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ABSTRACT

This NSF-sponsored conference on the teaching of mathematics at the junior high level involved mathematics teachers and teachers of the natural and social sciences. Papers written for the conference form the bulk of this report. Summaries of the papers and general discussions are organized into shorter reports to give some guidelines which could be useful to any group wishing to create curricular materials for junior high school mathematics or simply to get some ideas for direct application to the classroom. The topics of the report were: what mathematics junior high students do and do not know; new emphases in content; mathematics in geography, social science and biology; teaching strategies and styles; mathematics and language; and teaching training. (JP)

THE CAPE ANN CONFERENCE
ON
JUNIOR HIGH SCHOOL MATHEMATICS

SEPTEMBER 9 — 12, 1973

PHYSICAL SCIENCE GROUP
NEWTON COLLEGE

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THE ORGANIZATION OF THE CONFERENCE

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Even a brief analysis of school subjects shows that the relations among them is rather asymmetric. The subject matter and competencies developed in the social science class or natural science class have little effect on the success of the students in English or mathematics. However, what goes on in English and mathematics has a direct and dominant effect on what can be accomplished in the social and natural sciences. It is, therefore, not surprising that the mathematical curriculum in the schools has become the concern not only of mathematicians and mathematics teachers but also of the users of mathematics, i.e., teachers of the natural and social sciences. A Conference of members of all these professions took place on September 9-12, 1973, at Cape Ann, Massachusetts, to discuss the teaching of mathematics at the junior high school level. The complete papers written for the Conference form the bulk of this report. The purpose of this paper is to describe the way in which the Conference was organized.

Participation in the Conference was by invitation. Every participant (except John Lamb, the Recording Secretary of the Conference, Lauren G. Woodby and Michael M. Frodyma from the National Science Foundation, Judson B. Cross and myself) was asked to write a paper on a specified topic.

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To reserve maximum time for discussion of the papers and exchange of ideas, it was decided that the papers would not be read at the Conference but only discussed. This meant that all papers had to be submitted in time to be duplicated, mailed by us, and read by the participants before the Conference. Thus, lengthy presentations were avoided and discussion could proceed immediately. The papers had been grouped according to topics as shown in Table 1, and the respective authors formed panels to which questions and comments were directed. One member of each panel acted as chairman. Since there were only 24 participants, the discussions were on the informal side.

We succeeded in discussing 19 papers in plenary session in a little over two and a half days. The first two afternoons and part of the third morning were devoted to work in small groups for the purpose of summarizing the papers and the general discussion into group reports giving some guidelines which could be useful to any group wishing to create curricular materials for junior high school mathematics or simply getting some ideas for direct application to the mathematics classroom. These reports were then discussed in a final plenary session.

A summary of the conclusions reached by the various groups follows this article. The final reports submitted by the groups and the notes made by the recording secretary were edited and used in putting together the summary. Group B split into two groups, each submitting separate reports. There was no report written for Groups E and F.

The Conference was made possible through a grant from the National Science Foundation.

TABLE 1

Topical Grouping for Discussion

A - The Background: What Mathematics Junior High Students Do and Do Not Know

Fernand Prevost
Frederick Schippert
Glyn Sharpe
Thomas Dillon

B - New Emphases in Content

Romualdas Skvarcius
Leonard Nelson
Marion Walter
Byron Youtz

C - Mathematics in Geography, Social Science and Biology

Clyde Kohn
Irving Morrissett
Charles Walcott

D - Teaching Strategies and Styles

Jeanne Albert
Gwendolyn Steele
David Page
Earle Lomon

E - Mathematics and Language

James Rule
Peter Braunfeld
D. Newton Smith

F - Teacher Training

Peter Hilton

A SUMMARY OF THE DISCUSSIONS AND REPORTS OF THE VARIOUS GROUPS AT THE CAPE ANN MATHEMATICS CONFERENCE

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GROUP A: THE BACKGROUND: WHAT MATHEMATICS JUNIOR HIGH SCHOOL STUDENTS DO AND DO NOT KNOW

The primary objective of this group was the enumeration of mathematical skills of students entering the seventh-grade. These skills and the degree to which they have been mastered are:

Sets, numbers, and numeration: There exists a general understanding of these topics.

Operations and Properties: This is only partially mastered. In general, students have some computational skills involving whole numbers, but not all. There is little ability to work with fractions and decimals. It should be noted particularly that there is a lack of understanding in the areas of large numbers, reasonableness of answers, and estimation.

Similar deficiencies exist in the areas of: reading and problem solving, geometry and measurement, functions and relations, graphing, and probability and statistics.

For a more extensive explanation of the skills of entering seventh-graders, refer to Fernand Prevost's paper.

Much of the discussion of this group was concerned with what the non-mathematical world demands of a math curriculum - a topic closely

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related to the shortcomings of seventh-graders. A dichotomy emerged in the discussion between the "utilitarian" and the "aesthetic" aspects of mathematics. On the utilitarian side, parents, industry, teachers in other fields making use of mathematics (and even state legislatures) are demanding more emphasis on the "basics." The "new math" seemed to be related to the aesthetics aspect. There is, in other words, a shift in public desires away from the "understanding" and "theoretic" approaches of the "new math," in the direction of intense instruction in the "basics."

Although there are areas of computation that are not being mastered that should be mastered, seventh- and eighth-grade teachers trying to help their students gain the desired masteries should be strongly admonished not to repeat the same sterile drills that may have been used in grades 1 - 6. New meaningful, effective tools for learning must be provided.

Opinions were expressed against compartmentalizing math into individual units. A new curriculum should avoid this by repeatedly using important ideas throughout.

GROUP B1: APPROXIMATIONS AND ESTIMATIONS

Estimation, approximation, and the proper use of numbers obtained from measurements are generally unfamiliar not only to junior high school students but also to their teachers. Therefore, teacher training is deemed vital.

It was apparent in the course of the writing of this group's report that there was a need to clarify terminology (i.e., to distinguish between approximations and estimations). For this reason we decided to use the following definitions:

Estimation: The process of obtaining the approximate magnitude of some quantity when the necessary data (input numbers) are not readily available (e.g., estimate how many leaves are on a tree).

Approximation: The process of performing a "rough" calculation with given data (e.g., recognizing that

$$\begin{array}{r} 27 \\ \times\underline{39} \\ 243 \\ \underline{81} \\ 324 \end{array}$$

is obviously wrong because

$$27 \times 39 \approx 30 \times 40 = 1200).$$

The use of inequalities goes hand-in-hand with this area.

The practicing junior high school teacher in our group noted that approximation is mentioned in only three places in current mathematics programs. It comes up in connection with the algorithm for division, as a check to computation problems, and, simply, in a few problems in which students are directed to approximate quantities.

Too many students demonstrate the "exact answer syndrome" and hence are hesitant to work with approximations and estimations.

The need for increased attention to estimation and approximation arose more often than any other theme in this conference's discussions. Several conference participants noted that other recent mathematics conferences have also stressed this need.

Recommendations

It is recommended that increased attention be given to the following topics in the middle school curriculum: approximations, estimations, orders of magnitude, and numbers arising from measurement. These topics should be imbedded in units throughout the course as often as possible; they should not stand alone as isolated topics.

Initial estimation experiences should involve real objects so that results can be checked by counting (e.g., estimate the number of ping pong balls filling a shopping bag, rice in a jar, number of students in the school, etc.).

Continuing work on estimation should be done with circumstances which seem real and interesting to the student. This is where one can develop orders of magnitude and a feeling for large and small numbers (e.g., estimate the number of sixth-graders in the United States, or estimate the number of ping pong balls needed to fill the gymnasium). Care must be taken not to use examples involving numbers which are too large to be meaningful to the students (e.g., the number of drops of water in the oceans of the earth, the number of grains of sand needed to fill the earth sphere).

It is especially important that some numbers arising from measurements be introduced by having the students make the measurements themselves (e.g., using rulers, reading dials, reading graduated cylinders, etc.). Appropriate care should be taken in interpreting the results of computations with numbers arising from measurements. However, we do not at all suggest that a course in the theory of errors be introduced! Therefore, it makes no sense to generate hard and fast rules for tolerance or significant digits but simply to have some reasonable understanding of significant digits (e.g., don't give six significant digit answers when the data used contain only two significant numbers).

In view of the importance of approximation, it was noted that grading should reflect interest in "reasonableness" of answers. Hence, close answers should be given more credit than absurd responses.

GROUP B_2: MATHEMATICS AND AESTHETICS

The assignment for this group was to seek out and suggest topics and themes in mathematics which could reveal its structural and aesthetic qualities to junior high school students.

At the outset the practical and utilitarian roles of junior high school mathematics were recognized: the need for improved skills development, calculational ability, capacity to translate between the languages of English and mathematics. Any new curricular efforts <u>must</u> continue and intensify the search for better and more effective methods to teach these skills and improve these capabilities. Mathematics and science teachers share the concern of the general public on the importance of this learning.

There is a danger, however, that this concern could cause too great a reactive swing. In particular, it is essential that teachers continue to demonstrate to students that mathematics is structurally beautiful, powerful, and fun to do. We caution against a retreat from the intent of the newer mathematics curricula but acknowledge a need for substantial improvement in the methods used to accomplish that intent. Teachers must be trained in these arts so they do not inadvertently turn even the most elegant of concepts into hollow exercises in memory.

Aesthetic aspects of mathematics can be taught if, among other things, the mathematics curriculum provides more than one avenue for learning each concept and provides opportunity to investigate problems with more than one solution; affords lots of room for play and invention, proceeding at one's own pace; encourages honest discovery and local deduction in the classroom so far as possible; develops topics which contain surprises and challenges and when possible provides counter-examples to illustrate points; and deals with cases in which a student can verify the work for himself in order to build confidence and self-reliance.

Powerful unifying concepts, for example, the concept of function should be introduced early and used throughout the curriculum in a wide variety of contexts and situations and used in connection with topics which are intrinsically interesting and which get the student involved in the question for its own sake. (See the work of Frederique Papy for ideas and illustrations on symbolic representations, arrow diagrams, etc.)

Some Curricular Suggestions and Resources (No preferred order implied)

Much of the earlier curricular work should be reviewed and revitalized. Listed below are a number of possible topics.

Investigation of a variety of interesting and challenging geometrical ideas both in 2- and in 3-dimensions. These might include tessellations, symmetry, and projections, e.g., sphere on the plane. Use objects to manipulate and construct, when and for whom appropriate.

Exploration of geometries other than the standard Euclidean geometry (for example, "Taxi geometry" and "7-point geometry"). (See "Operational Systems" done for the Carbondale Project, Comprehensive School Mathematics Program.)

Graph theory. (Contact Jean Deskins, University of Pittsburgh, Mathematics Department, for references on introduction to graph theory.)

Combinatorial mathematics (see Engle's paper in Walter's bibliography).

Modular arithmetic to exhibit the arithmetic processes. Explore the differences between using a prime number vs. a non-prime as the modulus. Examine how we could reorder things (calendars, clocks, etc.) by choosing the modulus which would make the most ideal system. (See Carbondale Project, Book Zero.)

Elementary probability (It was recommended that sloppy "real" problems be avoided at first while developing concepts.) See probability trees as developed by Arthur Engles in another book produced for the Carbondale Project.

The curriculum should include a study of computers and algorithmic languages. The computer should be viewed both as a useful tool for further study in science and mathematics and as an interesting object of study in its own right.

Supplementary reading materials should be produced on the historical, cultural and biographical settings of various aspects of mathematics.

GROUP C: MATHEMATICS IN GEOGRAPHY, SOCIAL SCIENCE, AND BIOLOGY

The following six mathematical areas were identified as of major importance to geography, social science, and biology:

Estimation and approximation: Identifying relevant information and using it to arrive at approximate quantitative conclusions; example of estimation; estimate average family income; estimate what it would cost to build a school.

Geometry: Computing area, volume, perimeter; relationships between length, area, volume.

Graphing: Various forms of graphical representation of many different kinds of empirical data; reading off of ratios from graph: "eye-ball" fitting of curves to scattered data; scatter of data about line of best fit, linearity; comparison and addition of ratios; graphical representation of functional relations (analytical as well as empirical functions); qualitative reasoning from graphs; departures from linearity (quadratic and square root relationships).

Coordinates: Various kinds of coordinates (especially with regard to map function); latitude and longitude, world scales, standard projections; distance as a function of coordinates.

Functions: Basic notion of functional relations; functions as mathematical expression of causality; linear functions, equations; inverting functions (graphically).

Probability and statistics: Means, medians; random sample (intuitive feeling for randomness), tests of randomness, introduction to statistical inference; range and variance as measures of dispersion; compound probabilities (hence, familiarity with arithmetic of positive fractions); graphical representations of frequency distributions; concept of "reasonable behavior" based on probability estimates; decision-making and optimization.

All of the above topics should be integrated with each other as well as the rest of the curriculum.

GROUP D: TEACHING STRATEGIES AND STYLES*

It has been demonstrated that the mathematics classroom can be made into an environment where children learn from one another. Those who need more assistance can get it; those who need more challenge will find it.

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*Ed. Note: Much of this group's plenary "discussion" of teaching styles and strategies consisted of "live" demonstrations of the methods described below. For more details about these activities refer to the individual papers submitted by members of this group.

We can encourage playful competitiveness in which a great deal of desired learning takes place.

Creativity can be developed through student-initiated problems and their solution. The math classroom can act as a synthesizer of experience of varying lengths of time in and out of the classroom. Experiences involving a variety of solutions can be provided. Furthermore, given the opportunity students will approach the same problem in different ways, on different levels of sophistication.

Last, but not least, the junior high school mathematics classroom can be an area where it is fun to learn, by making judicial use of puzzles and games.

The kinds of active, vital classroom that include the above-mentioned activities do exist, but too infrequently. The goals of any new curriculum should include a recognition of the value of this kind of classroom. The structure of the text and teacher training should promote the appropriate learning atmosphere.

WHAT IS LEARNED IN ELEMENTARY SCHOOL MATHEMATICS?[1]

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It is currently fashionable to look at curriculum in terms of major strands. This has been especially true in mathematics and has been most extensively evident in elementary school mathematics. Of the many ways one can "break out" these strands let us consider these six.

1. Sets, Numbers, and Numeration
2. Operations and Properties
3. Sentences and Problem Solving
4. Geometry and Measurement
5. Functions and Relations
6. Probability, Statistics, and Graphing

These strands encompass the material generally covered in programs currently used in the schools of our country. With the possible exception of probability and statistics — not graphing — all major publishers do some work with these ideas in their series. The current series, copyrighted 1971 and after, have started to de-emphasize sets, do less with numerals to bases other than 10, incorporate some work with metric measures, and look at geometry from varied points of view.[2]

1 In order to analyze what is truly learned in elementary school mathematics one should be far more scientific than this paper intends to be. No claim is made for research, except as noted, and the bulk of the statements are made on the basis of observation, experience, and teacher interviews.

2 One must be aware that authors and publishers always include material that is outside of the main stream of the program. Final chapters are usually written with the brighter student in mind and other material is included for the purpose of "individualization."

The first three strands above cover material that society considers as fundamental. The latter three cover material with which the student should become familiar. Another way of looking at it is to say that strands 1, 2, and 3 should be taught for mastery while strands 4 through 6 are exposure strands — covering material to be mastered at a point beyond the time of initial introduction. Indeed, in these strands mastery may occur in high school or beyond.

Regardless of the intent of supervisors and authors, we are concerned here with what the evidence shows with regard to the learning in each strand. What does the average seventh-grader have as knowledge in mathematics when he steps into our classroom in September?

Strand 1

Generally the student is familiar with sets and the basic terminology associated with these. Union and intersection are familiar but not always clearly distinguished. A proper subset can be identified as a subset, but the empty set and the set itself are not yet accepted as subsets. If the terminology is to be used, review is urged.

The emphasis on concrete devices has led us to a better understanding of number and the "feel" for at least whole numbers is good. Integers are seen as plausible, especially in Northern climes, and fractions, as parts of wholes — not as quotients — are at least understood — in the sense of breaking something into equal parts and labeling each part. For some, the order of fractions has been learned and their position on the number line can be easily located. Decimals are generally less firmly learned, a function no doubt of our emphasis on fractions in Grade 5.

Numeration ideas, those of place value and the notion of a base, are often well established. Where they lack, concomitant problems arise in addition and subtraction.

Strand 2[3]

The criticism of lack of ability to compute — beginning in 1968 and continuing unabated — has led to some changes in our treatment of the algorithms. It is unclear if these changes will result in better computational ability on the part of students in the future. For the moment, the picture appears to be as follows.

Whole numbers: The student can add, subtract, and multiply with 80 per cent or better success. Division especially by three and four digits is generally poor. Students perform better in all four operations in non-test situations than they do in achievement batteries.

Fractions: Addition and subtraction of fractions with like denominators is generally done with success. Unlike denominators significantly reduce the success level. Multiplication is relatively easy but division, again, presents problems.

Decimals: If the decimals are only to two places, addition and subtraction are handled easily. More places and addends with an unlike number of places reduces the success level. Multiplication and especially division present difficulties to most students.

The basic properties of the operations are surprisingly well grounded. Students know and can give examples of the commutative (order) and associative (grouping) properties. The distributive property is not as well handled, but is recalled with little review. Unfortunately, we observe little ability to <u>apply</u> these properties in order to aid in calculations. The properties of closure, identity elements, and inverses are less well mastered.

<u>Strand 3</u>

The presence of "open sentences" in the programs from Grade 1 to Grade 6 has led to an intuitive ability to "solve" such sentences — when they

[3] The reader may wish to review the publication: Sixth Grade Mathematics: <u>A Needs Assessment Report</u>, Texas Education Agency, Austin, Texas, 1972. (A criterion referenced test report on 212 objectives administered to 22,000 students.)

contain but one operation. Some students can use the "inverse operation" procedure, but it is not a skill one can assume.

What does one say about problem solving? As was the case in earlier programs, some students can solve problems, others cannot. No "algorithm" is generally provided, no schema developed, and no common patterns of attack evident. One current series has attempted to develop a "Tell-Show-Solve-Answer" format. Though used successfully by some teachers it has yet to be tested in a large number of classrooms.

In general one can say, students in Grade 7 do not solve problems well.

Strands 4, 5, and 6

One can consider geometry, measurement, functions, graphing, and probability and statistics together — because little can be assumed in any of these areas.

Despite the presence of geometry in all series, it is the topic most often "skipped" in favor of more time to develop skills in computation. Certainly students know the names of the common geometric figures, both plane and solid, and have some ability to describe their properties. Also they can identify points, lines, planes, segments, rays, angles, right angles, and parallel lines. Notions of congruency, symmetry, similarity, and motions in the plane cannot be assumed. In the same manner, constructions cannot be assumed.

Generally, students can measure, but their experiences have been limited. Length is handled with greater ease than area and area most certainly has greater mastery than volume. Measure as a function or measure as comparison are both often lacking. Measure is seen as something you compute if you're lucky enough to remember a formula. Some students, of course, intrigued by the "application" of measurement are quite capable in the area.

At the moment, little familiarity can be expected with the metric system. It is to be hoped that this will change.

Liquid measure, weight, time, and other related topics are known but it is questionable as to whether or not the mathematics instruction has contributed most significantly to these.

Although "function machines" abound and "input" and "output" are terms familiar to the student, the concept of function is still developing — and may continue to "develop" through junior high.

Graphing ability is erratic. In the more activity-oriented programs it is well developed. In others, it may or may not be mastered. (We speak here of simple bar graphs, circle graphs, picto-graphs, and line graphs.) Test scores indicate an ability to read and interpret graphs. Little is known from these tests about the ability to construct a graph.

The graphing of ordered pairs and of functions is generally left to Grades 7 and 8.

And lastly, in the area of probability and statistics little, if anything at all, shows mastery. We would do well to assume no prior experience.

Finally, one must admit that teachers have had little guidance from us, mathematics educators, regarding specifically stated skills. Too often we have intermingled "mastery" topics among "familiarization" material. Without guidance, the whole has been taught for mastery or for exposure as the teacher felt was the intention of the authors. Accountability has to be shared by others and not exclusively by the teacher.

It is hoped that these observations of a suburban-rural area generalize to a larger — less parochial? — population.

MATHEMATICAL SKILLS OF SIXTH GRADERS IN DETROIT

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Enclosed is the mathematics portion of the Iowa Tests of Basic Skills Form 5, Grade 6 that is administered each April to all sixth graders in the Detroit Public Schools. It consists of two parts — Mathematics Concepts and Mathematics Problem Solving.

An item analysis is done each year. Since the results for the April 1973 testing will not be available until late 1973, the data shown reflects the results of the April, 1972 testing. They should prove adequate for the purpose of the Cape Ann Conference.

A total of 20,000 sixth grade Detroit students were tested city-wide. In addition to city-wide results, information regarding 3,000 inner-city sixth grade students is also reflected in the attached. Data is not developed in the Detroit testing program for the inner-city schools as a group as contrasted with the peripheral school. In view of the work that would have been involved to compile precise statements for all inner-city students a sampling of 3,000 was taken.

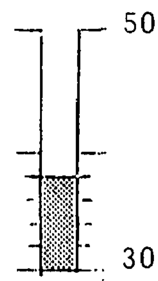
Although the first enclosure may appear cluttered, the data has been included with the test item for ease in analysis and discussion.

The key for the letters at the beginning of each item is given as a second enclosure. Two numerals precede each response — the first in a box indicates the per cent of inner-city Detroit sixth graders who gave that response, the second number is the per cent of sixth grade students city-wide who gave that response. The numeral of the correct response is circled. The per cent of the national norms group which gave the correct response is shown in a triangle. NR indicates no response was made by the student.

As an example, the third test item of Mathematics Concepts is marked as follows:

M - 3 3. What temperature does the thermometer show?

41	44	1	38° (63)
32	33	2)	39°
20	18	3)	41°
6	5	4)	42°
1	0	NR	



Meaning of Symbols

M - 3	Measurement - Temperature
$\boxed{41}$	41% of inner-city sixth graders selected the first response.
44	44% of the sixth graders city-wide selected the same response.
1	The correct response is No. 1.
63	63% of the national norms group selected the correct response.
$\boxed{1}$ NR	1% of inner-city students indiced no answer.

The readers attention is directed to the third enclosure in which the difference in per cent from the national norms group for the inner-city and city-wide is shown. The areas in which large variation exist should provide much discussion.

KEY FOR THE CATEGORY DESIGNATIONS

CONCEPTS

D - 1	Decimals - Reading and Writing
D - 2	Decimals - Relative Values
D - 3	Decimals - Rounding
D - 6	Decimals - Fundamental-Operations Estimating Results
E	Equations, Inequalities and Number Sentences
F - 2	Fractions - Relative Values
F - 3	Fractions - Equivalence
F - 4	Fractions - Terms
F - 5	Fractions - Fundamental Operations - Ways to Perform
F - 6	Fractions - Fundamental Operations - Estimating Results
G - 1	Geometry - Points, Lines and Planes
G - 2	Geometry - Recognizing Kinds and Parts of Geometric Figures
G - 3	Geometry - Angles and Triangles
G - 4	Geometry - Dimensions, Perimeters, and Areas of Polygons
G - 5	Geometry - Parts and Areas of Circles
M - 3	Measurement - Temperature
M - 5	Measurement - Length
M - 6	Measurement - Area and Volume
N - 3	Numeration - Place Value and Expanded Notation
N - 5	Numeration - Properties of Numeration and Number Systems
N - 6	Numeration - Special Subsets of the Real Numbers
R	Ratio and Proportional - General
S	Sets - General
W - 3	Whole Numbers - Rounding
W - 4	Whole Numbers - Partition and Measurement Average
W - 5	Whole Numbers - Fundamental Operations - Terms
W - 6	Whole Numbers - Fundamental Operations - Number Facts

PROBLEM SOLVING

C - A	Currency - Addition
C - S	Currency - Subtraction
C - M	Currency - Multiplication
C - D	Currency - Division
C - AS	Currency - Addition and Subtraction
C - MA	Currency - Multiplication and Addition
C - MS	Currency - Multiplication and Subtraction
C - DS	Currency - Division and Subtraction

D - S	Decimals - Subtraction
F - A	Fractions - Addition
F - S	Fractions - Subtraction
F - M	Fractions - Multiplication
F - AS	Fractions - Addition and Subtraction
F - MS	Fractions - Multiplication and Subtraction
P - MS	Per cents - Multiplication and Subtraction
R	Ratio and Proportion - General
W - A	Whole Numbers - Addition
W - S	Whole Numbers - Subtraction
W - D	Whole Numbers - Division
W - MA	Whole Numbers - Multiplication and Addition
W - MAS	Whole Numbers - Multiplication, Addition and Subtraction.

DIFFERENCE IN PER CENT FROM NATIONAL NORMS GROUP

CONCEPTS

Item	Inner-City	City-Wide	Item	Inner-City	City-Wide
1	23	16	23	15	13
2	6	5	24	8	4
3	22	19	25	19	13
4	10	6	26	10	3
5	15	4	27	22	18
6	8	6	28	19	14
7	22	19	29	16	14
8	19	20	30	16	15
9	12	13	31	22	18
10	13	6	32	9	5
11	20	16	33	6	6
12	15	11	34	21	17
13	28	20	35	16	8
14	26	22	36	25	22
15	18	14	37	29	22
16	12	11	38	18	12
17	18	19	39	17	14
18	8	8	40	19	14
19	11	2	41	11	10
20	14	6	42	9	8
21	14	8	43	13	10
22	15	8	44	6	0
			45	22	18

PROBLEM SOLVING

Item	Inner-City	City-Wide	Item	Inner-City	City-Wide
1	9	5	16	13	9
2	18	13	17	14	12
3	15	14	18	26	20
4	15	15	19	18	14
5	18	13	20	4	3
6	24	20	21	20	17
7	16	14	22	12	11
8	9	6	23	7	8
9	15	10	24	12	10
10	20	12	25	17	13
11	16	12	26	16	15
12	21	16	27	21	19
13	11	8	28	19	16
14	0	1	29	7	7
15	17	13	30	16	14
			31	5	5

Begin Here

N-3 **1.** What should replace the □ in the number sentence 73,642 = 70,000 + 3000 + □ + 40 + 2? 61

9 8 — 1) 6 15 13 — 2) 60 60 67 — (3) 600 15 11 — 4) 6000 1 1 NR

F-3 **2.** Which of these fractions is *not* equivalent to the other three?

8 8 — 1) $\frac{1}{2}$ 7 6 — 2) $\frac{2}{4}$ 9 9 — 3) $\frac{3}{6}$ 75 16 — (4) $\frac{5}{8}$ 81 — 1 1 NR

M-3 **3.** What temperature does this thermometer show?

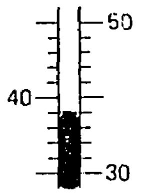
41 44 — (1) 38° 63

32 33 — 2) 39°

20 18 — 3) 41°

6 5 — 4) 42°

1 0 NR



N-6 **4.** Which numbers in the set {3, 5, 8, 9, 10, 11, 14, 15} are divisible by *neither* 3 nor 5?

13 14 — 1) {15}

15 13 — 2) {3, 9, 15}

16 13 — 3) {5, 10, 15}

55 59 — (4) {8, 11, 14} 65

1 1 NR

N-5 **5.** In which set below are both members factors of 16?

47 58 — (1) {2, 8} 62 — 6 6 — 3) {19, 3}

36 23 — 2) {1, 15} — 11 12 — 4) {32, 2}

0 1 NR

F-6 **6.** Which addition exercise can be explained by the picture below?

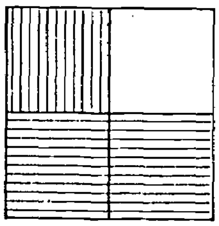
13 11 — 1) $\frac{1}{4} + \frac{2}{3} = \frac{11}{12}$

62 64 — (2) $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ 70

20 20 — 3) $\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

4 4 — 4) $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$

1 1 NR



E **7.** Bob saved 75¢ a week for 9 weeks. He bought a new light for his bicycle and had 90¢ left over. Which number sentence below can be used to find the cost of the light for his bicycle?

21 22 — 1) (9 × 75) + 90 = n

16 14 — 2) (9 × 90) − 75 = n

45 44 — (3) (9 × 75) − 90 = n 67

14 15 — 4) 90 − 75 = n

2 1 NR

N-5 **8.** What should replace the □ to make (96 × 24) ÷ □ = 96 a true number sentence? 61

11 12 — 1) 1 28 29 — 2) 4 42 41 — (3) 24 17 15 — 4) 96 — 2 3 NR

G-5 **9.** Which of the line segments in the figure below is a diameter?

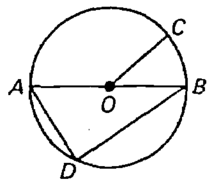
45 44 — (1) AB 57

30 33 — 2) OC

13 11 — 3) BD

10 10 — 4) AD

2 2 NR

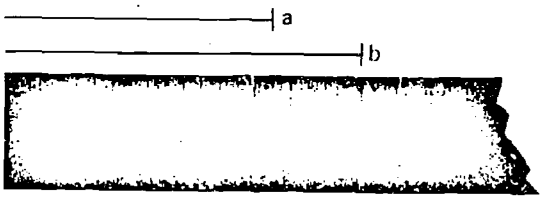


F-4 **10.** Which of these fractions can be expressed as a mixed numeral?

21 19 — 1) $\frac{7}{8}$ — 22 19 — 3) $\frac{0}{5}$ — 1 1 NR

18 16 — 2) $\frac{3}{3}$ — 38 45 — (4) $\frac{5}{4}$ 51

M-5 **11.** In the picture below how many inches longer is line segment *b* than line segment *a*?



32 28 — 1) $2\frac{1}{2}$ — 25 25 — 3) $\frac{1}{2}$

17 17 — 2) $1\frac{7}{8}$ — 25 29 — (4) $\frac{5}{8}$ 45

1 1 NR

N-5 **12.** Which pair of numerals below *cannot* be used as replacements for the □ and the △ in the number sentence 7 + (□ × △) = 12 + 7?

16 16 — 1) 2, 6 — 38 42 — (3) 5, 7 53

18 17 — 2) 3, 4 — 26 23 — 4) 1, 12

2 2 NR

F-2 **13.** Which of these fractional numbers is greater than one-half?

18 16 — 1) $\frac{1}{3}$ — 15 14 — 3) $\frac{1}{4}$

27 35 — (2) $\frac{2}{3}$ 55 — 39 34 — 4) $\frac{2}{5}$

1 1 NR

N-5 **14.** What should replace the □ in the number sentence 9 × (7 × 4) = 63 × □ ?

27 31 — (1) 4 53 — 19 18 — 2) 7 29 28 — 3) 9 24 22 — 4) 28 — 1 1 NR

Go on to next page ▶

Page 18

F-5

15. What should replace the □ in the equation
$\frac{5}{9} + \frac{\square}{9} = 1\frac{5}{9}$?

1) 1
2) 4
3) 9
4) 10

N-5 16. What should replace the □ in the number sentence
$(7 \times 12) + (7 \times 5) = \square \times 17$?

1) 5
2) 7
3) 12
4) 19

E 17. How could the equation $\square - 32 = 78$ be solved?

1) Add 32 to both sides of the equation.
2) Subtract 32 from both sides of the equation.
3) Multiply both sides of the equation by 32.
4) Divide both sides of the equation by 78.

G-4 18. Which of the following statements about rectangles is *not* always true?

1) A rectangle has 2 pairs of parallel sides.
2) The length of a rectangle is twice its width.
3) A rectangle is a parallelogram.
4) A rectangle has 4 right angles.

F-5 19. What should replace the □ in the number sentence
$\frac{2}{3} \times \frac{4}{7} = \frac{2 \times 4}{\square}$?

1) 6
2) 8
3) 10
4) 21

W-3 20. If 313 is rounded to the nearest ten, what is the resulting number?

1) 300
2) 310
3) 320
4) 400

G-4 21. Which graph shows the set of whole numbers greater than 5 and less than 7?

1)
0 1 2 3 4 5 6 7 8

2)
0 1 2 3 4 5 6 7 8

3)
0 1 2 3 4 5 6 7 8

4) 
0 1 2 3 4 5 6 7 8

D-1

22. How would you read 7.02?

1) 7 point 2
2) 7 and 2 tenths
3) 7 and 2 hundredths
4) 7 and 2 thousandths

23. The plans of a new house are drawn to the scale $\frac{1}{2}$ inch = 1 foot. The living room is 8 inches long. How long will the living room be in the new house?

1) 8 feet
2) 16 feet
3) 20 feet
4) 24 feet

W-5

24. Which of the following is a factor of $2 \times 2 \times 3 \times 5$?

1) 8
2) 9
3) 10
4) 16

M-6 25. Bob's room is 12 feet long and 8 feet wide. How many square feet of floor space does he have in his room?

1) 20
2) 40
3) 48
4) 96

N-6 26. What should replace the box in $27 + \square = 15$?

1) −42
2) −12
3) 12
4) 42

E 27. Which pair of numerals could be used as replacements for the □ and the △ in the equation
$64 = (\square \times 3) + \triangle$?

1) 15 and 4
2) 2 and 2
3) 20 and 1
4) 21 and 1

F-5

28. What should replace the □ in the equation
$\frac{7}{8} - \frac{2}{8} = \frac{7-2}{\square}$?

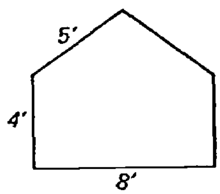
1) 0
2) 5
3) 8
4) 16

N-5 29. Which of the following expressions is *not* equivalent to $8 + (4 \times 3)$?

1) $(8 + 4) \times 3$
2) $8 + 4 + 4 + 4$
3) $(4 \times 3) + 8$
4) $8 + (3 \times 4)$

Go on to next page ▶

G-4 30. What is the perimeter of the figure below?



1) 17'
2) 18'
3) 24'
4) 26'

D-2 31. Which of these numerals represents the greatest number?

1) .307
2) .730
3) 3.07
4) 3.7

G-2 32. Which figure below is *not* a parallelogram?

1) A
2) B
3) C
4) D

Make no marks in this booklet.

E 33. In which equation would you divide to find the value of n?

1) $n \div 7 = 8$
2) $9n = 72$
3) $\frac{n}{7} = 6$
4) $n - 8 = 14$

F-3 34. Which of the following is a set of equivalent fractions?

1) $\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}\right\}$
2) $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$
3) $\left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$
4) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$

S 35. How many numbers are in the intersection of the sets {7, 8, 9, 10} and {8, 9, 10, 11}?

1) Exactly two
2) Exactly three
3) Exactly four
4) Exactly five

D-3 36. If 8.52 is rounded to the nearest whole number, what is the result?

1) 8
2) 8.5
3) 9
4) 10

E 37. Which numeral below will make $\square \times 8 < 56$ a true number sentence?

1) 6
2) 7
3) 9
4) None of these

W-4 38. The average age of 4 boys was 12 years. What was the sum of their ages?

1) 3 years
2) 16 years
3) 36 years
4) 48 years

D-6 39. Which of the following expressions would give the best estimate of $3.9 \times 5\frac{1}{6}$?

1) 3×5
2) 3×6
3) 4×5
4) 4×6

N-5 40. Which equation below does *not* have a solution?

1) $9 - n = 0$
2) $9 + 0 = n$
3) $9 \times 0 = n$
4) $0 \times n = 9$

E 41. What numeral should replace n in the number sentence $\frac{3}{4} \times n = \frac{2}{3}$?

1) $\frac{1}{2}$
2) $\frac{5}{7}$
3) $\frac{8}{9}$
4) $1\frac{5}{12}$

F-5 42. What should replace the $\square$ in the equation $\frac{3}{4} + \frac{1}{2} = \frac{3 + \square}{4}$?

1) 1
2) 2
3) 3
4) 4

G-3 43. At which of these times do the hands of a clock form an angle of 90°?

1) 3 o'clock
2) 4 o'clock
3) 6 o'clock
4) 12 o'clock

N-6 44. Which of the following represents a prime number?

1) 6
2) 7
3) 8
4) 9

D-2 45. Which of the following is greater in value than .761?

1) .8
2) .699
3) .7603
4) .716

STOP Here

IOWA TESTS OF BASIC SKILLS

Test M-2: Mathematics Problem Solving

Directions: This is a test of your skill in solving mathematics problems. The exercises in the test are like the samples shown at the right. After each exercise are three possible answers and a "Not given" — meaning that the correct answer is not given.

Work each exercise and compare your answer with the three possible answers. If the correct answer is given, fill in the answer space on the answer sheet that has the same number as the right answer. If the correct answer is not given, fill in the fourth answer space.

The sample exercises show you what to do.

You will have 30 minutes to complete the test.

SAMPLE EXERCISES

S1. Peg has 1 sister and 2 brothers. How many brothers and sisters does she have?
1) **2** 3) **4**
2) **3** 4) (Not given)

S2. Ben had 5 butterflies in a jar. He opened the jar and 4 flew away. How many did he have left?
1) **5** 3) **2**
2) **4** 4) (Not given)

ANSWERS

S1. 1 ▮ 3 4

S2. 1 2 3 ▮

Begin Here

1. At Lincoln Park on June 18, there were 33 boys who wanted to play in the Little League baseball club. If they needed 9 players for each team, how many complete teams could they have?

W-D

7 7 1) 2 27 26 3) 4
50 54 2) 3 59 16 13 4) 6
0 0 NR

2. By the end of June enough boys had joined the Little League so that there were 4 teams of 9 players each, and 4 boys who served as stand-ins. How many boys were in the Little League by the end of June?

WMA

5 4 1) 32 32 37 3) 40 50
30 29 2) 36 32 29 4) (Not given)
1 1 NR

3. The distance between bases on a baseball diamond for older players is 90 feet. The distance between bases on a Little League diamond is $\frac{2}{3}$ as long. How many feet apart are the bases on a Little League diamond?

F-M

20 21 1) 30 13 11 3) 135
33 34 2) 60 47 33 33 4) (Not given)
1 1 NR

4. For the games on the Fourth of July, the Little Leaguers sold 50 tickets to adults and 46 tickets to children. The tickets cost 25¢ each for adults and 10¢ for children. How much in all did the club receive for the tickets?

C-MA

6 5 1) $4.60 36 36 3) $17.10 51
15 13 2) $12.50 42 44 4) (Not given)
1 2 NR

5. Trees are the oldest of all green plants. A giant sequoia tree cut down recently was estimated to be 3700 years old. A bristlecone pine tree was found that was 4600 years old. How many years older was the pine tree than the giant sequoia?

W-S

42 47 1) 2900 60 12 11 3) 3900
17 17 2) 3100 28 24 4) (Not given)
1 0 NR

6. The General Sherman, a famous tree in California, is almost 300 feet tall. This is 5 times as tall as a fully grown maple tree. About how many feet tall is a fully grown maple tree?

W-D

13 12 1) 50 19 16 3) 80
32 36 2) 60 56 35 34 4) 150
1 2 NR

7. Maple syrup is made from the sap of maple trees. If it takes 30 gallons of sap to make one gallon of maple syrup, how many gallons of maple syrup can be made from a tree that gives up 20 gallons of sap in one season?

R

10 11 1) $\frac{1}{2}$ 23 21 3) $1\frac{1}{2}$
28 2) $\frac{2}{3}$ 48 37 37 4) (Not given)
2 1 NR

Go on to next page ▶

8. The Page family drove to Montreal for their vacation. By the route they took, it was 693 miles from Tulsa to Chicago and 848 miles from Chicago to Montreal. How many miles was it from Tulsa to Montreal by this route?

W-A

1) 1541 3) 1441
2) 1531 4) (Not given)

9. Mr. Page had 3 weeks of vacation. The family spent 10 days in Montreal on their trip and 2 days in Detroit. How many days did they have left for travel?

W-MAS

1) 21 3) 9
2) 12 4) (Not given)

10. The Page family stayed at motels for 7 nights on their trip, and the rest of the time with friends or relatives. The total cost for motels was about $100. About what was the average cost for each night they stayed in a motel?

W-D

1) $7 2) $10 3) $11 4) $14

11. Ron had saved $15 to spend on the trip. In the first 6 days he spent one-third of the $15. How much did he have left to spend?

F-MS

1) $3.00 3) $10.00
2) $5.00 4) (Not given)

12. Ron wanted to drive home by way of Detroit and Denver. Mr. Page said the distance that way would be 2550 miles, but it would be 1618 if they drove from Montreal to Detroit to Tulsa. How many miles longer was the route by way of Denver?

W-S

1) 832 3) 948
2) 932 4) (Not given)

C-S

13. Mike and his friend Van found a walkie-talkie set in a store for $2.59. In a catalog they found a toy telephone set they liked better for $7.95. How much more did the telephone set cost than the walkie-talkie?

1) $4.36 3) $5.46
2) $5.44 4) (Not given)

14. The catalog gave the shipping weight of the telephone set as 35 ounces. It said the Post Office charged postage for a full pound on any part of a pound. For how many pounds would the boys have to pay postage on the telephone set? (16 ounces = 1 pound)

W-D

1) 1[illegible] 2) 2[illegible] 3) 3 4) (Not given)

15. The boys wanted to keep a telephone in each of their houses. A line 50 feet long was included with the telephone set. They measured 42 feet between their houses, $4\frac{1}{2}$ feet from a window to a table in Mike's room, and 6 feet from a window to a table in Van's room. How many more feet of line did they need to reach from the table in Van's room to the table in Mike's room?

F-AS

1) $2\frac{1}{2}$ 3) $10\frac{1}{2}$
2) 8 4) (Not given)

16. Helen's mother made a matching jacket and skirt for Helen to wear to school. She used $1\frac{3}{4}$ yards of woolen cloth for the jacket and $1\frac{1}{8}$ yards for the skirt. How many yards of woolen cloth in all did she use?

F-A

1) $1\frac{5}{8}$ 3) $2\frac{7}{8}$
2) $2\frac{3}{4}$ 4) (Not given)

17. Helen's mother bought $1\frac{1}{2}$ yards of rayon fabric to line the jacket. It cost 54¢ a yard. How much did she pay for the $1\frac{1}{2}$ yards of lining?

F-M

1) 27¢ 3) 81¢
2) 76¢ 4) (Not given)

18. Last year she made Helen a jacket from $1\frac{1}{2}$ yards of material. This year she used $1\frac{3}{4}$ yards to make a jacket. How many more yards of material did she use this year for the jacket?

FS

1) $\frac{1}{4}$ 2) $\frac{1}{2}$ 3) $\frac{3}{4}$ 4) (Not given)

19. Helen's mother had 2 yards of cotton material. She used $1\frac{3}{8}$ yards of it to make Helen a blouse to go with the skirt and jacket. How many yards of cotton material did she have left?

F-S

1) $\frac{3}{8}$ 2) $\frac{3}{4}$ 3) $1\frac{3}{8}$ 4) (Not given)

20. In a recent census the population of Alaska was 226,167. In the same census the population of Anchorage, the largest city, was 44,237. At the time of the census about what fraction of the people in Alaska lived in Anchorage?

R

1) $\frac{1}{2}$ 2) $\frac{1}{4}$ 3) $\frac{1}{5}$ 4) $\frac{1}{6}$

21. Mount McKinley, the highest point in Alaska, is 20,320 feet above sea level. Before Alaska became a state, the highest point in the United States was Mount Whitney in California, which is 14,495 feet above sea level. How many feet higher is Mount McKinley than Mount Whitney?

W-S

1) 5,825 3) 6,825
2) 6,175 4) (Not given)

22. Alaska has an area of nearly 600,000 square miles. About $\frac{1}{4}$ of Alaska is north of the Arctic Circle. About how many square miles of Alaska are north of the Arctic Circle?

F-M

1) 120,000 3) 200,000
2) 150,000 4) 240,000

23. The United States bought Alaska from Russia for 7.2 million dollars. The Louisiana Territory was purchased from France for 23 million dollars. How many million dollars less did the United States pay for Alaska than for the Louisiana Territory?

D-S

1) 4.9 3) 30.2
2) 15.8 4) (Not given)

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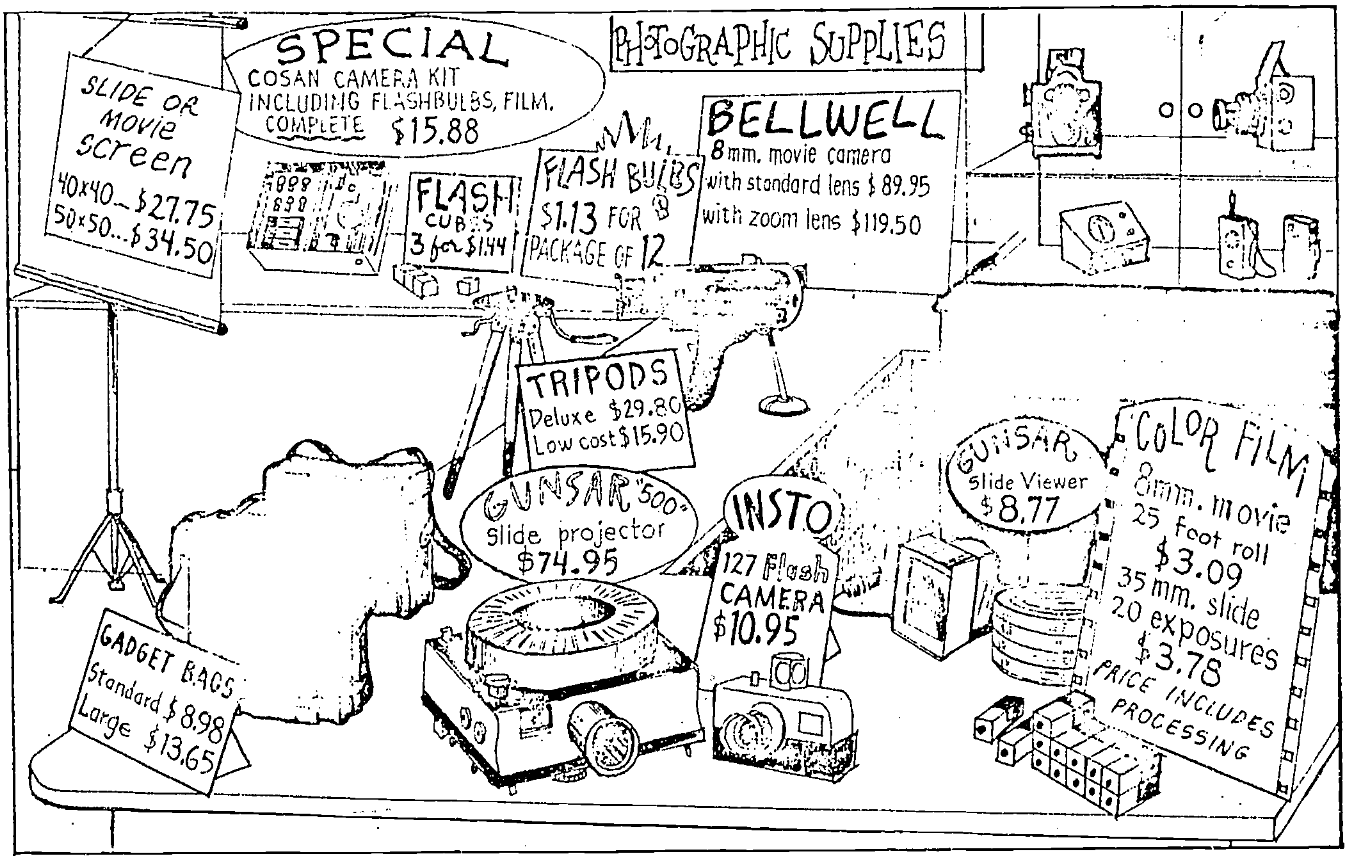

To work problems 24–31, look at the picture above to find the price of things. Do not allow for sales tax.

C-A 24. Mr. Douglas bought the following items: Gunsar "500" slide projector, 40" by 40" movie screen, and a 25' roll of 8 mm. movie film. What was his total bill?
1) **$95.79** 3) **$105.79**
2) **$102.70** 4) (Not given)

C-M 25. Before going to Glacier Park the Dennis family bought 7 rolls of 8 mm. movie film. How much did they pay?
1) **$21.63** 3) **$27.03**
2) **$22.63** 4) (Not given)

C-MS 26. Don bought 6 flash cubes and paid for them with a $10 bill. How much change should he have received?
1) **$2.88** 3) **$8.12**
2) **$7.12** 4) (Not given)

C-D 27. Roger wanted to buy a large gadget bag. If he saved 65¢ a week, how many weeks would it take for him to save enough money?
1) **20** 2) **21** 3) **23** 4) (Not given)

C-AS 28. Doug bought the Cosan camera kit. If he had bought the same items separately, the flash camera would have cost $14.95, the flash bulbs $1.25, and the film $1.98. How much did he save by buying the kit?
1) **$18.18** 3) **$2.30**
2) **$3.30** 4) (Not given)

F-M 29. Rick found the same "deluxe" tripod at a discount store for $\frac{1}{4}$ less. How much would he save on this tripod by buying at the discount store?
1) **$7.40** 3) **$11.92**
2) **$7.45** 4) (Not given)

C-DS 30. Every Saturday the store has a "special" on flash cubes at 39¢ each. This is how much saving over the regular price on one cube?
1) **9¢** 2) **27¢** 3) **$1.05** 4) (Not given)

P-MS 31. Next week the 50" movie screens are going on sale at 20 per cent off. What will one of these movie screens cost next week?
1) **$6.90** 3) **$27.60**
2) **$17.25** 4) (Not given)

Stop Here

MINIMAL ESSENTIAL PROFICIENCIES OF MATHEMATICS TO BE ACCOMPLISHED IN JUNIOR HIGH SCHOOL

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Overview

An earlier listing of more comprehensive objectives and topics was advocated for students in the junior high grades. These are felt to be desirable for average and above average students.

The listing indicated in this paper advocates a minimal or essential listing of objectives and topics deemed essential for all students. The position reflects strongly the point of view recently stated by the NCTM Committee on Basic Competencies in Mathematics of which the writer was a member.

Introduction

Few educators will deny that past efforts in mathematics education have produced significant gains for many pupils. There are too many cases, however, where many pupils leave our schools without the necessary skills to make them employable and to allow them to apply mathematics to help them solve the problems of daily life.

Some decisions are desperately needed to give guidance and direction to teachers and administrators concerning:

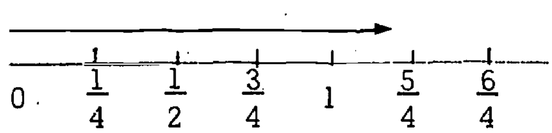
- what mathematics must every pupil master to "barely get by" in contemporary society?
- what mathematics is essential for full participation of an individual in contemporary society?
- what teaching techniques and processes not only assure the acquisition of mathematical skills and competencies deemed

essential but also convince pupils to be willing to use these skills once acquired.

Individuals often find the need to use mathematics in everyday life and in many jobs that frequently require some technical application of mathematics. The following outline of content gives some indication, under each heading, as to what minimum "doing skills" are needed by the enlightened citizen.

Grade Six - Seven

Pupils can describe areas of plane geometric figures by measuring bases and heights and applying-a-rule to determine areas.	Area
Pupils can apply a rule for determining the area of a circle.	Area of a circle
Pupils can describe the volume of solids by applying a rule to determine volume in standard units.	Volume
Pupils can describe volume of liquid measure of a given container to the nearest whole unit using both English and metric units.	Liquid Measure
Pupils can describe time measures in years, decades, centuries, time zones, and relate distances in light years.	Time
Pupils can describe by reading temperatures from either centigrade or Fahrenheit scales.	Temperature
Pupils can identify standard units used to measure length, area, and volume. He can describe and use arbitrary units to measure length and area.	Arbitrary and Standard
Pupils can describe approximate measures to nearest 10,100,1000, etc. or as being between two known measures.	Rounding off to approximate measures.



$\frac{5}{4} >$ actual length $> 1''$

Pupils can apply a rule to determine speed. e.g.,

$$D = R\ T$$
$$R = D/T$$
$$T = D/R$$

Speed and Velocity

Pupils can describe angle measures in degrees using a protractor.

Angle measurement

Pupils can describe sums, differences, products, and quotients using denominate numbers. e.g.,

$$\square = \frac{214 \text{ mph}}{642 \text{ miles}}$$

$$\frac{1}{3 \text{ hr.}} = \frac{214 \text{ miles/hr.}}{\text{miles}}$$

rectangle

$\square$ = area

base = 8 in.
height = 4 in.
area = 32 sq. in.

Computation with measurement

Pupils can describe sums and differences for money problems.

Computation

Pupils can describe products and quotients of decimal fractions.

Basic Facts of Multiplication and Division of Decimal Fractions.

Pupils can describe sums, differences products, and quotients of two fractional numbers with like and unlike deonimators.

Basic Facts for Adding, Subtracting, Multiplying, and Dividing Fractional numbers.

Given numbers like 1/5, pupils can convert the rational number to a decimal.

Decimal Notation

Pupils can describe the products, quotients or missing factors for problems like the following:

$$20\% \text{ of } 140 = \square$$
$$40 \times \square\% = 10$$
$$\square \times 30\% - 21$$

Percent Notation

Pupils can construct a set of equivalent fractions from a given fraction.	Equivalent Fractions
Pupils can interpret models like  as 6/4 or 1-2/4 or 1-1/2 by writing the improper fraction or mixed number associated with the model.	Improper Fractions and mixed numbers.
Pupils can identify and describe fractions whose denominators are 10, 100 and 1000. e.g., 3 5/10 = 3.5 or 2 7/1000 = 2.007.	Decimals
Pupils can sequence a set of rational numbers like 1/8, 1/4, .3, 16, 1/2, 7/8 in order from smallest to largest.	Order Relations
Pupils can convert percentage to decimal notation and conversely. e.g., 47% = .47; .35 = 35%	Percents
Pupils can convert percents to common fractions and conversely. e.g., 40% = 2/5; 1/4 = 25%	Percents
Pupils can convert numbers like 8 to exponential form. e.g., $8 = 2^3$	Exponential Notation
Pupils can tabulate information and interpret trends like population growth, product costs and cost of living.	Interpretation of Data
Pupils can demonstrate or display data collected by constructing histograms, circle graphs or line graphs.	Demonstrating Data
Pupils can apply a rule for finding perimeter or circumference of plane figures.	Perimeter and Circumference
By using the number line, pupils can demonstrate which of two integers is greater. $a > b$, $a < b$, $a = b$	Inequality of Integers
Pupils can convert a given measure into both smaller and larger units in both the English and Metric systems. (No conversion from one system to another) e.g., 39 in. = 3.25 ft. 100000 mg. = .1 kg.	Conversion of Measures

Pupils can demonstrate the definition of a ratio using physical models. e.g., 4 pennies to 3 nickels at 4/15	Definition of Rational Numbers
Pupils can demonstrate indicated division by renaming numbers like 13/4 as 3 1/4 or 3.25.	Names for Rational Numbers
Pupils can illustrate the position of integers like 3, -4, or -2 by graphing them on the number line.	Integers as Directed Numbers
Pupils can describe that division by zero is not allowed because there is no unique number $\triangle$ such that $\triangle/0 = \square$ because $0 \cdot \square \neq \triangle$	Definition of not dividing by zero.
Pupils can apply a rule to solve long division problems of four place dividends and two place divisors.	Long Division Algorithm
Given a sequence of numbers like 63, 71, 75, 86, and 90, pupils can identify the median and solve for the arithmetic mean (average).	Average Determination
Pupils can describe answers to multiplication and division problems by using the inverse relation of multiplication and division. e.g., $63 \times 41 = 2583$; $\frac{2583}{63} = 41$ or $\frac{2583}{41} = 63$	Inverse relation of Multiplication and Division

Grades Eight - Nine

Students can apply rules to write the products of any two integers.	Multiplying fractions
Students can state rules to write the quotients of any two integers in terms of multiplication.	Dividing fractions
Students can describe the sums of any two real numbers.	Addition of real numbers
Students can describe the differences of any two real numbers.	Subtraction of real numbers
Students can describe products of any two real numbers.	Multiplying real numbers
Students can describe quotients of any two real numbers.	Dividing real numbers
Pupils can identify period values of fifteen digit numbers.	Place Value

235	612	223	819	045
Trillions	Billions	Millions	Thousand	One

Pupils can apply the Pythagorean Theorem to solve right triangle problems.	Pythagorean Theorem
Pupils can convert measures from English to Metric measures and conversely. (Basic units of length, area, volume, liquids, temperature, and mass.)	Conversion of Measurement
Pupils can describe lengths in both English and Metric units.	Length
Pupils can describe numbers like: $.124 = (1 \times 1/10) + (2 \times 1/100) + (4 + 1/1000)$.	Expanded Notation for Decimals
Pupils can describe answers for problems like: $3^4 = 3 \times 3 \times 3 \times 3 = 81$	Exponential Notation
Pupils can describe any number as a number between 1 and 10 times some power of 10. e.g.,	Scientific Notation Algorithm

$$643 = 6.43 \times 10^4$$
$$45 = 4.5 \times 10$$
$$42763 = 4.2763 \times 10^4$$

Pupils can <u>apply rules</u> to determine surface area and volume of geometric solids.	Area, Volume
Pupils can <u>describe</u> by computing measures with needed regrouping to larger or smaller units.	Computation in Measurement
Pupils can <u>interpret</u> percentage problems by <u>describing</u> a mathematical sentence, and its solution. e.g., 60% of ___ is 120. $(.60 \times \square = 120)$. __% of 10 is 5. $(\square \times 10 = 5)$. 20% of 80 is ___. $(.20 \times 80 = \square)$.	Percents
Pupils can <u>interpret</u> gauges that measure some required quantity. e.g., utility meters.	Measuring Devices
Pupils can <u>interpret</u> graphical information by constructing a table showing the relationship displayed by the graph.	Graphing
The pupil can <u>identify</u> a required measure from scale drawing.	Scale Drawings
Pupils can <u>construct</u> a simple budget for a given income alloting amounts for necessities and contingency.	Budget
Pupils can <u>describe</u> by determining total cost from given unit costs of items.	Costs
Pupils can <u>construct</u> a scale drawing of a simple object, building, or site.	Scale Drawings
Pupils can <u>construct</u> a frequency table from given data.	Frequency Table
Pupils can <u>manipulate</u> variables by applying properties of equality to solve mathematical problems.	Equation Solving
Pupils can <u>describe</u> sums, differences, products, and quotients of two integers.	Operations on Integers
Pupils can <u>describe</u> by computing the sum, difference, product, and quotient for any two rational numbers in fractional or decimal form.	Computation on Integers
Pupils can <u>construct</u> a diagram and ratios to solve indirect measurement problems (similar triangles).	Indirect Measurement

Pupils can <u>construct</u> the Euclidean standard type construction.	Construction
Pupils can <u>demonstrate</u> a method of computing perimeters and areas of plane geometric figures which can be divided into triangles and rectangles.	Perimeters
Given a number n, pupils can <u>construct</u> the next few consecutive integers as n + 1, n + 2, n + 3, etc.	Name for numbers
Given a number, the pupil can <u>identify</u> whether the number is divisible by 2, 3, 5, 9, or 10.	Divisibility
Pupils can <u>construct</u> true sentences using =, >, and < to show the relationship between two integers.	<u>Equations and Inequalities</u> Inequalities
Pupils can <u>construct</u> mathematical sentences to solve verbal or written problems.	Word Problems
Pupils can <u>apply rules</u> using prime factorization of two given numbers to find the greatest common factor and least common multiple of the numbers.	<u>Numbers and Numerations</u> Common factors and Multiples
Pupils can <u>sequence</u> rationals from least to greatest by the method of equivalent denominators.	Value of Rationals
A given positive rational number can be <u>converted</u> by pupils to either decimal, common, mixed numeral, percent, or scientific notation form.	Equivalent names
The pupils can <u>convert</u> numbers to exponential form and conversely. e.g., $81 = 3^4$ or $2^5 = 32$.	Exponents
Pupils can <u>identify</u> rationals by mapping on the number line.	Number line
Pupils can <u>describe</u> by computing interest, time, rate, or principal from the formula I = PRT.	<u>Relations and Functions</u> Interest

Pupils can apply the rule for implications like: if a > b and b > c then a > c to indicate understanding of the substitution principle.	Substitution principle
Pupils can apply rules for solving proportions to compute answers to indirect measurement problems.	Indirect Measurement
Pupils can identify and describe distances on maps.	Distances on maps
Pupils can interpret regions of a graphed relation to identify minimum and maximum values.	Probability, Statistics, and Graphing Graphs
Pupils can describe the chance of events occuring by computing simple combinations.	Combinations
Pupils can interpret data to identify measures of central tendency.	Measure of central tendency
Pupils can state a rule to predict the probability of an event occurring from such activities as a coin flip or the roll of a die.	Probability
Pupils can demonstrate his competence to use a wide variety of measuring instruments for distance, time, temperature, velocity, capacity, etc.	Measuring instruments
Pupils can apply rules to determine the total cost of items with a fixed interest rate and a given time.	Interest
Pupils can interpret advertisements as to total consumer costs regardless of payment schedules indicated.	Consumer costs
Pupils can demonstrate ability to compute electrical or other utilities bills from given data on cost per unit and meter readings.	Utility bills
Pupils can state rules to determine surface areas and volume of geometric figures.	Measurement Surface Area and Volume
Pupils can demonstrate that two given triangles are similar by writing the appropriate proportion.	Similar triangles

Pupils can <u>demonstrate</u> data by any two of three possible illustrations (graph, table, mathematical sentence).	<u>Consumer Mathematics</u> Data Interpretation
For two given composite numbers, the pupils can <u>state the rules</u> for finding the greatest common factor and least common multiple.	Greatest Common Factor and Least Common Multiple
Pupils can <u>state a rule</u> to define whether any number is odd or even. 2n = always even 2n + 1 = always odd	Odd or Even
Pupils can <u>demonstrate</u> computational ability by writing solutions of mathematical sentences.	Equation solving
Pupils can <u>describe</u> the order relationship between two or more numbers in sentence form. e.g., $5 > 3$ or $8 > 3 > 1$ or $6 > a > 1$.	Inequalities
Pupils can <u>manipulate</u> variables to solve for any term for problems like: $\frac{a}{b} = \frac{c}{d}$ e.g., $a = \frac{b \times c}{d}$ $b = \frac{a \times c}{d}$ $c = \frac{a \times d}{b}$ $d = \frac{c \times b}{a}$	Equation solving in terms of variables
Pupils can <u>state a rule</u> indicating the decimal form of any given rational number.	Decimal names

MATHEMATICS AND THE SCIENCE STUDENT

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One often hears high school science teachers say that they can teach their students the math they will need at the appropriate point in the science course. The math they refer to, however, is usually a specific topic such as the laws of exponents, ratios and proportions, graphical techniques, or the use of the slide rule. Indeed, math topics such as these are often presented in a science course, and sometimes, with success. In some instances the student has "had" this material before and this treatment in the science class provides the appropriate review. In most cases, however, although the student may have met these ideas before, he never really understood them. It is with these students that the science teacher feels he can introduce physical applications to make the math more meaningful and, therefore, understandable. He is often rudely shocked. No matter how hard he tries, the science teacher soon accepts the fact that some students will never get these ideas straight. And the reason for this soon becomes obvious.

At the heart of the problem is the uncontestable fact that too many high school students demonstrate a widespread lack of basic skills in elementary arithmetic. Because the subject is so broad and because so many valuable years have passed where the concepts should have been maturating, the science teacher is usually powerless (if not ill-equipped) to correct the situation. I am referring to elementary school topics such as long division, decimals, and fractions. To illustrate the point, here are some examples taken from student papers in a sophomore science class:

(a) $\begin{array}{r} 1.9 \\ 27\overline{)\,5.30} \\ \underline{2\,7.} \\ 2\,6.0 \\ \underline{2\,4.3} \end{array}$ (b) $\frac{\cancel{3}^{\,1}}{\cancel{12}_{\,4}} = 4$ (c) $\frac{4}{.5} = 8.0$

Not only are many of the basic skills missing, but students generally demonstrate almost total refusal to do mental arithmetic. It is almost as if there was no way to get an answer unless one followed a specific method. Examples of this phenomena from freshman and sophomore science papers include such as these:

(a) $\begin{array}{r} 17.6 \\ 10\overline{)\,176.0} \\ \underline{10} \\ 76 \\ \underline{70} \\ 60 \\ 60 \end{array}$ (b) $\frac{2}{1} \times 1 = \frac{2}{1} = 2$ (c) $\begin{array}{l} 2t = 6 \\ \frac{\cancel{2}t}{\cancel{2}} = \frac{6}{2} \\ t = \frac{6}{2} \\ t = 3 \end{array}$

When confronted with the obvious mental solutions to these examples, the student exhibits a fear that if he tries to do it in "his head" he might make a mistake. Clearly, one cannot criticize a student's attempt to be careful or thorough, but not to the extent shown by some. A basic awareness is lacking.

Because many students seem to demonstrate a blind allegiance to the rules and methods of arithmetic, they seldom bother to read what they have written. Apply the rule, that's all you have to do. It is as if by magic the answer appears as long as the rule has been applied. For example, "how do you find the average of a list of numbers?" Answer, add them up and divide by the number in the list. That's the rule. So after collecting several ammeter readings in a science lab that fall between a low value of 0.50 amps and a high value of 0.76 amps, students report average values of 0.94 amps or (even worse) 6.5 amps. They have applied the rule but are completely unaware that their answer makes no sense. They do not bother to read what they have written. Again, a lack of basic awareness.

Which mathematical operation to use in a given situation is very often a source of confusion. Year after year I can count on the class to be split down the middle on their answer to a question like this: To change feet to inches do you divide by 12 or multiply by 12?

Whenever a slight change is made in a familiar situation, students are often unable to "transfer" a technique. For example: solving a proportion with the unknown in the numerator usually meets with success. However, place the unknown in the denominator of a proportion and then watch the fireworks. "We never did this kind before!"

The rough estimate is a very helpful tool in science. This idea is foreign to many in-coming students. Which is larger, $\frac{7^2}{50}$ or $\frac{1}{3}$? They usually will answer only after both have been reduced to decimals. Estimate your height in centimeters. To some students this question is of the same order of difficulty as estimating their weight in grains of sand. Evaluate mentally:

$$\frac{35}{49 \times 5}$$

One could go on, the list is a long one. And all of these examples will point to a basic lack of skills, the inability to reason, the inability to detect their own mistakes, a fear of mental arithmetic, and, in general, a lack of awareness of what arithmetic is all about. However, I do detect one very positive phenomena among my students. Perhaps this could be used as one of the starting points for those who would address themselves to the problem of middle school mathematics. And that phenomena is this: In spite of poor math backgrounds exhibited by some students, almost all of them are able to demonstrate an ability to reason mathematically but they are usually unaware of it. That is, they can get an answer sometimes but cannot tell you how they got it. For example:

> From a science text, "If a wheel makes seven revolutions each second, how long does it take to go around once?" Complete silence!
>
> One then asks the following question, "If you can wash two cars in one hours, how long would it take to wash one car?" Immediate response — one-half hour.

Replacing "revolution," "seven," and "second" with "wash," "two," and "hour," that is, the unfamiliar with the familiar, appears to make all the difference. Again, from a science text:

> If one calorie = 4.2 joules, two joules = how many calories?
>
> No response!
>
> "Go ahead, use your pencils." After several minutes of doodling, a few of them come up with the answer.
>
> Restated: If one foot = 12 inches, six inches = how many feet?
>
> Unanimous response.

But in each of these examples, when you ask them how they got the "easy" answer, they cannot tell you. They are unable to identify the mental operation used. They would reply, instead, "well, if it takes one hour to wash two cars, then it takes only one-half an hour to wash one car." They seem surprised that you do not see how simple the solution is. They did not use any "mathematical operation" to solve the problem, they simply "did it in my head."

This, to me, is an encouraging starting point. They are able to reason as long as the language is familiar to them. This, then, should allow us to concentrate on the operations. When the language is unfamiliar, it appears to block any attempt on our part to introduce the even more unfamiliar language of mathematics. But identifying the mathematical operation is the second half of the problem. They must first learn how to perform the operation. This brings me back to my earlier comments, and this, I suppose, is the reason for this conference.

THE PLACE OF ESTIMATION IN THE MATHEMATICS CURRICULUM OF THE JUNIOR HIGH SCHOOL

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This paper will attempt to look at three aspects of the relationship of estimation to the mathematics curriculum of the junior high school.[1]

i) What advantages to students can reasonably be expected to come from studying estimation

ii) What are the processes and skills needed to learn how to estimate.

iii) What are some appropriate ways by which estimation may be taught.

This may already sound like a paper that promises to fill an important gap in mathematics education literature. Such is not the case. The goal of this paper is two-fold. First it attempts to focus the reader's attention on major issues of teaching estimation by illustrating, in many cases with examples familiar to th ..ader, problems related to the place of estimation in the mathematical education of children. Secondly, it is hoped that it may prove useful in generating ideas on how to integrate estimation into the mathematics curriculum of the junior high school.

1. The Advantages of Estimation

Consider the following problems taken from actual work of elementary children.[2]

(a) $\begin{array}{r} 74 \\ +\ 56 \\ \hline 1210 \end{array}$ (b) $\begin{array}{r} {}^{1}\ \ \\ 59 \\ +\ \ 6 \\ \hline 125 \end{array}$ (c) $\begin{array}{r} 135 \\ -\ 67 \\ \hline 132 \end{array}$

(d) $\begin{array}{r} {}^{8}\ {}^{1} \\ 1\cancel{9}7 \\ -\ \ 43 \\ \hline 1414 \end{array}$ (e) $\begin{array}{r} {}^{1}\ \ \\ 433 \\ \times\ 226 \\ \hline 878 \end{array}$ (f) $2\overline{)176}$ with quotient 233

Although each of the above errors is interesting in and of itself and can be precisely diagnosed as a specific failure in understanding and correctly using the appropriate algorithm (problem (a) for example involves improper carrying, (b) confusion as to the meaning of place value, and (c) handling of each place value as an independent problem as well as always subtracting the smaller number from the larger, (d), (e), and (f) involve similar difficulties) the one common and striking feature of all is the lack of reasonableness of the obtained answers. The child appears to be so involved in the formal processes of computing that even the complete lack of sensibility of the answer is totally overlooked.

Realizing that the obtained answer does not make sense will not in and of itself correct the error, but there is very little doubt that children who have the habit of considering the reasonableness of their answers are not as prone to adopt incorrect computational procedures. Therein lies one of the greatest benefits of acquiring the ability to estimate.

As one proceeds into either more complex algorithms such as extraction of square roots or more difficult numerical situations such as operations involving numbers with decimals, the value of estimating increases. There is, for example, ample evidence[3] that even at the high school level students often experience uncertainty over the proper placement of the decimal point in products, quotients or square roots of numbers that involve decimal fractions. This, of course, is a serious matter since misplacement of a decimal point is a non-trivial error indeed — much worse than many students seem to realize. Here again, there is very little doubt that if students become accustomed to making rough estimates of their answers, they are not as likely to make this type of error. The estimates, in this case, do not even have to be very good ones, and they can generally be made mentally with radically rounded-off numbers. For instance, consider an example[4] as follows. The product 0.327×49.2 yields an answer consisting of the sequence of digits 160884. Where is the decimal point to be placed to yield the correct answer? If the student realizes that 0.327 is close to 3/10 and that 49.2 is just about 50, it will be immediately apparent to that student that the product is somehwere near 15, which is three-tenths of 50. This, of course, leads to the conclusion that the only place where the decimal point makes sense is after the 16. In other words the answer must be 16.0884. The use of this type of procedure to check the answer or even to place the decimal[5] should greatly alleviate misunderstanding of the formal rules involved. A student needs to find the square root of 29.6. He has tables which provide the square root for the whole number N where $1 \leq N \leq 1000$ as well as the square root for 10N. He can easily locate the row labeled 296, and suspects that he can use this to obtain the square root of 29.6 but which of the two columns $\sqrt{N}$ or $\sqrt{10N}$ is he to use? What is he to do with the decimal? The answer is readily obtained with an estimation. Since 29.6 is close to 30 and therefore, between 25 and 36, the square root of 29.6 will be somewhere between 5 and 6. This information indicates that the column of interest is $\sqrt{10N}$ since it contains 54.4059 versus 17.2047 in the $\sqrt{N}$ column. Furthermore, the decimal must go after the 5. That is, the square root of 29.6, as listed in this table, is 5.44059. A more rigorous algebraic verification of this result using the relationship between 29.6 and 2960 is of course possible but it is actually not that important since the obtained answer is the only possibility that even makes sense under the circumstances. Many additional examples of this type are possible but even the few so far presented illustrate the usefulness of estimating as a most powerful procedure to check computational results. As a matter of fact, as the last example illustrates, and as any user of the slide rule will attest to, many situations call for uses of estimating procedures not only as means to check the reasonableness of answers but as a fundamental part of the procedure itself.

Estimation is the very backbone of the most common algorithms of what to most children is the most difficult operation of them all: division. There are many ways by which children learn how to divide, however, most procedures involve estimating either a product or a quotient, depending on one's point of view. As some authors[6] point out almost all procedures used in estimating quotients are either a one-rule or a two-rule method. According to the one-rule method, the divisor is always rounded downward to the nearest 10 while in the two-rule method the divisor is rounded both downward and upward. The second plan follows the usual pattern for rounding numbers. The key point here, however, is the fact that both methods involve estimating a quotient as a first step to actually finding the true quotient. For example, in the problem $32\overline{)96}$ a typical reasoning process might be the following: "It looks like the number of 32's in 96 is the same as the number of 3's in 9 or the number of 30's in 90. Therefore the answer must be 3." Only after this type of reasoning is an actual multiplication performed and values compared to determine if the estimate is correct. Although the example is a simple one, similar processes are used in all division problems the exact details differing mostly along the lines by which quotients or partial quotients are estimated. It would seem reasonable to assume that the more competent a child is at estimating the better he will be in the use of the division algorithm. Similar examples from 9th grade algebra are quite easy to obtain. A popular Algebra text[7] describes three methods of obtaining square roots of real numbers. With the exception of method 3 which requires the use of tables[8] the remaining two methods[9] are based on estimating the square root in one manner or another. The reader, I am sure, is familiar with the following general procedure which is one of the outlined methods.

1. Estimate the square root of the number as best you can.
2. Divide the number by the estimate.

3. Find the average of the quotient found in Step 2 and the estimated square root. This is a closer approximation of the square root than the original estimate. Repeat Steps 2 and 3 if necessary.

From the above it is quite obvious that fundamental to the whole process is the ability to estimate well. Even though a poor estimate will work, an accurate one will provide a more precise answer more readily. In many ways a procedure such as the above is to be preferred over more complex, even if more efficient, algorithms since the procedure is relatively easy to understand as to its mechanics and purpose.

The examples so far described indicate two major uses of estimation in mathematics. An idea of the gains to be made through estimation can be obtained by contrasting it to traditional algorithmic processes. Williams[10] in an article discussing aspects of calculative thinking, pointed out that mathematical calculations, especially those based on the use of algorithms, often involve many stages and that procedures are usually prescribed in every detail. This makes the process of learning how to compute quite difficult. Often the only alternative to complete mastery is complete failure. The detection of errors by a careful examination of parts of a calculation is not an easy process for a child in elementary school or in junior high school. This is usually further compounded by the fact that due to the highly prescribed nature of most algorithms, the child is a passive learner in a process over which he has no influence. Estimation provides some relief in both areas. By allowing the student to evaluate the reasonableness of his answer without having to retrace his steps through the many strands inherent to most algorithmic processes, estimation reduces at least some of the inability of students to check their work. By being incorporated into an algorithm, estimation by the virtue of usually reducing the number of steps required in the algorithm and by making the remaining steps more "intuitively obvious" does not force the student into as passive a role as highly formulated algorithms do.

Yet another benefit of estimation which is often overlooked probably because of its very obviousness is the relationship between estimation and mental arithmetic. Some investigators have indicated that about 75 percent of adult non-occupational uses of arithmetic are mental.[11] This further underscores the value of estimating in that it is an ideal vehicle for mental work. It is through this relationship with mental arithmetic that estimation can provide a much needed link between the formal structures and patterns which are so heavily stressed in current mathematics curricula and the skills which are so sorely lacking in the mathematical repertory of most children. Two simple illustrations should make this quite apparent.

How shall we quickly obtain the answer to the following addition problem: 14 + 17 + 8 + 23 + 16? Through judicious use of the commutative and associative properties of addition the problem can easily be solved without paper and pencil. 14 + 17 + 8 + 23 +16 becomes 14 + 16 + 17 + 23 + 8 which in turn becomes 30 + 40 + 8 and finally 78.

Another slightly more complex example follows. Compute the answer to 3×398 without using paper and pencil. A straightforward application of the distributive property of multiplication over addition to a slightly rewritten problem produces a ready solution. 3×398 becomes $3\times(400-2)$ which is the same as $3 \times 400 - 3 \times 2$ or 1200−6 or 1194.

Although the above examples are not exactly estimations they nonetheless are excellent illustrations of how properties of numbers and operations may be used in mental arithmetic.

Examples more complex in nature and more closely related to estimation are readily obtainable. How could one estimate the answer to say 712 − 448 or to $616 \div 22$?[12] Both of these problems are excellent vehicles to demonstrate relationships between the operations under consideration and inequality. The fact that 723 is between 700 and 800 and that 448 is between 400 and 500 does not immediately allow us to determine what numbers 723 − 448 is between. As a matter of fact the most "obvious" attempt to bracket the difference as between 700 − 400 and 800 − 500 leads to total nonsense. A much closer look at the relationship between subtraction and inequality is needed.

This eventually involves considerations of upper and lower estimates, the resulting argument being similar to the following: Since 700 is a lower estimate for 723 while 500 is an upper estimate for 423, 700 - 500 is a lower estimate for 723 - 448 and, similarly, 800 - 400 is an upper estimate. Not surprisingly, therefore, $200 \leq 723 - 448 \leq 400$. A similar argument can be used to estimate $616 \div 22$. Of course better estimates are possible and not difficult to obtain, however, the point here is that the process of estimating a difference brings into focus an important relationship between inequality and subtraction. This should not only lead to an improved performance with respect to subtraction by providing a tool to estimate how reasonable answers are but at the same time it should enhance the student's understanding of inequality — a difficult area indeed. This has taken us somewhat afield from the intent of this section of the paper, but it does illustrate how estimation, properly taught, can serve an integrating function between various topics of mathematics. It does so even more forcefully if it remains rooted within the common sense experiences of the student.

Finally, it must be pointed out that estimating is in and of itself indispensable to the operation and planning of many eminently practical everyday projects. Budgets, tax rates, construction bids, merchandising, manufacturing and a host of many other activities are based on estimates; without intelligent estimates, they cannot be planned or executed. If students could be kept aware of this, it should help to enrich their mathematical experience and deepen their insight.

2. <u>The Estimating Process</u>

In view of its importance, there is no phase of arithmetic which is more neglected than estimation. Much of this is probably due to the fact that even though at almost every meeting of mathematics teachers, uniform agreement is reached as to the value of estimation little beyond telling children "You should learn to estimate!" is done in the classroom.[13]

Estimating is itself a complex of skills, any one of which may require instruction and practice apart from the more general question "Is your answer reasonable?" Some of the relationships between estimation, operations and inequalities have already been pointed out, other skills can be identified with relative ease. Included among these are at least the following[14]

i) The ability to round a whole number to the nearest ten, hundred, etc. . . .

ii) The ability to multiply by powers of ten in a single step.

iii) The ability to add, subtract, and multiply two numbers each of which is a multiple of a power of ten.

All of the above should be done mentally without the use of algorithms. Other skills which would aid in the estimation processes can, of course, be easily found, however they are probably not as peculiar to the estimation process as the above. There is no doubt that an ability to handle inequalities is important, however, inequalities are already studied in many other contexts and such skills are therefore not unique to estimation.

The ability to round whole numbers and decimals is perhaps the skill that is most unique and fundamental to estimating. It is because this ability is so closely tied to a "feel" for how large or small numbers are that it is so crucial. This is also the ability that is the most similar, from a numerical point of view, to skills associated with measuring physical objects. If an object is 3/4" long what would it measure to the nearest inch? The answer to this question is akin to rounding .75 to the nearest whole number. It must be kept in mind, however, that what we are interested in is the ability to round off a number without having recourse to a physical process such as measuring lengths. The latter, however, does indicate possible avenues through which such a skill may be taught.

To be able to multiply by 10, 100, etc. . . . with ease is a necessary prerequisite to any estimation involving an operation. Taken together with the ability to handle numbers which are multiples of powers of ten it allows the student to use number facts for single digit numbers in a greatly expanded set of situations. For instance, $20 + 50$ can easily be handled from simple knowledge of the addition fact $2 + 5 = 7$. Similarly, 20×30 is easily obtained from $2 \times 3 = 6$ and from the fact that multiplying by 10 is tantamount to tacking-on a zero. Thus, $2 \times 3 = 6$ leads to $20 \times 3 = 60$ and finally to $20 \times 30 = 600$.

There are, of course, other ways of approaching the problem but the above solution does serve to illustrate the fact that estimations are possible based on a relatively small number of skills.

If additionally one includes knowledge of structural properties of number systems with the above described skills then more sophisticated estimates become possible. For example, 233×19 can be readily estimated by rounding off the numbers to 230×20 then applying the distributive property to $(200 + 30) \times 20$ which becomes $4000 + 600$ or 4600. Note that in addition to the distributive property the only skills needed are the ones previously described. The actual answer to the problem is 4427. The answer that would have been obtained by rounding 233 to 200 would have been 4000. Although not as good as the previous one, it is still not a bad estimate especially when one considers the small number of skills required.

Estimation involving decimals are, of course, more complex and do involve some "feel" for fractional equivalences of decimals, as well as the ability to perform operations on these fractions. They are, however, essentially the same as the ones described above if we include 1/10, 1/100, etc. . . . in our definition of powers of 10 and 3/10, 4/100, etc. . . . in our definition of multiples of powers of 10. This is not to say that the skills involved in handling fractions are exactly the same as those for whole numbers, but it does point out that the similarity is greater than apparent at first glance.

There are other situations for which the skills thus far ennumerated appear not to suffice. For example, in estimating a square root it would appear that skills of a different order and specific to the process of extracting square roots are needed. In the final analysis, however, even in this situation, the estimation problem is very similar to one related to estimating products and hence can be handled within the context of the processes thus far examined. If we further restrict ourselves to the kind of estimates that are generally made by junior high school students, then the ability to round-off coupled with the ability to handle multiples of powers of ten will probably cover most situations.

It does appear, therefore, that the actual skills involved in making estimates are few in number and simple in nature. Coupled with an intuitive feel of how numbers behave under certain operations they should improve students' quantitative reasoning by removing some of their dependence on formalized symbol manipulations.

3. The Teaching of Estimation

A survey of texts used in grades 6 through 9 will rapidly convince one that work in estimation is either totally absent or at best minimal. A look at the professional literature in mathematics education will only serve to reinforce the point. Suggestions about methods of teaching estimation are few and far between. Some small articles suggesting such practices as use of the number line in facilitating mental computations[15] or the use of the greatest integer function as a means of providing specific practice in estimation[16] may be found. But even these articles are primarily aimed at the elementary level and are highly specific in scope. The last of these has some possibilities as an interesting way to practice some skills which in fact are highly related to estimation processes, however, it is questionable as a substitute for the process of estimation itself.

In view of the apparent simplicity of the skills required to make estimates, what is perhaps needed is an approach to integrate estimation into the mathematics curriculum rather than some new clever ways of teaching it.

First and foremost children should be encouraged to guess and should be given the opportunity to do so. The practice of recording an estimated answer before computations are performed should be highly encouraged. By guessing, trying their guesses, and revising these guesses when needed the children should become more and more able to determine when an answer is reasonable.

What, then, is advocated here is the inclusion of estimation into other mathematical work instead of teaching it as a self contained activity. Some more or less "pure" estimation problems should be considered such as

"Estimate the number of ping-pong balls that it would take to fill this room completely"[17] if for no other reason than the fact that such problems are in many cases interesting in their own right. Most of the work on estimation, however, should probably be done as an adjunct to other topics, either simply to check answers to computations or to practice using the skills learned in a familiar and useful context.

In order to accomplish this a deliberate effort must be made to include estimation topics where ever such an addition may prove beneficial. This is most readily accomplished by the inclusion of problems which would require a child to estimate an answer without or prior to computing it. Thus work on addition, subtraction, multiplication and division whether with whole numbers, decimals or fractions should have estimation problems embedded in it. Similarly in the study of such topics as the distributive, associative, or commutative properties exercises should be included to illustrate how these topics can be used to simplify computations and aid in estimating answers.

Throughout practical and independent applications of estimation to such areas as business and science should be presented. In this manner estimation will not only enrich a student's understanding of the mathematical topics but will in turn be enriched by the application of these topics to realistic situations, including areas outside mathematics.

In summary, it can be seen that the ability to estimate is intimately related to many fundamental mathematical skills not the least of which is the evaluation of the reasonableness of answers to mathematical computations. Additionally, the skills involved in making estimates are not very complex and can probably be easily mastered by most students. Thus maximum benefit from studying estimation processes can probably be best obtained by incorporating such study directly into the presentation of other topics. Such meshing of estimation into other topics, where its study can be of benefit, can probably be accomplished quite easily in view of the small number of additional skills required.

FOOTNOTES

1. The junior high school level includes the middle school, i.e., grades 6 through 9.

2. Ashlock, Robert B., Error patterns in computation. Columbus, Ohio: Charles E. Merrill, 1972.

3. Butler, Charles H., Wren, F. Lynwood, & Banks, J. Houston. The teaching of secondary mathematics. New York: McGraw-Hill, 1970.

4. Ibid.

5. Ibid.

6. Grossnickle, Foster E., & Reckzeh, John. Discovering meanings in elementary school mathematics. New York: Holt, Rinehart and Winston, 1973.

7. Clarkson, Donald R., Douglas, Edwin C., Eade, Arthur, Olson, Joyce F., & Glass, Elizabeth. Algebra one. Englewood Cliffs, N.J.: Prentice-Hall, 1966.

8. In a strict sense the use of tables also involves an estimation process, as is apparent from the previous example.

9. The methods are: (1) Trial and error method; (2) Approximation process method.

10. Williams, John D. Some peculiarities of calculative thinking and their consequences. In Lamon, William E. (Ed.), Learning & the nature of mathematics. Chicago: Science Research Associates, 1972.

11. Kramer, Klass. Adding and subtracting without pencil and paper. In Kramer, Klass (Ed.), Problems in the teaching of elementary school mathematics. Boston: Allyn and Bacon, 1970.

12. Kelley, John L., & Richert, Donald. Elementary mathematics for teachers. San Francisco: Holden-Day, 1970.

13. Page, David A. Do something about estimation. In Kramer, Klass (Ed.), Problems in the teaching of elementary school mathematics.

14. Ashlock, Robert B. Error patterns in computation. Columbus, Ohio: Charles E. Merrill, 1972.

15. Kramer, Klass. Adding and subtracting without pencil and paper. In Kramer, Klass (Ed.), Problems in the teaching of elementary school mathematics. Boston: Allyn and Bacon, 1970.

16. Page, David A. Do something about estimation. In Kramer, Klass (Ed.), Problems in the teaching of elementary school mathematics. Boston: Allyn and Bacon, 1970.

17. Buckeye, Donald A., Ewbank, William A., & Ginther, John L. A cloudburst of math lab experiments (4 Vols.). Birmingham, Mich.: Midwest Publications, 1971.

APPROXIMATIONS AND ORDERS OF MAGNITUDE

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In this paper I would like to present four areas that I believe need emphasis in the middle-school mathematics curriculum: estimation, physical numbers, approximate calculations, and orders of magnitude. Although the easiest way to approach these topics would be to present them as chapters or units, I believe that the efficacy of these ideas will be enhanced if they can be woven throughout the middle-school mathematics curriculum.

The middle-school mathematics program which I envision is a dynamic approach to mathematics which incorporates student involvement through experimentation and exploration. It utilizes the inquiry approach. It is a program in which students encounter "real" situations before moving to hypothetical situations — real in the sense that they are meaningful, interesting, and the students want to explore them. Hopefully, this program will have more situations "to think about" and fewer problems "to solve".

Estimation

Estimation can be thought of as the process of obtaining an approximate magnitude for some quantity when exact values are not readily available. For example, how many bottles and cans are discarded each year in the United States? A facility for estimating is lacking in students at all levels. The ability to estimate seems to be the result of experience and needs a certain degree of practice. A perusal of elementary and junior high school mathematics curriculum materials indicates that very little has been done to enhance children's ability to make estimations.

While attempting to teach certain science units, members of the Elementary Science Study Project found that students had difficulty in making estimates and grasping large numbers. In order to remedy this situation they developed a unit, Peas and Particles, which they have classroom-tested in the early middle-school years. They start their unit with many concrete objects (dry peas, corn, tennis balls, frozen juice cans, etc.) and have students estimate and then count handfuls, bagfuls, and jarfuls. Not only do the students develop their ability to estimate but they discover some ingenious counting methods — taking a sample, counting by area and volume, counting by halving or doubling, counting by weighing, and counting by ratios. After these initial experiences the students moved to more difficult problems which could not be so easily counted. In fact, they suggested many new challenges for themselves (e.g., How many grains of sand in a sandbox? How many hairs on a person's head? How many gallons of water in the Atlantic Ocean? etc.)

Estimation might also be closely associated with measurement. As soon as units of measurement are known the students should be encouraged to make estimates before measuring (this is a good way to teach — not teach about — the metric system). By estimating and then measuring students are able to determine if their estimates are good or not. This also gives them the opportunity to consider the size of their error in relation to the size of the quantity measured (estimating errors can also be introduced at this point). Students soon discover that it is more difficult to estimate large entities than small and that time, weight, and volume are more difficult to estimate than length and area.

In short, the process of estimating is important for many reasons: it insures that numbers for measurement and counting are used more meaningfully; units of measurement are made more meaningful; it leads to a better understanding of large numbers; it is useful in everyday life when measuring or counting are either inconvenient or impossible.

Physical Numbers

Often when counting and always when measuring, our results can only be attained approximately. Let us call such numbers physical numbers. There are several ways to introduce physical numbers to students. Currently, the most common approach is, simply, to tell the students — as illustrated in the following passage from one of the more popular junior high school textbooks.

> Can you ever find the exact length of a given line segment? The answer is "No," for no matter how accurate your measuring instrument or how carefully you use them, a physical measurement is necessarily only approximate. You can measure only to the smallest unit of measure available on your measuring instrument. While smaller and smaller units of measure may be used to give you more and more precise measurements, you obtain closer and closer approximations to the length.

This passage is followed by three definitions (greatest possible error, precision, more precise) three examples, and then three pages of drill problems (paper and pencil type). The students are never encouraged to use any real measuring instruments to experience the problems of measuring. They are outside observers reading an account of someone else's experiences. Wouldn't it be more meaningful to have them engaged in projects where measurements are needed — to have them use several different measuring devices to experience the approximate nature of measuring directly, to discover precision and to discuss significant digits in connection with these activities? Then they can move to hypothetical situations. The approximate nature of counting problems is not even considered. By using pictures of crowds, jars of rice, and discussing census-taking or polls they can experience the approximate nature of situations other than measuring.

At the present time it may be more important to talk about physical numbers and computations with physical numbers than ever before, because more and more students have access to computers and calculating machines. The computational power of these machines many times overwhelms the student. For example, by pressing a couple of buttons, including a π button, it is so easy to take a circle whose radius measures two inches and find the area to be 12.5663701 square inches – increasing the accuracy of their data through computation.

The current emphasis of middle-school mathematics focuses on the development of mathematical number systems — usually through an introduction to the real number system. Although I have no hard evidence upon which to base this opinion — other than my own experience and reports from other teachers — our students are very confused when they switch to problems dealing with physical numbers.

Approximate Calculations

There are times when one is not concerned with exactness in computation, but only rough approximations. This may occur when we are estimating or when we want to check the result of a computation. Consider the following calculations:

$$\text{a)}\ \begin{array}{r} 357 \\ \underline{-289} \\ 168 \end{array} \qquad \text{b)}\ \begin{array}{r} 579.2 \\ \underline{+324.1} \\ 893.3 \end{array} \qquad \text{c)}\ \begin{array}{r} 705 \\ \underline{\times\ 63} \\ 4230 \\ \underline{2115} \\ 6345 \end{array} \qquad \text{d)}\ 37.8\overline{)\,452.7}^{\;1.19}$$

Do these results seem reasonable? Do most students pause to consider the results of their computations? How can we develop the ability to do quick mental checks on computational results? We might begin by having the students:

(I) Round each number to one significant digit, then perform the operation. The approximate calculation can then be compared to the original computation to see if it is "reasonable".

(II) If the results of (I) are not convincing, round to two significant digits and repeat the process.

	(I)	(II)
$\begin{array}{r} 357 \\ \underline{-289} \\ 168 \end{array}$	$\begin{array}{r} 400 \\ \underline{-300} \\ 100 \end{array}$	$\begin{array}{r} 360 \\ \underline{-290} \\ 70 \end{array}$
$\begin{array}{r} 579.2 \\ \underline{+324.1} \\ 893.3 \end{array}$	$\begin{array}{r} 600 \\ \underline{+300} \\ 900 \end{array}$	$\begin{array}{r} 580 \\ \underline{+320} \\ 900 \end{array}$
$\begin{array}{r} 705 \\ \underline{\times\ 63} \\ 6345 \end{array}$	$\begin{array}{r} 700 \\ \underline{\times\ 60} \\ 42{,}000 \end{array}$	
$37.8\overline{)\,452.7}^{\;1.19}$	$40\overline{)\,500}^{\;12}$	

Of course, this process is not going to catch minor errors in computation but it should prevent gross errors and provide practice for quick mental calculations. In place of written rules and individual drill problems it is suggested that many of these problems should be analyzed orally with much class discussion. The nature of problems involving approximate calculations can be varied — consider, for example, the following:

A. Using the symbol, ≈, to mean "approximately equal to," explain how the answers to the following statements were arrived at.

a. $19.8 \times 42.3 \approx 20 \times 40 = 800$

b. $324.5 \div 18.9 \approx 300 \div 20 = 15$

c. $173.6 + 243.2 \approx 200 + 200 = 400$

d. $784.62 - 397.86 \approx 800 - 400 = 400$

e. $\dfrac{31.2 \times 42.7}{163} \approx \dfrac{30 \times 40}{160} = \dfrac{30}{4} \approx 7$

f. $29 \div .72 = \dfrac{29}{.72} \approx \dfrac{30 \times 10}{7} \approx 4 \times 10 = 40$

B. Using approximate calculations, as above, which of the following are obviously wrong.

a. $27.30 \times 41.67 = 1726$

b. $243.3 + 79.1 + 105.6 = 319.0$

c. $\dfrac{28.83}{.93} = 3.1$

C. Using approximate calculations, locate the decimal point in each of the following answers.

a. $\dfrac{.726}{.154} = 471$

b. $6.23 \times 17.91 \times 0.131 = 144$

Once the students have developed their ability to make approximate calculations, they are able to move on to estimation problems that illustrate the power of their new calculating abilities. (I will assume that the students are familiar with exponential notation.) As an example, let us consider a problem that I have asked several different groups: About how many revolutions does an automobile wheel make on a trip from New York to Los Angeles? It doesn't take long to agree that the diameter of an automobile wheel is about two and one-half feet in diameter and, thus, about seven feet in circumference. Since it is about 3,000 miles from Los Angeles to New York, we can make the calculation

$$\frac{3000 \text{ miles} \times 5280 \text{ feet/mile}}{7 \text{ feet}} \approx \frac{3000 \text{ miles} \times 5000 \text{ feet/mile}}{7 \text{ feet}} = \frac{15 \times 10^6 \text{ feet}}{7 \text{ feet}} \approx 2 \times 10^6$$

to obtain about 2 million revolutions.

Many junior high school students are interested in environmental problems. Consider the following statement made by the well-known environmentalist, Paul Ehrlich:

> Each day American cars exhaust into our atmosphere a variety of pollutants weighing more than a bumper-to-bumper line of cars stretching from Chicago to New York.

Is this a reasonable statement? How many people take the time to challenge such a statement? Estimating the distance from New York to Chicago at 1000 miles, the average car length to be 15 feet, and the average car weight to be about 3000 pounds we can make the following calculation

$$\left(\frac{1000 \text{ miles} \times 5280 \text{ feet/mile}}{15 \text{ feet}}\right) \times 3000 \text{ lbs} \approx \frac{1000 \text{ miles} \times 5000 \text{ ft/mile} \times 3000 \text{ lbs}}{15 \text{ feet}} = 10^9 \text{ lbs.}$$

to obtain the approximate weight of pollutants. Since "a pint is a pound the world around" we can convert the pounds to gallons, distribute the gallons among the cars in the United States, and determine the reasonableness of the statement.

There is no end to such problems, in fact, there is a problem for almost any student interest. Notice that these problems involve many skills in addition to estimation and approximate calculations. For example, the above statement by Ehrlich contains no numbers, does not ask a question, must be analyzed and translated from English to mathematics, and the students have to decide what arithmetical operations to use.

Orders of Magnitude

For all the time and effort spent on the development of number systems and number properties in the contemporary school mathematics curricula, one would think that students should develop a good number "sense." However, in my experience this is not the case and I am in full agreement with Buckminster Fuller who said that "like parrots, we learn to recite numbers without any sensorial appreciation of their significance. We have yielded so completely to specialization that we disregard the comprehensive significance of information." This lack of number sense is especially obvious in the realm of large and small numbers. Most students are fascinated by very large and very small magnitudes and we should be able to capitalize on this built-in motivation factor.

A few nights ago the headlines in the local newspaper declared that personal income in the United States is expected to exceed $1 trillion this year. I couldn't help but wonder how many people read that same statement and what went through their minds as they read "$1 trillion." (A few million people must have been exposed to that figure because it was repeated that evening on a national network news program.) Skimming through that newspaper I found over twenty articles which contained large numbers — the numbers ranged from Henry Ford II's 1972 salary ($874,567) through many millions to the projected cost of automobile anti-pollution devices over the next decade ($147 billion). Are such numbers meaningless magnitudes imposed upon faulty number senses? Is it possible, and important, to reify such large magnitudes in terms of personal life experiences? I believe that we can, and should strive to enhance a sensorial appreciation for such numbers.

To develop a meaningful sense of large numbers we might start with something that is quantifiable and personal like time or money. Time is especially meaningful because an average healthy heart beats about once every second and heartbeats are very personal. Looking at the common everyday large numbers — millions, billions, and trillions — in terms of seconds we see that:

$$1 \text{ day} = 24 \times 60 \times 60 \text{ sec} \approx 25 \times 3600 \text{ sec} = \frac{100}{4} \times 3600 \text{ sec}$$
$$= 90{,}000 \text{ sec.}$$

Thus, a million seconds is about 11 days, a billion seconds is about 30 years $\left(11{,}000 \text{ day} \approx \frac{11{,}000}{360} \text{ years} \approx 30 \text{ years}\right)$, and a trillion seconds is about 30,000 years. This approach may illustrate the relative sizes of these three numbers and correct the often mistaken idea that the difference between a million and a billion is the same as the difference between a billion and a trillion. An interesting project might be to have the students construct a historical time line in seconds — including their own two and one-half billion second life expectancy.

Some people make large quantities, especially money, meaningful to themselves by distributing the amount among a given group. For example, the U.S. defense budget is about 80 billion dollars. A quick calculation shows that this is about 400 dollars for every man, woman, and child in the country. However, this may not be meaningful unless one has an idea how large 200 million (our approximate population) is. (Of course, if the average family size is about four, this means about 1600 dollars for each family is contributed for defense — almost as much as the family spends for food each year.) Almost everyone has seen a large group of people gathered — either in person or on television. Many of the nationally televised sports events have 100 thousand people in attendance. How 80 billion dollars distributed among this group means 800,000 dollars for each person!

Students can also be encouraged to appreciate large numbers by going outside to identify a million of something. This could include things like the area of lawn needed to encompass one million blades of grass. (How much area is needed to include a billion blades? a trillion?) How many grains of sand in a sandbox? How many jars of rice (that we worked with in our estimation unit) are needed to have a million pieces of rice? a billion?

Once we have established an appreciation for these large numbers through meaningful experiences, we can attempt to extend appreciation to the very large and very small numbers of science. One way to facilitate learning about orders of magnitude might be to construct a large classroom power-of-ten chart and have the students place particular magnitudes on the chart as they encounter them in their studies. For example:

$10^3 = 1{,}000$	number of grains of sand in cubic inch (coarse sand) visible stars in the sky
$10^4 = 10{,}000$	
$10^5 = 100{,}000$	
$10^6 = 1{,}000{,}000$	population of a big city
$10^7 = 10{,}000{,}000$	
$10^8 = 100{,}000{,}000$	population of U.S.
$10^9 = 1{,}000{,}000{,}000$	population of world; age of earth in years
$10^{10} = 10{,}000{,}000{,}000$	number of grains of sand to fill classroom
$10^{11} = 100{,}000{,}000{,}000$	number of grains of sand to fill school
$10^{12} = 1{,}000{,}000{,}000{,}000$	
.	
.	
.	
$10^{28} = 10{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$	(number of grains of sand to fill earth sphere)

When the students have gained a feeling for the very small — for example, after examining small objects under a microscope and discussing the small things of science — a similar chart could be constructed for negative powers of ten.

Curriculum Materials and Projects

In order to introduce positive changes in the teaching of middle-school mathematics it seems to me to be absolutely essential to change the format of the instructional materials. Instead of asking questions, answering those questions, working several examples, and then providing exercises, we need to provide more open-ended questions to promote the inquiry method. Successful materials with this format have been developed for junior high science programs.

There is also a need for well-conceived projects which are interesting to the students and which illustrate the need for mathematics. Some of the projects should involve physical as well as mental involvement. As an example, SMSG (in Mathematics and Living Things) constructed a mathematical unit around transpiration in trees. Very briefly, this unit involved:

(i) Physical numbers - measurement of the length and width of leaves; tracing leaves on graph paper and measuring their surface area; finding average area of leaf.

(ii) Ratios and graphing - computing the ratio of length to width of various leaves from same tree; graphing length vs. width of leaves from same tree.

(iii) Physical Activities - placing plastic bag around leaves on a tree branch to collect water loss; weigh water collected.

(iv) Estimation - use microscope to count stomates on portion of leaf and then estimate total number of stomates on a leaf; estimate number of leaves on a particular tree.

(v) Approximate calculation - amount of water loss per leaf, per tree and per stomate.

Very rarely do mathematics teachers have their students read materials other than the text. Newspapers and magazines have an abundant supply of statements that could be collected and discussed. For example, National Wildlife Magazine, (October-November 1970) claims that 76 billion bottles and cans are thrown away each year in the United States. About how many bottles and cans does this amount to for each family in a year? in a day? Does this seem reasonable? If 76 billion bottles and cans were laid end-to-end, about how long would the line be in feet? in miles? in circumferences of the earth?

Maybe the mathematics curriculum materials should contain short articles from areas like astronomy, social science, and science (appropriately selected for middle-school students) to serve as a background source for "applied" problems. Such articles may contain mathematical statements that could be discussed, interpreted, or challenged. There are many articles which distort number facts in order to prove their point of view (as illustrated in Huff's How To Lie with Statistics). Advertisements, on television and in print, provide numerous examples of statements which can be analyzed for their assumptions and "logical" arguments.

Lastly, there has been vast increase in the production of games and manipulative devices in the past three or four years. Some of these devices do illustrate and embody mathematical concepts and others provide practice for certain tasks in novel ways. The better materials should be integrated into the middle-school mathematics curriculum.

There is quite a difference between the way numbers are used in the current mathematics programs and the way they are used in daily life or in studies other than mathematics. Most mathematics courses concentrate on numbers in the "exact" sense — problems and computations lead to exact answers and there is heavy emphasis on the structure of number systems. However, when we measure we are using physical numbers and when we pick up a newspaper or magazine we are confronted with estimates, approximate calculations and very large numbers. It seems to me that students should be able to deal with exact numbers, understand and perform operations with physical numbers, deal loosely with numbers through approximate calculations and estimation, and, in general, develop a number "sense".

VISUALIZING IN TWO AND THREE DIMENSIONS

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I will start with a brief outline justifying both the inclusion of more visual work in mathematics classes, and more generally the development of students' geometric intuition. I shall then indicate by a very few short examples what type of work I have in mind. Many of the references in the bibliography give more detailed examples.

A. Why is such work needed?

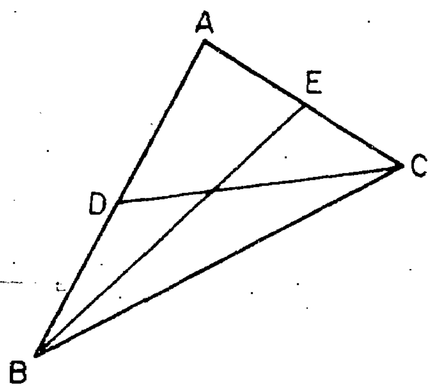
1. General reasons

The fact that the ability to visualize is required in any profession such as architecture, chrystalography, plumbing, highway construction, medicine, engineering, and ordinary household chores and everyday living is not disputed. I suspect it is also needed in such activities as driving especially while looking in the mirror, parking, or backing. Unless one assumes that these skills develop naturally these would be reasons enough why more visual work should be done in schools.

2. Reasons pertaining to the learning of mathematics

i. From experience that I have had in schools with children doing mathematics that is highly visual, tactile, and non-verbal, I have noticed that many children classed as failures, especially in mathematics, have been exceedingly successful with this visual type of work. Introduction of this type of visual work into the mathematics curriculum could give these children a new lease on life. Although I believe such activity has transfer value, even if the evidence should be negative, this activity is worthwhile in its own right.

ii. We place much emphasis on children being able to carry out certain algorithms, on being able to do certain kinds of formal reasoning, on being able to follow verbal rules, on being able to manipulate symbols but little or no emphasis is placed on visual ability. We check whether children are ready to do certain kinds of logical operations, whether they are ready for certain kinds of abstractions but we don't concern ourselves with adequate visual preparation. Here are two small examples where visualizing is necessary. Three-dimensional diagrams are drawn in textbooks but usually one does not check that children can interpret these diagrams. Yet the crux of a particular explanation or understanding may hang on being able to interpret the diagram. It may be exceedingly difficult for a child to interpret a diagram unless he has already seen or constructed the actual figure. Or as a second example, a problem in geometry may deal with a figure such as the one below and may involve using $\triangle BEC$ and $\triangle BDC$ but the child may not see these embedded triangles. Of course such visual activities are needed for other subject matter as well.



3. <u>Reasons dealing more directly with specific mathematical topics or concepts</u>

Visual facility is necessary for the successful solving of or even understanding of problems. I will give just three specific examples starting with the most sophisticated.

(a) Volume problems in Calculus. Students often have difficulty in setting up the double integrals. For example, what does the intersection of two right circular cylinders look like? I know of several teachers including myself, who just before teaching a class, practice drawing the appropriate diagram. If a student can't picture the appropriate cross-section he won't be able to set up the double integral. I am not suggesting that junior high school students should be doing calculus problems but that having appropriate visual tactile experiences will prevent difficulties later on.

(b) Locus problems lend themselves to concrete visual experience. That is, at an early stage children can plot loci by various means before proving anything theoretically.

(c) Then there are those endless algebraic problems of the kind that deal for example, with a flat sheet of paper out of which pieces should be cut to make a box. Their difficulty could be avoided or lessened if students could visualize what they are doing. Students often can't solve such and other problems because they can not set up the equation, they can't set up the equations because they can't visualize the problem.

Clearly there are many more basic mathematical topics, for example, those dealing with symmetry and transformations related to visualizing. These will be inherent in specific examples given later.

4. <u>Reasons relating to learning and teaching strategies</u>

i. The type of visual work that I will describe lends itself to students asking their own questions and providing their own problems — a much neglected activity.[20]

ii. Much of the standard work requires a <u>particular</u> algorithm, one way of doing a problem, one answer. Usually children have no opportunity to decide in what manner to attack a problem and they usually get little time to explore a situation. This work lends itself to solving problems in a variety of ways, to providing problems that have several or even an infinite number of different solutions, and to work that requires exploration.

iii. Visual type of work with concrete material lends itself to the student checking himself whether he is right or wrong rather than always relying on the teacher to make the judgment.

iv. Of course when I emphasize visual work I am not ruling out written work and computation. This kind of work gives rise to some honest calculations and problems rather than the so often dull, cooked-up, verbal problems or endless repetitious exercises.

v. Too often concepts are given verbal labels first (often causing discomfort and lack of confidence and understanding) and then experience follows. I feel it is important to <u>use</u> the objects, or situation first and define them when necessary and when children become familiar and comfortable with them (which may be five minutes or five years later).

vi. Students are usually given examples of how to solve particular types or classes of problems. In this type of visual work students will often have to decide on not only <u>what</u> to do but on <u>how</u> to do it.

B. <u>A few examples of topics or situations which lend themselves to the type of visual work referred to above.</u>

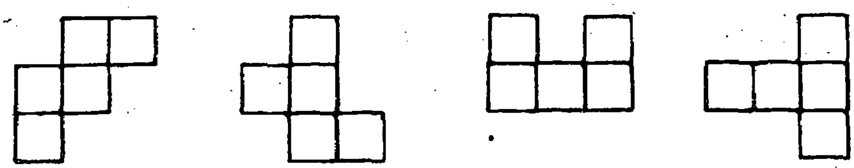
In each example I have indicated just one or two suggestions out of a much larger number of possibilities. The references have more details.

1. <u>Going from two to three dimensions and vice versa</u>

Visualize a cubical box without a top. Visualize how it looks flattened out. Draw it. How many different ways can you flatten it.

How many different patterns can you obtain using five squares regardless of whether they fold into boxes?

The students discuss what is meant by "different" when the need arises. They also discuss what, if any, restrictions should be placed on the arrangement of the five squares.



Which of the above patterns fold into boxes without tops? What other questions can one ask?

Given a milk carton cut to be a cube without a top, tear it to obtain the patterns below for example.

Repeat these type of problems with rectangular boxes, six-sided boxes, or with triangles. For detailed discussion of this see reference (18).

2. Cutting things into different number of parts

Construct a square. How many different ways can you do this? Can you do it without ruler and compass?

How many different ways can you cut it in half? (Adults usually produce only four ways.) Draw some of them. In cutting them in half, did you use a ruler and compass? If so, what geometrical constructions did you use? What do you notice about your halves?

How many different ways can you cut a cube in half? Again students decide on their own restrictions. Examine theorems or facts used. For one simple interesting half of a cube, see reference (9c).

Can you cut a cube in half so that the cross-section is a square? A rectangle, a parallelogram, a regular hexagon? What other cross-sections could you look for? What other questions can you ask?

How can you be sure you have cut a cube in half.

3. A combinational coloring game

How many different ways can you cut an equilateral triangle in half? Into three congruent parts? What do you notice about your work? Make a set of equilateral triangles, cut them out. You have four colors. Divide each triangle into three congruent parts in some interesting way. Use the same method for each triangle. Now color them. Each triangle can have one, two, or three of the four different colors you have and they all must look different. How many different triangles can you get.

Make up games using the triangles.

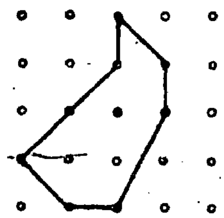
For other combinational problems see reference (11).

4. Tessellation problems

See references (15a), (16), and (8) in the bibliography.

5. Geoboard work

How many different ways can you find the area of



Assume that a small square has unit area. What other questions can you ask about the shape?

How many different shapes of area 3 can you make on 5 × 5 board. There are numerous references on geoboards (including (1), (2), (7), (8), and (19)) and many non-trivial 6th - 9th grade problems.

6. Combinational activities

A rich source can be found in reference (11). I particularly like problems of the type — Into how many regions can two rectangles in the plane subdivide the plane?

Closing Remarks

I have only sketched a few ideas above. Most important, many of the activities lend themselves to the looking for numerical patterns and generalizations. Many functions are obtained and they can be on a very sophisticated or elementary level, depending on the problem. We also learn to be cautious in making generalizations. That is, although it may seem that all the samples are elementary, the math involved can become as high-powered as one wants. Incidentally, the problem of how many shapes there are of area n on a 5 × 5 geoboard is unsolved, and other problems are non-trivial even for much older children. To sum up, this type of visual work, not only gives needed visual experience, a fresh chance to some children who are failing in other type of work, but also opportunity for posing their own problems, finding ways of attacking these problems, and checking on their own work.

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MATHEMATICS AND AESTHETICS

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I begin with three propositions concerning the nature of mathematics which I believe to be of fundamental concern in the design of a curriculum in mathematics for all ages:

1. Mathematics is beautiful. It is this aspect of the subject which has attracted most of us to the study of mathematics, both as physicists and as mathematicians;
2. Mathematics is fun. For those young people not frightened off at an early age in school mathematics classes, the wide range of mathematical puzzles and games available and popular speaks for itself.
3. Mathematics is utilitarian. Included here is everything from balancing the check book and fencing the garden to the remarkable, apparent fact that the laws of the natural universe seem to correspond ever so closely to the laws of mathematics.

The "old math" of my youth was principally concerned with Proposition 3. Lots of fine examples were provided which enabled us to turn the crank on our rote methods and rules. We got lots of practice in setting up rather impractical and uninteresting word problems, learning the methods of translation from English into Algebra or Geometry. The best of us developed an intuition of why the "rules" worked as they did and began to see some of the beauty and the logic and the power of mathematics in spite of the methods of teaching.

The "new math" of our children has concerned itself mostly with Proposition 1. The emphasis now seems to be very much on the fundamental bases of the logical systems. For those students who are willing to suspend the questions of relevance, a good bit of the beauty of the system is being exhibited, the reasons behind the rules are shown through a variety of clever examples, and the potential for a lot more "fun" is available. But it really is a bit difficult to convince the majority of students in the lower and middle schools of the importance of Unions and Intersections of Sets, of counting in Base 7, etc. And precious little is done in the way of developing practical uses of the mathematics which is emphasized, largely perhaps because there are very few practical uses for such mathematics, dealing as they do with foundations.

What is needed then is a sensible balance between "old" and "new", between 1. and 3., with a much more substantial eye on Proposition 2, which can be the rallying point. The major pleas of this paper are twofold then: that we not go overboard in correcting the ills of the new math, and that we keep an eagle eye on the necessity for making mathematics fun. In this latter context, I have been trying for some years to interest particularly elementary school teachers in the idea of "free math" in much the way such teachers now use the bribe of "free reading" as inducement for finishing an assignment early. The wide assortment of commercially available mathematical games and puzzles, and also many of the possibilities available in the "new math" offer some real opportunities in this direction. To my knowledge, only one of my teacher acquaintances has fallen for my suggestion, but she reports some measure of success with some of her students in improving attitudes toward mathematics. We should keep something of this sort in mind as a possible vehicle for making math fun and also for providing some of the insights and extensions which the new math has generated and made available to young people.

My thesis and my examples for mathematics and aesthetics, then, center on the notion that the "new math" movement was generally a good development which was overdone; that the correction is not to throw out the baby with the bathwater but rather to dip the baby in the bath; and that these "new math" techniques continue to offer the best way for making math fun and imaginative and captivating to the best students if embedded in a proper context of applicability and relevance. Following out my assignment, then, I will speak to some of the aesthetically satisfying and beautiful aspects of "new math" which seem to me worth including in any/every math curriculum, at least as seductive examples of what the "handmaiden of the sciences" can really do.

<u>Geometry</u>: One of the trends in the new math curricula has been the displacement of a considerable amount of geometry by algebra, mostly abstract.[1] As one who was bored by an excess of Euclidian proofs in school mathematics, I am not arguing for a return to a bulk of formalism (although some amount of this is in order). But I would argue that it is natural to think geometrically, and that we can hope to go a long ways further in developing "intuition" about mathematics <u>and</u> its applications if we do more geometry. Geometric examples, in fact, can then lead us to develop algebraic expressions of some interest as a means for stretching our intuition.

1. Geometric constructions afford an <u>activity</u> which is appealing to both the mathematically bright and the nearly illiterate. Starting with Euclidian constructions using compass and ruler, one can work up to construction of three dimensional paper-folded objects of great beauty.[2] Along the way, one can develop many questions on why things work as they do according to the rules of Euclid, two and three space, etc. Using construction methods based on a cellular approach[2] in two and three dimensions brings one very close to nature's way of constructing objects and gives access to a number of such natural or physical questions. Analysis of geometric patterns leads one to the possibility of extrapolating from a few simple examples to more general and less intuitive cases. (e.g., given 2, 3, 4, 5, etc. points in a plane interconnected by line segments, how many

segments are at each point? How many total line segments are there for each case? How many would there be for "n" points? What if the points were not in a plane? etc.)

2. Exercises in spacial visualization are enormously fun and challenging, and represent an area where intuition can probably be developed. Such exercises are often a part of an engineering aptitude test, I.Q. test, etc. and are almost universally considered as one of the pleasurable parts of such tests. These develop (or test) one's ability to think logically, geometrically and to express a result of this thinking in a simple quantitative way. As they are now used, almost exclusively in the test format, there is little opportunity to use these exercises in a teaching format, where they could be both fun and challenging.

3. Symmetry concepts represent one of the most elegant, and at the same time functional and natural sets of ideas available. Precious little has been done with such principles in the school curricula, in spite of the fact that they are so powerful in science and in art, and so readily available as examples in nature. Many of the exercises and examples in Martin Gardner's little book[3] could serve as beginning material for such developments. These can readily be carried over from the mathematics class into the biology, chemistry and physics classes as well (leaf and flower patterns, crystal structure, molecular models, etc.). And, of course, the subject can be taken to almost any degree of complexity one wishes to use, including considerable group theory.[4]

4. The Golden Section of the Greeks (the ratio of the sides of a rectangle which can be divided into a square and a rectangle with sides of that same ratio) is a geometric property of almost mystical quality in terms of its versatility: art, architecture (the Parthenon), some biological examples, even molecular structure! Pure aesthetics ..... or is it? A lovely little film titled, The Golden Section,[5] may provide some points of departure. Parenthetical clause: even the ratio of successive Fibonaci numbers (the sequence 1,1,2,3,5,8,13,21,...) approaches the Golden Section for large

numbers of the sequence!, thus giving a possible entrance into number theory.

5. Finally, under geometry, it should be possible to develop the basic ideas and some practical applications of elementary trigonometry at a much earlier stage and in more intuitive fashion than is common now. Back to surveying!

Topology: While I would not recommend a large amount of this in a curriculum, some elementary considerations in topology form a logical extension to geometry and there are some cute examples which young people enjoy. The Königsberg Bridge problem and its extensions into network theory represent one such track.[6] Another, related track is represented by the Descartes-Euler Formula[7] connecting the number of faces-edges-vertices for polyhedra. This is a lovely example, Pythagorean-Euclidian intertwining, representing the height of Greek aesthetics in mathematics; and one of the happy by-products is a proof that there exist only five regular polyhedra. Here we are, then, back into three dimensional geometry and constructions using D-Stix and connectors.[8]

Number Theory: A whole variety of pretty examples exist in number theory, many of which are really quite simple. The motive here should not be to develop a whole set of rigorous and connected theorems in the real numbers, as has been the tendency sometimes in the past. Rather, attention should be given to developing the methods of thinking inherent in number theory. Principal among these is the method of mathematical induction, so simple and yet so powerful. By its very nature, this method stimulates a person to investigate a number of cases on exploratory and intuitive grounds, then develop a generalization, and then test it by extension from "n" to "n+1". Also, number theory allows one to perform a collection of playful acts with numbers, some of which are quite useful in other work, others are more curiosities, all provide lots of practice with numerical calculations. Among these are: sums of consecutive integers = $n(n+1)/2$; sums of odd integers = n^2; prime numbers and their theorems; magic squares and their design; multiplication by Napier's rods; numbers in base 2 and their extension into many kinds of binary systems; factorials; and many other possibilities.[7] The possible combinations of beauty, fun and practicality here are numerous.

Probability: Picking up from factorials and their uses, one can go into some intriguing and potentially useful discussions of probability, permutations, combinations, exponential growth and decay, dice and other games of chance. Again there is opportunity for activities as well as logic and thought to give the study more life and interest. If tied into biological growth, chemical rates, radioactive decay, etc., there is good background built for future use, as well as elegance in concepts and methods.

Algebra: I will not put in a plea for preservation of all the "new math" work in abstract algebra and the theory of sets. All too often, this work turns out to be nothing but the empty set as far as meaning, value and retention are concerned. But let's do develop more of cartesian geometry, graphical analysis, concepts of functions and slopes at an early and intuitive stage and continue that development right through into algebraic and analytic forms. This goes for the trigonometric functions, exponentials, logarithmics, conic sections, polynomials, etc. A number of 8mm film loops exist to help this along, and what is lacking in such teaching aids should be made so that these all important concepts are more readily available in the general populace. The current ecological crisis just goes to show that most people don't appreciate the nature of the exponential function, for example.

In conclusion, let me assert that the above is a mere scratching of the surface of all those concepts and materials which are now readily available and accessible to young people, which reveal the beauty and elegance of mathematics. In our great concern for the functional and the practical, let us not neglect the sense of fun and beauty which has served as a strong magnet in attracting some of the best minds in history (including our own, of course) to a love of mathematics.

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QUANTITATIVE REASONING IN GEOGRAPHY

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I. Background Information

A. The Recent Quantitative Revolution in Geography

During the decade 1955-1965, geography underwent a radical transformation of spirit and purposes, best described as the "quantitative revolution." The consequences of the revolution are still being worked out and are likely to involve the "mathematization" of much of the discipline, with an attendant emphasis on the construction and testing of theoretical models.

B. Long before the mid-1950's, however, geographers were interested in the application of mathematics to geographic problems, but this interest was mainly limited to the use of data in geographic analysis, to cartographic developments, and to the use of maps.

C. Many of the recent learning activities being introduced into the elementary and secondary schools, as well as into college programs, calls for quantitative reasoning as students seek solution to problems from a geographic point of view.

II. Some Traditional Uses of Basic Arithmetical Skills in Geography

A. Furnishing Evidence

Data serve as evidence for statements made by students or teachers. Their effectiveness for this purpose depends in some degree on the arrangement and grouping of the statistical items. Often this calls for adding (to ascertain totals of groups), subtraction (in deducting the white population from a total population to find the non-white population), dividing (to find the density of an area given its size and numbers of people), and multiplying (e.g., if one dot equals 5000 people, how many are represented by "x" number of dots?).

B. Expressing Changes Over Time

In various connections, geographic work involves dealing with facts as limited by time relations. Changes are frequent in the patterns with which human geography deals. Designated variety in space and variation in time are two concepts that should be emphasized in the study of geography. Variety in space, together with orderliness in arrangement, is the keynote of world pattern. Variation in time belongs to the story of how the existing pattern came into being. Any number of changing phenomena are of interest to geographers: population changes, land-use trends, commodity trade movements, rainfall and temperature changes by months, or by years, changes in sizes of cities, etc.

The translation of these changes into graphic form is a familiar practice. A simple curve makes the general trend clear, and its construction is simple enough not to try the patience of even young students. In fact, the mechanical work of constructing even a simple graph is likely to absorb the attention of the beginner to such an extent that he misses the facts portrayed. This tendency may be overcome, however, by a little carefully directed study of the completed graph. Attention might be called, for example, to comparison of stated periods of time or other comparisons demanding skills in reading a completed graph.

Other types of graphs may be constructed, including overlapping squares, often used to represent areas of larger size, bar graphs, climatic charts which combine both line and bar graphs, proportional circles, etc.

In dealing with data, the concept of "means" might be developed, as for example, annual mean temperatures, or mean temperatures for given months.

C. Discovering Facts of Area Pattern

The plotting of statistical data on maps is a ~~familiar procedure~~ for showing the areal distribution of various phenomena. Used in this way, statistics contribute to knowledge of areal pattern, thus performing a definitely geographic function. The degree to which statistics can reveal details of pattern depends on the size of the statistical units, on the method of plotting the data, on the value of a symbol, and on accuracy of judgment enlightened by knowledge of the area involved. One principle to keep in mind in plotting statistical data on maps is that the smaller the statistical division selected as a base, the closer can the statistical pattern approach the true pattern of distribution. Also the dot-value is selected which, for the particular data involved, will give a maximum of contrast without overcrowding an area of concentration, and with some indication of pattern in minor areas.

III. Basic Mathematical Skills Needed in Reading and Making Maps

A. Maps are representations of the earth or parts of the earth drawn to scale on a flat surface. The degree to which a student can interpret maps depends on his ability to: (1) orient the map and note directions, (2) recognize the scale of a map and compute distance, (3) locate places on maps and globes by means of grid systems, (4) recognize and express relative locations, (5) read symbols and look through maps to see the realities for which the symbols stand, and (6) correlate patterns that appear on maps and make inferences concerning the association of people and things in particular areas.

B. Obviously, arithmetic skills are needed to perform most of these activities, but especially numbers (2), (3), and (4).

C. Recognizing the Scale of a Map and Computing Distance

The definition of a map includes the phrase "drawn to scale." A map, therefore, is more than a sketch, or diagram. It represents, in reduced size, a portion or all of the earth's surface, a surface which may be many thousand or even many million times larger than the area of the paper on which it is drawn. In other words, all maps are graphic reductions of real areas, and the area on the map is proportional to the area in reality.

Graphic reduction is, in itself, not a difficult problem in map reading, for the human mind accepts readily many reductions without hesitation. The child recognizes houses or trees in a picture because he is familiar with their shape and structure and has no difficulty in accepting the degree of their reduction. On the other hand, even relatively small parts of the earth's surface are beyond the range of the child's sensory experience. Hence, their graphic representation on maps in a reduced form is an abstraction because it cannot be directly related to any such experience. This poses one of the major problems of map reading. Because the reduction of areas on maps is difficult for the child to comprehend, map scales are correspondingly difficult. Yet, he must learn to read the map by scale and to understand the degree of reduction. Here the mathematics teacher can be extremely useful.

1. Ground and Map Distances

The first step in understanding scale is to develop a knowledge and a sense of ground and map distances. Scale is an expression of the relation between these two types of linear measurements. The development of understanding involves: (a) a knowledge of the terminology and the mathematical relationship among the linear units themselves: inches, feet, yards, blocks, miles, and degrees of latitude and longitude, and in the future components of the metric system; (b) experience in observing and recognizing these linear units or in acquiring mental impressions of their lengths: and (c) the ability to compare linear units applicable to the map with those applicable to the ground. Children's world should be graded with respect to each of these steps, but work in all three steps may be carried on simultaneously. Usually, however, instruction in recognizing the scale of maps and in computing distances cannot be undertaken in the primary grades. It must wait for the development of certain arithmetical skills. Normally, the first large scale maps introduced into the first or second grade rooms should be drawn to scale by the teacher. Children of primary age can then be led to express and compute scales in terms of number of blocks, amount of time required to walk, ride, or fly from one place to another. They can use relative terms such as "nearer" or "farther" in comparing distances.

In the intermediate grades, children can easily be asked to draw a plan of a school desk to scale, make a plan of the school room with accurate measurements, make a map of the home neighborhood, and then compute and express longer linear distances which cannot be directly observed or measured. These maps might be made at three or four different scales, and in so doing the pupil can begin to grasp the ideal of large-scale and small-scale maps — that large-scale maps show much about a small area and small-scale maps show larger areas but not as much detail. This concept will need reinforcing at almost every grade level. It is often very difficult for students to grasp, and the teacher must not become impatient.

2. Ways of Expressing Scale

There are three ways of expressing the concept of scale: the graphic, the inch-to-mile statement, and the representative fraction. They vary in difficulty and are used for certain special purposes. The three means of expressing scale need to be taught with care and simultaneously with the teaching of ground and map distances.

The graphic scale is the easiest and simplest. It is a line subdivided into actual units of map distance but marked to represent any of a variety of ground distances. The relationship (proportion) between the unit of map distance and the unit of ground distance is arbitrarily decided upon by the map maker. In drawing a plan of a desk or the room, for example, children can assume that any specific map distance represents any other specific ground unit. For instance, a half inch can represent a foot and a graphic scale using those proportions can be constructed to use with the plan.

Exercises may be designed in which students use the scales as rulers to read directly from maps, the ground distances between points. For example, on a desk map of the United States, the class may determine the number of inches which two cities are apart. They can then mark off this distance on a strip of paper. Then by applying this measurement or the graphic scale, they can read off the ground distance. Again, the students might be asked to do this on maps of different scales (using distance between same cities) to develop their concepts of large and small scale maps.

The statement scale is a more advanced way of expressing scale, for example, "one inch equals one foot," or "one inch equals four miles." Whereas the statement of scale appears to be a simple concept, it presents difficulties in visualization because it is a mixture of linear units. One does not say that one inch represents 253,400 inches, since long distances are not expressed in inches. Instead, the statement scale is expressed in inches to miles, as one inch equals four miles. The student is forced to try to visualize four miles for each inch of linear measurement on the map.

The representative fraction is the third way to express size and is also the most advanced mathematically. It is a ratio of a single unit of map distance to the ground distance represented, measured in the same units. It requires a knowledge of fractions, so for the most part is delayed until the junior or senior high school. The advantages of the representative fraction are its universality and precision. Such precision is possible because most fractional map scales are very small fractions such as 1/10,000 or 1/2,500,000. Sinch it is almost impossible to visualize a 1/2,500,000 relationship, the smallest of such fractions (in itself a difficult concept) makes for lack of concreteness as compared to such simple large fractions as 1/2 or 1/10. To help overcome this difficulty, the learner must develop ma... matical ability in converting the fractional scale to a statement of scale or a graphic scale which he can understand. For example, 1/62,500 is approximately one inch to a mile; 1/500,000 is one inch to eight miles; and 1/10,000,000 is approximately one inch to 16 miles. The scale of 16-inch globes is 1/31,680,000 or one inch to 500 miles.

For more precise purposes, the pupil needs to learn to use a simple equation, first reducing all values to the same linear units and expressing known values in these units. The unknown is represented by X. For example, two cities are five inches apart on a map which has a scale of 1/153,250. What is the ground distance between the two cities?

$$\frac{\text{Map Distance}}{\text{Ground Distance}} = \frac{\text{Map Distance}}{\text{Ground Distance}}$$

$$\frac{1}{153,250} = \frac{5}{X}$$

$$X = 766,250 \text{ inches}$$

$$766,250 \text{ inches} = \frac{766,250}{63,360} \text{ or } 12.09 \text{ miles}$$

Learners need practice with similar problems giving either ground or map distances and using either fractional or statement scales to give them facility in converting from one type of scale to another.

3. Visualization of Scale

Understanding map scales requires more than the mechanical ability of mathematical manipulation. Since a map is a graphic reduction, intelligent interpretation of it depends on the ability to visualize map distances in terms of ground distances in the landscape, or the reverse process. This ability is easy in terms of large scale maps (relative small areas showing detail), but it becomes difficult with small scale maps.

That it is difficult even for the adult is demonstrated by the failure of many teachers who must take examinations for certification. The following three questions on a recent "Certification Examination for Teachers of High School Geography" administered in one of our big city school systems proved not only to be very difficult, but lacking in power to discriminate between those who obtained high scores and those with low scores.

(A) If a given map has a fractional scale of 1/63,360 and two points on that map are six inches apart, how far apart in miles are those two points? (1) 1 mile, (2) 2 miles, (3) 4 miles, (4) 6 miles, (5) 63,360 miles.

(B) If you were to measure distances on a globe whose scale was 1/16,000,000 an inch would represent about: (1) 500 miles, (2) 250 miles, (3) 125 miles, (4) 50 miles, (5) 5 miles.

(C) If the scale of a map is 1/125,000, then one square inch on the map represents how many square miles on the earth? (1) 2 square miles, (2) 4 square miles, (3) 8 square miles, (4) 12 square miles, (5) 16 square miles.

(D) What Can a Map Show (Resolution)?

Once that a student understand scales, he is able to understand the differences between large and small scale maps, and what can be shown on a map of a specific scale. It is necessary, for example, for the student to understand that as the map scale becomes smaller (holding the page size constant), the size of the area covered can be increased, but the detail which can be expressed must be reduced. In other words, the resolution is not so good. Resolution is here defined as the concept of separating a pattern into its components. The limit of resolution is reached when the components merge together to an unacceptable degree and cannot be separated. For example, at one scale, many details of a specific city can be shown — its street pattern, even individual buildings. But at a smaller scale, these merge together and only the city as a whole can be seen. At a world scale, even the city disappears and its location cannot be accurately depicted because of the scale of the map.

Perhaps an illustration can be used to clarify the meaning of resolution. On a scale of 1:62,500 (approximately one inch to one mile), the street pattern and the location of individual government building can be mapped. One could use such a map to find one's way about the capital. If the scale of the map were reduced however to 1:250,000 (approximately one inch to four miles), only main arteries (and these greatly exaggerated in size) could be depicted. One could not locate specific buildings very readily. At still a smaller scale, 1:1,000,000 (approximately one inch to 16 miles), the details are even fewer. At a scale of 1:4,000,000 (one inch to 64 miles), Washington, D.C. can be shown only symbolically.

The term large scale map means one on which most of the details, or small things, can be show... A large scale map shows streets, roads, small rivers, and sometimes even buildings. A medium scale map shows some of the more important details but not all. It may cover several degrees of latitude and longitude. A small scale map does not show much detail; in fact, often does not even show all the important things. A map of the world on a piece of paper this size would very small scale. No detail could be shown. The resolution of such a map is said to be low.

E. The Ability to Locate Places on Maps and Globes

Any point on the earth's surface may be located exactly be determining its longitude, i.e., the point of the intersection of a parallel and a meridian. This exact location may be expressed in degrees of latitude and longitude. The learner should understand that latitude is a measure of the angle between the plane of the equator and lines projected from the center of the earth. Lines on latitude connect points on the face of the earth whose projections to the center of the earth form a 30 degree angle with the plane of the equator. The latitude of the equator is zero degrees. Lines of latitude north and south of the equator are numbered to 90 degrees because a line drawn from the pole to the center of the earth forms a 90 degree angle with the plane of the equator. There is no latitude higher than 90 degrees.

Longitude is the measure of the angle between the planes of two meridian circles, one of which is the prime meridian. For example, the plane of the 90th line of longitude, on which New Orleans is located, forms a 90 degree angle with the plane of the prime meridian. All places on the 90th line of longitude west of the prime meridian, there, are at 90 degrees west longitude.

Understanding what a system of lines of latitude and longitude are, students can examine a globe to discover important facts about the system, e.g., all lines of longitude are great circles which converge at both poles and bisect the equator and every line of latitude; only the equator, on the other hand, of all lines of latitude is a great circle, and all lines of latitude are parallel to teach other and are true east-west lines. Other facts can also be observed.

With these facts obtained from studying the system of latitude and longitude lines on a globe, students can then examine systems of lines of latitude and longitude on world and larger scale maps. When they do not agree with the global network, they know that the map contains some kind of distortion as to size, shapes, distances, etc.

Simpler grid systems can be introduced in the elementary grades, and students can locate places on maps by rows and columns that are numbered or lettered.

F. The Ability to Recognize and Express Relative Location

The concept of "relative location" is an important one in geography. It is an expression of the distance between any two or more places, and the direction in which anyone lies in respect to others. It also depends upon certain features that make one more accessible to another.

It is important, therefore, for students to know how to read and give directions, and how to measure distances before the relative location of two or more places or objects can be stated.

With the use of aerial photographs, the term resolution has been introduced into the vocabulary of those interested in reading and interpreting them. Rather simply, it might be defined as the minimum separation at which two objects can be distinguished on a photograph or map. In aerial photography, the term "spatial resolution" is used and applies to both imaging and non-imaging sensors. It refers to their capability of recording terrain patches of limited size distinctly from other adjacent terrain patches. The resolution of a sensor is given by the smallest distance between two equal objects that still permits one to recognize these objects as separate entities. It can be expressed by an angle (angular resolution). This concept, however, is quite difficult, and is usually left to the experts to understand and utilize.

IV. Some Modern Applications of Mathematics to Geographic Analysis

As indicated during the late 1950's and early 1960's, the study of geography underwent a veritable revolution, sometimes referred to as the "Quantitative Revolution in Geography." Two stages can be recognized in this revolution. First came the analysis of the data according to recognized statistical procedures. Second, the use of the formal logic of mathematics was introduced to state propositions and develop logical structures. The first has since been introduced into learning activities at the high school level; the second, insofar as is known, has not yet been introduced at the pre-collegiate level.

Statistical Applications

The bulk of statistical procedures utilized by geographers in the period 1954 - 1965 were of the "regression-correlation" type. This development was due to the accepted philosophical position that area co-variation was a central problem in geography, and that geographers sought morphological laws derived from pattern co-variation. It gave rise to such hypotheses and laws as, "where x - there y."

A second organizing concept which is currently dominating geographic research and instruction at the collegiate level is that of "spatial interaction." Examples of spatial interaction models introduced early in the literature included the use of gravity models. More complex decision-making models have now replaced these earlier models.

Use of Statistical Techniques in Learning Activities

It is possible, especially at the high school level, to introduce activities in which learners must work out simple correlation problems involving area association or spatial interaction. Students might also study distribution patterns to discover their trend surfaces, measures of centrality, intensity of distribution, and other attributes.

SOME THOUGHTS ON SCHOOL MATHEMATICS, ESPECIALLY FOR THE MIDDLE GRADES

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Mathematics, like all other academic subjects (or, like all other subjects taught academically — that is, in schools), suffers from being isolated. It is isolated from other subjects and it is isolated from "the real world." Efforts to overcome this isolation, such as introducing into mathematics illustrations from other subjects (for example, sociological data) and from the real world (for example, making change in the grocery store), point in the right direction, but are usually too little and too late. They are like a little sweetening for a bitter pill, or like Band-Aids for a badly wounded patient. Far too many students are turned off to mathematics early in their school careers, often with the aid of teachers (especially elementary teachers, who must teach mathematics whether they like it or not) who were also turned off to mathematics early in their academic careers.

The cry in recent years for more relevance in the schools points to a very real problem; but the solution to the problem has not been helped much by the diagnosis: "lack of relevance." It helps a little to break "lack of relevance" down into what usually turn out to be its chief components, lack of interest and lack of apparent usefulness. Intrinsic interest and apparent usefulness are the chief motivators for any kind of learning. Both should be appealed to, but not necessarily at the same time, with the same students, or with the same emphasis.

The suggestions given here are intended to provide the beginnings of an overall framework for the teaching of mathematics and, within that framework, some specific ideas that are compatible with the framework. Most of the suggestions apply to the teaching of mathematics throughout elementary and high school beginning, of course, with the first grade. The emphasis on mathematics as a part of the knowledge and skills needed for coping with the real world should be particularly useful in the middle school years, when youth are particularly concerned with their identity, their place in the world, and how to cope with the world.

The overall framework is intended to:

1. Put mathematics into a broader context than is usually used.

2. Make this broader context available to the teacher.

3. Make the broader context available to students as they feel a need or desire for it.

4. Emphasize the usefulness of mathematics for coping with the world, without denigrating the joy of mathematics for its own sake.

The specific suggestions, within the framework, are intended to:

1. Relate mathematics more closely to students' goals.

2. Relate mathematics more closely to the real world.

3. Reduce some of the forbidding aspects of mathematics.

The suggestions, not necessarily in logical order, follow.

1. Mathematics should be viewed as a part of the knowledge and skill needed to cope with the world. An important part of every individual's effort is directed toward explaining, predicting, and controlling events in the world about him -- whether this is done intuitively or with scientific rigor. Logic, mathematics, statistics, and scientific theorizing are inter-related approaches or tools to this common goal of explaining, predicting, and controlling events.

2. Language, or linguistic thinking, has many parallels with scientific and mathematical thinking, mostly at an intuitive or subconscious level. We do much quantitative, logical, theoretical thinking with everyday language. The teaching of mathematics could profit from linking mathematics more closely to language, taking advantage of the facts that (a) language has already been learned by all school-age children because of its

demonstrated usefulness in coping with the individual's world, and (b) embedded in language is much of the structure and set of concepts that are basic to mathematics, statistics, logic, and theory.

3. Mathematics is not identical with perfect precision, and the highest level of precision is not necessarily the best state of affairs. Approximations are not a necessary evil (as imperfect approaches to a perfect state of affairs), but useful and acceptable means of measurement. As a general rule, it is more useful to ask "How accurate does this measurement need to be?" rather than "How accurate can I make this measurement?" It is unfortunate that the term "error" as used in mathematics and statistics carries over from common language a heavy moralistic taint. As antidotes to the goodness of "precision" and the badness of "error," we may need courses in "Sloppy Mathematics," and "The Virtue of Error."

4. The concept of inequality is simple and powerful. In school mathematics it is underused; yet in everyday life it is probably the most used concept, since it is the basic ingredient in choosing, or decision making. Many decisions require only a statement of inequality.

5. Students need early and constant practice with the concept of orders of magnitude, a practice that has probably been inhibited by the emphasis on precision with which mathematics is commonly associated. The concept is tremendously useful in everyday coping, (a) for estimating quantities or magnitudes on the basis of inadequate data, and (b) for checking on the approximate correctness of (exact) answers.

6. Mathematical, logical, and scientific thinking are considered by many to be less applicable to the social sciences than to the natural sciences. This is probably due to the belief that the measurement of quantities and relationships is generally much more accurate in the natural sciences than in the social sciences. Leaving aside the question of what "accuracy" means when applied to the measurement of different objects or events with different kinds of measuring instruments, let us assume as correct the common belief that the "hard" sciences are much more amenable to precise

measurement than the "soft" sciences. Why should this consideration make the social sciences less amenable to scientific thinking about cause and effect -- to explanation, prediction, and control? While a high degree of precision is often important, and sometimes crucial, in scientific endeavors, it is not the essence of scientific thinking.

7. Problem solving or decision making with incomplete information is a common problem in the real world which is given little attention in school mathematics. It should receive more attention within the context of early school mathematics.

8. The cost of securing data for problem solving or decision making is largely ignored in school mathematics; most mathematical problems give sufficient data and train and test students only in data manipulation.

Some of the ideas and relationships described above are illustrated in Figure 1, which is quite preliminary and incomplete. The figure shows some parallels among linguistic, logical, mathematical, statistical, and theoretical thinking; elements on the same horizontal level are comparable to each other. In general, the more elementary modes of thought are at the bottom, the more comprehensive at the top, leading to the main purpose of it all -- coping, controlling, and managing.

Turning to more specific curriculum suggestions, drawn from the social sciences and related at least in part to some of the ideas presented above, the following items are suggested.

1. The social sciences abound with data, most of which are dull and/or meaningless until and unless you have an important question that might be answered by some of the data. The Statistical Abstract is a source of such data -- easily available, massive, and dull or useless unless an important question is posed to which it might have an answer. The telephone book is another source of much dull and useless data -- unless an appropriate question is posed. Many problems about how to use a telephone book

COPING
(Controlling, Managing)

LINGUISTIC THINKING	LOGICAL THINKING	MATHEMATICAL AND STATISTICAL THINKING: STATISTICAL	MATHEMATICAL AND STATISTICAL THINKING: MATHEMATICAL	THEORETICAL THINKING
Prediction	Prediction (Deduction)			Theory
Explanation	Explanation (Induction)	Correlation	Functional relations	Generalizations
				Structure (Taxonomies)
Estimation		Estimation	Measurement Scaling Approximations Counting	
Equality Order Inequality			Equality Order Inequality	Concepts
Identification of things		Sampling	Classification	Singular events
		Probability theory		

Figure 1

can be posed, excluding the simple problem in which you know the full and correct name and address of a party known to be listed in the book, whom you wish to call. Suppose, for example, the problem is how to find the number of a person whom you very much want to call (a pretty girl, a handsome boy, an electronics expert, a person named Chuck who knows about a big record buy, etc.) and about whom you have incomplete information. What are some good search strategies? Can logical or mathematical or statistical or scientific reasoning help? Are the known or probable costs worth the known or probable benefits?

2. All of us carry around in our heads a great number of conclusions about correlations among traits of individuals: unpleasant voices go with unpleasant dispositions, girls are better English students than boys, boys are better science and math students than girls, Jewish families are more clannish than non-Jewish families, students and teachers who dress conservatively have stricter moral standards than those who dress more casually, etc. How can such generalizations be tested? What data are needed? Are some such data available? Must new data be gathered to answer such questions? How are appropriate data handled to give useful answers?

3. A very important concept in human behavior, developed most completely in economics but applicable to many situations, is the concept of trade-off ratios. There is almost nothing -- including life and the most precious tangible and intangible possessions -- that an individual will not trade for something else -- if the price (the trade-off ratio) is right. Most commonly, money is traded for goods and services. But we also trade time for entertainment, an inconvenience for the approval of a friend, health for pleasure, discomfort now for comfort in the future, and so on. There are two important kinds of trade-offs -- trade-offs preferred (by the individual) and trade-offs available (to the individual). Much of life's decision making consists of comparing preferred and available trade-offs and selecting from among the available trade-offs those to be acted upon. What determines the preferred trade-offs? How can information about available and preferred

trade-offs be obtained? How can selections be made from among the available trade-offs?

4. Related to (3), what causes changes in available trade-off ratios? What causes changes in preferred trade-off ratios?

5. There is a wide range of phenomena in the physical and social sciences that can be compared under the headings of equilibrium and disequilibrium. Equilibrium is a state in which all opposing forces are in balance so that no movement is taking place in the relevant variables. Disequilibrium is a state in which movement in relevant variables is taking place due to an imbalance of forces acting upon them. Social examples of equilibrium include many kinds of social conservatism, market prices under some conditions, and stalemates of various kinds. Social examples of disequilibrium include escalation and de-escalation of personal and social conflicts, inflation, stock market fluctuations, and rumors.

6. System analysis has had an interesting parallel development in the natural and social sciences. Many aspects of systems analysis can be developed for the early and middle grades, concentrating either on the natural sciences, the social sciences, or parallels between the two. These aspects include signals, noise, and signal-noise ratios; communication; information; positive and negative feedback; cybernetic systems (detector-selector-effector); preference systems; transactions; and organizations.

BASIC MATHEMATICS NEEDS FOR LIFE SCIENCE AT THE JUNIOR HIGH SCHOOL LEVEL

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Life Sciences can probably be taught without any quantitative skills whatever, at least at the Junior High School level. At this level biology should ideally involve a great deal of observation and simple experiment, neither of which necessarily involves a heavy use of mathematics. But I believe that quantitative skills are essential for a well-educated person and that one can use biology to help students understand some basic ideas. In particular I see five major areas that seem to me to be important and which I believe would fit naturally into a biological framework. These are: Graphing, Geometry, Measurement, Estimation and Probability. I assume that students know the basic arithmetic operations and need only practice to become proficient, unless of course, they're dunces like me and have to resort to slide-rules and calculations to keep the numbers straight!

Graphing and the interpretation of graphs is one of the more generally useful tools that can come out of biology. The growth of a plant plotted as height against days makes a natural graph. Days that were skipped offer the chance for interpolation. Bets on how high it will be next Wednesday lead to notions of extrapolation and its problems. One can relate two variables graphically; using the same example one could relate the amount of water to the height of the plant on any given day or the amount of nitrogen or whatever Using graphical techniques with living material is bound to lead to all sorts of causes. There should be linear, as well as other curve shapes, and students should have the chance to work from graphs back to real data; looking at a graph they should be able to describe what went on to produce the curve they see. The main issue here, of course, is that graphs should not be a sterile exercise by itself — a graph is a description of reality and unless students have the reality I don't believe that they will ever really understand graphical representation.

Measurement is another central topic. Whether it is math or science I don't know. But it has a habit of leading to numbers which are then added, multiplied, or whatever. It certainly is something that had better be understood. Measurements are almost always tied up with geometry so some basic ideas about area, volume mass, etc. are essential. Again I don't think these are things that one should make a major topic by themselves. They are tools for answering questions — How big a plant do I have? Area? or Volume? How do you begin to answer the questions that arise naturally from children's work with living materials? At the end we want students to understand how to measure and how to connect measurements from one form to another. This involves fractions, decimals, and fudge factors (^{o}F to ^{o}C). Somewhere there should be a simple introduction to shapes, areas, volumes, and formulas for calculating what you need. But please, no long lists of formulas to memorize and then forget. This does no one any good, or perhaps I should say that it never did me any good. One needs real things and a reason for measuring and calculating. This again comes naturally — one can measure leaf area, plant volume, plant weight, etc.

Thirdly, I believe that some basic ideas of probability will be helpful. I don't propose an analysis of variance or even an χ^2 but rather some notion of chance. This is probably coin flipping, seed sorting, and the like. It will come up when discussing class results in an experiment, it leads to a crucial idea in basic Mendelian genetics and it lies at the heart of what science is all about, namely predictability. I'm not sure how this can be taught in Junior High or how it fits with biology. But it is well worth exploring. I suspect that it might well come out of a "Peas and Particles" approach to estimation which is the fifth area in need of attention. He we are dealing with the five-gallon jar full of beans. How many are there? One can't really count, at least not if you're lazy. Basically kids develop ways of sampling and estimating. This experience is very important because there has been so much emphasis on "exactly the right answer" that kids have no feeling for guesses. 673×12 is more than 6730, in fact about 1400 more. Kids need to feel free to approximate and make guesses. Such guesses may indeed greatly help accuracy rather than the opposite. One can also use the rather fun game of estimating all sorts of wild things. For example, if one has been growing beans in the classroom, suppose one estimated the amount of water used by a 100-acre bean field. How many beans would you expect to get? How many people would that feed at one meal? And what are likely to be the high and low limits? The aim here is not exactitude, but a feel for numbers and size; the ability to make an intelligent guess.

Looking over what I've written, I'm not sure I've answered the question. These skills are not really needed for life science, they are needed for understanding the world. Life science makes a good context in which to develop them.

TEACHING JUNIOR HIGH MATHEMATICS IN A HETEROGENEOUS CLASS

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A "heterogenous class" is a class where the range of student performance and/or ability is significantly greater than that commonly found in a homogeneous class. In addition to social benefits, students in heterogeneous classes gain in other ways. The class tends to be more exciting as there are more children who serve as examples of motiviation, interest, and behavior than one generally finds in a low or average homogeneous class. As the heterogeneous class does not carry an ability label, the students with low or average performance records are less likely to develop or reinforce a negative self-image. Also, the level at which a child is taught is more likely to depend on his knowledge of a particular concept rather than be predetermined by his permanent grouping. In addition, low performers can participate in the regular mathematics curricula as well as receive remedial help.

Given a broad spectrum of students, ranging from the high performers to those needing remedial help, the goal is to best provide both for the class to work as a single group and to accommodate the needs of each individual student. Features of good teaching — a good learning environment, opportunities for learning in a variety of ways, and enabling each student to maximize his growth in mathematics — must not be ignored. In addition, through heterogenous class organization, one hopes to avoid premature "ability labeling" of students and to provide class members of varying performance and ability levels many opportunities to work with one another. Students teaching other students is a crucial aspect of a heterogeneous class. Small groups within the class continually change their composition, and thus it is not only the better students who are helping the slower ones. Sometimes it is better students helping other good students; and sometimes it is slower students working with other slow students.

Two features appear essential to achieve the above goals: (1) various grouping patterns, including "full-class, teacher-led" sessions, small heterogeneous groups, and small homogeneous groups; and (2) use of manipulative materials. The amount of time given to particular grouping patterns varies relative to the topic, and progress of the students, the availability of aides, and the use of materials.

It is assumed that not all students will proceed in a linear fashion through the same experiences, varying only in speed. Rather, it is felt that a variety of learning avenues should be provided. Manipulative materials often seem to provide for the necessary alternatives. For the above-average students, manipulative devices can be used to extend the understanding of a concept. For the average student, experiences with materials allow the grasp of a concept on a more concrete and understandable level. The low-average student needs the materials to express ideas and to aid him in computation and problem-solving.

Following are three days in my heterogeneous math class which illustrate how these ideas work.

DAY 1: (1) Review with whole class definition of a rectangle, making sure students understand that a square is a rectangle. Review idea of perimeter. Draw rectangle on the board — give its base and height, have students tell what perimeter is.

(2) Students sit around tables in groups of four to six, mixed heterogeneously. Pass out worksheets and scissors and paper. Students to go work. (See Worksheet 1 — page 117.)

Things to notice:

(1) Most of the period is spent by students working, not teacher talking.

(2) Students are working individually, but they are all working on the same problem. They are working around a table and encouraged to talk about their work, and help each other. At the end of class, however, each turns in his individual solution.

(3) The worksheet is open-ended and so even the brightest student does not find the assignment trivial. The sheet assumes that students will cut out squares and so the slowest student has something to manipulate, yet a student who can think abstractly is free to complete the assignment without moving squares if he doesn't need to. Each student is free to "think" the problem instead of "moving squares around" at the stage he's ready for it. (Often I find that students will request graph paper, and shade in squares to help solve the problem.)

(4) Concepts are reviewed before students begin working — I can check that each pupil knows the basic ideas by the judicious asking of questions. While working, students are seated in heterogeneous groups and so can question each other. At the end of the lesson — or at the start of the next day — a summary of the basic ideas students should have found is discussed. That is, it is not expected that students will make all the generalizations for themselves — time is allowed for this to be done on a class basis. Of course, some of the brighter students might go beyond this, but for the majority of the students there is a definite need for class summary of major ideas.

(5) Class time is very non-threatening. Students can check their answers by moving squares and counting. And when questioned by the teacher can be asked, "Show me how you got that answer." "Why?" "Prove it." The type of questions teachers should be asking. Students are free to talk to each other — and do. (Conversations often overheard — "Why did you write that?" "Explain to me how to do this." "You're not right — teacher!") Thus the amount of learning that takes place is greatly increased as I cannot possibly ask all the students all the questions I'd like to. And making the class more enjoyable for the students, for one characteristic of junior high pupils is that they are very sociable.

DAY 2: (1) Students are separated into two groups — one group is composed of six to eight better youngsters (based on observations of teacher during Day 1) and the other group is the rest of the class.

(2) "Top" group has assignment to do — a continuation of the previous day's work — and they are given a table in the corner to work at. (See Worksheet 2 — page 118.)

(3) Rest of class talk with the teacher about what they found the previous day — make generalizations which these students do not do automatically by themselves. Then they are given the next assignment to do, and may work together as usual. (See Worksheet 3 — page 119.)

Things to notice:

(1) Students who made generalizations by themselves were <u>not</u> required to sit still while teacher helped others realize them.

(2) "Top" class was chosen on the basis of the previous day's work, and is a flexible group — membership varies based on the concept. Of course, there are some students who are generally in this group,* and there are others who fluctuate between groups.

(3) "Top" group works in the room and so students not involved can see them — they are not separated from the majority of the class.

(4) Some days the "separate" group is the bottom four to six students and some days it's an arbitrary group (maybe using a game or material only available in limited quantity), so each pupil has the opportunity of being a part of a special group periodically. Teacher's membership fluctuates from group to group.

(5) Teacher must talk with members of each group every day. Junior high pupils do not seem able to sustain themselves without knowing that the teacher will be asking what they've discovered, what they've learned, or some such question each day. In the lesson described above, the teacher can check in with the top group at the end of the class when others start on their assignment.

(6) Students are taught to rely on group members — they must learn early in the school year (after explanations by the teacher and much practice) that when the teacher is involved with another group, they must resolve problems for themselves — i.e., don't interrupt. The work students are asked to do must be carefully planned so problems are not likely to arise which a group member cannot solve.

DAY 3: (1) Teacher introduces new idea to whole class by use of the overhead projector, film loop, or some such device (the blackboard can even be used).

*Truly gifted students who always grasp concepts first and enter junior high with facility in arithmetic seem to need their own class. They do not seem to benefit from the heterogeneity and may be handicapped by it — i.e., they are ready for a much more abstract curriculum.

(2) Discussion by whole class and then problem assigned — students work individually or in pairs, but each turns in assignment.

(3) Possible game relating to idea being introduced available for use when worksheet is finished (or reverse the order — all students play game in pairs for about 10 minutes, and then work on worksheet).

The above outline of three days in a heterogeneous math class gives you a sample of what I consider desirable education in math on the junior high level. The structure seems quite effective — I've been using it for two years now. However, there are problems:

(1) The organization is complex — grouping changes constantly — and so daily evaluation of individual students is needed. Bookkeeping is complicated, including keeping track of which student does which assignment.

(2) Grading is difficult if your school requires letter grades. Tests need to be given on several levels. But then how to decide which students gets an A for a report card? And how to report to parents on what a student seems to be learning that doesn't show up on tests — or how to write better tests that reveal all the students are learning?

(3) There is a dearth of non-worksheet activities. S teachers need to create these on their own — and it's quite difficult to write appropriate activities that are open-ended. Projects, building things, etc. seem like they would be appropriate, but are difficult when limited to 40 minutes a day and a shared classroom.

(4) Relating activities to algorithms is hard. Algebra teachers want students who are proficient in arithmetic, and use standard algorithms.

WORKSHEET 1

1. Cut out 24 one-inch squares (use those below).

2. How many different ways can you fit them together to make a rectangle? (Use all 24 each time.) Write down the length of the base, the height, and the perimeter of each rectangle you make. Record this information in a chart.

3. Are any of these rectangles also squares? Which?

4. How many of the 24 one-inch squares would you use to make the largest square possible? How many more one-inch squares than the 24 would you need to make the next largest square? And the next largest? And the next? Can you continue this pattern?

5. How many more rectangles could you make if you used one more one-inch square than the 24 allowed above? List the rectangles. How about if you used 26 one-inch squares? List them. How about if you used 27 one-inch squares? List them. Can you find a pattern? How about if you had 100 one-inch squares — can you predict how many rectangles can be formed?

WORKSHEET 2

All the rectangles you made yesterday of 24 one-inch squares have the same area: 24 square inches. You found that their perimeters varied. Thus we could say, if two rectangles have the same area their perimeters vary.

Write the converse of this statement: ______________________________

____

Prove the above converse TRUE or FALSE.

Can you find an example where the perimeter and the area of a rectangle are the same?

If two rectangles have the same area and the same perimeter, are they congruent? Prove it.

Is the converse of this statement true?

WORKSHEET 3

Create all possible rectangles whose perimeter is 36 inches.

Make a chart showing the base, height, and area of each of the rectangles.

THE "LIGHTER" SIDE OF MATHEMATICS

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"I can't do math," or "I hate math," or "Math was always my worst subject" are not uncommon statements made by many adults. These attitudes are not formed in the adult stage, however, but early in life, probably as a result of unhappy or unsuccessful experiences with mathematics. As a math teacher in an inner-city junior high school, I am trying to prevent or lessen these negative attitudes toward mathematics by showing the student that math can be fun and that he can be successful with it.

Math games have proven to be an excellent means of doing so. Learning occurs most effectively when students are eager and willing to learn; games are a means of motivating students to learn mathematics. Games are fun, challenging, competitive and yet there is no great fear of failure. During normal instruction the right answer is expected, but when playing a game, losing is acceptable. Consequently in game-playing, the junior high school student responds more freely without fear of ridicule from peers. Remember peer groups play an important role in the adolescent stage.

Games, with their various designs and complexities, serve many functions in my classroom. For example "Mancala" is a very simple game which involves two players dropping beads in successive pits on a board. There are only a couple of rules for the game. The game goes fast. Students can apply strategy to win. But most of all, mature thinkers and immature thinkers, too, enjoy the game. Thus, it is an excellent game for getting students to class and many times even before class actually begins. "Quinto," a game which deals with summing multiples of five is also good in attracting students to class. The better math student is very much attracted to "Krypto." "Krypto" is a game which involves combining five numerals (dealt from a check of Krypto cards) with any of the fundamental operations to obtain a final numeral (goal card). The Tower of Hanoi, which requires use of exponents, is also a drawing card. Thus, not only is the student getting to class and enjoying the games, he is also learning and using mathematics. "Numo," "Quizmo," "Baseball," and other similar games are good exercises of recall. Students enjoy them and the game can be varied depending on the students' mathematical abilities. All of the previously mentioned games can be employed at the end of the class period, or when individuals finish regular assignments, or sometimes for remedial work.

The last game I'd like to discuss in somewhat more detail because of the tremendous affect it had on so many of my students is called "Equations." Once the game rules are learned, the student can concentrate on the mathematics. The aim in playing is to win by correctly challenging your opponents or to have written such a complex solution for the goal that you are incorrectly challenged. In my room I had 15 hierarchy tables with three players at each. Students with similar mathematics ability competed against each other. Thus, there is a winner at every table. Consequently, a student of low ability could experience success as well as a student of greater ability. A "bumping" system which shifted a player from a lower or higher table maintained the competitiveness of the game and kept students with similar abilities playing each other. So, if a student grasped a mathematical concept that others at his table hadn't, he could employ it in the game and possibly move to a higher hierarchy table. Thus students come to me asking for additional help on an old concept or to be taught a new concept which they could use to win the game. My teaching lessons were actually taught as strategies to be used in playing "Equations." For example, my lesson on exponents involved writing solutions with exponents. The students were very attentive because they knew that they would be using these expressions in the game. If they could use this tool well, it meant a "win" for them. Other topics: integers, fractions, roots, etc. were developed similarly.

Not only did the "Equations" game help my students develop academically, but socially as well. The student was constantly interacting with his peers. He was able to experience success whether at the top of the class or the bottom. He was a member of a team. So, he was trying for the team as well as self esteem. Students were helping each other; a top player on a team would show a bottom player on that same team some strategy which might be helpful in the next playing period. Students looked forward to the next playing period and were very disappointed if for some reason play was not possible. Attendance picked up because students didn't want to drop down to a lower table nor lose points for their team. Students even asked to borrow equations games so that they could practice outside of class. When a student begins coming to a math class early, working in class, and asking to take math materials out of class, these are signs that he is motivated to learn mathematics.

SOME OTHER EQUATIONS

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One of the common activities in school mathematics is solving equations. It often seems important to point out to students that an equation may have no roots, some roots, or that every number (among the numbers under consideration) may be a root. After the extreme cases of "no numbers are roots" and "all numbers are roots" are investigated the situations in between where only some numbers work are the interesting ones. However, such equations are often quite difficult to solve.

A linear equation has one well behaved root. Quadratic equations will have two roots but they are not necessarily well behaved unless the equation is constructed with care. If a student were so bold as to supply his own coefficients for a quadratic equation, he would be likely to get unpleasant irrational roots or even complex roots.

Of course to deal with equations having several roots one can restrict the equations to factored form. Thus the equation

$$x(x-2)(x+\tfrac{1}{2})(x+7) = 0$$

has roots 0, 2, $-\frac{1}{2}$, -7. This is an interesting technique for "making up" equations but woe to the class where a student constructs an equation in this fashion with a half dozen roots and then multiplies out the factors and presents it to the class for solution!

The theory of polynomial equations involving various astute detective methods for searching out roots has been a commonly studied subject at the high school and early college level but its popularity seems to be waning.

Nowadays, of course, one can have the friendly neighborhood computer chase down the roots of such equations.

* * *

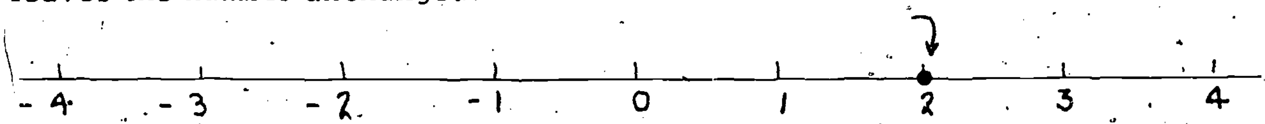
It is the purpose of this paper to suggest that equations of manageable difficulty but with interesting collections of roots can be introduced into the curriculum by the use of a common mathematical function not ordinarily dealt with at the junior high school level.

Consider the "greatest integer not greater than" function. Perhaps a less twisty name for this function is "the integer part of" although it is not so clear what the integer part of a negative number is. Whatever words are used to name it otherwise, we shall here call it "Square Brackets" and use brackets like these [ ] to symbolize it. The whole notion of this function would be communication to students by numerous examples such as:

$$[5] = 5 \qquad \left[3\tfrac{1}{3}\right] = 3 \qquad \left[4\tfrac{9}{10}\right] = 4$$

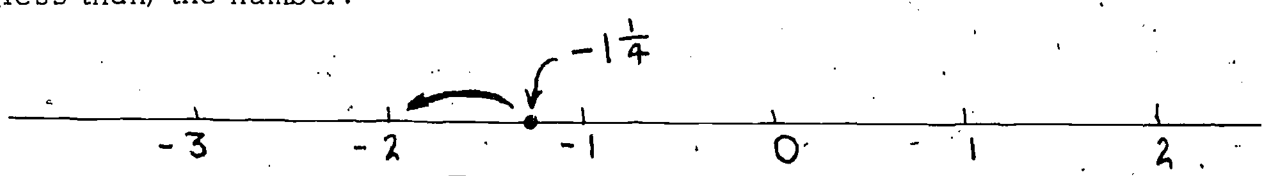
$$\left[-1\tfrac{1}{4}\right] = -2 \qquad \left[-9\tfrac{4}{5}\right] = -10 \qquad \left[-\tfrac{1}{2}\right] = -1$$

A reasonable description of this function for a school student would involve a number line. There are two cases. Square Brackets of an integer leaves the number unchanged.



$[2] = 2$

Square Brackets of a number between integers gives the first integer left of (less than) the number.



$\left[-1\frac{1}{4}\right] = -2$

* * *

The main reason for suggesting the Square Bracket function here is that interesting equations can be constructed using it. However, it might be pointed out that Square Brackets has mundane, "practical" uses also.

If you have x dollars and want to buy as many quarts of milk as possible at 35 cents per quart, how many can you buy? Answer:

$$\left[\frac{100x}{35}\right]$$

How much change do you have left? Answer in dollars:

$$x - .35\left[\frac{100x}{35}\right]$$

At a slightly fancier level, a first class letter weighs exactly x ounces. How much postage is required? Answer in cents:

$$-8\left[-x\right]$$

* * *

Now we will consider some equations and their roots. Instead of dealing with such equations in mathematical generality we will skip quickly among specific examples to hint at how one might present such problems to students. Most discussion of pedagogy will be omitted in the interest of brevity.

What are the roots of

Equation I $\qquad [x] = x$

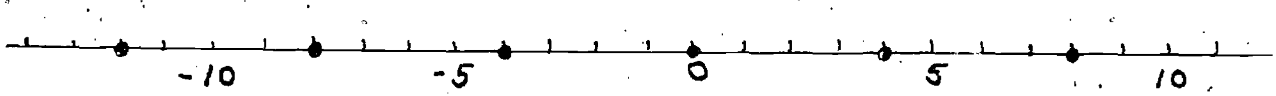
Answer:

All of the integers (and only the integers).

From here on we will give a graph of the roots instead of describing them verbally. Many of the following equations have infinitely many distinct roots and the diagram shown will only suggest the entire set of roots.

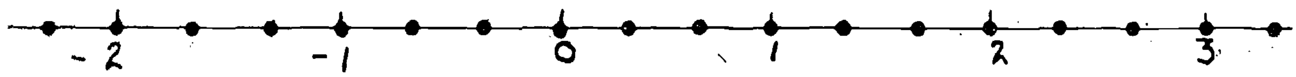
Equation II $\qquad 4\left[\frac{x}{4}\right] = x$

Roots:



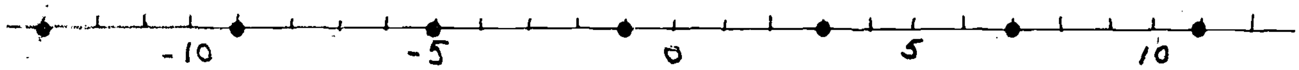
Equation III $$\frac{[3x]}{3} = x$$

Roots:



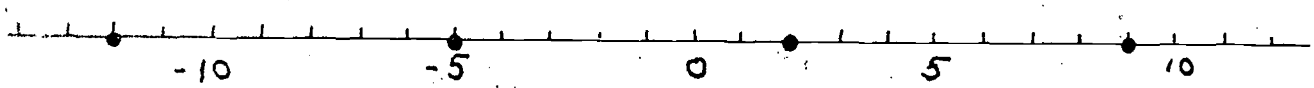
Equation IV $$4\left[\frac{x+1}{4}\right] = x + 1$$

Roots:



Equation V $$7\left[\frac{x-2}{7}\right] = x - 2$$

Roots:



Given any constantly spaced infinite collection of roots (dots), an equation like I - V above can be constructed which will be satisfied by them.

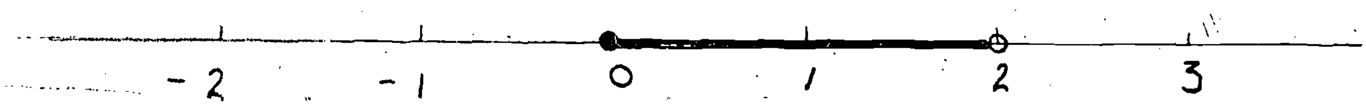
Through a variety of problems like these the student comes to realize that the multiplication-division in the equation determines the spacing between roots and the addition-subtraction determines the actual left-right position of the roots. If one wants to pursue it, the student can begin to get a feeling for transformations which are dilation-contractions and transformations which are (rigid) translations. The whole area of functions, transformations, or mappings which are constructed using Square Brackets is a right extension of this work.

* * *

Another class of equations.

Equation VI $$\left[\frac{x}{2}\right] = 0$$

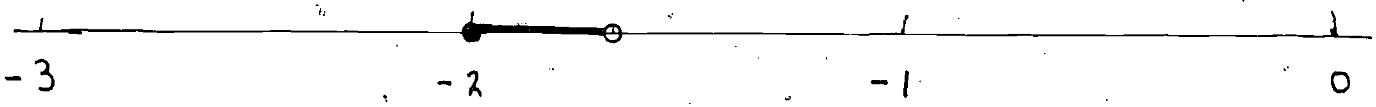
Roots:



The symbols on the diagram above mean that the number 0 <u>is</u> a root and the number 2 <u>is</u> <u>not</u> a root (and, of course, all the numbers between 0 and 2 are roots).

Equation VII
$$\frac{[3x]}{3} = -2$$

Roots:



Given any halfway decent interval an equation along the lines of VI and VII can be constructed which will have that interval as roots (as its "solution set" but we are not sure here that we want to allude to any extensive use of conscious set theory).

* * *

The preceding examples are intended to suggest a considerable body of material that can be covered by average junior high school students. There are many more roads to go down with Square Brackets than have even been hinted at here. Furthermore by combining Square Brackets with Absolute Value (and possibly Signum) another wealth of material arises.

This function <u>can</u> lead to problems which are probably suited only to a very bright and motivated student. Here is one such problem. Its solution will be given without comment. How we stumbled upon equations like this is an interesting story which will be supplied upon request.

Equation VIII
$$x\left[\frac{425}{x}\right] = 400$$

Roots (listed rather than graphed):

$400,\ 200,\ 133\frac{1}{3},\ 100,\ 80,\ 66\frac{2}{3},\ 57\frac{1}{7},\ 50,\ 44\frac{4}{9},\ 40$
$36\frac{4}{11},\ 33\frac{1}{3},\ 30\frac{10}{13},\ 28\frac{4}{7},\ 26\frac{2}{3}.$

(The author does not guarantee that there has not been a typographical error in the listing of these roots but this is the general idea in any case!)

USE OF DATA FROM REAL PROBLEMS IN THE MATHEMATICS CURRICULUM AT THE JUNIOR HIGH SCHOOL LEVEL

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The objectives of the conference — to improve the math skills, quantitative reasoning and the ability to apply mathematics can be approached by introducing curricular materials in which the problems addressed and the means of dealing with them are considerably more real (that is, such as those that individuals and society really face) than those typically used in math courses. A detailed rationale for this was expressed at the 1967 Cambridge Conference and has been applied in a fully real problem-oriented, integrated form by the Unified Science and Mathematics for Elementary Schools Project (USMES). USMES does not expect all of mathematics (or science or social science, or language skills, etc.) to be learned through this fully unified strategy. In secondary schools in particular, the present organizational modes and specialization of teachers inhibits an intense application of that strategy.

An intermediate strategy, always useful, and more readily applicable now, is to use real data as sources for quantitative considerations, without fully intending to treat all as parts of the contextual problem (nor to try to find practical ameliorations of the problem). It is my intention to discuss this possibility here by presenting an example of this mode. Many of these types of activities arise in the context of USMES units. Some have been considered for possible USMES units, but have been put aside for lack of being able to put them into a complete USMES "challenge" context.

Among the examples to be discussed are:

1. Taking sample polls on important local issues (school bussing, cafeteria food, a new road or rapid transit, etc.). The students must decide

on the questions (important variables), wording (reaction and bias) and sampling. They can follow the first sample with others (variance, long run stability, etc.) and at times with the whole relevant population (predictive power). Addition, division and graphing are among the skills involved. Formal elements of probability and statistics may be introduced and tested. Functional relations (variance and sample size) may be developed and used.

2. Making scale drawings to simulate situations to study their modification (streets, intersections and parking lots, assembly halls, cafeterias, playgrounds, etc.). Measurements can be made and remade (obtaining distributions and accuracy estimates). The scaling calculations are done (arithmetic) and the drawing made (angles, lengths, triangular relations, symmetry, etc.). On the drawing, modifications can be made in movable components — (lights, car spaces, chairs, tables, swings, basketball fields, etc. — to observe improvements such as increases in crossing or parking space, seating or corridors. (Arithmetic and geometry, optimization problems.)

VERBALIZATION AS A MEANS OF IMPROVING PROBLEM-SOLVING SKILLS IN JUNIOR HIGH MATH

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It is my position that problem solving must begin in elementary school as soon as the student begins the study of arithmetic. Oral problems can be given to the students; they in turn may tell how to find a suitable solution, or write an appropriate mathematical sentence. Later they can make stories to fit given equations or sets of data. It is interesting to see that many different stories fit the same equation, but all of the stories have a common thread. Children write fine stories — much better in many ways than those that appear in their books.

Oral work in problem solving should also be a part of middle school mathematics. Students should be encouraged to make hypotheses: Will the solution be greater or less? What operations must I use? Do I have enough information? What mathematical sentence may I write? Is my answer reasonable? In the beginning one must be willing to accept all hypotheses without undue criticism. After all, one must begin somewhere. If the problem situations are presented orally, the students may respond by making a note on their paper, so that they may check themselves as some appropriate responses are given. As students become more proficient, or less embarrassed, more interesting situations can be devised.

Problem situations should be presented to groups of students, with written as well as oral reports given to the class on the strategies used in resolving the situation. Other members of the class should be encouraged to offer alternate methods or strategies. Being able to explain — verbalize — not only what you did but why you did it is often more difficult than just cranking out a solution. How many times have you heard a student say, "I don't know how I did it. I just did it." As groups of students discuss possible solutions, and listen to how other groups worked, some kind of learning must be going on.

Another technique I use is to "give dictation." The idea is to give the students practice in translating English sentences into mathematical sentences. I usually start with expressions like "two more than a certain number," "five less than twice a certain number," "three times a number plus five." Notice that the last one depends upon inflection in the voice, or a suitable pause. While these are not "problem" situations, they do resemble the kinds of problems found in many books.

Unfortunately the current emphasis on the structure of mathematics has produced textbook materials that do not offer the student opportunities to practice problem-solving skills. Students do not have the opportunity to try to solve "original" problems — problems that could be a part of their world, their concerns, their interests. Consequently junior high teachers must spend inordinate amount of time searching for interesting situations (they do exist).

Perhaps the "lab-approach" to junior high mathematics is the solution, or at least a step in the right direction. Seeing students get involved solving problem situations, working with geoboards, tangrams, Cuisenaire Rods, Attribute Blocks, mirror cards, etc., will make a believer out of you.

SOME THOUGHTS ON ALGORITHMS AND COMPUTERS

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The idea of an algorithm, and interest in algorithms, predates the invention of computers by several thousands of years. Indeed, it is amusing to note that one of the earliest extant mathematical texts (the Ahmes Papyrus) is essentially a collection of algorithms for solving certain problems. In fact, it appears that prior to the Greeks — with their concern for deduction and abstraction — most mathematics consisted of a search for algorithms. Obviously, most of "arithmetic" is algorithmic in nature. Indeed, the glory (and utility) of our Arabic system of place notation for numbers lies precisely in the fact that it makes the algorithms for elementary computations reasonably easy and manageable. It is, of course, possible to state algorithms for performing the elementary arithmetic operations with Roman numerals, but such algorithms would have an order of complexity far greater than the "Arabic" algorithms we currently struggle to teach children in the elementary school. We might mention two non-computational algorithms that were known to the Greeks: First, the Sieve of Eratosthenes is nothing more than an algorithm for finding all primes between 1 and some fixed number N. Second, consider the Euclidian algorithm for finding the greatest common measure of two given line segments. This algorithm is particularly interesting. It will terminate if and only if the two given line segments are commensurable, and, in that case, it will produce their greatest common measure. On the other hand, if the given segments are incommensurable, then it will not terminate. Thus, as early as the Greeks, it was known that there are algorithmic procedures that, for given inputs, may go on forever.

With the advent of the digital computer, our interest in algorithms has taken a quantum jump upward. The reasons for this are simple: (1) computers are used for an enormous variety of tasks in contemporary society, and (2) computers can only carry out algorithmic procedures — they cannot do anything else. We can characterize our situation vis-a-vis computers somewhat more picturesquely as follows: Computers constitute a very large and important servant class in our society. If we are to use this servant class effectively we must first know what they can and what they cannot do (i.e., we need to understand the nature of algorithmic procedures). Secondly, we must be able to "talk" to the computer-servants in order to convey to them what they are to do for us (i.e., we must have a facility for using algorithmic languages). Our linguistic situation vis-a-vis computers is somewhat analogous to the situation of the Russian aristocracy before World War I. That aristocracy rather disdained the use of the Russian tongue and much preferred to talk French among themselves. Nevertheless, if they wanted to convey their wishes and instructions to their often illiterate servants, they were required to do so in Russian. Thus a working knowledge of Russian was probably indispensible even to the most "effete" francophilic Russian nobleman. We are in a somewhat similar position today: As much as we may dislike talking to the computer, we must do so if we are to make it perform its chores. Our only alternative is to give the problem to an expert (computer programmer) and have him do the talking for us. But it is generally a bad business to give a problem to someone else unless we have at least a general idea of how he is going to solve it for us. Without such an idea it is difficult to know is reasonable to expect, and it is also often difficult to interpret the answers we get.

In addition to the very direct practical reasons discussed above for teaching people to deal with algorithmic processes, there are intellectual advantages to learning about algorithmic languages. To learn a computer language — whichever one it may be — is to learn a new language. Now, we have believed for a very long time that learning a foreign language is not only a directly practical thing, but is also intellectually broadening. The

French language, for example, forces us into quite different categories and modes of expression than we are accustomed to in English. These different categories and modes of expression will certainly result in our looking at the world from a new and different perspective. Perhaps most important, learning French will give us insight into our own language, English. By getting outside of English, and looking at it from the perspective of, say, French, we shall begin to understand more deeply our mother tongue. I believe that what I have said here about French also applies to any non-trivial computer language. As we try to talk to the computer in its language, we shall begin to see things in a quite different way. Since English is not an algorithmic language, we shall soon discover that things simple to say in English may become very complex in the computer language, and things complex in English may become much simpler. Our whole perspective on problems, and how big problems are put together from smaller problems, will change as we translate from ordinary English to an algorithmic language. Thus, I would argue that the learning of algorithmic languages — quite apart from any immediate practical utility — should serve as a very broadening intellectual experience, very much like the learning of French. It is certainly true that no computer language comes even close in sophistication to a natural language like French. But, in a way, I think that therein lies precisely the strength of such languages to serve as the vehicle for important intellectual experiences. Computer languages are characterized by paucity and precision when compared to natural languages. What is interesting and surprising about computer languages is two-fold: On the one hand, how much one can get the computer to do, if one is sufficiently clever and patient; on the other hand, how incredibly complex and difficult it is to state certain tasks in these languages that are almost trivial to state in English. Thus, I submit that by teaching children to use and cope with simple algorithmic languages we shall be providing them not only with useful tools, but with important broadening intellectual experiences.

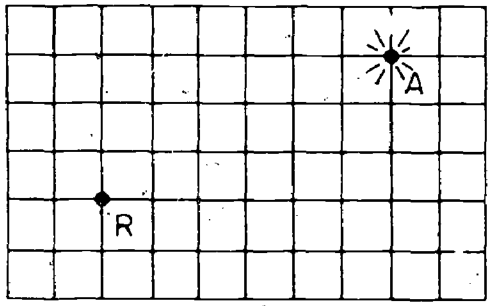
Let us look now a little more closely at what is involved in programming a computer. A computer is a device that can execute (i.e., perform) a certain number of fixed instructions (stated in a prescribed language) that are fed to it by means of some input device. A programmer, on the other hand, is a human being, who has some more or less complex task in mind that he would like the computer to execute for him. The programming task is now two-fold. First, the human being (programmer) must make it completely clear to himself what the task is. Frequently, what we think we want to have done is not really what we want. Perhaps the most telling examples of this proposition are the many fairy tales dealing with the story of the three wishes. Our first problem then as programmers is to make clear to ourselves precisely and unambiguously what the task is. This is often a surprisingly difficult thing to do because in our ordinary intercourse with other human beings we rely on an enormous body of unspoken presuppositions. For example, if there is a knock on the door, and I say "come in," I do not expect the hearer simply to batter down the door. I need not say "turn the door knob, open the door, and come in," because there are well-established, unspoken conventions that govern the proper reaction to "come in." As soon as we wish to talk to computers, however, this whole enormous set of unspoken, often unconscious, presuppositions is no longer at our disposal. In the world of computer algorithms, nothing "goes without saying" -- everything must be clearly defined and spelled out, all possible non-desirable alternatives must be disposed of.

The clarification and specification of the task is computer-independent. It must be done, and can, at least in principle, be done independent of the computer that is to carry out the task. Now the programmer faces his second problem -- to specify the defined task in the particular language(s) his computer can "understand." This is a problem of task-analysis and comes essentially to this: to break down the given task into a series of subtasks, each of which the computer can execute. Whether this can or cannot be done depends on the given task and the richness of the given computer language. For example, suppose we have a computer that that can only add integers (i.e., given a and b it can form a + b). If we give it two integers a and b and ask it to find their difference, a - b, this is impossible. On the other hand, if we have a computer that can only subtract, then we can use it to find sums, since $a + b = a - ((b - b) - b)$. It seems to me that at some time in the curriculum it might be appropriate to give students tasks that cannot, in principle, be done on a given machine. I suggest this because I believe that the idea "in principle, impossible" is an intriguing intellectual concept that certainly plays an important role in mathematics. Of course, in addition to questions of principle, there are also practical considerations: Supposing that a given task is in principle programmable on a given computer, there remains the question whether it is practically possible to do so, e.g., how long will the computer require to do the given problem? It is also important to note that the same task can be programmed in many different ways, some of which will be much more "efficient" than others. The art of programming consists, of course, precisely in finding "efficient" programs. I believe that the kinds of task-analysis required for programming are an important and useful intellectual exercise and should thus have a place in the school curriculum.

As a teacher of mathematics, I am most particularly interested in the utility of teaching algorithmic thinking for mathematics itself. I think there are reasons to hope that the "pay-off" might be very great. One of the most important things we ultimately try to teach in mathematics is the idea of proof. A mathematical proof is, of course, a convincing argument, but let us look at the idea of a strictly formal proof. Here we have a sequence of steps, each of which can be obtained from the preceding steps by clearly stated, logical rules of inference. In this sense, a proof is just like an algorithmic procedure. The purpose of a formal proof is to get from the hypothesis to the conclusion — using only previously agreed on rules of inference. The correctness of a formal proof can be checked by a computer; indeed, if we list the "reasons" for each step, then the sequence of "reasons" is essentially an algorithm for generating the proof itself. Now we know that many students find the idea of proof very difficult. I am not speaking of their ingenuity at finding proofs for difficult theorems; I am simply talking about the idea of what is involved in a proof. I suspect that one reason for this difficulty goes back to something I said before: we are just not in the habit of talking in the nit-picking detail that mathematics requires. It is not natural to think logically; rather it is a habit of mind to be acquired. But now, in the case of writing proofs, the student sees no need for all the fuss. The proof is, at best, read by his teacher, and he can claim with some injustice, that "she knows what he means" even if he hasn't said it. Moreover, to submit a wrong proof has no practical consequences whatever (except, perhaps, a low grade in the course!). All these motivational problems disappear in teaching programming. The student can easily accept the fact that the computer "doesn't understand" — it is not at all like the teacher "playing dumb." Further, if the program is incorrect, the computer will not perform the desired task — there is no room whatever for quibbling about what is right and what is wrong. Finally, the art of "debugging" an incorrect program is in itself a valuable learning experience. To summarize: I have great hope that early experiences with algorithmic procedures will serve to make the idea of mathematical proof much more accessible to students.

To conclude, I think it is very important to gather together a wide variety of interesting and challenging problems for the students to do. I think there is no point whatever in talking to students about these matters in the abstract. Their feeling for algorithmic procedures, for task-analysis, for precision of language will come only as a function of doing problems. By way of illustration, I cite two amusing problems that I heard about several years ago from my colleagues in computer science.

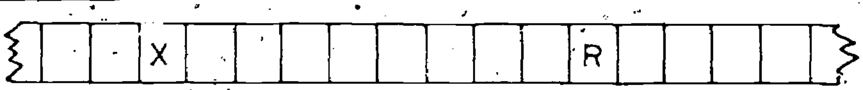
Problem 1: Consider an infinite, rectangular grid as indicated below:



At point A of the grid is located a light bulb. At an arbitrary point R is located a robot. The task is to get the robot to walk to the light bulb at A and stop there. The robot has the following instruction repertory:

N:	Take one step north
W:	Take one step west
E:	Take one step east
S:	Take one step south
I(k):	If light is more intense now than at preceding location. jump to instruction at memory location k. Otherwise, go to next instruction.
J(k):	Jump to instruction at memory location k.
S:	Stop

Problem 2: Consider an infinite strip of tape as indicated below:



One cell there has been marked with an "X". On another cell is a robot R. The task is for the robot to find the cell marked "X" and stop. As shown below the robot can make and erase its own X's. For reasons of elegance, the problem further requires that at the end no X's other than the original one remain on the tape. The robot has the following instruction repertory:

L:	Take one step to the left
R:	Take one step to the right
X:	Make an X on the current cell
Y:	Erase any X on the current cell
J(k):	Jump to instruction at memory location k
I(k):	If there is an X on the cell, jump to instruction at memory location k. Otherwise, go to next instruction.
F:	Stop

Remark: the interesting part of this problem is that the robot does not know whether the X is to the left or the right of his starting position.

NOTES FOR A COLLOQUIUM ON JUNIOR HIGH MATHEMATICS

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I. Most of what's wrong with junior high mathematics is the same thing that's wrong with junior high.

My first memory of math in school goes back to the 4th grade. By then I had already sized up what school was all about. It was a game which the teacher played with us — the old "I've got the answer" routine. The way she played it was to ask some kind of silly question that she already knew the answer to, and we were supposed to act interested and give her the answer. It was no different in math, history, geography or reading. When was America discovered, what is the capital of South Africa, who wrote Kim and 2 + 2 is what, are all the same game. Kids don't believe in their hearts that the questions are serious; the only reason they play it is for the sake of social or psychological comfort.

Certainly there is no quick cure for the American educational system; it seems to be suffering from a terminal disease. The artificial leaning situation of fake questions and answers ignores what seems to me to be a basic axiom: an answer given to a student before he asks the question himself is for him useless information. Providing questions that we would like for him to ask is pretty suspect too. The student sees the whole situation rigged so that in the end the teacher gets to what he already knew was the answer. The answer then becomes useless, and we are surrounded by information all the time — most of it as useless as the fillers in the newspaper. School is like trying to discover the news and only getting fillers.

The problem is how does one so arouse students that they generate their own questions. I do not know the answer. Perhaps only master teachers can manage it. But it does seem that if students generate their own questions they must be permitted to ask them in whatever order they wish. The textbook makes this virtually impossible. "We can't take that up now. You'll get to it next year," I seem to hear echoing from my past.

II. Mathematics is not normally presented as a way of structuring the world or one of the tools to use in making sense of our experience.

I ran away from school the second day of the 1st grade. Something was wrong with the whole set-up. How could you learn anything that mattered behind brick walls and chain link fences? What I wanted to know was how to make sense of my experiences and the world around me. Watching carpenters build a house seemed, at the time, more educational than learning about Dick and Jane. Not only that but there was a magic about the carpenters rule and square that held my curiosity. Learning to write numbers down and add up columns of figures made no sense out of anything at the time. This is perhaps the crux of the whole issue. Once you are inside the classroom, math is no longer operating in the world.

It seems to me that all our systems of knowledge are simply alternate means of naming and structuring the world. Each system has its own features that make it attractive and only partial. And of course each system overlaps the others since it is, after all, one world. For example, one of the few primary means of knowing something about our world is by measuring it. This is not the exclusive domain of mathematics. Historians measure time and events by dates and "before" and "after". Poets have their own sense of measure — meters and feet — overlapping into the world of dance.

Here, I am reminded of what opportunities were lost while I was in junior high mathematics. Then measurements and proportions were matters of anguish and desire. I grew out of that period without ever discovering that I was fascinated by the world in a way that mathematics could make sensible. That seems to be part of what is missing in math education. Numbers have a magical relationship to the world. They are metaphors in the same way that roses are in poetry. When you put together a string of metaphors, you make a model of the world that tells you more than you thought to ask. When you put together a series of numbers or equations, here too is another model for the world. A model always says more than was intended; it speaks as much about itself as it does about what it replicates. It is a spell or an incantation that orders the things and events of the world according to the desires and powers of the wizard and whatever magic he controls.

On this point it seems to me mathematics teaching could be improved if other disciplines were brought in as parallels to show that math has been with us in the world as an alternate tool a long time. What the Greeks and the Egyptians and Arabs discovered and believed about the relationship between numbers or geometric shapes and the world would not only stimulate curiosity, but perhaps make history, art, poetry, philosophy and math sister arts as they once were. Imagine a few weeks investigating Pythagoras — the triangle, the proportions, music, art and Greek civilization.

But beyond the purely pedagogical concern of making math relevant is the question of the use of the mind. To solve a problem requires creating a structure such that something falls out of the whole that will satisfy us as being sufficient to stand for the whole. Of course most problems are not new to the human race, and the structures already accumulated over the centuries to solve them operate quite well. As a result we usually hurry to provide the solutions, the answers, knowing that these work and work well. But what is lost for the student is the creative act and the satisfaction that comes of bringing one's mind into the structure and order of the universe. The associative law is a remarkable structure, but to require a student to memorize it before he has discovered it is to deprive him of one of the pleasures of being human. In fact, it seems that Math teachers should incorporate some of the findings of creativity studies in their teaching. "Synetics," for example, suggests that in problem-solving better solutions come if you probe into the problem rather than struggling for the solution. I remember geometry in this way. If you looked at your triangle and thought about how it was made, all sorts of things became apparent. Geometry was the high point of my math career, for in that class it was possible to create a "proof" the teacher had never seen. That was a thrill! I even read a book about Pythagoras and Euclid.

If a student is allowed or encouraged to creatively engage in mathematics, he will soon learn the peculiar orders and properties of that model of the world. Math is one way of speaking about the world, and like any language it has characteristics implicit within it that carry meaning over and above what it says about the world. In terms of linguistics, the syntax or the rules of operation, carry meaning as well as the semantic elements. Here it seems math teachers could make use of Chomsky's generative or transformation grammar. To break a complex sentence down from its surface structure to its deep structure, whether the sentence be the written language or the language of Mathematics, reveals the elegance, the continuity, the creative powers of that language. In junior high I would have been intrigued to discover that beneath the algebra I was puzzled by lay the simple structures of addition, multiplication, division, and beneath that 1, 2, 3, and beneath that my fingers and toes.

III. When the math program ignores that students are physical bodies living in a physical world, it neglects the major portal of knowledge the students possess.

School years, for me, were always interminable. Every summer I dreaded the thought of those long hours of sitting still in the classroom looking at the same walls or out the same windows. In math class you at least got to get up from your seat, but it was only to face trial by chalk. As a result most of my childhood days in class were spent in daydreams. I always dreamed I was <u>doing</u> something. In my dreams my body was of use; in class I never knew what to do with it.

This lies at the heart of education's failure in America. We as teachers are all too anxious to go directly to the abstract, forgetting that the abstract by definition is derived from the concrete. For most of us analysis is the key to learning. To consume a novel so that it changes one's life has no proper entry in the midterm quiz, yet we insist one must be able to abstract the plot, analyze the style and capsule other critical analyses.

The majority of our teaching seems directed at denying the most persistent fact of our existence — our bodies. It was through the 5 senses that we gained consciousness and began to distinguish the things around us. Indeed, all knowledge, no matter how abstruse or abstract, derives from our sensory perceptions, yet in the classroom, and even in the teaching machines, all that is addressed in instruction is the brain. It's a sci-fi horror story.

I believe that teaching by means of abstract concepts is begun far too soon in the schools. We are told that by the time students reach junior high they have reached the stage where they can learn by this method. If my experience was in any way typical, the opposite is true. I could hardly get my mind off my body and the bodies of those around me. It is true that at this age I was shy and awkward, but that was all the more reason for my exploring the world for the surety I wanted. What is needed is a teaching that is grounded in perceptions — sight, sound, touch and perhaps smell and taste.

Math is particularly culpable. It is perhaps the most abstract of all systems of knowledge, but it too is rooted in its origins in perception. As I understand it in most languages numbers of things were at first inseparable from the things — two cows was a different concept from two apples — and only later did man arrive at the concept of pure number. Men built dwellings and ritual structures long before there were texts on geometry and trigonemetric tables. The problem with math teaching is that it suggests to the students that unless you can work it out on paper it can't be done. Fractions, for example, are matters of ultimate concern to a younger brother or sister. He has to know how things are divided if he is going to get his share of the koolaide. Yet when he see "6/12 = ?" he is apt to write his answer "2".

Although improving the writing of math texts may help, the real problem is their approach. Perhaps the best learning device I know of in educational circles is the Cusinaire Rods. They weren't around when I came through, but they are remarkable kits. They are a palpable and visible way of learning math. The color scheme alone is ingenious as well as stimulating. This suggests that the game kit approach has the more to offer than practically any text at the junior high level. A kit with mirrors to a girl who spends hours before them might say more about parallel lines than all of Euclid. To return to geometry, why should geometry be reserved until high school? For students so intrigued with shapes and proportions, angles and curves, the subject of geometry might find an entrance by means of the tape measure. Furthermore, saving geometry for so long tends to limit students to linear or algebraic thinking. Most of us have been at one time or another stumped at solving something algebraically when the geometric approach would have seemed simple.

The advantage of approaching math in this way in these years is that without paper students would be forced to derive a feel for mathematical operations. They would develop a head mathematics, a means of "guess-timation" that would serve them far better than the formulae they usually memorize. Along this line I think some investigation should be made into what is called the "Nuiffield Project" in England. Much of their time is spent in developing this "head math". Students might be asked to measure a soccer field's area but not told or given any standards of measurement. They come back in with a wide variety of answers "333 × 150 of these sticks", or "220 × 85 Cindy's" etc. Area becomes real for them for they have walked it, measured it, and discovered that it was necessary to agree on unit standard of measurement if they wanted to communicate.

IV. Math textbooks usually are difficult to read, offer no relationships to other subjects, have only precarious connections with the living world, seem to be directed at solving quantities of problems rather than understanding, rarely acknowledge that alternate approaches exist or that much can be learned from inconclusive attempts, and concentrate too narrowly on present concerns rather than demonstrating the connectedness of the whole field of mathematics.

Why math textbooks, in the prose parts, have to be so difficult to read I could never understand. Perhaps in the higher reaches as math attempts to become more systematic and inclusive, a philosopher's turgid style has its place, but on the junior high level there is no reason. If my information is correct, math textbooks as a rule are written in language which is designed for that grade level or above while most other texts are designed to achieve the designated grade level (readability) by the end of the term if at all. Most students reading skills fall below the level they have achieved in school, and any reading on their own grade level requires a struggle. Yet math textbooks, while introducing difficult concepts, fail to put these concepts into a language the students can understand. The purpose of a math textbook is to teach mathematics not reading. It would seem that if the prose text were written in a language one grade level below the one the text was designed for, many of what were presumed to be poor math students would be successful.

In English classes, the history of an era is often introduced into the study of a work of literature. In the same way history might refer to a Shakespeare or a Descartes or a Newton. Though it is poorly done, as a rule, there is a general attempt at providing a cross-disciplinary context for most courses. The exception is mathematics. As it is taught, the student might come to believe that mathematics was born like Minerva, fully developed out of the head of Zeus. The pressure of events has in some ways shaped the development of mathematics. In the same way mathematics has had a considerable impact on the world as it developed. To teach a purely conceived math course is to presume that the student's mind has a well defined compartment in his brain that overlaps nothing. Math ought to be frequently related to other subjects, for these associations might improve retention.

One possibility in providing a context for math might come from simple vocabulary studies. When students are introduced to a new term, a moment might be taken to study the Greek and Latin roots and other English cognates. It might also be possible to design a method of diagramming

English sentences so that the "what to do" becomes apparent. However, such a system is probably limited to the artificial sentences in the problems at the end of Sections in the typical math book.

The word problems were for me the most difficult. They were an attempt by the authors to show that math was applicable in the practical world. The attempt failed. The real world does not present its problems in sentence form. The man on the street has to make up the sentences and then solve the problem. The task is to show how one breaks experience down into sentences that are useful in mathematical solutions. Again, I think the problem-solving techniques developed by researchers in creativity are most valuable. These techniques show how one goes about sorting the data of the world for solutions. In order to make such training meaningful very real problems for which no answers are possible in the back of the book, must be presented. This, to me, means that elementary statistics and probability theory should be a part of junior high. Problems of this sort might come from the daily newspaper (school bond issue and its cost) or typical student life (allowances, cost of education, or royalty earnings for a million record hit).

Teachers of mathematics have a sado-masochist tendency. They assign far too many problems causing the student to revert to the answer hunt. With a few well-chosen problems requiring perhaps a descriptive narrative of what and why something was done in arriving at a solution, the teacher can produce better results.

Because of their rigidity textbooks make mathematics a lock-step drill. One concept, since it is written that way, necessarily follows another. Logarithms cannot be taken up until high school yet exponents come much earlier. There appears to be only one way to solve a specific problem, because that is the only way discussed in the book. The method is discussed in the first of the chapter and problems to test if that method has been mastered come at the end. There is no room for curiosity or exploration. Math texts never take dead ends in their problem solving. The student, however, does take dead ends and will for the rest of his life, but he learns nothing from these dead ends except that they are wrong. He is never asked to think for himself; he is only tested. No wonder logical reasoning is not developed. Furthermore, because there exists a "correct" answer in either his or his teacher's book, the student doesn't learn to reason quantitatively to see if his answer is "in the ball park."

Finally, what little I understood of Principia Mathematica by Russell and Whitehead led me to believe that within the elementary operations of mathematics are the means of developing complex operations. Yet, I don't remember sensing any sort of continuity in my math education. Going back occasionally to see the implications of our basic number system is quite helpful in understanding more complex operations. For example, the concept of place value is learned very early, yet I was completely baffled by polynomials. A teacher could have shown me that $ax^3 + bx^2 + cx^1 + dx^0$ could be related to the base 10 system. Then if $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $X = 10$ that polynomial is 1,234 or $1(10 \times 10 \times 10) + 2(10 \times 10) + 3(10) + 4(1)$.

V. Transference from mathematics to other disciplines rarely occurs because math teaching generally has few points of contact with the phenomenal world where all disciplines meet.

As a final word and a summary I would like to consider the problem of transference. In literature translating or transferring a poem from one language to another presents some problems that are quite revealing. Although some elements of the poem can be transferred successfully, it is virtually impossible to carry the full experience of the poem across language barriers. Images are the easiest — "the frail, pink, cherry blossoms." Images are easy because they are made of perceptions, pictures constructed out of sensory impressions of the world. All humans have the language of the senses in common. Actions too are not that difficult — "float like kites in the wind." These are also based on perceptions with the kinesthetic and spatial relations added. Again, this is common to all. But there is always an area somehow inaccessible to the translator which is carried in the very structure of the original language and not in the second language. I would call it a mode or perhaps a mood or model. Whatever it is called, there is a kind meaning inherent in every language, and its very syntax and phonology, that is imbued with the collective lives and meaning of all those who spoke that tongue.

Those who have learned French and German can intuit that in matters of love French is more appropriate, for in the very tones and tenses of French there is an erotic perception of the world. And every language carries in it an approach to the world uniquely different from all others.

If mathematics is a language, as I believe it is, then in teaching it we should remember what language is and how it usually is learned. A language is born of man's attempt to project himself into the world; it is a response to what he finds in the world. He breathes in the air of his world and breathes it out again adding his own voice, his own mark. A language is made of billions of such responses, but all altered so that when he speaks his comrades know to what he is responding to. And as they all speak, each begins to take into cognizance the ways and degrees his neighbor responds to the world. Soon there is a community of language that carries in it the collective meaning of the lives of those who spoke that tongue. When it is taught, the Mother or Father points to the world, calls attention to the sensory perception and then repeats his word. In that experience we are drawn together, there are sensory perceptions, images, even symbols we share, and to acknowledge that we use the same name, a name common to the whole language group.

In teaching math, it seems we should, as much as possible, return to the roots of experience, the sensory perceptions, for there all systems of knowledge, all languages, begin. There we discover that transference is possible. It is difficult to learn from abstractions exclusively, for their relationship to the phenomenonal world we know is dim, and discerning the structures that language has added to phenomenon is even more difficult. The best way to learn French is to go somewhere where French is spoken and try to carry on your life with that language. It is hard to learn French from a book. To learn math best, you must bring its language into your daily life.

TEACHING APPLIED MATHEMATICS AND TRAINING TEACHERS

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My concern is with the special problems arising in designing a teacher training program to teach children who are eventually going to be able and willing to use mathematics. Thus, the emphasis is rather on teaching future users of mathematics than teaching merely for the understanding of mathematics itself.

The starting-off point of my considerations will be an attempt to describe the equipment and training of an applied mathematician. I will then endeavor to deduce from this description an appropriate way of educating children in mathematics, and finally, I will try to infer from that an appropriate way of training their teachers.

I start from the proposition than an applied mathematician is a mathematician. From this, it follows, first, than an applied mathematician is more than a mathematician, and, second, that the training of an applied mathematician is more difficult than the training of a pure mathematician. It also follows from this that I distinguish between an applied mathematician and, for example, a theoretical physicist, even when the applied mathematician is applying his mathematics to physics. The difference between the mathematician and the scientist is most clearly seen when the problem is in the process of being solved. The mathematician may very well wish to take the mathematics further than is required by the strict demands of the problem. The scientist may very well wish to take his speculation further than is justified by the mathematical theory. In this way, each is behaving perfectly properly and each should be encouraged.

The training of an applied mathematician cannot be predicated on the assumption that there exists some proper subset of mathematical knowledge which is proper for the applied mathematician, and whose complement is the exclusive concern of the pure mathematician. This view, though current, is utter nonsense. Of course, the theoretical physicist may very well wish to circumscribe his knowledge of mathematics and is justified in doing so. So, for example, is the econometrician. But the applied mathematician should have available at his disposal a very broad knowledge of mathematics, and certainly should not ignore certain fields of mathematics because currently the main thrust in their development is coming from within mathematics itself. An excellent example of this principle is the recent use by Thom of the theory of fibre bundles and the global theory of differentiable manifolds in his investigations in biology.

Thus, the inference to be drawn is that the student who wishes to have some understanding of the applications of mathematics, must, whereever appropriate, be presented with applications in the course of his study of mathematics. He must, moreover, become familiar with the methodology of applying mathematics. It is probably more important that he understand this methodology than that he learn some area of science, physical or social or biological, to a sufficient level of sophistication to be able to tackle a real problem with the help of his mathematics. Certainly, it is quite inadequate to present applications of mathematical ideas simply as illustrations of the most recently acquired mathematics. This is the conventional procedure, and it is wrong on two counts. First, it falsifies the process of applying mathematics, and second it fails to distinguish between the role of illustration and that of application.

It is, of course, necessary to develop further and more systematically this description of the appropriate way in which applications of mathematics should enter into the curriculum. However, in this talk, I believe I should give greater emphasis to the teacher training problem, and I believe that enough has been said about the end result which is to be achieved to indicate that the problem is a very difficult one. It is, I believe, far easier to train a person to be able to teach mathematics for its own sake than somebody who can teach mathematics with a view to its being used. This comment, while true, might appear ironical to those familiar with the extreme difficulty of training effective teachers of mathematics.

The first remark I would wish to make is that, throughout the teaching of mathematics, procedures can be used which would in fact favor the understanding of the methodology of applying mathematics. That is to say, even where a topic is being taught simply as part of the mathematics curriculum, it can be taught in such a way that its relevance to other parts of mathematics is emphasized. For example, the theory of quadratic polynomials in one variable may be applied to geometrical problems and to maxima and minima problems. These are genuine applications of mathematics, even though the applications happen to be to topics within mathematics itself. By stressing these interrelationships, the teacher will be preparing the student to understand the role played by mathematics in comprehending the world around him.

The second point to be emphasized is that the teacher must be at home with the idea of applications himself. Very many teachers are extremely unsure of themselves when it comes to applications of mathematics. It was the experience of the teacher training panel of CUPM, when it made its site visits recently in order to explain and discuss the recently amended recommendations for the training of teachers of mathematics that the teachers present at those meetings were extremely apprehensive of the responsibility that would fall on their shoulders if they were to carry out the recommendation of the panel to emphasize applications. We were constantly asked to provide examples of applications. When we mentioned the forthcoming publication by Henry Pollak and Gail Young which would provide a compendium of applications appropriate at high school level, many teachers asked when it would appear and whether copies could be made immediately available to the teachers. It was clear that the intention would be simply to use the applications listed by Pollak and Young, and that none of the teachers had in mind the absolute necessity of updating applications, and of finding applications themselves which would be appropriate to the particular interests of the children under their charge.

Third, the teaching of mathematics with a view to applications certainly involves some sort of cooperation with other departments within one school. It is clear that the children must have some knowledge in a scientific area if they are to be able to make an application of interesting mathematics to interesting problems. It similarly follows that the teacher trainee must also be made familiar with certain scientific ideas in order to discuss applications.

Of course, these scientific ideas do not have to be terribly sophisticated, particularly at the more elementary levels. In this respect, I would suppose that the junior high school constitutes the first occasion upon which it is legitimate to try to motivate mathematics by reference to ideas which do not correspond to the immediate interest of the child. That is to say, my own view would be that, although elementary mathematics should also be informed and enriched by frequent reference to nonmathematical situations, those situations should be of natural and intrinsic interest to the child. On the other hand, I would suppose that, starting at the junior high school level, scientific ideas could be introduced which would be substantially beyond the capacity of the child to invent. For example, I would expect that conservation notions could appropriately be introduced at the junior high school level together with such concepts as momentum and energy which would not, I believe, occur in their precise form to even the most intelligent child.

To sum up, it appears that the conclusion is the following. The recommendations of the teacher training panel of CUPM specify the content of courses appropriate to the teachers of mathematics at the elementary level, at the junior high school level, and at the high school level. Already, the achievement of those levels of knowledge and understanding impose considerable difficulties since they are held to require a commitment of time which it is unrealistic to expect with the present situation obtaining in the universities where teachers are trained. Nevertheless, the claim is being made that if the teachers are to be able to teach mathematics in such a way that it will be really useful to the student, and in such a way that the student himself, at the end of his mathematical education, be in the position to use his mathematics in an intelligent way, then something more is required of the future teacher than a mere grasp of the content of the courses which he will receive. In particular, even the junior high school teacher will have to be familiar with the rudiments of some branch of science. My own recommendation would be that the best possible branch of science here is biology, since it concerns matters of clear interest to the child and since fairly elementary mathematics can be brought to bear on it. It is crucial that the future teacher acquire the right attitude towards mathematics and its applications and that he should then be able to communicate his attitude to his own students.

It is not possible for me to be prescriptive as to precisely how this should be done since I lack the necessary experience. On the other hand, it has frequently been argued that the recommendations of the teacher training panel are inadequate precisely because they concentrate on content and do not discuss the many other problems which beset us in trying to design an effective way of teaching mathematics. I do not believe that this criticism is a fair one, since the teacher training panel was asked to concern itself with content and fully recognized the existence of these other very difficult problems. On the other hand, the time is clearly ripe now to give attention to these other problems, and I hope that, by ventilating this particular problem of the teaching of applicable mathematics, I have contributed to the initiation of this "second round." At least it should be clear that it is far harder to teach mathematics which is going to be applied than to teach mathematics as a self-consistent discipline, since the former includes the latter. Thus the naive idea that we can somehow simplify the teaching of mathematics by concentrating on "relevant" mathematics is revealed as opportunistic nonsense.