send *one of the arithmetic groups,* say 10–12 children, for special help in some area. These children were taught by the math lab teacher or by a capable college student teacher. Sometimes, the classroom teacher arranged for coverage of the rest of the class in order to come too.

Very often, *small groups* of students worked in the lab on anything that the math lab teacher, or a college student teacher, or a high school student teacher, or a parent volunteer[5] wished to introduce or that they might choose. The games became very popular, as we constantly changed the rules to make them more mathematical and more challenging.

Nongraded sessions, with 4–6 children from three grades at once, or with three children from each grade, proved very fruitful. Sometimes many units of work were introduced; while at other times, the students of various ages worked on the same thing, going as far as ability will allow. (Most of the materials are so open-ended that they are appropriate for students, kindergarten–grade 6 and higher.) We were in for many surprises as to what children can do when given concrete situations—it became clear that many of the common conceptions about age and ability grouping need to be challenged seriously.

After the program got started, the children often acted as *lab assistants.* Children from all grades acted as teachers in an area for which they had been checked as being well versed. Eventually, we accumulated enough materials in the lab to start a loan service to each classroom. Many of the teachers established a math lab day once or twice a week in their classrooms and used children to help.

The buddy system

Another aspect of the program was the buddy system, in which sixth-grade students volunteered to work in math with

5. Through math lab sessions with parents and invitations for them to visit the math lab in session with children, several parents became interested in helping in various ways.

first graders. A workshop (four sessions) was given to these sixth graders in the use of materials, with the stress on the meaning of number and number relationships. The first-grade teachers identified those children who needed help or enrichment, as the case might be. Each sixth grader was assigned one of these first graders with whom to work throughout the year. They worked for about one-half hour at a time, usually three afternoons a week. The improvement in attitude of all of the children was remarkable. A couple of youngsters who no longer needed help did not want to give up their "buddies," and every first grader wanted to be included. We were never lacking in teachers (sixth graders); students kept volunteering as those involved shared their experiences. The comments of the sixth graders were interesting: "I didn't realize teaching was so difficult." "It is such fun to see him begin to understand it." "I didn't think I could find so many ways to show 3 + 4 = 7." "It is like having a real sister," (said an only child).

The buddy system gave the sixth graders a real purpose, a feeling of importance, and a sense of responsibility. The children initiated the system that if a sixth grader was to be absent, he would call a friend on the phone to arrange for a substitute who was to report to the lab teacher in the morning. In handling all of the materials and games, the sixth graders really improved their own backgrounds—"to teach is to really know it."

The mathematics lab fair

The year culminated in a Math Lab Fair at which more than 350 children in the school were at work with materials at the same time in nine classrooms, the corridors, the gym, and the math lab. The pictures illustrate some of the activities. In one evening, over 500 parents came to observe and to work with the children. The Fair was also held on a Thursday afternoon for teachers from the other 22 elementary schools in the city and for visitors from communities. During another morning, the Oak Hill children helped groups of children from other schools in the city which were planning for a math lab program.

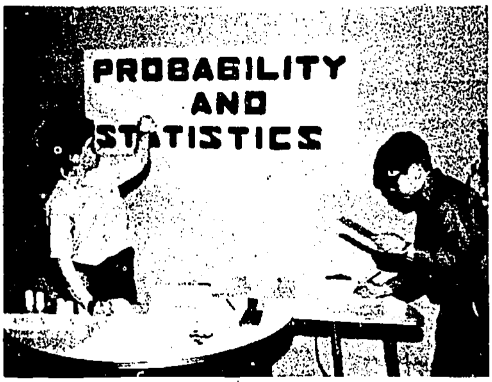
How to make a start

Starting a math lab program in a school is no easy adventure. The success of the program at Oak Hill is highly dependent on the cooperation and support of the principal, Mr. Samuel Turner, and the hard work and enthusiasm of the entire staff. From a small beginning, the program is now growing and expanding in ways that could not possibly have been anticipated. You must begin at whatever stage is appropriate to your present mathematics program.

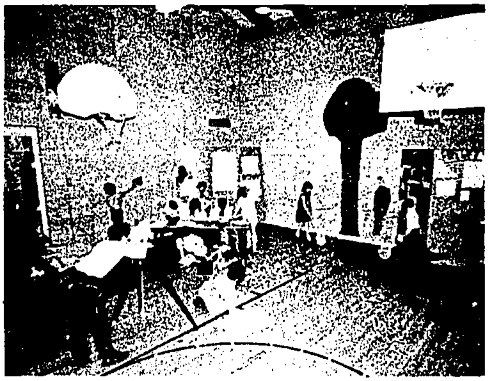
Original games

In deciding what to purchase initially, consider each large category of manipulative materials and get some from each, as much as money allows. The materials specifically mentioned in this article have been particularly useful for our program. Materials like geoboards, Cuisenaire rods, balances, chips, etc., should be bought in small quantities, but there is no need to start with more than one each of numerical games, strategy games, and puzzles. See which ones the children like before ordering more, and also use your own ingenuity in making materials. Once you start, you will find that through improvising, you often can add many improvements that "hand tailor" the materials to the needs of your particular students. The children can make a great many things and will enjoy creating some of their own games and problems. A must is adequate space for proper care of the materi ls.

But most of all you need one large bottle of vitamin pills, unlimited vitality and enthusiasm, and the patience of Job, for you will not only be the mathematics coordinator for your building and the resource person for every teacher in the school but you will be expected to know: every game and how to "win"; the answer to every puzzle presented to you by kindergarteners through sixth graders; where all the missing pieces are; and how to mend every broken game.



Probability and statistics



The mathematics fair

But when you sense the enthusiasm and see the growth in your students, you no doubt will share the feelings of the teachers at Oak Hill—that the laboratory approach to teaching mathematics is exciting and rewar ng.