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ABSTRACT

This report considers a university faculty which is divided into k grades. The total size is to remain fixed, but the proportions in the grades may vary. The problem is to find control strategies which will bring about desired changes in these proportions. This report is confined to investigating what can be achieved by controlling the numbers of new appointments made into each grade. It is assumed that movements within the system and to the outside would be governed by time-homogeneous transition probabilities. A number of theorems are presented showing that not all structures can be attained and that some which are attainable cannot be maintained. Some bounds are given for the length of time needed to achieve the goal when this is possible. A number of suboptimal strategies are proposed and their performance is studied empirically. Suggestions are made for further research. Finally, a nontechnical summary of this analysis is given at the beginning of this report. (Author)

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A MATHEMATICAL ANALYSIS OF STRUCTURAL CONTROL IN A GRADED MANPOWER SYSTEM

D.J. Bartholomew

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PREFACE

This is one of a continuing series of reports of the Ford Foundation sponsored Research Program in University Administration at the University of California, Berkeley. The guiding purpose of this program is to undertake quantitative research which will assist university administrators and other individuals seriously concerned with the management of university systems both to understand the basic functions of their complex systems and to utilize effectively the tools of modern management in the allocation of educational resources.

This paper presents a Markov model of a graded faculty system and investigates preferred policies for achieving a desired balance between faculty of various ranks. The problems associated with faculty promotion and other attrition are a complex confluence of many political, social, economic, and traditional forces which are greatly abstracted for the descriptive purposes of this model.

The research reported in the Paper is not a complete analysis of the vital problem of faculty retention and advancement. However, this is an illustration of the conceptual and computational feasibility of applying sophisticated operations research techniques to important aspects of university management.

ABSTRACT

This report considers a university faculty which is divided into k grades. The total size is to remain fixed but the proportions in the grades may vary. The problem is to find control strategies which will bring about desired changes in these proportions. This report is confined to investigating what can be achieved by controlling the numbers of new appointments made into each grade. It is assumed that movements within the system and to the outside would be governed by time homogeneous transition probabilities. A number of theorems are presented showing that not all structures can be attained and that some which are attainable cannot be maintained. Some bounds are given for the length of time needed to achieve the goal when this is possible. A number of sub-optimal strategies are proposed and their performance is studied empirically. Suggestions are made for further research. Finally, a nontechnical summary of this analysis is given at the beginning of this report.

I. SUMMARY OBSERVATIONS

1. Introduction

The problem addressed by this report is that of maintaining a satisfactory balance between the numbers of faculty in the various grades in an institution of higher education. It has been found, for example, in the College of Engineering at Berkeley, that if present promotion and attrition rates are continued into the future then an unduly top-heavy structure will result. This has been a conclusion reached in many manpower systems and it commonly results from a history of expansion. The relatively high promotion rates which were necessary and desirable during a period of expansion are found to be too large to maintain a satisfactory and stable structure once the expansion has ceased. The management problem which this situation poses is how to maintain the required balance between the grades and, at the same time, maintain adequate promotion prospects to attract and retain high quality staff.

There are three main aspects of the system which are amenable to control as follows:

(a) The promotion and demotion policies.

(b) The losses due to resignation (through depressed salary scales, early retirement benefits, etc.)

(c) The proportions recruited into each grade.

Of these the last involves the least disruption of the system and is thus an obvious candidate for initial investigation. The work reported here is thus concerned with showing what changes in structure can be produced by making changes in the appointment pattern. All other parameters of the system are assumed to be constant.

2. Attainability and Maintainability

A principal conclusion, with important practical implications, is that not every desired structure can be attained. Furthermore, there are some structures which, though they can be attained, cannot be maintained. A good deal of effort has been devoted to trying to characterize these structures and answers have been found to some of the most important questions. For example, a set of structures has been found which cannot be reached no matter where we start. Another set can be reached from any starting point, and there are yet others which can only be reached from some starting points but not others.

3. Optimal Strategies

If a desired structure can be attained, it is necessary to know what appointment strategy should be used. This analysis shows that there is usually a choice of strategies and this raises the question of whether some are better than others. The main criterion of optimality adopted in this report is that of the number of steps (years in the academic context) needed to reach the goal. In general, it does not seem possible to find the minimum number of steps needed but some useful results have been found which enable a lower bound to be calculated. In certain cases it is possible to find an upper bound as well.

4. Sub-Optimal Strategies

In the absence of a general method of finding optimal strategies, it seemed advisable to find ways of achieving more limited objectives. The first class investigated consisted of those which aim to get as near as possible to the goal at each step. There is no single measure of "distance" so a class of distance measures was investigated, each member leading to a different strategy. The performance of these strategies was investigated by making calculations for a three-grade and a five-grade organization. The tentative conclusions to be drawn from these calculations are as follows:

(a) The definition of what is meant by "distance" is not crucial. All the strategies considered produce very similar results.

(b) All of the strategies lead to points which can be maintained. If the goal is not maintainable we shall have to be satisfied with a nearby maintainable structure.

(c) The more grades in the system the longer it takes to reach the goal.

(d) The rate of approach to the goal is very dependent on the size of the loss rates. The higher the loss rate the easier it is to effect changes in the structure.

In order to give an idea of what is involved in implementing the strategies which have been discussed, a verbal description of one of the most successful is given. The Study first computes how many of the existing members of the system will still be present in one year's time and how they will be distributed throughout the system. Next, it calculates the difference between the number desired in each grade and the number who will be left. In general, some of these differences will be positive and some negative. The positive differences are called the "recruitment needs" for those grades. The final stage is to distribute the new entrants to those grades with recruitment needs in proportion to those needs.

Better strategies, which look farther ahead, have been proposed but not investigated in detail.

5. Further Work

The work carried out so far is incomplete at almost every point. Some of the topics in which further work is needed are listed below.

(a) This study assumes that the faculty size is constant in time. The theory should be extended to cover organizations whose size changes over time.

(b) It is not known whether or not the strategies which look one step ahead are near to the optimum. A first step toward the elucidation of this point would be a comparison with those which look two steps ahead.

(c) Some structures are not maintainable but it may be possible to remain near to them for a sufficiently long time. This point needs investigation.

(d) The calculations in this report are based on typical but artificial data. Suitable data collected from various colleges and campuses should be used to test the methods in practice.

6. Implications for Policy

The work described in the report is a step toward a general theory of control for Markov chain models of graded manpower systems. As such, it is primarily a theoretical exercise intended to give some insight into the dynamics of such systems. It was motivated by the problem facing a university whose faculty structure was moving in an unwanted direction and there are several conclusions which can be drawn which have important implications for those faced with that problem. For any organization in which the assumptions of the model apply, the following conclusions hold:

(a) Not all structures are attainable or maintainable by controlling the appointment policy alone. Those which are can be found from results given in the report. Those which are not can only be reached by making changes in the promotion rates or by contriving to alter the leaving rate. The kind of changes needed can also be determined from the results given.

(b) Some changes in structure can be brought about by appropriate variations in the appointment policy. The best way to do this is still an open question but several strategies have been proposed which seem to give satisfactory results.

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II. MATHEMATICAL ANALYSIS

1. Statement of the Problem

The management of a graded manpower system such as a university faculty involves making repeated decisions on appointments and promotions. This paper is primarily concerned with h[illegible] rol the "shape" of the hierarchy by manipulating the pattern of appointments.

The problem may present itself to the management of the system in a variety of ways. Perhaps the commonest arises when a period of expansion comes to an end. It is then discovered that if past promotion rates are continued in the future, the system will show unacceptable growth at the top. This phenomenon has been observed repeatedly in many kinds of organizations. At Berkeley, Branchflower's calculations for the College of Engineering [illegible] the same tendency. The management problem which then has to be faced is what changes to make to bring the system under control. A second situation is where one wishes to prepare in advance to cope with some change in the external environment. For example, if a new campus were to be established which would be expected to attract senior faculty from an existing campus, then the latter would wish to ensure in advance that sufficient people were available who were capable of being promoted to fill the gaps. In both examples, the problem may be stated as follows. At the present, we have a system with a certain grade structure. At time T in the future, we require it to have some other structure. What appointment and/or recruitment policy should we adopt between now and T to ensure that the goal is achieved?

The main parameters of the system which we subject to management control are the following:

(a) The numbers of people recruited into each grade at each point in time.

(b) The numbers of people promoted (or demoted) between grades of the system.

(c) The numbers of people dismissed or induced to leave the system.

In most organizations there is no control over members who decide to leave of their own free will. Control by (a) alone has many attractions. Chief among these is that it has fewer adverse repercussions on those who are already in the system than the other two methods. To control unwanted growth at the top by method (b) involves cutting promotion rates; method (c) is usually reserved for use when all else has failed. It is for these reasons that our main effort will be directed to (a) as a means of control. We shall show that it is a fundamental control variable but that there are severe limitations on what can be achieved by using it. This will lead us to consider how methods (a) and (b) can be combined to arrive at our objectives. We shall not discuss method (c). If this is available, our problem is trivial since a combination of (a) and (c) can always be found to achieve any goal as soon as desired.

2. The Model

To analyze the problem as stated above, a mathematical model of the system is constructed. We shall use the Markov chain model first introduced by Gani [1963] and Young [1961] for use in industrial and educational organizations. These models have been used successfully on many occasions and they include all the basic factures required to describe a university faculty system. Although the Markov chain model is a stochastic model we shall treat it deterministically in this paper. The elements of the model are as follows. We have a system consisting of k grades. At time zero there are N members of the system of which a proportion $x_i(0)$ are in grade i. Changes take place at discrete points in time (yearly in a faculty system). The porportion who move from grade i to grade j at any time is denoted by p_{ij} and the porportion who leave is w_i ; p_{ii} is the proportion who remain in the same grade. It follows at once that

$$\sum_{j=1}^{k} p_{ij} + w_i = 1 \quad \text{for} \quad i = 1, 2, \ldots k \quad .$$

For simplicity we shall assume that the system is maintained at a constant size N by recruiting as many new members at any time as there are leavers. We shall let $p_i(T)$ denote the proportion of new recruits at time T who go into grade i. Finally, let $x_i(T)$ denote the proportion of the N members of the system who are in grade i at time T. Then there is a simple recurrence formula connecting $x_i(T+1)$ and $x_i(T)$ which may be expressed in matrix notation as follows:

$$\underset{\sim}{x}(T+1) = \underset{\sim}{x}(T)\,\underset{\sim}{P} + \underset{\sim}{x}(T)\underset{\sim}{w}'\underset{\sim}{p}(T+1) \qquad (1)$$

where

$$\underset{\sim}{x}(T) = \left(x_1(T),\ x_2(T),\ .\ .\ .\ x_k(T)\right)$$

$$\underset{\sim}{w} = \left(w_1,\ w_2,\ .\ .\ .\ w_k\right)$$

$$\underset{\sim}{p}(T+1) = \left(p_1(T+1),\ p_2(T+1),\ .\ .\ .\ p_k(T+1)\right)$$

$$\underset{\sim}{P} = \begin{bmatrix} p_{11}p_{12} & \cdots & p_{1k} \\ p_{21}p_{22} & \cdots & p_{2k} \\ p_{k1}p_{k2} & \cdots & p_{kk} \end{bmatrix}$$

It is important to note that (1) can also be written in the form

$$\underset{\sim}{x}(T+1) = \underset{\sim}{x}(T)\left(\underset{\sim}{P} + \underset{\sim}{w}'\underset{\sim}{p}(T+1)\right) = \underset{\sim}{x}(T)\underset{\sim}{Q}(T+1) \tag{2}$$

where $\underset{\sim}{Q}(T+1)$ is a stochastic matrix.

The elements of $\underset{\sim}{P}$ and $\underset{\sim}{w}$ could be made functions of T but this would only be appropriate if we wished to treat them as control parameters. We will assume that they are fixed at their current values and see what can be achieved by exercising control over $\underset{\sim}{p}(T+1)$ only. Our problem may now be stated more precisely as follows.

To find a sequence of vectors $\underset{\sim}{p}(T+1)$ such that we pass from the initial structure $\underset{\sim}{x}(0)$ to the goal structure $\underset{\sim}{x}^*$, say, in an "acceptable" way. We deliberately leave open what is meant by "acceptable". (I am not aware of any other work on this problem.) The problem of a stationary $\underset{\sim}{p}$, rather than $P(T)$, was discussed in Bartholomew, [1967]. The results given there have since been extended by A. F. Forbes in a paper to be published.

The elements of the vectors $\underset{\sim}{x}(T)$ must all be positive and sum to one. It is convenient for purposes of exposition to visualize what is happening in geometrical terms. The structure at time T may be represented by a point in k-dimensional Euclidean space. It will lie in the positive orthant on the hyperplane $x_1 + x_2 + \ldots x_k = 1$. In three dimensions this region is an equilateral triangle in the positive octant. This representation leads us to speak of moving from the point $\underset{\sim}{x}(0)$ to the point $\underset{\sim}{x}^*$ and to refer to the path or trajectory of the movement.

3. Two Basic Questions About $\underset{\sim}{x}^*$

The foregoing discussion should not be taken to imply that the problem we have set ourselves is always soluble. Before attempting to find optimum sequences $\{p(T)\}$ we shall first attempt to discover when and whether a solution is possible. We consider this question in two parts:

(a) Is $\underset{\sim}{x}^*$ attainable?

(b) Is it maintainable? That is can we remain at $\underset{\sim}{x}^*$ once we have arrived there?

For practical purposes these questions are unnecessarily restrictive: it will usually suffice to answer

(a') Can we get near enough to $\underset{\sim}{x}^*$?

(b') Can we remain near enough to $\underset{\sim}{x}^*$ for a sufficiently long period of time?

The second two questions are less easy to answer so we shall concentrate on the former but bear the latter in mind.

4. Condition for Attainability

We shall show that some points cannot be reached at all. Hence if $\underset{\sim}{x}^*$ belongs to this set our goal is certainly unattainable. Some other points are reachable from some but not all points. Finally, there are points which can be reached (or, at least, approached arbitrarily closely) from any other point.

We first delineate the set of points which cannot be reached no matter where we start. We do this by determining its complement which we denote by A . The result is contained in the following theorem.

THEOREM 1

$$A = \{\underset{\sim}{x} \mid \underset{\sim}{x} \geq \underset{\sim}{y}\,\underset{\sim}{P} \quad \underset{\sim}{x},\ y \in X\}$$

$$\text{where} \quad X = \{\underset{\sim}{x} \mid \underset{\sim}{x} \geq 0, \quad \underset{\sim}{x}\underset{\sim}{1}' = 1\}$$

Proof

If $\underset{\sim}{x}^*$ can be reached at all it must be reachable from at least one other point in X . That is there must be a $\underset{\sim}{y}$ such that

$$\underset{\sim}{x}^* = \underset{\sim}{y}(\underset{\sim}{P} + \underset{\sim}{w}'\underset{\sim}{p}) .$$

for any such point $\underset{\sim}{x}^* \geq \underset{\sim}{y}\underset{\sim}{P}$ since $\underset{\sim}{y}\underset{\sim}{w}'\underset{\sim}{p} \geq 0$. A is thus the set of $\underset{\sim}{x}^*$'s for which at least one such $\underset{\sim}{y}$ can be found. □

Although this theorem characterizes A it does not make it very easy to determine the boundaries of the set in any particular case. Theorem 2 is more explicit on this point.

THEOREM 2

A is the convex hull of the points with co-ordinates

$$\underset{\sim}{p}^{(i)} + w_i \underset{\sim}{e}_j \quad (j = 1, 2, \ldots k; \quad i = 1, \quad \ldots k)$$

where $\underset{\sim}{p}^{(i)}$ is the vector of elements in the i^{th} row of $\underset{\sim}{P}$ and $\underset{\sim}{e}_j$ is a vector of zeros with a 1 in the j^{th} position.

Proof

Consider first the set of points which can be reached in one step from $\underset{\sim}{e}_i$ $(i = 1, 2, \ldots k)$. Clearly, all of these belong to A.

The k sets reachable from each $\underset{\sim}{e}_i$ will be termed primary regions.
Substituting $T = 0$, $\underset{\sim}{x}(T) = \underset{\sim}{e}_i$ in (2)

$$\underset{\sim}{x}(1) = \underset{\sim}{e}_i\underset{\sim}{P} + \underset{\sim}{e}_i\underset{\sim}{w}'\underset{\sim}{p}(1) \tag{3}$$

$$= \underset{\sim}{p}^{(i)} + w_i\sum_{j=1}^{k} p_j(1)\underset{\sim}{e}_j \quad . \tag{4}$$

The primary region is thus

$$\{\underset{\sim}{x} \mid \underset{\sim}{x} = \underset{\sim}{p}^{(i)} + w_i \sum_{j=1}^{k}\alpha_j\underset{\sim}{e}_j \ , \ \sum_j \alpha_j = 1 \quad \alpha_j \geq 0 \ , \quad i = 1, 2, \ldots k\} \ .$$

Consider next the set of points which can be reached from $(y_1, y_2, \ldots y_k)$.
These co-ordinates may be written $\sum_{i=1}^{k} y_i\underset{\sim}{e}_i$.
Substituting this for $\underset{\sim}{x}(1)$ in (3) now gives

$$\begin{aligned}\underset{\sim}{x}(1) &= \sum_{i=1}^{k} y_i\underset{\sim}{e}_i\underset{\sim}{P} + \sum_{i=1}^{k} y_i\underset{\sim}{e}_i\underset{\sim}{w}'\underset{\sim}{p}(1) \\ &= \sum_{i=1}^{k} y_i\underset{\sim}{p}^{(i)} + \sum_{i=1}^{k} y_i w_i \sum_{j=1}^{k} p_j(1)\underset{\sim}{e}_j \\ &= \sum_{i=1}^{k} y_i\{\underset{\sim}{p}^{(i)} + w_i \sum_{j=1}^{k} p_j(1)\underset{\sim}{e}_j\} \quad .\end{aligned}$$

Hence, the set of points which can be reached from $\underset{\sim}{y}$ is

$$\left\{\underset{\sim}{x} \mid \underset{\sim}{x} = \sum_{i=1}^{k} y_i\{\underset{\sim}{p}^{(i)} + w_i \sum_{j=1}^{k} \alpha_j\underset{\sim}{e}\},\ \alpha_j \geq 0,\ \Sigma\alpha_j = 1,\ \underset{\sim}{y} \geq 0,\ \Sigma y_i = 1\right\} \ .$$

But any point in this set is a convex combination of points in the primary region. Hence, the totality of points in A is the convex hull of the vertices of the primary region. These vertices are obtained by setting $\underset{\sim}{\alpha} = \underset{\sim}{e}_i$ in (4) for each value of i . This gives the set of points

$$\underset{\sim}{p}^{(i)} + w_i \underset{\sim}{e}_j \quad (i, j = 1, 2, \ldots k). \qquad \square$$

The number of points here is k^2 and not all of them will be extreme points of the convex hull. However, for k small it is a simple matter to determine the region.

EXAMPLE 1

Consider a 3-grade system with the following parameters

$$\underset{\sim}{P} = \begin{pmatrix} 0.5 & 0.4 & 0 \\ 0 & 0.6 & 0.3 \\ 0 & 0 & 0.8 \end{pmatrix} \quad \underset{\sim}{w} = (0.1, 0.1, 0.2) \quad .$$

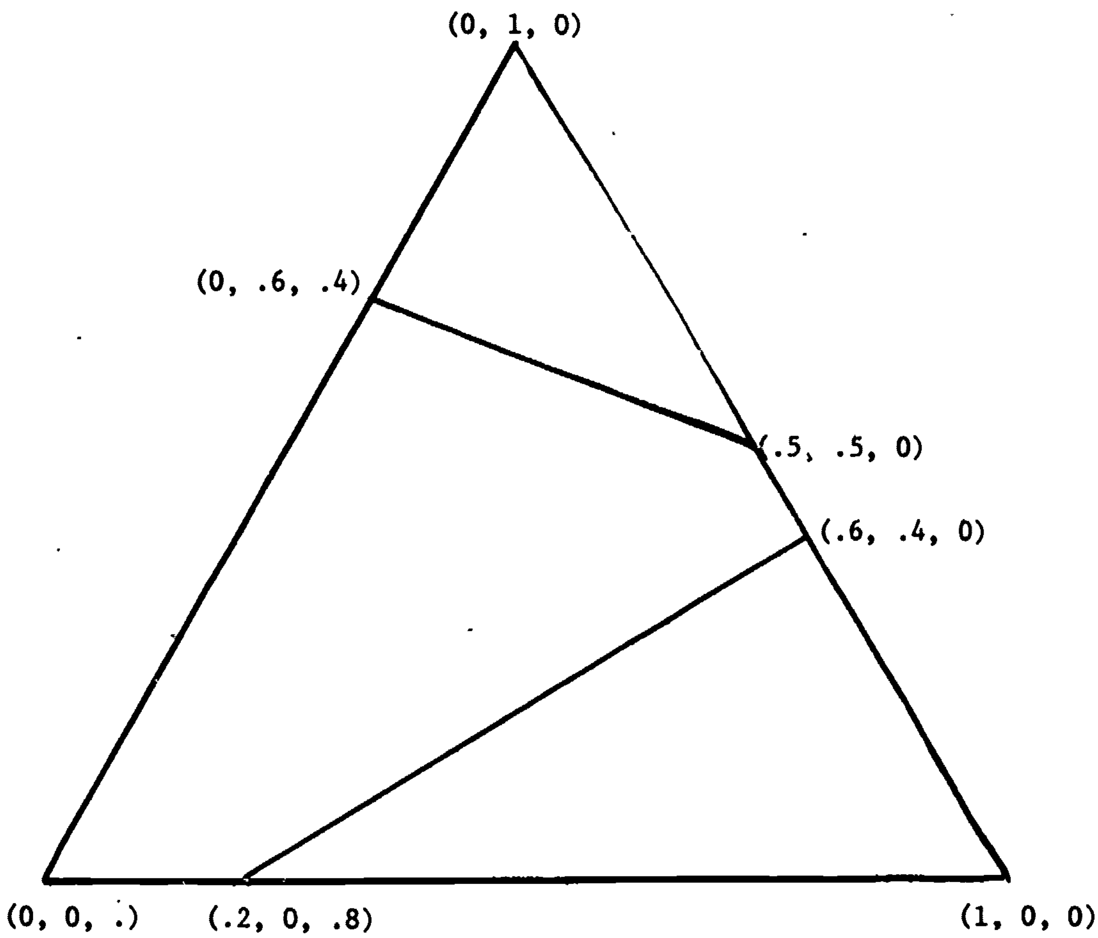
Theorem 2 says that the region A will be the convex hull of the points with co-ordinates

$$(.5 + .1, .4, 0)^*, \ (0 + .1, .6, .3), \ (0 + .2, 0, .8)^*$$
$$(.5, .4 + .1, 0)^*, \ (0, .6 + .1, .3), \ (0, 0 + .2, .8)$$
$$(.5, .4, 0 + .1) \ , \ (0, .6, .3 + .1)^*, (0, 0, 0.8 + .2)^* \quad .$$

In this case the extreme points, which are marked with an asterisk, are easily recognized by plotting them on the triangular plane of the region X as in Figure 1. In higher dimensions the recognition of extreme points is more difficult although algorithms are available. However, the mere inspection of the set of co-ordinates given by Theorem 2 gives an idea of the kind of structures which are attainable.

Condition for Maintainability

This is easily derived as a special case of Theorem 1. If a point



The region A when $k = 3$ and $\underset{\sim}{P} = \begin{pmatrix} .5 & .4 & 0 \\ 0 & .6 & .3 \\ 0 & 0 & .8 \end{pmatrix}$

Figure 1

can be reached from itself then the system can remain there as long as desired. Let S denote the set of maintainable points, then we have

THEOREM 3

$$S = \{x \mid x \geq xP,\ x \varepsilon X\}$$

The recruitment vector required to hold the system at a particular x is easily obtained from (1).

THEOREM 4

If $x \varepsilon S$ then the system can be kept at that point by choosing

$$p = x(I - P)/L$$

where $L = xw'$, the proportion of leavers.

This result leads to a more explicit description of the region S which is given in the following theorem.

THEOREM 5

The vertices of S have co-ordinates proportional to

$$e_i(I - P)^{-1} \quad (i = 1, 2, \ldots k)$$

with the constant of proportionality being determined so that the elements sum to one.

Proof: From Theorem 4.

$$x = xw'p(I - P)^{-1}$$
$$= xw' \sum_{i=1}^{k} p_i e_i (I - P)^{-1}$$

for all points $x \varepsilon S$. Hence, any point in S can be written as a convex ination of the points $e_i(I - P)^{-1}$ multiplied by a scaling factor xw'

to make the elements sum to one. The vertices of S thus have co-ordinates proportional to $\underset{\sim}{e}_i(\underset{\sim}{I} - \underset{\sim}{P})^{-1}$. $\square$

COROLLARY

If $w_i = w$ $(i = 1, 2, \ldots k)$ then the co-ordinates of the vertices are

$$\underset{\sim}{e}_i(\underset{\sim}{I} - \underset{\sim}{P})^{-1}, \ (i = 1, 2, \ldots k) \quad .$$

Proof

The result follows at once on noting that, in this case $\underset{\sim}{x}\underset{\sim}{w}' = 1$ for all $\underset{\sim}{x}$.

THEOREM 6

If $\underset{\sim}{x}^* \in S$ then we can get arbitrarily close to $\underset{\sim}{x}^*$, no matter what our starting point, by adopting the recruitment policy

$$\underset{\sim}{p} = \underset{\sim}{x}^*(\underset{\sim}{I} - \underset{\sim}{P})/\underset{\sim}{x}^*\underset{\sim}{w}' \quad \text{for all} \quad T.$$

Proof

If we apply a recruitment policy which is constant over time we may write (2) in the form

$$\underset{\sim}{x}(T + 1) = \underset{\sim}{x}(T)\underset{\sim}{Q} \quad .$$

$\underset{\sim}{Q}$ is a regular stochastic matrix so that the system will have a limiting structure satisfying $\underset{\sim}{x} = \underset{\sim}{x}\underset{\sim}{Q}$. The limiting structure will therefore be $\underset{\sim}{x}^*$ if we can find a $\underset{\sim}{p}$ such that

$$\underset{\sim}{x}^* = \underset{\sim}{x}^*(\underset{\sim}{P} + \underset{\sim}{w}'\underset{\sim}{p}) \quad \text{with} \quad \underset{\sim}{x}^*\underset{\sim}{1}' = 1 \quad .$$

This equation is satisfied if we choose $\underset{\sim}{p}$ as given in the theorem. $\square$

At this point it is convenient to identify the practical implications of these theorems as listed below:

(a) If our goal vector $\underset{\sim}{x}^*$ is in A it can never be attained by varying the appointment vector alone (Theorems 1 and 2).

(b) If our goal vector $\underset{\sim}{x}^*$ is in $A - S$ it may be attainable but it will not be possible to remain at $\underset{\sim}{x}^*$ once it is attained (Theorem 3). Whether or not we can remain sufficiently near to $\underset{\sim}{x}^*$ remains an open question.

(c) If $\underset{\sim}{x}^* \varepsilon S$ we can ultimately get as close as we please to $\underset{\sim}{x}^*$, no matter where we start. Further we can reach the goal by using a constant appointment policy (Theorems 3, 4, 5 and 6).

In the following section we shall investigate the best way of reaching $\underset{\sim}{x}^*$ given that it is (or may be) attainable. If our goal is not in S there are only two courses open to us. Either we must change the goal so that it can be reached or we must change the parameters of the problem. This amounts to resorting to methods (b) or (c) of Section 1. The means of doing this are provided by Theorems 3 and 4. We must adjust the elements of $\underset{\sim}{P}$, and by implication, the value of $\underset{\sim}{w}$ so that $\underset{\sim}{x}^*$ lies within the new S.

It may happen that the appointment vector needed to maintain the desired structure is itself unacceptable. For example, it may require that appointments be made only at the lowest level, so preventing the recruitment of distinguished faculty at higher levels. In such an eventuality, it must be accepted that the two requirements are incompatible and that one of them, at least, must be relaxed.

5. The Problem of Determining Optimal Appointment Policies

A successful appointment policy is one which achieves the desired goal. An optimal policy is one which does so in the most "satisfactory" way. It will help to clarify the attributes of an optimal policy if we first identify some of the characteristics of an unsatisfactory policy. We suggest that the following features are undesirable in any appointment policy:

(a) a large number of steps to reach the goal.

(b) a series of abrupt changes from one time period to the next.

(c) passing through some undesired structures on the way to the goal.

(d) high cost (in terms of salaries, etc.) ,

The last point may simply be another aspect of (c) since high cost would make some structures undesirable. We shall proceed on the assumption that (a) is the most serious disadvantage and so try to devise strategies which attain the goal in a small number of steps. We shall use (b) and (c) as guides in choosing between strategies which take similar times to reach the goal.

The mathematical problem which we shall try to solve is thus the following: To find the smallest T , denoted by T^* , such that there exists a sequence of vectors $\{\underset{\sim}{p}(T)\}$ with non-negative elements satisfying

$$\underset{\sim}{x}^* = \underset{\sim}{x}(0) \prod_{T=1}^{T^*} \{\underset{\sim}{P} + \underset{\sim}{w}'\underset{\sim}{p}(T)\} \tag{5}$$

If the goal is attainable in a finite number of steps, T^* can always be found, in principle, by direct computation. We would have to set $T^* = 1, 2, \ldots$ in turn and determine at each stage whether or not the equations (5) had a solution in non-negative $\underset{\sim}{p}(T)$'s. The smallest value of T^* for which this was true would be the one required. In practice this is a formidable task so we shall aim here to gain some insight into the nature of the solution by elementary mathematical analysis.

Our results are in the form of bounds on T^* which, in exceptional cases, may serve to determine its value. No computations have been made to investigate the closeness of these bounds; this should have high priority in future work. The following theorems and their corollaries contain the main results.

THEOREM 7

$T^* \geq T'$ where T' is the smallest T for which

$$\underset{\sim}{x}^* \geq \underset{\sim}{x}(0)\underset{\sim}{P}^T \qquad .$$

Proof

If we are to have

$$\underset{\sim}{x}^* = \underset{\sim}{x}(0) \prod_{j=1}^{T} \underset{\sim}{P} + \underset{\sim}{w}'\underset{\sim}{p}(j)$$

for some T then the first term in the expansion of the product on the right hand side cannot exceed $\underset{\sim}{x}^*$ since all the remaining terms are necessarily non-negative. $\square$

It should be noted that $\underset{\sim}{x}(0)\underset{\sim}{P}^T$ is the vector of the proportions of the original members of the system who remain at time T . Hence it is intuitively obvious that the goal cannot be reached until all of these have decreased below the target levels. The result of Theorem 7 is quite general but from this point onwards we shall have to make a further assumption in order to make progress. We shall assume that $w_i = w$ for all i . This means that an individual's chance of leaving is the same no matter where he is in the hierarchy. In practice leaving rates tend to decrease as we move from the lower to the higher ranks. In spite of this restriction we may hope to gain some insight into the nature of the solution which will pave the way for further generalization. After Theorem 8 we shall make the additional assumption that the promotion matrix $\underset{\sim}{P}$ is upper triangular. There is no lack of realism in this assumption since demotions rarely, if ever, occur.

THEOREM 8

$$\underset{\sim}{x}(T) = \underset{\sim}{x}(0)\underset{\sim}{P}^T + w \sum_{j=0}^{T-1} \underset{\sim}{p}(T - j)\underset{\sim}{P}^j \quad .$$

Proof

The proportion of leavers who have to be replaced at time T is $\underset{\sim}{x}(T)\underset{\sim}{w}' = w$ because all the elements of $\underset{\sim}{w}$ are equal to w. Hence, $w\underset{\sim}{p}(T - j)\underset{\sim}{P}^j$ is the vector of the proportions of people recruited at time $T - j$ who survive to time T. The expression on the right hand side in the statement of the theorem is thus the cumulative sum of all survivors, including those who were present at $T = 0$. $\square$

The point of this theorem is that it shows that $\underset{\sim}{x}(T)$ can be expressed as a linear function of the appointment vectors when the loss probabilities are equal. This represents a very considerable simplification which makes it possible to arrive at the following results.

THEOREM 9

$T^* \leq T''$ where T'' is the smallest value of T such that

$$\underset{\sim}{x}^*\underset{\sim}{P}^{-(T - 1)} - \underset{\sim}{x}(0)\underset{\sim}{P} \geq 0 \quad .$$

Proof

The method of proof is to show that $\underset{\sim}{x}^*$ can be reached in T''

steps if the condition given holds. The smallest possible number of steps obviously cannot exceed T" . If the goal is reached after T steps then

$$\underset{\sim}{x}^* = \underset{\sim}{x}(0)\underset{\sim}{P}^T + w \sum_{j=0}^{T-1} \underset{\sim}{p}(T - j)\underset{\sim}{P}^j \quad \text{(by Theorem 8).}$$

Re-write this equation in the form

$$\sum_{j=0}^{T-1} \underset{\sim}{p}(T - j)\underset{\sim}{P}^j = \frac{\underset{\sim}{x}^* - \underset{\sim}{x}(0)\underset{\sim}{P}^T}{w} = \underset{\sim}{y} \quad . \tag{6}$$

Then it may easily be verified that (6) is satisfied by

$$\underset{\sim}{p}(T - j) = \underset{\sim\sim}{yP}^{-j}\left\{\frac{w(1 - w)^j}{1 - (1 - w)^T}\right\} \quad \text{for } j = 0, ., ., . . T \quad . \tag{7}$$

For this to be an admissible solution it is also necessary that all of the vectors $\underset{\sim}{p}(T - j)$ should have non-negative elements summing to one. We first show that

$$\underset{\sim}{p}(T - j)\underset{\sim}{1}' = 1 \quad \text{where } \underset{\sim}{1}' \text{ is the vector } (1, 1, . . 1) \quad .$$

Let $\underset{\sim}{u} = \underset{\sim\sim}{yP}^{-j}$ then $\underset{\sim}{u}$ is the vector which when post-multiplied j times by $\underset{\sim}{P}$ gives $\underset{\sim}{y}$. Since the loss rates are equal, each multiplication by $\underset{\sim}{P}$ reduces the total size by a factor $(1 - w)$. Therefore,

$$\underset{\sim\sim}{u1}' = (1 - w)^{-j}\, \underset{\sim\sim}{y1}' \quad .$$

But

$$\underset{\sim\sim}{y1}' = \left(\underset{\sim}{x}^*1' - \underset{\sim}{x}(0)\underset{\sim}{P}^T1'\right)/w = \{1 - (1 - w)^T\}/w \quad .$$

Therefore,

$$\underset{\sim}{y}\underset{\sim}{1}' = \{1 - (1 - w)^T\}/w(1 - w)^j \quad .$$

Substitution in (7) now yields the required result.

The requirement that the elements should be non-negative is that

$$\underset{\sim}{y}\underset{\sim}{P}^{-j} \geq \underset{\sim}{0} \qquad \text{for } j = 0, 1, 2, \ldots T - 1$$

which, substituting for $\underset{\sim}{y}$, may be written

$$\underset{\sim}{x}^*\underset{\sim}{P}^{-j} - \underset{\sim}{x}(0)\underset{\sim}{P}^{T-j} \geq 0 \quad , \quad j = 0, 1, 2, \ldots T - 1 \quad . \tag{8}$$

Suppose that these inequalities are satisfied for $j = j_o$ then it follows that they are also satisfied for $j = j_o - 1$ since a vector of non-negative elements multiplied by a matrix with non-negative elements yields another non-negative vector. Hence it is sufficient for (8) to be satisfied with $j = T - 1$ and this gives the condition of the theorem. $\square$

Note that condition (8) is always satisfied for $j = 0$ if $T \geq T''$ (see Theorem 7). Further insight into the nature of the conditions can be obtained by writing out the first two elements in the vector inequality

$$\underset{\sim}{x}^*\underset{\sim}{P}^{-(T - 1)} - \underset{\sim}{x}(0)\underset{\sim}{P} \geq 0 \qquad .$$

These are

$$\left.\begin{aligned} \underset{\sim}{x}_1^* p_{11}^{-(T - 1)} - \underset{\sim}{x}_1(\theta)p_{11} &\geq 0 \\ \underset{\sim}{x}_1^* q_{12}^{[T - 1]} + x_2^* p_{22}^{-(T - 1)} - x_1(0)p_{12} - x_2(0)p_{22} &\geq 0 \end{aligned}\right\} \tag{9}$$

where

$$q_{12}^{[T-1]} = -p_{12} \sum_{j=0}^{T-2} p_{11}^{j-T+1} p_{22}^{-j-1}$$

is the $(1, 2)^{th}$ element in the $(T-1)^{th}$ power of $\underset{\sim}{P}^{-1}$. The first inequality can obviously be satisfied by making T sufficiently large. However, the same is not true in general for the second inequality because x_1^* and x_2^* have coefficients of opposite sign and comparable magnitude.

If a value of T'' can be found then (7) gives a sequence of appointment vectors which will achieve the goal in T'' steps. This may not be an optimum sequence but if T'' is acceptably small the strategy is well worth considering. It is the only one we have for which the number of steps to reach the goal is known in advance but it relates only to the case of equal loss probabilities.

The strategy implied by (7) has the following interpretation. It requires us to choose $\underset{\sim}{p}(T - j)$ in such a way that the proportions of those recruited at time $T - j$ who will remain at time T is proportional to $\underset{\sim}{y}$ for all j. The vector $w\underset{\sim}{y}$ may be described as the "recruitment needs" at time T and so the strategy requires us to select appointment vectors in the light of what the recruitment needs will be at time T.

Our remaining result is much more limited in its scope. Instead of trying to reach the goal in every grade we merely try to meet the goal in the lowest grade only. In this special case a complete solution can be achieved if $\underset{\sim}{P}$ is upper triangular and $w_i = w$ for all i.

THEOREM 10

Let T_1^* be the smallest number of steps in which the target can be reached in grade 1, then T_1^* is the smallest T for which

$$0 \leq \frac{x_1^* - p_{11}^T x_1(0)}{w} \leq (1 - p_{11}^T)/(1 - p_{11}) .$$

Proof From Theorem 8

$$x_1(T) = p_{11}^T x_1(0) + w\{p_{11}^{T-1} p_1(1) + p_{11}^{T-1} p_1(2) + \ldots + p_1(T)\} .$$

We can only make $x_1(T) = x_1^*$ if x_1^* is between the minimum and the maximum of the right hand side of the equation. It is obvious that the minimum occurs when

$$p_1(1) = p_1(2) = \ldots = p_1(T) = 0$$

and the maximum when

$$p_1(1) = p_1(2) = \ldots = p_1(T) = 1 .$$

Hence, at least one solution in positive $p_i(j)$'s exists if

$$p_{11}^T x_1(0) \leq x_1^* \leq p_{11}^T x_1(0) + w\{p_{11}^{T-1} + p_{11}^{T-2} + \ldots + 1\} \tag{10}$$

which is equivalent to the inequalities given in the statement of the theorem. $\square$

Corollary 10.1

The goal in grade 1 can ultimately be reached if

$$\underset{\sim}{x}_1^* \leq \max\left(w + p_{11} x_1(0),\ w/(1 - p_{11})\right) .$$

Proof

Re-write (10) in the form

$$p_{11}^{T} x_1(0) \le x_1^* \le w/(1 - p_{11}) + p_{11}^{T}\{x_1(0) - w/(1 - p_{11})\} \quad .$$

Then if $x_1(0) > w/(1 - p_{11})$ the upper limit is a decreasing function of T and so if the inequality is satisfied at all it is satisfied for $T = 1$. This gives

$$x_1^* \le w + p_{11} x_1(0) \quad .$$

If $x_1(0) \le w/(1 - p_{11})$ the upper limit is a non-decreasing function of T with maximum value, occurring in the limit as $T \to \infty$, on $w/(1 - p_{11})$. In the limit, the lower inequality is always satisfied and so the result follows. $\square$

Corollary 10.2

Theorem 10 provides a complete solution to the problem of finding T^* when $k = 2$ and shows when a solution exists. (Triangular $\underset{\sim}{P}$ and $w_i = w$ for all i).

The geometric form of the two limits of Corollary 10.2 shows that the limiting values will be attained quite rapidly for typical values of $p_{11}(0.6 - 0.7)$. This means that if the goal can be achieved at all it is likely to be reached quickly.

Corollary 10.3

$$T^* \ge T_1^* \quad .$$

In this section we have defined an optimal policy as one which reaches the goal in the smallest number of steps. At the present time it does not appear computationally feasible to determine the optimal policy in this sense in any but very special cases. Nevertheless, the theorems which we have given provide some information about the value of T^* and hence provide a yardstick with which proposed policies may be compared. In the remainder of this report we shall be considering some strategies which seem intuitively reasonable. In our suggestions for further work in Section 7 we shall propose that they be evaluated in part by the criterion which forms the basis of this section.

6. Strategies Which Look One Step Ahead

We have seen that it is not possible, in general, to find strategies which will make the desired change of structure in the minimum number of steps. However, it is possible to devise strategies which have certain sub-optimal properties. In this and the following section we shall describe some such strategies and investigate their performance.

The first class of strategies is designed to get as near as possible to the goal in one step. In the short term this is an attractive property but it may have to be paid for in the later stages of the transition. For a system starting at $\underset{\sim}{x}(0)$ we have to find a point $\underset{\sim}{x}(1)$ which can be reached from $\underset{\sim}{x}(0)$ and which is such that the distance from $\underset{\sim}{x}(1)$ to $\underset{\sim}{x}^*$ is a minimum. Having reached $\underset{\sim}{x}(1)$ so that then becomes the starting point for another step, and so on. There is, of course, no unique measure of "distance" in this context, and different measures will lead to different strategies. A class of reasonable measures of the distance from $\underset{\sim}{x}(1)$ to $\underset{\sim}{x}^*$ is given by

$$D = \sum_{i=1}^{k} | x_i^* - x_i(1) |^a , \quad a > 0 .$$

We shall investigate the case $a = 1$ and $a = 2$. First we prove two theorems about all members of this class.

THEOREM 11

If $\underset{\sim}{x}^*$ can be reached in one step from $\underset{\sim}{x}(0)$ then a strategy minimizing D will do this.

Proof

The result is obvious because if $\underset{\sim}{x}^*$ can be reached in one step there must exist at least one $\underset{\sim}{x}(1)$ which makes D zero. $\square$

THEOREM 12

The vector $\underset{\sim}{p}$ which minimizes D also minimizes

$$D' = \sum_{i=1}^{k} |y_i - p_i|^a , \; a > 0$$

where

$$\underset{\sim}{y} = \left(\underset{\sim}{x}^* - \underset{\sim}{x}(0)\underset{\sim}{P}\right)/\underset{\sim}{x}(0)\underset{\sim}{w}' \quad .$$

Proof

$$\begin{aligned}
\underset{\sim}{x}^* - \underset{\sim}{x}(1) &= \underset{\sim}{x}^* - \underset{\sim}{x}(0)(\underset{\sim}{P} + \underset{\sim}{w}'p) \\
&= \underset{\sim}{x}(0)w\left\{\frac{\underset{\sim}{x}^* - \underset{\sim}{x}(0)\underset{\sim}{P}}{\underset{\sim}{x}(0)\underset{\sim}{w}'} - \underset{\sim}{p}\right\} \\
&= \text{constant } \{\underset{\sim}{y} - \underset{\sim}{p}\}
\end{aligned}$$

Hence, any function of the differences $x_i^* - x_i(1)$ will be proportional to the same function of the differences $y_i - p_i$. $\square$

This theorem has an interesting interpretation. If we were to take $\underset{\sim}{p} = \underset{\sim}{y}$ then it may easily be verified that the goal would be reached in one step. However, we cannot do this in general because the elements of $\underset{\sim}{y}$ are not necessarily all positive. Theorem 12 leads us to choose an admissible $\underset{\sim}{p}$ which in a certain sense is nearest to the inadmissible $\underset{\sim}{p}$ which would have taken us to the goal in one step.

It would be useful to know something about the conditions under which strategies minimizing D will reach or converge on $\underset{\sim}{x}^*$. The following conjecture is supported by intuition and calculation but a satisfactory proof is lacking.

CONJECTURE

If $\underset{\sim}{x}^* \varepsilon S$, strategies minimizing D will ensure that $\lim_{T\to\infty} \underset{\sim}{x}(T) = \underset{\sim}{x}^*$. If $\underset{\sim}{x}^* \notin S$, $\underset{\sim}{x}(T)$ will reach, or converge to the point in S nearest (in the sense of D) to $\underset{\sim}{x}^*$.

The minimization of D subject to the restraints $\underset{\sim}{p} > \underset{\sim}{0}$ and $\underset{\sim}{p}\underset{\sim}{1}' = 1$ is a problem in mathematical programming which is discussed in the Appendix. When $a = 1$ it turns out that there is a whole class of strategies which will give the same value of D . It does not follow that, when applied repeatedly, every member of this class will produce the same trajectory because there are infinitely many steps which can be taken from $\underset{\sim}{x}(0)$ which yield the same value of D . When $a = 2$ the solution is unique. It also minimizes D with $a = 1$ and the argument of the Appendix may easily be extended to show that it minimizes D for any even positive integer $a > 2$. The strategies which we have selected to examine in detail are the following.

The Strategy S_1

This is one of those which minimizes D with $a = 1$. It is defined as follows. Compute $\underset{\sim}{y}$ and replace all negative elements by zero. Rescale the remainder so that their sum is 1 . This strategy is thus one in which we allocate to each grade which is below strength an equal proportion of their needs.

The Strategy S_2

This is the strategy obtained by minimizing D with $a = 2$. It may be compared iteratively as follows. Compute

$$y_i' = 0 \quad \text{if} \quad y_i < 0$$
$$y_i' = y_i - \frac{1}{m}\{Y - 1\} \quad \text{if} \quad y_i \geq 0$$

where m is the number of non-negative y's and Y is their sum. If $\underset{\sim}{y}' \geq \underset{\sim}{0}$ then $\underset{\sim}{p} = \underset{\sim}{y}'$, otherwise treat the y_i''s as original y_i's and repeat the cycle as often as necessary to produce a positive vector.

The Strategy S_3

In each time period a proportion $\underset{\sim}{x}(T)\underset{\sim}{w}'$ of members leave the system. An equal number of replacements are allocated to the various grades according to the chosen $\underset{\sim}{p}$. With this strategy we select $\max\limits_i y_i$ and if $\max y_i \geq 1$, all of the recruits go into that grade. If $\max y_i < 1$, we bring the number in this grade up to its target by allocating a proportion $\max y_i$ to this grade. We then move on to the grade showing the next largest shortfall and treat that in the same way. This procedure is continued until all of the recruits have been dealt with.

It is easy to see that S_3 also minimizes D with $a = 1$ and hence will reach the goal in one step if that is possible.

The Strategy S_4

Here we put all recruits into the grade having the largest y_i. In general, this does not seem a very desirable policy since it might involve appointing all assistant professors in one year followed by all full professors the next. Nevertheless it is useful to include it for purposes of comparison with the other strategies.

The final strategy which we shall consider was specifically designed to ensure a smooth transition from $\underset{\sim}{x}(0)$ to $\underset{\sim}{x}^*$. We restrict our attention to points $\underset{\sim}{x}(T)$ satisfying

$$\underset{\sim}{x}(T + 1) = \alpha\underset{\sim}{x}^* + (1 - \alpha)\,\underset{\sim}{x}(T), \quad 0 \leq \alpha \leq 1, \quad T \geq 0 \quad . \tag{11}$$

The trajectory for such a policy is thus a straight line joining the two points $\underset{\sim}{x}(0)$ and $\underset{\sim}{x}^*$. Substituting (11) in the basic equation

$$\alpha\underset{\sim}{x}^* + (1 - \alpha)\underset{\sim}{x}(T) = \underset{\sim}{x}(T)\left(\underset{\sim}{P} + \underset{\sim}{w}'\underset{\sim}{p}(T + 1)\right) \quad .$$

Solving this equation for $\underset{\sim}{p}(T + 1)$ we find

$$\underset{\sim}{p}(T + 1) = \left(\alpha\underset{\sim}{x}^* + (1 - \alpha)\underset{\sim}{x}(T) - \underset{\sim}{x}(T)\underset{\sim}{P}\right)/\underset{\sim}{x}(T)\underset{\sim}{w}' \quad . \tag{12}$$

The formal analysis just carried out may not yield a non-negative $\underset{\sim}{p}$ for any α. In this case no progress will be possible and the strategy must be abandoned. Otherwise S_5 is defined as follows.

The Strategy S_5

Select the largest value of $\alpha(0 \leq \alpha \leq 1)$ for which $\underset{\sim}{p}(T + 1)$ as given by (12) is non-negative.

This strategy thus moves as far as possible in the direction of $\underset{\sim}{x}^*$ at each step. Like S_1, S_2 and S_3 (but unlike S_4) it will therefore reach $\underset{\sim}{x}^*$ in one step if that is possible. In practice we have rarely found it possible to make any progress at all with this strategy. When it can be used it is either equivalent to or no better than one of the other strategies. For this reason it only appears in one of the tables.

An obvious strategy if $\underset{\sim}{x}^* \in S$ is the one suggested by Theorem 6 which uses a constant appointment vector given by

$$\underset{\sim}{p} = \underset{\sim}{x}^*(\underset{\sim}{I} - \underset{\sim}{P})/\underset{\sim}{x}^*\underset{\sim}{w}' \quad .$$

Although this strategy does not appear in our calculations we shall be able to compare it with the others in certain cases.

Tables 1-5 summarize the results of calculations made using the strategies S_1-S_5 for two promotion matrixes of dimension 3 and 5 . The calculations may not be fully representative and the conclusions set out below must be interpreted with this remark in mind. We shall deal with the tables one by one but the following remarks apply to them all.

(a) S_3 and S_4 are almost identical for these examples.

(b) S_2 tends to spread the recruits more widely over the grades than does S_1 but otherwise the strategies are similar.

(c) The goal is reached within 10 steps on only one occasion but when k = 3 the structure is often very close to the goal after 5 steps. Rather more steps seem to be needed when k = 5 .

Table 1

The goal here is the vertex of S which would be reached, in the limit, by using the constant appointment vector p = (1 0 0).

The strategies S_3 and S_4 starting from (0 1 0) turn out to be identical with this constant strategy. In the case of the starts from (1 0 0) and (0 0 1) the sequence $\underset{\sim}{p}(T)$ becomes equal or very close to this in every case as the structure approaches the goal. In this example, all of the strategies produce a structure close to that required in a very few steps.

Table 2

Here the goal is outside S and all strategies appear to be converging on a point at or near the vertex used as the goal for Table 1. None of the strategies takes us very near to the goal but all get near to their limiting structure quite quickly. There are interesting differences between the sequences of appointment vectors in this case. Again, they appear to be converging to a limit.

Table 3

The starting points in this table are not the five extreme points of X but three structures chosen to represent the spectrum of possibilities. As when k = 3 there is little difference between the strategies except now it is taking much longer to get close to the goal. This is particularly true of the example in the first section of the table. S_2', S_3, and S_4 are either near or equal to the pure strategy (1 0 0 0 0) throughout or quickly converge to it but S_1 allows a much greater spread.

Table 4

This is our only example in which the goal is exactly attained. Only S_1 and S_2 are successful but it would be premature on such limited evidence to conclude that they will always show this superiority. The goal cannot be maintained and beyond $T = 4$ there seems little to choose between the strategies. This is a case where a sequence of pure strategies would obviously be unsuccessful, in the early stages at least.

Table 5

This table is included to show a case where none of the strategies is very successful. The two illustrated are never near to the goal in every grade and matters are getting worse rather than better beyond $T = 5$.

ERIC

TABLE 1a: A COMPARISON OF CONTROL STRATEGIES FOR $k = 3$

$$\underset{\sim}{P} = \begin{bmatrix} 0.5 & 0.4 & 0 \\ 0 & 0.6 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix} \qquad \underset{\sim}{w} = (0.1\ 0.1\ 0.2)\ .$$

The goal is a vertex of S .

T	Start	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
	Start	1	0	0	0	1	0	0	0	1
2	S_1	.265	.440	.295	.168	.388	.445	.151	.179	.670
	S_2	.250	.440	.310	.180	.400	.4[illegible]0	.165	.165	.670
	S_3							.180	.120	.700
	S_4	.250	.440	.310	.180	.400	.420	.100	.260	.640
5	S_1	.270	.289	.441	.258	.283	.459	.258	.283	.459
	S_2	.273	.295	.432	.275	.275	.451	.275	.275	.451
	S_3							.285	.240	.474
	S_4	.274	.299	.427	.277	.273	.451	.277	.273	.451
10	S_1	.281	.286	.424	.277	.285	.437	.277	.285	.437
	S_2	.285	.285	.430	.284	.284	.431	.284	.284	.431
	S_3							.285	.284	.430
	S_4	.286	.285	.429	.286	.285	.429	.286	.285	.429
	Goal	.286	.286	.429	.286	.286	.429	.286	.236	.429

TABLE 1b: SEQUENCES OF APPOINTMENT VECTORS USED TO ACHIEVE THE RESULTS OF TABLE 1a

Appointment Vectors for start at (1 0 0)

S_1	0	.135	.717	1	1	.970	.970	.976	.981	.985	p_1
	0	0	0	0	0	.030	.030	.024	.019	.015	p_2
	1	.865	.238	0	0	0	0	0	0	0	p_3
S_2	0	0	.928	1	1	1	1	.995	.996	.998	
	0	0	0	0	0	0	0	.005	.004	.002	
	1	1	.072	0	0	0	0	0	0	0	
S_3 and S_4	0	0	1	.	.	.	.	.	.	1	
	0	0	0	.	.	.	.	.	.	0	
	1	1	0	.	.	.	.	.	.	0	

Appointment Vectors for start at (0 1 0)

S_1	.690	1	1	.921	.908	.925	.941	.953	.963	.970
	0	0	0	.099	.092	.075	.059	.047	.037	.030
	.310	0	0	0	0	0	0	0	0	0
S_2	1	1	1	.986	.973	.983	.989	.993	.995	.995
	0	0	0	.014	.027	.017	.011	.007	.005	.005
	0	0	0	0	0	0	0	0	0	0
S_3 and S_4	1	.	.	.	.	.	.	.	.	1
	0	.	.	.	.	.	.	.	.	0
	0	.	.	.	.	.	.	.	.	0

TABLE 1b (Continued)

Appointment Vectors for start at (0 0 1)

S_1	.500	.559	.641	.719	.782	.831	.868	.896	.918	.935
	.500	.441	.359	.281	.218	.169	.132	.104	.082	.005
	0	0	0	0	0	0	0	0	0	0
S_2	.500	.639	.747	.827	.883	.922	.949	.966	.978	.986
	.500	.361	.253	.173	.117	.078	.051	.034	.022	.014
	0	0	0	0	0	0	0	0	0	0
S_3	0	1	1	.956	.940	.969	.984	.992	.996	[illegible]
	1	0	0	.024	.060	.031	.016	.008	.004	.002
	0	0	0	0	0	0	0	0	0	0
S_4	1	0	1	.	.	.	.	.	.	1
	0	1	0	.	.	.	.	.	.	0
	0	0	0	.	.	.	.	.	.	0

Note: At $T = 1$ $x_1^* - x_1(0) = x_2^* - x_2(0) = 0.286$ when starting from (0 0 1) . S_3 treated grade 1 as having the largest difference and S_4 chose grade 2 . This also accounts for the big difference between the S_3 and S_4 rows in the (0 0 1) part of Table 1a for $T = 2$.

TABLE 2a: A COMPARISON OF CONTROL STRATEGIES FOR $k = 3$

$$\underset{\sim}{P} = \begin{bmatrix} 0.5 & 0.4 & 0 \\ 0 & 0.6 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix} , \quad \underset{\sim}{w} = (0.1\ 0.1\ 0.2) .$$ The goal is in A but not in S .

T		x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
	Start	1	0	0	0	1	0	0	0	1
2	S_1	.266	.440	.294	.144	.384	.472	.186	.150	.664
	S_2	.250	.440	.310	.180	.400	.420	.223	.123	.655
	S_3							.280	.080	.640
	S_4	.250	.440	.310	.180	.400	.420	.280	.080	.640
5	S_1	.239	.269	.492	.253	.254	.493	.270	.234	.496
	S_2	.259	.282	.459	.264	.272	.464	.279	.242	.479
	S_3							.300	.252	.448
	S_4	.274	.299	.427	.277	.273	.451	.300	.253	.447
10	S_1	.254	.254	.492	.254	.254	.492	.254	.254	.492
	S_2	.267	.269	.467	.267	.267	.467	.267	.266	.467
	S_3							.286	.286	.428
	S_4	.286	.285	.429	.286	.285	.429	.286	.286	.427
	Goal	.300	.200	.500	.300	.200	.500	.300	.200	.500

TABLE 2b: SEQUENCES OF APPOINTMENT VECTORS USED TO ACHIEVE THE RESULTS OF TABLE 2a

Appointment Vectors for start at (1 0 0)

S_1	0	.143	.558	.801	.883	.875	.860	.853	.852	.851
	0	0	0	0	0	0	0	0	0	0
	1	.857	.442	.199	.117	.125	.140	.147	.148	.149
S_2	0	0	.710	.975	.935	.919	.913	.911	.910	.909
	0	0	0	0	0	0	0	0	0	0
	1	1	.290	.025	.005	.081	.087	.089	.090	.091
S_3 and S_4	0	0	1	.	.	.	.	.	.	1
	0	0	0	.	.	.	.	.	.	0
	1	1	0	.	.	.	.	.	.	0

Appointment Vectors for start at (0 1 0)

S_1	.600	.849	.970	.925	.872	.855	.851	.851	.851	.851
	0	0	0	0	0	0	0	0	0	0
	.400	.151	.030	.075	.128	.145	.149	.149	.149	.149
S_2	1	1	1	.989	.918	.913	.910	.910	.909	.909
	0	0	0	0	0	0	0	0	0	0
	0	0	0	.011	.082	.087	.090	.090	.091	.091
S_3 and S_4	1	.	.	.	.	.	.	.	.	1
	0	.	.	.	.	.	.	.	.	0
	0	.	.	.	.	.	.	.	.	0

TABLE 2b (Continued)

Appointment Vectors for start at (0 0 1)

S_1	.600	.698	.834	1	.873	.833	.842	.848	.851	.851
	.400	.302	.146	0	0	0	0	0	0	0
	0	0	0	0	.127	.167	.158	.152	.149	.149
S_2	.750	.819	.957	1	.890	.893	.903	.907	.908	.909
	.250	.181	.043	0	0	0	0	0	0	0
	0	0	0	0	.110	.107	.097	.093	.092	.091
S_3	1	1	.976	1	.	.	.	.	.	1
	0	0	.024	0	.	.	.	.	.	0
	0	0	0	0	.	.	.	.	.	0
S_4	1	.	.	.	.	.	.	.	.	1
	0	.	.	.	.	.	.	.	.	0
	0	.	.	.	.	.	.	.	.	0

Tables 3 , 4 and 5 relate to a system with the following parameters

$k = 5$

$$\underset{\sim}{P} = \begin{pmatrix} .65 & .20 & 0 & 0 & 0 \\ 0 & .70 & .15 & 0 & 0 \\ 0 & 0 & .75 & .15 & 0 \\ 0 & 0 & 0 & .85 & .10 \\ 0 & 0 & 0 & 0 & .95 \end{pmatrix}$$

$$\underset{\sim}{w} = (.15\ .15\ .10\ .05\ .05)$$

A point is in S if

$$x_2 \geq (2/3)x_1 \ , \ x_3 \geq (3/5)x_2 \ , \ x_4 \geq x_3 \ , \ x_5 \geq 2x_4$$

The goal in Table 3 is obtained by taking the equality sign in each case. This gives the structure with the greatest degree of "tapering" towards the top which can be maintained. The other goals in Tables
4 an
4 and 5 lie outside S .

TABLE 3a: EXAMPLE SHOWING AN ATTEMPT TO REACH A VERTEX OF S WHEN $k = 5$

T		x_1	x_2	x_3	x_4	x_5	x_1	x_2	x_3	x_4	x_5	x_1	x_2	x_3	x_4	x_5
	Start	.05	.10	.15	.30	.40	.20	.20	.20	.20	.20	.40	.30	.15	.10	.05
2	S_1	.103	.111	.113	.255	.417	.213	.189	.164	.197	.239	.251	.260	.162	.122	.205
	S_2	.143	.077	.108	.255	.417	.250	.172	.162	.197	.220	.265	.255	.162	.115	.204
	S_3 and S_4	.143	.077	.108	.255	.417	.250	.172	.162	.197	.220	.282	.255	.162	.115	.187
5	S_1	.137	.138	.103	.198	.424	.234	.193	.135	.179	.259	.287	.203	.148	.136	.244
	S_2	.205	.103	.075	.192	.424	.277	.175	.128	.178	.243	.287	.208	.148	.132	.255
	S_3 and S_4	.206	.102	.075	.192	.424	.277	.174	.128	.178	.243	.307	.217	.150	.132	.195
10	S_1	.171	.164	.111	.147	.407	.254	.198	.122	.150	.276	.282	.203	.128	.136	.251
	S_2	.250	.148	.077	.126	.400	.291	.189	.114	.145	.261	.300	.199	.127	.134	.239
	S_3 and S_4	.251	.147	.076	.126	.400	.292	.188	.114	.145	.261	.311	.209	.132	.136	.212
	Goal	.306	.204	.122	.122	.245	.306	.204	.122	.122	.245	.306	.204	.122	.122	.245

TABLE 3b: THE SEQUENCES OF APPOINTMENT VECTORS LEADING TO THE STRUCTURES GIVEN IN TABLE 3a

<u>Appointment Vectors starting at (.05 .10 .15 .30 .40)</u>

S_1	p_1	.668	.665	.654	.655	.663	.674	.687	.700	.713	.725
	p_2	.312	.305	.289	.274	.261	.250	.240	.231	.223	.216
	p_3	0	.030	.057	.071	.076	.076	[illegible]	[illegible]	.064	.059
S_2	p_1	1	1	1	1	.989	.984	.989	.992	.994	.996
	p_2	0	0	0	0	.011	.016	.011	.008	.006	.004
S_3 and S_4 :	p_1	1	1	1	1	1	1	1	1	1	1

<u>Appointment Vectors starting at (.2 .2 .2 .2 .2)</u>

S_1	p_1	.749	.808	.848	.855	.864	.873	.882	.889	.896	.898
	p_2	.102	.140	.152	.145	.136	.127	.118	.111	.104	.098
	p_3	0	0	0	0	0	0	0	0	0	.005
	p_5	0	0	0	0	0	0	0	0	0	0
S_2	p_1	1	1	1	1	.995	.992	.994	.996	.997	.998
	p_2	0	0	0	0	.005	.008	.006	.004	.003	.002
S_3 and S_4 :	p_1	1	1	1	1	1	1	1	1	1	1

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TABLE 3b (Continued)

Appointment Vectors starting at (.40 .30 .15 .10 .05)

S_2	p_1	.186	.580	.792	.914	.954	.942	.941	.945	.949	.953
	p_2	0	0	0	0	.025	.058	.059	.055	.051	.047
	p_4	.060	.044	0	0	0	0	0	0	0	0
	p_5	.754	.416	.208	.086	.022	0	0	0	0	0
S_2	p_1	0	.849	.927	.949	.966	.978	.987	.993	.997	1
	p_5	1	.151	.073	.051	.034	.022	.013	.007	.003	0
S_3 and S_4	p_1	0	1	.	.	.	.	.	.	.	1
	p_5	1	0	.	.	.	.	.	.	.	0

TABLE 4a: AN EXAMPLE IN WHICH THE GOAL IS ACHIEVED AT T = 4 BY S_1 AND S_2 WHERE THE GOAL IS NOT MAINTAINABLE

T		x_1	x_2	x_3	x_4	x_5
	Start	.2	.2	.2	.2	.2
2	S_1	.085	.152	.162	.261	.340
	S_2	.085	.152	.162	.251	.351
	S_3, S_4	.085	.152	.162	.287	.315
	S_5	.140	.160	.180	.240	.280
5	S_1	.048	.098	.148	.297	.410
	S_2	.048	.098	.148	.298	.410
	S_3	.029	.101	.151	.301	.417
	S_4	.023	.075	.179	.295	.427
	S_5	.052	.101	.151	.299	.397
10	S_1	.041	.088	.136	.286	.449
	S_2	.038	.088	.138	.288	.450
	S_3	.021	.100	.148	.276	.455
	S_4	.031	.121	.130	.254	.464
	S_5	................Cannot proceed beyond T = 5.......				
	Goal	.05	.10	.15	.30	.40

TABLE 4b: THE SEQUENCES OF APPOINTMENT VECTORS LEADING TO THE STRUCTURES GIVEN IN TABLE 4a

S_1	p_1	0	0	0	.190*	.212	.207	.203	.199	.195	.192
	p_2	0	0	0	.036	.242	.243	.243	.243	.243	.242
	p_3	0	0	.052	.267	.273	.273	.274	.274	.275	.275
	p_4	.345	.391	.486	.457	.273	.277	.281	.284	[illegible]	[illegible]
	p_5	.655	.609	.462	.050	0	0	0	0	0	0
S_2	p_1	0	0	0	.191*	.207	.199	.192	.185	.179	.172
	p_2	0	0	0	.036	.241	.243	.244	.245	.246	.247
	p_3	0	0	0	.310	.276	.277	.279	.280	.281	.283
	p_4	.050	.550	.628	.439	.276	.281	.286	.290	.294	.298
	p_5	.950	.450	.372	.024	0	0	0	0	0	0
S_3	p_1	0	0	0	.123	0	.439	0	.417	0	0
	p_2	0	0	0	0	.335	.331	.060	.498	0	.549
	p_3	0	0	0	.310	.335	.230	.366	.085	.378	.451
	p_4	0	1	.113	.566	.329	0	.574	0	.622	0
	p_5	1	0	.887	0	0	0	0	0	0	0
S_4	p_1	0	0	0	0	0	0	0	1	0	0
	p_2	0	0	0	0	0	1	0	0	0	1
	p_3	0	0	0	0	1	0	0	0	1	0
	p_4	0	1	0	1	0	0	1	0	0	0
	p_5	1	0	1	0	0	0	0	0	0	0
S_5	p_1	.400	.313	.215	.102	0					
	p_2	0	0	0	0	.017					
	p_3	.100	.111	.124	.138	.162					
	p_4	.200	.259	.325	.401	.470					
	p_5	.300	.316	.336	.359	.351					

Strategy Terminates at T = 5

* Indicates the point at which the goal is achieved

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TABLE 5a: TWO EXAMPLES WHERE S_1 AND S_2 ARE UNSUCCESSFUL IN REACHING THE GOAL

In both cases the goal is outside S

T		x_1	x_2	x_3	x_4	x_5	x_1	x_2	x_3	x_4	x_5
T	Start	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
2	S_1	.200	.217	.167	.197	.220	.085	.152	.283	.258	.223
	S_2	.250	.172	.162	.197	.220	.085	.152	.322	.222	.220
5	S_1	.203	.224	.148	.182	.243	.034	.111	.293	.292	.269
	S_2	.276	.176	.128	.178	.243	.023	.097	.347	.275	.258
10	S_1	.203	.225	.144	.161	.266	.024	.088	.255	.291	.341
	S_2	.290	.190	.114	.145	.261	.003	.056	.306	.304	.331
	Goal	.40	.30	.15	.10	.05	.05	.15	.40	.30	.10

TABLE 5b: THE SEQUENCES OF APPOINTMENT VECTORS LEADING TO THE STRUCTURES GIVEN IN TABLE 5a

Appointment Vectors when the Goal is (.40 .30 .15 .10 .05)

S_1	p_1	.692	.706	.714	.718	.715	.712	.711	.710	.710	.710
	p_2	.308	.294	.286	.282	.276	.273	.271	.270	.269	.269
	p_3	0	0	0	0	.009	.015	.018	.020	.021	.021
S_1	p_1	1	1	1	1	.985	.987	.990	.991	.992	.993
	p_2	0	0	0	0	.015	.013	.010	.009	.008	.007

Appointment Vectors when the Goal is (.05 .15 .40 .[illegible]0 .10)

S_1	p_1	0	0	0	.060	.094	.105	.109	.109	.107	.106
	p_2	0	0	.116	.192	.230	.248	.255	.258	.258	.257
	p_3	.688	.738	.719	.661	.631	.619	.612	.608	.604	.601
	p_4	.313	.262	.165	.086	.045	.029	.024	.026	.030	.047
S_2	p_2	0	0	0	.091	.203	.200	.191	.179	.165	.150
	p_3	1	.895	.905	.909	.797	.800	.809	.821	.835	.850
	p_4	0	.105	.095	0	0	0	0	0	0	0

7. Strategies Which Look Two Steps Ahead

Although the strategies investigated so far perform reasonably well we might hope to improve upon them by looking farther ahead. We shall therefore suggest two further classes of strategies which take into account what the position will be after the next step has been taken.

The Class of Strategies S'

This class is a natural extension of that which gave rise to S_1 and S_2. Instead of seeking to minimize, at the start, the distance between $\underset{\sim}{x}(1)$ and $\underset{\sim}{x}^*$, we now consider the distance between $\underset{\sim}{x}(2)$ and $\underset{\sim}{x}^*$. The problem is then one of minimizing a function such as

$$\sum_{i=1}^{k} |x_i^* - x_i(2)|^a \ , \ a > 0$$

with respect to $\underset{\sim}{p}(1)$ and $\underset{\sim}{p}(2)$ subject to the usual restraints.

The Class of Strategies S''

Here the object is to aim initially not for $\underset{\sim}{x}^*$ itself but for some other point from which $\underset{\sim}{x}^*$ can be reached in one step. The motivation for this is that there may well be points from which $\underset{\sim}{x}^*$ is reachable in one step which are much more reasonable than $\underset{\sim}{x}^*$. Let X^* denote the set of points from which $\underset{\sim}{x}^*$ can be reached in one step; then the following would be a possible way to proceed. At time T use one of the strategies S_1-S_5 to aim for the point in X^* nearest to $\underset{\sim}{x}(T)$. If this point can be reached then $\underset{\sim}{x}^*$ can be reached in one further step.

Other classes of strategies can easily be devised by extension of these ideas but it would be better to explore those already proposed before proceeding further.

8. Suggestions for Further Work

The work described in this report is incomplete at almost every point. Below we list some of the topics on which further work is needed if the control problem is to be fully understood.

(a) Throughout we have assumed that the total size of the organization was fixed. The theory needs to be developed for organizations whose size is changing in time.

(b) The class of strategies S' and S'' require further investigation. A comparison between the original class S and S' or S'' would throw some light on whether it is worthwhile to look beyond the next step when choosing $\underset{\sim}{p}$.

(c) The question of how one can stay near to $\underset{\sim}{x}^*$ when $\underset{\sim}{x}^*$ is outside S needs further investigation. It is known that a trajectory can move outside S infinitely often but it is not clear whether it is possible to pursue a path which is "nearer" to $\underset{\sim}{x}^*$ than the nearest point on the boundary of S.

(d) The strategies should be tried out using data for the larger colleges on the various campuses.

9. Implications for Policy

The work described in this report is a step towards a general theory of control for Markov chain models of graded manpower systems. As such it is primarily a theoretical exercise intended to give some insight into the dynamics of such systems. It was, however, motivated by the problem facing a university whose faculty structure was moving in an unwanted direction and there are several conclusions which can be drawn which have important implications for those faced with that problem. For any organization, in which the assumptions of the model apply, the following conclusions hold.

(a) Not all structures are attainable or maintainable by controlling the appointment policy alone. Those which are can be found from results given in this report. Those which are not can only be reached by making changes in the promotion rates or by contriving to alter the leaving rate. The kind of changes needed can also be determined from the results given here.

(b) Some changes in structure can be brought about by appropriate variations in the appointment policy. The best way to do this is still an open question but several strategies have been proposed which seem to give satisfactory results. Of these S_1 strategy described in Section 5 is both easy to determine and satisfactory in its performance.

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10. References

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APPENDIX

The implementation of the strategies S_1 and S_2 requires the minimization of

$$D = \sum_{i=1}^{k} |y_i - p_i|^a$$

for $a = 1$ and 2 where $p_i \geq 0$ $(i = 1, 2, . . . k)$ and $\sum_{i=1}^{k} p_i = 1$. We show here how the results given in the text may be justified taking first the case $a = 1$.

The problem of minimizing D when $a = 1$ can be converted into a problem in linear programming by a well known device as follows. Transform to new variables x_i^+ and x_i^- given by

$$x_i^+ = y_i - p_i \quad \text{if} \quad y_i \geq p_i$$
$$= 0 \quad \text{otherwise,}$$

$$x_i^- = -(y_i - p_i) \quad \text{if} \quad y_i < p_i$$
$$= 0 \quad \text{otherwise.}$$

It follows that

$$|y_i - p_i| = x_i^+ + x_i^-$$

and so the function to be minimized is

$$D = \sum_{i=1}^{k} (x_i^+ + x_i^-)$$

The restriction

$$\sum_{i=1}^{k} p_i = 1$$

becomes

$$\sum_{i=1}^{k} x_i^+ = \sum_{i=1}^{k} x_i^- \tag{A1}$$

and the inequalities $p_i \geq 0$ become

$$x_i^+ \leq y_i \quad \text{for those } i \text{ for which } y_i \geq 0$$

$$x_i^- \geq -y_i \quad \text{for those } i \text{ for which } y_i < 0 \ .$$

Introduce slack variables $v_i \geq 0$ such that $x_i^- = v_i - y_i$ and let Σ^+ and Σ^- denote summation with respect to i over those values for which $y_i \geq 0$ and $y_i < 0$ respectively. Then eliminating Σx_i^+ from D using (A1) and substituting $x_i^- = v_i - y_i$ we have

$$D = -2\Sigma^+ y_i + 2\Sigma^+ v_i + 2\ ^- x_i^- \quad .$$

This clearly achieves a minimum value of $-2\Sigma^- y_i$ when $x_i^- = -y_i$ if $y_i < 0$. This implies $p_i = 0$ if $y_i < 0$. The complete solution can easily be found by revetting to the original notation for now we have to find p_i's (for $y_i \geq 0$) which make

$$\Sigma^+ |y_i - p_i| - \Sigma^- y_i = -2\Sigma^- y_i \quad .$$

It is clear that any set of p's satisfying $y_i \geq p_i$ will yield this minimum value. In particular the $\underset{\sim}{p}$'s adopted for S_1 and for S_3 satisfy this equation and, as we shall now see, so do those of S_2.

When $a = 2$ the problem can be interpreted geometrically. D can be thought of as the squared distance from $\underset{\sim}{y}$ to the point $\underset{\sim}{p}$. It must lie on the hyperplane $p_1 + p_2 + \ldots p_k = 1$ in the orthant $\underset{\sim}{p} \geq 0$; call this region P. The vector $\underset{\sim}{y}$ lies on the same hyperplane but not, in general, in the positive orthant. The problem is thus one of finding the nearest point in P to the given point $\underset{\sim}{y}$. Let

$$\phi = \sum_{i=1}^{k} (y_i - p_i)^2 + 2\alpha \sum_{i=1}^{k} p_i$$

where α is an undetermined multiplier. Then we have to find the minimum of ϕ subject to $\underset{\sim}{p} \geq \underset{\sim}{0}$. Denote the minimizing value of $\underset{\sim}{p}$ by $\underset{\sim}{p}'$ then at this point

$$\frac{d\phi}{dp_i} = 0 \quad \text{if} \quad p_i' > 0$$

$$\frac{d\phi}{dp_i} > 0 \quad \text{if} \quad p_i' = 0$$

which implies

$$\begin{aligned} p_i' &= y_i - \alpha \quad && \text{if} \quad y_i > \alpha \\ &= 0 && \text{if} \quad y_i \leq \alpha \end{aligned}$$

for some α which will be a function of the y's . The multiplier α is determined by the condition

$$\sum_{i=1}^{k} p_i = 1 \quad .$$

The iterative procedure described in Section 5 is one way of finding p_i' and is will suited to automatic computation. A simple method with pencil and paper is to plot

$$\sum_{i=1}^{k} < y_i - \alpha >$$

where

$$\begin{aligned} <x> &= x \quad \text{if} \quad x \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

as a function of α , finding the point at which the sum is one. This gives the value of α to substitute in

$$p_i' = y_i - \alpha \quad .$$

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