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ABSTRACT

A theory of psychophysics is discussed that enlarges the classical theory in three general ways: (1) the multidimensional nature of perception is made explicit; (2) the transformations of the theory are interpreted geometrically; and (3) attributes are distinguished from sensations and only partially ordered. It is shown that, with the enlarged theory and some elementary geometry, one may use a few ideas to explain many qualitative results, to give a credible and precise account of quantitative properties of data, and to design experiments which seem worth doing whether the theory is valid or not. In developing the new theory, the major criticisms of the classical theory are answered. This paper is limited to one of three aspects of the typical psychophysical measurement experiment--the relation between perceptions and attributes. Although the theory has been extended to category and rating scales, only magnitude estimation and related measurement procedures are discussed here. The scope of this paper is further limited to phenomena which have analogues in several sense modalities. (Author/KM)

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GEOMETRIC INTERPRETATIONS OF SOME PSYCHOPHYSICAL RESULTS

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GEOMETRIC INTERPRETATIONS OF SOME PSYCHOPHYSICAL RESULTS

Michael V. Levine[1]

Part I: Introduction

Many people, especially certain psychologists, speak and act as if their impressions, perceptions and opinions were like numbers. Otherwise reasonable people say things like, "The football team seemed twice as strong when Jackson was coach." They behave as if addition, subtraction, multiplication and division were defined for mental objects. The articles in our technical journals have ample evidence. Psychophysicists commonly set willing subjects to such tasks as adjusting a light until it looks half as bright or deciding whether one tone is louder than the difference in loudness of two other tones.

These curious phenomena are taken as the starting point for a highly speculative theory of psychophysics. The theory extends the commonsense, classical mapping theories which recently have been sharply attacked (Krantz, 1972).

The new theory enlarges the classical theory in three general ways: (1) The multidimensional nature of perception is made explicit. (2) The transformations of the theory are interpreted geometrically. (3) Attributes are distinguished from sensations and only partially ordered.

I will try to show that with the enlarged theory and some elementary geometry one may use a few ideas to explain many qualitative results, to give a credible and precise account of quantitative properties of data and to design experiments which seem worth doing whether the theory is valid or not. In the process of developing the new theory, the major criticisms of the classical theory will be answered.

I.1 Comments on Goals and Scope of the Theory

There are many replicable experimental findings in psychophysical measurement. The goal of this theory is to show that some of the most important ones can be related to a few geometric principles. The object is simply to better understand the psychological findings by deducing them from a small number of psychological ideas which, for convenience and clarity, are geometrically expressed.

To define the scope of this chapter it is necessary to distinguish three aspects of the typical psychophysical measurement experiment. For concreteness, consider an observer viewing a small illuminated panel and judging brightness. The three aspects are (1) the relation between physical stimuli and perceptions, (2) the relation between complex perceptions (for example, the subjective representation of an illuminated panel having a definite texture, distance from the observer, area, etc.) and simpler attributes (for example, brightness or perceived size), and (3) the relation between attributes and the responses the observer uses to report the level of an attribute. This paper is limited to (2), the relation between perceptions and attributes. The other aspects are deliberately treated in a schematic, noncommittal way.

Although the theory has been extended to category and rating scales (Levine, 1973c), only magnitude estimation and related measurement procedures are discussed in this paper. The scope of the paper is further limited to phenomena which have analogues in several sense modalities.

I.2 Theoretical Background: Mapping Theories of Psychophysics

The essence of our classical,commonsense theories of direct measurement seems to be this. There is a functional relationship between stimuli and the observer's internal states. Numbers, or more properly number names, are treated as stimuli. When performing in a measurement experiment, the observer responds by selecting stimuli mapping onto approximately the same internal state. Following Krantz, these theories will be called mapping theories.[2]

When details of these theories are made explicit, they often seem highly implausible. For example, consider the following analysis of an observer selecting a light appearing half as bright as a given light. The observer first selects a number mapping onto approximately the same internal state or magnitude as the light. He divides the number by two. Finally he selects a light with the same magnitude as the halved number.

In spite of their implausibility, these theories have been remarkably fruitful. An impressive number of replicable findings have been discovered by experimenters evidently using the theories. It seems that some of our most fruitful and widely held psychophysical ideas are inadequate.

A detailed and penetrating analysis of this situation is given in D. H. Krantz's recent theoretical study of magnitude estimation (Krantz, 1972). Krantz lists a number of empirical generalizations which are based on the direct measurement experiments. He then surveys the available classical interpretations of the generalizations. Each interpretation involves unlikely mental arithmetic, complicated sequences of matches or is otherwise unacceptable.

Krantz's solution has been to discard the classical framework and develop a fundamentally different conceptualization of psychophysics called relation theory. For a clear and persuasive exposition of relation theory with references to Kristof's and Shepard's earlier formulations, see the Krantz paper.

The solution offered in this paper retains the classical framework. By appropriately specifying the relationship between perceptions and attributes, one obtains a number of simple, intuitive interpretations of the generalizations. See Section III.7 for further details.

These interpretations are free of all the defects of the earlier theories. They are part of a theory which seems able to integrate a wide range of data. This new theory, as a part of mapping theory, seems to be a natural development of a vigorous, highly successful line of research.

In the remaining sections of Part I some informal arguments are used to introduce the theory.

I.3 Some Informal Considerations Introducing Geometric Interpretations of Arithmetic

Ordinarily when we think of arithmetic, we think of the operations on whole numbers we were taught in school. However, various experimental results have led psychologists to conjecture that people are able to solve arithmetic problems as analogue computers solve them: by representing numbers as continuous quantities such as lengths or electrical potentials and operating on them.

In this section, one experimental finding is used to introduce geometry as a language for reasoning about the sort of analogue devices and phenomena that interest psychologists. The goal is to introduce the geometric ideas needed for psychophysics rather than to explain the experimental result. Consequently, details about experiments are omitted.

One of the simplest results to consider geometrically is an unpublished study of Restle (1969a). A related published study is Restle (1969b). Restle gave his subjects some carefully chosen mental arithmetic problems, placed them under time pressure and measured their accuracy and rates of responding. He found that the speed and accuracy of deciding whether a number b is closer to a (equal to zero in this study) or c depends largely upon the proportion $\frac{b - a}{c - b}$ rather than upon the digits of b. This suggests (at least to Restle and the present writer) that the numbers were encoded like lengths and that the encoded numbers rather than the digits were manipulated and compared.

All of the elementary statements of arithmetic can be translated into geometric constructions. These constructions[3] are well understood. A few examples will be given in this section and the next.

From the psychologist's point of view, it is suggestive that a multidimensional space is required for the constructions and that there is little correlation between the complexity of the arithmetic statement and the complexity of a geometric interpretation. For example, in most constructions the distinction between rational and irrational numbers is irrelevant.

Here is a construction relevant to the Restle study. Consider three parallel lines called vertex, quantity and intensity arranged as in Figure

One. Although the middle line will be used to represent numbers, all of the metric properties on all of the lines will be deliberately ignored. The points on each of the lines (vertices, quantities, intensities) will be regarded as ordered from left to right. After this discussion it should be clear that only the ordering on one of the lines, say the intensity line, need be ordered. The transformations we consider will induce the orderings on the other lines.

Let two points F and V (for Fixed and Variable) be selected on the vertex line. One can define a transformation of the quantity line by passing a line through V and y so that the intersection x' of these lines lies on the intensity line. It can be shown that the length of the segment yx is independent of the point x . (Use similar triangles FVx' and xyx' to show the ratio of the lengths of segments xy to FV is independent of x .) Consequently, we may regard the mapping $x \to y$ as the addition of a constant mapping, $x \to x + k$. By elaborating the diagram we can interpret various statements about addition and subtraction.

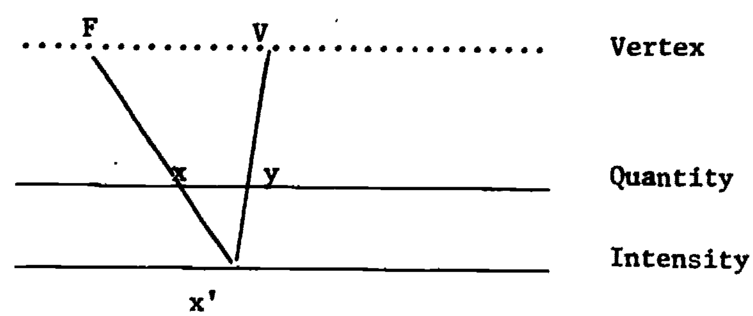

Figure One: Adding a Constant to x .

One way to use this construction to understand the Restle result is as follows: Each of the stimuli a,b,c are encoded as points on the quantity line also denoted a,b,c . The fixed vertex F places each of the quantities in correspondence with intensities a',b',c' as in Figure Two. The observer under time pressure quickly selects the vertex V on the same line as a' and b . By projecting through c he creates an intensity x . Simply by comparing b' and x he is able to decide whether b - a is greater than c - b . For c - b is greater if and only if x is to the right of b' . In this way the observer could behave as if he were doing arithmetic without actually counting or manipulating digits.

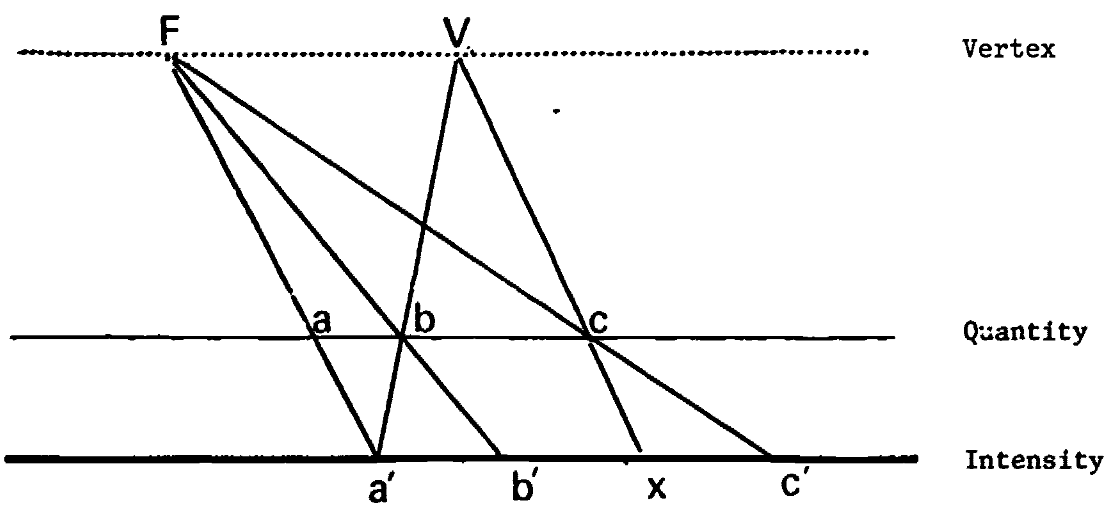

Figure Two: Geometric Interpretation of an Experiment

Before relating these observations to psychophysics there are several qualifying remarks which must be made. (1) Many other constructions might have been used to formulate process models for mental arithmetic.

The one given above was chosen for its simplicity and usefulness in introducing considerations needed in the discussions of psychophysics. For more plausible models of mental arithmetic, see Restle (1969b) and his references. (2) Some of the alternative constructions are compatible with the relation theories now being proposed as alternatives for mapping theory. For details see the concluding paragraph of this section. (3) Whether or not these geometric constructions can be realized by the components of hypothetical neural nets now accepted by physiologists and anatomists seems to be as irrelevant a consideration for psychologists as, for example, the physiological realizability of Hering's[4] now acceptable, previously rejected, psychological ideas on color. The relationship between attributes and perceptions seems to be a purely psychological problem. However, to make the calculations required by the constructions little more is needed than neural nets with output $y(t)$ accepting positive bounded signals $x(t)$ such that if $x(t)$ rapidly converges to $\underline{x}$, then $y(t)$ rapidly converges to $\ell(\underline{x})$, where ℓ is a linear fractional transformation.

Some details follow. They will be indented and printed in smaller type. This device will be used throughout the paper to indicate material which can be skipped on first reading.

> On alternative constructions: As indicated in the discussion of Figure One, each vertex V to the right of F is associated with an increment of quantity of length k . (k depends on V and the ordering of these k 's is the same as the ordering from left to right of the V 's.) In this sense it may be legitimate to interpret V as a sense distance of size k . Then each pair of quantities is associated with a unique sense distance $V(a,b)$. To solve Restle's problem the subject need only check to see whether

sense distance $V(a,b)$ is to the left of $V(b,c)$. To modify this construction in order to obtain the sense ratios which play an important role in the Shepard-Krantz relation theory, simply rotate the vertex line through F until it passes through the point on the quantity line used to represent the number zero. Then the transformations analogous to those in the preceding discussion, except with vertices on the rotated line, will define multiplicative transformations $x \to kx$. Just as vertices on the parallel line were interpreted as sense distances, these new vertices behave like sense ratios. More is said about this construction and geometric interpretations of multiplication in the next section.

I.4 The Central Hypothesis of the Theory

Consider an experiment in which an observer adjusts a light until it appears half as bright. There are many published studies of this kind. One of the earliest careful fractionation studies is Hanes (1949). For an exceptional bibliography and a description of current experimental procedures, results and theories of direct measurement, see L. E. Marks' The New Psychophysics of Sensory Processes.

In the opinion of many experimental psychologists, subjects can behave consistently and reliably *as if* they were converting stimuli to numbers, multiplying these numbers by .5 and finding a stimulus which maps onto the product. Experimental evidence for number-like encodings with interpretable ratios has been cited for many different kinds of physical stimuli and also for stimuli (such as crimes of varying seriousness) for which there is no established physical measurement procedure (S. S. Stevens, 1966).

Here is one way to consider the abili.y to perform in bisection experiments. Suppose people represent numbers on a quantity line parallel to an intensity line in a plane having a fixed vertex F placing quantities in correspondence with intensities as in the discussion of Figure One. However, now consider the vertex line passing through the encoding z of zero and its corresponding intensity i_o as in Figure Three. For a vertex M on the vertex line, a correspondence $x \to x' \to y$ can be defined as illustrated in Figure Three. This new correspondence $x \to y$ can be considered multiplication by a constant in the sense that the segment between y and the representation of zero is easily shown to be a constant fraction of the segment xz . (Use similar triangles Mzy , Mi_ox' and Fi_ox' , Fzx with common segment i_oy .) Changing the vertex M only changes the multiplicative constant. There is exactly one M such that the constant is one half.

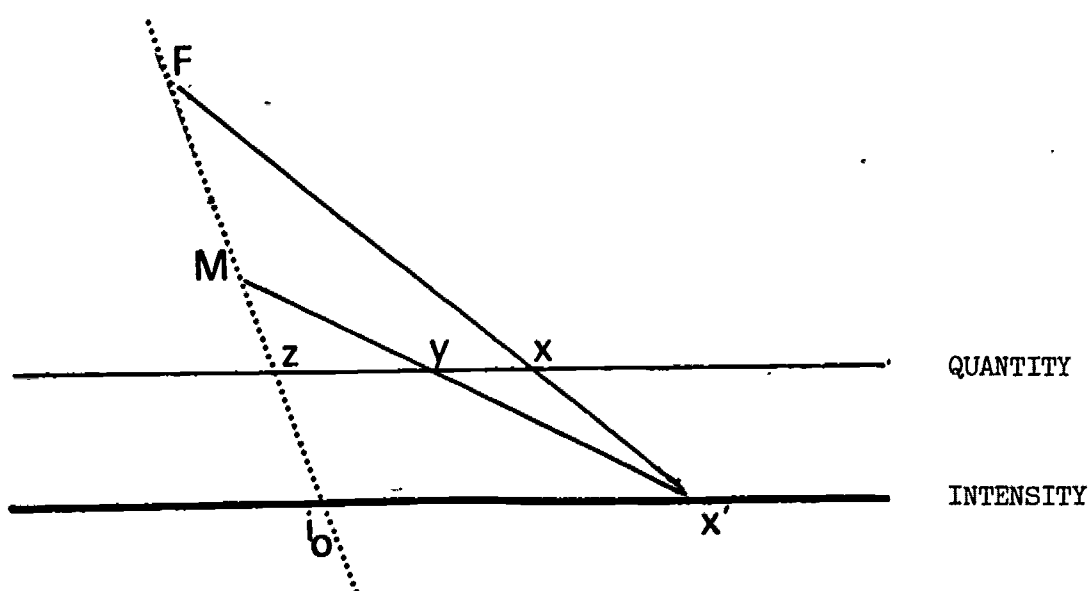

Figure Three: A Geometric Interpretation of Fractionation

To relate this construction to bisection experiments, suppose that the experimental series of lights is represented by the observer as a series of points in the same plane. Then the vertices F and M induce a transformation of lights analogous to the halving transformation just discussed.

Notice that once the vertices F and M are specified, the observer is independent of numbers. There is absolutely no reason for him to map lights to numbers, halve the numbers and return to the corresponding lights. The transformation of lights is produced by an observer operating on lights and their intensities only.

I propose that we acquire[5] the ability to halve sensations by first learning to do this geometric arithmetic. Then we use the same apparatus for operating on sensory continua. There is nothing special about the number continuum. The psychological continuum of length or time for example could play the same role. The important point is that the vertices, once defined for any dimension in which arithmetic is natural, become available for an arithmetic of visual brightness or of seriousness of criminal offenses.

The main hypothesis of this paper is that we have essentially multidimensional perceptions, that we use processes like the projections considered above to abstract unidimensional attributes and that the regularities of data suggesting mental calculation actually appear as a consequence of these processes.

Part II: The Basic Psychophysical Theory

The theory is called projective theory. Its components are:

Physical Space

physical events

physical continua

parameterizations

Psychological Space

appearances

straight psychological continua

intensities

projections

Magnitudes

points of view

intensity curves and measures

In this section they are briefly discussed. The purpose is to provide a vocabulary for mapping theory interpretations of psychophysical phenomena. Enough structure is specified to provide geometric interpretations and information processing schemes for arithmetic tasks with sensations and with numbers.

The most closely related alternative theory seems to be Ross and Di Lollo's. In their 1968 paper[6] they use multidimensional representations and parallel projections for some of the same purposes as multidimensional representations and not necessarily parallel projections are used here.

The main problem in classical psychophysics is to describe the relationship between physical events and their mental representations. The view implicit in mapping theory, namely that the psychological representation is a function of the physical event, is used here. However, an attempt will be made to regard all psychophysical regularities as imposed by the observer rather than transmitted from the physical world. For this reason only the crudest physical relations will be acknowledged. This is made explicit in the discussion of physical space.

II.1 Physical Space

Physical space is simply the set of all physical environments that the experimenter may wish to present to his subject. The points in the set are called physical events.

In a typical experiment the psychophysicist adjusts a bit of hardware to obtain a one parameter subset of physical space. For example, when studying brightness, he may turn a dial through λ degrees to control the amount of electricity passing through the filament of a light bulb. These one-dimensional subsets will be called physical continua. More precisely, a physical continuum is a parameterized set of physical events $\{X_\lambda\}$ where the index λ ranges over a set (generally interval) of numbers. The mapping $\lambda \to X_\lambda$ is called a parameterization.

From the point of view adhered to in this paper, physical space is just a set. All of the quantitative regularities of data will be studied from a psychological point of view.

II.2 Psychological Space

Psychological space, the space in which physical events are represented by the observer, is taken as a multidimensional space (euclidean space) of low dimensionality. Each physical event is represented by the observer as a point in psychological space called an appearance. At any given time, for any given observer, there is exactly one appearance for one physical event. Some transformations and extra structure will be defined in later sections. The usual topology in euclidean space will be used.

It will not be necessary to refer to coordinates in psychological space. The psychological considerations that are ordinarily dealt with by coordinate systems are handled by projections (Section II.4) and points of view (Section II.6).

The set of appearances corresponding to a physical continuum is called a psychological continuum. The function between a physical continuum and its corresponding psychological continuum is called a psychophysical function. Since every physical continuum has a real parameterization, every psychological continuum also has a real parameterization. Although this may occasionally be confusing, the real parameterization of a psychological continuum will also be called a psychophysical function. To restrict attention to the sort of parameterizations psychophysicists study, only continuous psychophysical functions are considered.

If the appearances in a psychological continuum fall on a line segment in psychological space, then the continuum is called straight. The main geometric fact dictating the use of this word is the fact that two lines can meet in at most one point. However a curve can cross a line twice.

Not every psychological continuum will be straight. The example which follows will be clearer after the sections on projections and points of view have been read. The reader may wish to return to this example after reading those sections.

Since Newton's time, spectral colors have often been represented by points on a curve. Consider the physical continuum that one gets by selecting a narrow segment of sunlight analyzed by a prism. If one were to use this theory to understand the scaling of redness one would be forced to conclude that the psychological continuum corresponding to these stimuli was not straight. There would be equally read-appearing stimuli at both ends of the spectrum. This can only happen if some line intersects the psychological continuum twice.

Some continua--those which appear to vary in exactly one salient attribute--may be considered straight. Finding suitable continua seems to be a matter of experimental skill and psychological sophistication. After the continuum has been chosen there will be experimental implications of straightness which can be tested.

It is curious that a great many psychophysical experiments can be analyzed without considering spaces of high dimensionality. This may reveal a significant limitation on the observer's ability to process information or it may simply be a consequence of the experimenter's successful attempts to design experiments in which all but a few dimensions of experience are irrelevant. In the absence of direct experimental evidence the more conservative latter alternative is used, and the low dimensional spaces are considered to be quotient spaces of higher dimensional spaces.

II.3 Intensities and the Interpretation of Matching

In cross modality matching experiments, subjects attempt to equate levels of different attributes. For example, they may attempt to adjust a light until it appears as bright as a sound is loud. Responses are variable, both within and between observers. But averaged results have a consistency which has strengthened belief in the following interpretation:

Traditional mapping theory interpretation of matching: An appearance has several qualities such as brightness or yellowness and has each quality to a definite extent. The extent to which an appearance has brightness can be represented by a number or something number-like such as neural excitation or length. The same number-like continuum is used for brightness and for loudness. The observer matches brightness to loudness by equating neural excitation, length, magnitude or whatever the continuum which carries ordinal information for attributes is called.

I do not accept this interpretation, except as an approximation. Criticism and an alternative interpretation are offered in Sections II.5, II.6 and III.8. However the traditional interpretation will be used frequently for two reasons. It gives a convenient approximation of the central tendencies predicted by the alternative interpretation. And, being familiar to most readers, it simplifies the exposition of the parts of the theory for which the alternative and traditional views agree. To prepare to use it, a special line is considered in the next assertion about psychological space.

There is a distinguished line in psychological space, the points of which are called <u>intensities</u>. No appearance[7] is an intensity.

II.4 Projections

To relate this theory to the traditional mapping theory interpretation of matching, appearances must somehow be placed in correspondence with intensities. There are certain qualitative properties demanded by data. For example, the whiteness ordering on a psychological continuum must be the opposite of the blackness ordering. The simplest mappings from a vector space to one of its lines with all of the appropriate qualitative properties and with sufficient generality to include the analogue arithmetic schemes of Part I are the projections.

Projections give this theory its geometric and intuitive character. Although they are a part of classical algebraic geometry, a few illustrations and definitions will be included to show some of the qualitative phenomena they describe and how they function like coordinate systems.

Projections in a two-dimensional space were illustrated in Sections I.2 and I.3. To define a projection from an appearance in a two dimensional space onto the intensity line, one begins by selecting a point in the space not on the intensity line called a <u>vertex</u>. If an appearance is not on the line through the vertex parallel to the intensity line then the projection of the appearance is the unique intensity on the line through the vertex and the appearance. Just as the ratio of some numbers is undefined, the projection of an appearance on the parallel line is not defined.

Suppose a continuum of lights is presented. To make it easy to draw pictures, suppose the experiment is adequately represented in a two dimensional space. (See Figure Four.) A vertex B (for Brightness) will induce one ordering on the appearances, when appearances with greater projected intensities are regarded as greater. Another vertex Y (for Yellowness) induces the opposite ordering for these appearances. As Figure Four is drawn, there is very little change in the projection through the yellowness vertex for one extreme of the psychological continuum.

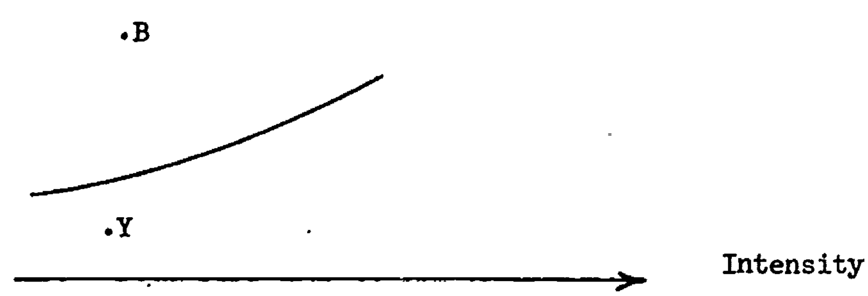

Figure Four: Two Projections

The two projections function like coordinates in the sense that each appearance x can be specified exactly by a pair of intensities x_Y, x_B where x_Y is the projection through the yellowness vertex and x_B is the projection through the brightness vertex. This is true not only for appearances on the continuum but also for all appearances in the same plane, provided the projections are defined.

In three dimensions, instead of taking a point as a vertex to define a projection, one takes a line (not in a plane with the intensity line) as vertex. Then the projection of an appearance not in the plane containing the vertex parallel to the intensity line is the unique intensity on the same plane as the vertex line and the appearance. To develop intuitions quickly, it may be helpful to visualize a large book with its spine on the vertex. The projection of an appearance is the unique intensity on the same page as the appearance. The same remarks about coordinates remain valid, however three rather than two projections are needed to locate an appearance.

> There is no need to consider more than three dimensions in this paper, however for completeness a vertex in psychological n-space is an n-2 affine subspace not in the same hyperplane as the intensity line. A projection onto the intensity continuum is the mapping from appearances to intensities such that corresponding points lie on the same hyperplane containing the vertex.

II. Motivation for Points of View

This section contains some informal remarks needed to motivate an alternative to the traditional mapping theory interpretation of matching.

Both intensities and appearances are points in psychological space. Yet there seems to be a fundamental difference between the appearance of an object and the degree to which it possesses an attribute. One intuitively obvious difference is the definiteness of perception and the uncertainty of psychophysical judgment. It is common to see an object clearly and still be very uncertain about one of its attributes.

For a simple demonstration, I scattered seven ennies on a sheet of paper and eight on another. Two quarters, three nickels and a dime were added to both displays to break up easily counted clusters. People passing my office were invited to guess which paper had more pennies. They were asked not to count or place the coins in one-to-one correspondence. Although it is easy to see a few coins on a white sheet of paper in a well lighted room, four out of ten first guesses were wrong. (With practice people became very good at the task.)

In the currently fashionable jargon, "perception is categorical." If the perceptual system is functioning normally and processing familiar objects, then, when we see, we see something in particular. But when we are forced to abstract and judge the intensity of an attribute such as numerosity, a vagueness becomes evident.

Consider matching brightnesses and loudnesses once again. A moderately loud tone is held constant. A very bright light is clearly more intense and a very weak light is clearly less intense. Brightness appears to change continuously with luminance, so we expect to find one light which exactly matches the sound. Instead, there is a broad range of equally acceptable lights. Within the range, one light matches the tone as well as another.

It is not sufficient to dispatch this uncertainty with the usual observation that there are always errors of measurement. These errors have an orderliness which indicates that something basic has been omitted from the traditional interpretation. For example, with myself as subject in informal experiments I have observed that the acceptable range of lights matched to lights is considerably narrower than the range of lights matched to tones. A probabilistic device unreliably ordering points on an intensity line would give equal ranges. A semiorder structure (Suppes and Zinnes, 1963, Section 3.2) on the intensity line again would give equal ranges.

Undoubtedly there are many ways to account for these aspects of the intensity of experience. I wish to do so without losing two attractive features of the traditional mapping interpretation: (1) The traditional interpretation is deterministic rather than probabilistic and (2) magnitudes of different qualities are comparable in the traditional interpretation. My solution, given in the next section, is to make appearances and the degree to which an appearance has an attribute fundamentally different entities. The appearance of a light shall remain a point in psychological space. But the brightness will be a curve in a plane. In addition to yielding an intuitive theory capable of dealing with the considerations above, the analysis gives a new explanation (see Section III.8) of a systematic departure from transitivity observed in matching.

II.6 Points of View and Magnitudes

This section may be skimmed on first reading. The ideas presented are not needed for predicting the main effects in most experiments.

The traditional mapping theory interpretation of matching has each appearance in correspondence with a definite point on the intensity line prior to matching. The alternative to the traditional view offered in this section has each appearance associated with a definite curve. The goal is to let the shape of these curves (henceforth called intensity curves) carry the information which intensity points failed to carry.

Each intensity curve will be the graph of a number-valued function of intensity. The values of the function may be thought of as the extent to which an intensity is characteristic of an appearance. Once these curves are defined, it will be easy to reason with precision about the problems raised in the preceding section.

In the following paragraphs intensity curves will be discussed informally. Then a generalization of the vertices used to define correspondences between appearances and intensities will be introduced. The generalization, called points of view, will be used to place appearances in correspondence with intensity curves.

Intensity curves will be specified in such a way as to be generally unimodal and zero except on an interval. Physically intense stimuli will have modes over a high intensity. Assumptions will be made to force the shape of the curve and the length of the supporting interval to change smoothly as an appearance is varied along a psychological continuum.

Instead of comparing points as in the traditional interpretation the observer will be considered to be comparing curves. As discussed in the final section of J. C. Falmagne's contribution to this volume, there are many alternative ways to order curves. The details of the actual processing carried out by the observer will depend on minor experimental details such as instructions. For example, an observer instructed to bracket (see Section III.8) is likely to do so. Fortunately, it is possible to derive many predictions by simultaneously considering a large class of plausible processing schemes. The constraints[8] defining these schemes are given in the next paragraph.

It will be assumed that the observer is equipped with a device for comparing intensity curves. When given a pair of curves u,v the device responds that u is greater than v , that v is greater than u or that it is unable to choose. For the purposes of this section it will only be assumed that the device has the following properties: (see footnote 8 for a discussion of these properties). If the supporting intervals of u and v are disjoint then it always responds that the curve with the interval on the right is greater. If the curves have exactly the same shape in the sense that it is possible to obtain the curve u by shifting the curve v to the right, then it always chooses u . If the curves have approximately the same shape, then it applies some complicated rule which generally selects the curve to the right as greater. When the curves have very different shapes and overlapping supports, the device responds that it is unable to choose.

To see some of the implications of assuming that the observer uses a device with these properties, consider matching experiments with lights and tones. A very bright light with supporting interval to the right of a moderately loud tone will be judged more intense. Lights with approximately the same appearance will have approximately the same shape intensity curves and so slightly different lights can be reliably ordered. Since the loudness curves need not have the same shape as brightness curves, the ordering device would be expected to fail to choose over a larger range for matching lights to tones than matching lights to lights. Some other qualitative phenomena are considered in Section III.8.

The alternative to the traditional view will be entirely deterministic. There is no need for probability at this level of the theory. However, the quickest way to communicate the alternative theory is to appeal temporarily to the reader's intuitions about probability.

Suppose that instead of specifying the vertex of a projection precisely, a probability distribution over vertices is given with most of its mass concentrated on a particular vertex. Then each appearance defines a distribution of intensities in the following way. The probability of set of intensities is the probability of the set of vertices projecting the ppearance into the set. If suitable restrictions are made on the probability distribution over the vertices, then the intensity of the appearance can be thought of as a random variable with a continuous density.

A point of view is simply a measure defined on vertices. Only measures sufficiently well behaved to permit the definition of the analogue of the density in the preceding paragraph will be considered. But measures other than probability measures will be permitted. The extra generality (signed measures, measures with total mass not equal to one) is not especially important. What is important is to retain a deterministic theory with a partially ordered space of magnitudes sufficiently rich to account for effects like those considered in the last section.

Magnitudes in the traditional interpretation were zero dimensional intensity points. In this alternative they are taken to be the one dimensional intensity curves or, equivalently, the measure on the intensity line used to define the intensity curve. The intensity curves will be used when it is desirable to reason geometrically about psychological phenomena, and the measure will be used when it is desirable to consider properties which are invariant under smooth transformations of the intensity line.

The brightness of an appearance is simply the intensity curve (or measure) induced by a point of view characteristically associated with brightness, and the loudness of an appearance is simply that induced by a loudness point of view. When magnitudes are defined in this way, there is no difference in kind between the brightness and the loudness of appearances. They differ only in the manner in which they are generated. To the extent that curves with different shapes can be ordered, these brightnesses and loudnesses can be ordered. In particular, when overlapping curves of different shapes are compared, the observer's behavior is expected to be variable.

> To return to the distinction between a probabilistic and deterministic theory, a magnitude is an intensity curve or measure. It is not a random variable having a particular measure or density. An analogous physical situation is described in the next paragraph to show that this is more than a verbal distinction.
>
> A stubborn man might refuse to acknowledge the fact that his right index finger is extended in space. In order to avoid being incorrect when asked questions about its location, he might explain that the location is not <u>really</u> a point in space, but a <u>random</u> point in space. We know enough about electricity to quickly enlighten such a man. Not even a random point can be in two places at the same time, and it is safe for an ungrounded man to touch one of the terminals of a light socket.

II.7 Judgmental and Purely Sensory Effects

To recapitulate, physical events are represented by psychophysical functions as appearances in psychological space. When the observer wishes to judge the degree to which an appearance has an attribute he selects an appropriate point of view and generates (by a process like projection) an intensity measure or curve.

This leaves only two ways for accounting for the effects of experimental manipulations which change the relation between stimuli and magnitudes. There are purely sensory changes in which only the psychophysical functions change, and there are purely judgmental changes in which points of view change.

In simple studies the two kinds of effects are likely to be confounded. For example, if an observer is asked to judge brightness and then is asked to judge the yellowness of a series of lights, the most obvious theoretical interpretation is a change from brightness point of view to a yellowness point of view. However, the data may indicate that asking the subject to attend to a different aspect of the lights changes their appearance. This is an empirical question which can be translated into a standard statistical question of the form: Does a model give a significantly better fit of data than a submodel?

For an opposite example, suppose the subject is exposed to an intense blue light prior to viewing the stimuli for yellowness judgments. The appearances of the lights will certainly be affected by the blue light. But, as discussed in Section III.7 there may be judgmental effects in addition to the purely sensory effects. Even if the yellowness point of view remains constant, there are other points of view involved in producing the response. Some of these may change when the distribution of appearances in psychological space changes. These considerations are considered in greater detail in Section III.7.

In the remainder of the chapter, only effects of changes of point of view are considered in detail. The calculation of psychophysical functions and the quantification of changes in psychophysical functions are outside the scope of this paper. Some references to the author's work in this area and some comments on results being prepared for publication are included in Footnotes 9 and 12.

Part III: Elaborations of the Basic Theory and Some Illustrative Applications

In this section the basic theory is elaborated and applied to published data. The elaborations needed are:

1. degenerate points of view for the approximation of central tendencies in published data
2. embedding functions to generate graphical displays of experimental results
3. number appearances to place magnitude estimation, cross modality matching and rating experiments in the same framework
4. inhomogeneities in the intensity continuum to study the consequences of assuming psychological space is bounded
5. general factors influencing the subject's selection of a point of view
6. symmetry in points of view to simplify the calculation of regression effect and Ekman's law.

Most of the elaborations will be incorporated in the discussion of experimental results.

In the following paragraphs it will be seen that the theory is consistent with several essentially different explanations of published experimental findings. The data for choosing between alternative explanations is not yet available. But each of the alternative explanations has testable implications. In Section III.6 I will try to show that in at least one case the experiments suggested by the theory are intrinsically interesting.

The theory is formulated as a description of individual information processing. Most of the data considered are averaged over groups of observers. Fortunately, there are recent theoretical and experimental developments which should help overcome some of the problems[10] involved in obtaining useful data from individual subjects. At this time appropriate individual data are not available.

III.1 Degenerate Points of View

Points of view in their full generality are needed only in the final applications of this section. In the other applications a trivialization of the point of view called a degenerate point of view is used to approximate central tendencies in published data.

If a point of view has all but a very small amount of its mass concentrated on a single vertex, then the intensity curve for an appearance will be a sharp spike over a narrow range of intensities. The limiting case, a point of view with all of its mass concentrated on a single vertex, will be called a _degenerate point of view_. The vertex induces a projective correspondence between intensities and appearances. When the point of view is degenerate the distinction between magnitudes and intensities will be ignored.

III.2 Embedding Functions

The appearance of a physical event is a point in a vector space rather than a number or a number-measured sensation. And so it seems that the convenience of real functions in alternative mapping theories is lost. In this section it is shown that this is not generally so.

Consider a physical continuum $\{x_t\}$ which is carried by its psychophysical function to a straight continuum $\{X_t\}$. For example, consider dial settings t parameterizing the luminance of a light source in a brightness study. To define a real function simply select two different dial settings t and s having appearances A equal to X_t and B equal to X_s. Since every point on the line through A and B can be written in the form $[1 - \alpha]A + \alpha B$ for exactly one number α, we can represent the psychophysical function $t \to X_t$ by a real function $t \to u(t)$ with $X_t = [1 - u(t)]A + u(t)B$. Any such function is called an <u>embedding function</u>.

The definition of an embedding function for a psychophysical function $t \to X_t$ depended upon the choice of arbitrary parameter values t and s. Embedding functions are interval scales in the following sense. A different embedding function v is defined if different t and s are selected. But it is easy to show that there will always be a pair of numbers a and b such that for all t, $v(t) = au(t) + b$. In particular, when s and t are interchanged v is $1 - u$. For this reason every strictly monotonic embedding function can and will be assumed to be increasing.

As illustrated in the sequel, many questions can be considered in detail in terms of the number-valued embedding functions rather than the vector-valued psychophysical functions. It is in this sense that the convenience of real functions is retained in this version of mapping theory.

III.3 Some Qualitative Facts about Projective Transformations

To move between qualitative descriptions of psychological processes and quantitative aspects of data a few easily proven, elementary geometric facts are needed. Proofs[11] and detailed discussions can be found in introductory textbooks.

A projective or linear fractional transformation is a function f defined for all numbers and the symbol ∞. It is defined by four numbers a,b,c,d such that ad and bc are different. For the numbers x such that $cx + d$ is not zero, $f(x)$ is defined by

$$f(x) = (ax + b)/(cx + d).$$

If $cx + d$ is zero, then $f(x)$ is equal to ∞. If c is zero, then $f(\infty)$ is zero. If c is not zero, then $f(\infty)$ is equal to a/c. The word "infinity" will sometimes be used instead of ∞ below.

The transformation f defined with numbers a,b,c,d has an inverse denoted by f^{-1} and defined with numbers $d,-b,-c,a$ in the place of a,b,c,d. The projective transformations form a group with products fg defined by function composition

$$(fg)(x) = f[g(x)].$$

Projective transformations enter the theory in the following way. Suppose a psychological continuum and vertex are given. When another continuum and vertex are chosen, a correspondence between appearances on the two continua is obtained by pairing appearances projecting onto the same intensity. If the first continuum has embedding function u and the second has embedding function v then there will always be a unique projective transformation f relating u and v in the following sense. If the appearance of x on the first continuum is paired with the appearance of y on the second, then $v(y)$ equals $f[u(x)]$.

To expedite the study of variations of points of view, f can be decomposed into a product $f = pq$. This can always be done so that a change of the first point of view only changes q and a change of the second only changes p.

The decomposition of f into p and q can be done in many ways. A device to force uniqueness upon the decomposition is to label the intensities with numbers in the manner that embedding functions were used to label appearances with numbers. For a fixed pair of intensities A,B any intensity can be written uniquely in the form $iA + (1 - i)B$ for exactly one number i.

The factors p and q are specified exactly by the following condition: if the appearances of both x and y project to the intensity $iA + (1 - i)B$ then $qu(x)$ equals i .

> Some statements using these number assignments on the following pages may strike the reader as a retrogression to the earlier theories which failed to distinguish numbers from nonnumerical psychological processes. Expressions such as "the intensity i_o " are written in place of more precise or clearly nonnumerical expressions such as "the intensity $i_oA + (1 - i_o)B$ " or "the projected intensity of the appearance of the weakest stimulus." However, the numbers are never used in an essential way. Every argument with numbers can be replaced by a nonnumerical statement. The translation is routine.

A basic property of projective transformations is that if $f(x_n)$ equals $g(x_n)$ for three different values of x_n then f equals g. One implication is that unless f is the identity transformation, $x \to x$, there can be no more than two solutions to the equation, $f(x) = x$.

A number or infinity is a fixed point of f if it satisfies the equation $f(x) = x$. If infinity is a fixed point of f , then f is called an affine transformation and can be written in the form $f(x) = ax + b$ for some real constants a and b .

The affine transformations f are the only projective transformations satisfying the monotonicity condition

> x less than y implies $f(x)$ less than $f(y)$
> for all real numbers x and y .

If f and g have exactly the same fixed points then f and g commute, i.e., fg equals gf . As a special case, two affine transformations with a common real fixed point commute.

As another special case, if f and g both have zero and infinity as fixed points, then they are both in the commutative group of similarity transformations $\{ x \to ax \}$. In particular, if f and g both carry x_o to zero and x_+ to infinity, then for some constant a, $g(x) = (gf^{-1})[f(x)]$ equals $af(x)$.

III.4 Magnitude Estimation as a Matching Experiment

In this section a simple interpretation of magnitude estimation is offered. The goal is to introduce some ideas needed in the sequel. In the process, an interpretation of Stevens's power law is given. Later (Section III.7) the evidence for the power law is reconsidered and a different interpretation is given. This section is concluded with some comments on some extra assumptions used to relate projective theory to the power law.

As is now common among psychologists, perceived numbers are given approximately the same theoretical status as perceptions of traditional stimuli. The physical events representing particular numbers are assumed to give rise to appearances falling on a straight line in psychological space and an increasing embedding function u is assumed to relate numbers to number appearances.

> Some complications--~~tangential~~ tangential to the interpretation of the magnitude estimation experiment as a matching experiment are avoided by treating the set of number appearances as if it were topologically equivalent to a line segment. No matter what topological structure the set of number appearances is assumed to have, a subject with a fixed set of rules for naming appearances and a definite rate of speaking must select his responses in an experiment from a finite set of number names. In Section III.7 the discreteness of the set of responses is acknowledged and used.

Numbers are assumed to have a special feature as a consequence of being extremely familiar and abstract. The observer can generate and operate upon a number appearance in the absence of an obviously appropriate stimulating physical event.

In the magnitude estimation experiment, the observer attempts to choose numbers with the same magnitude as auditory, visual or other appearances. For concreteness, consider a continuum of lines of varying length. Magnitudes are obtained from number appearances with a number point of view and from line appearances with a length point of view. If both points of view are degenerate, then it will be possible to match magnitudes exactly.

To make contact with the power law, suppose the length continuum is straight with embedding function v . Then, as noted in the preceding section, there will be constants a,b,c,d such that number y matches line x if and only if

(1) $$u(y) = [av(x) + b]/[cv(x) + d] \quad .$$

S. S. Stevens's power law asserts that for matching pairs (x,y), y is a power function of x; i.e., y equals ax^n for real constants a and n. Stevens and many others get good fits of data with linear functions when mean $\log y$ is plotted against $\log x$. This generalization holds for length and a very large number of other physical continua. The fit is best when x is large.

The most straightforward way to relate the power law to projective theory is to adjoin assumptions like those presently being made by Stevens and his associates:

1. The number embedding function u is linear.
2. The physical continuum embedding function v is a power function.
3. (Monotonicity) If x matches y, x' matches y' and x' is greater than x, then y' is greater than y.

The power law data can be easily deduced from these assumptions. Since affine functions are the only monotonic projective functions, c must be zero in equation (1). Consequently, for large x, $\log y$ is approximately equal to a linear function of $\log x$.

None of these assumptions are essential parts of the theory. They are intended as convenient by expendable specializations of the theory. The monotonicity assumption 3. is relaxed in Section III.7. Some experimental evidence is considered there.

The assumptions 1. and 2. restricting the form of the embedding functions have been systematically studied by D. Curtis and S. Rule in the context of another mapping theory. Some of their findings are summarized in a recent symposium paper (Curtis and Rule, 1972). Using a general curve fitting algorithm of Kruskal they have computed the best fitting monotonic embedding functions for direct measurement data. In partial support of the assumptions used here, they find the physical continuum embedding function to be a power function. The number embedding function is also a power function, but with exponent close to one.

Curtis and Rule's work shows that it is not necessary to assume particular functional forms for the embedding functions. The functions can be calculated from experimental data. Some alternative methods for obtaining embedding functions from psychological data will be published[9] separately.

Projective theory is neutral with respect to the controversy over the validity of the power law and the competing refinements of the law. The functional form of embedding functions is also a peripheral concern for the theory. The theory is concerned with the quantification of the relationships between functions and the psychological processes generating functions rather than with the shape of any particular curve[12].

III.5 On the Size Weight Illusion and Concurrent Graphs

J. C. Stevens and L. L. Rubin (1970) have recently published a remarkably simple and precise study of the size-weight illusion. Magnitude estimates of weights of various size were made under carefully controlled conditions. To a very high degree of accuracy the logarithm of the (geometric mean of) estimates of weights of fixed size were linear functions of the logarithm of weight measured in grams. When the slope and intercept of the lines were estimated by the standard least squares procedure applied to each of the sizes separately, a rather odd relation was discovered. The various slopes were a definite function of the intercepts. Geometrically, the extrapolations of the separately fitted lines all intersected at one point. This point, coincidentally, corresponded to the heaviest weight the subject could lift in the experimental position.

J. C. Stevens (1972) recently showed that this is not an isolated occurrence. Concurrent magnitude estimation functions have been observed many times, in many laboratories, with many dimensions of stimulation. In nearly all of the published cases cited by Stevens the point of intersection was interpretable.

In order to further elaborate the theory and illustrate its application, an interpretation of the size-weight illusion will be attempted. No attempt will be made to deduce the fact that the curves are linear on logarithmic paper. Instead we will attempt to deduce the existence of some transformations having the property of logarithm transformation. In the concluding paragraph of the section it will be shown that the Stevens-Rubin finding can be expressed algebraically and tested without fitting special functions or extrapolating.

An essential part of the theory needed in the interpretation of the Stevens-Rubin finding is the boundedness of psychological space. Nearly all psychological continua definitely seem bounded. Few of us have any conception of the heaviness of a 500 pound weight. The range of intensities is also bounded. It is probably unnecessary to consider negative intensities. The bounds of the intensity continuum are probably not sharp. More likely our ability to use intensities varies over the range of intensities so that only in the middle range do we process and compare quickly and accurately. The adjustments of point of view considered below and in greater detail in III.7 are a sort of calibration enabling us to operate over ranges where we are most efficient.

The subject's behavior in the experiment is considered as a process with five stages: estimation of heaviness prior to contact, the initial attempt to lift, revised calculation of heaviness, compensatory muscular adjustment and judgment.

One idealization suggested by the Stevens-Rubin finding is this: There is a useable range of intensities extending from a lower intensity i_o to an upper intensity i_+. There is a straight effort continuum in psychological space corresponding to the appearance of muscular effort in the lifting position ranging from e_o for no effort to e_+ for the greatest effort the subject can or will produce. The number continuum is straight and its point of view constant throughout the experiment. The effort point of view is also constant. All points of view are assumed degenerate. Effort e_o projects to i_o ; e_+ projects to i_+ .

The subject views the stimulus. From its size and his guess about its density he estimates its appearance on a straight continuum. He selects his heaviness point of view such that he will be able to deal with whatever stimulus the experimenter offers. In particular, he chooses so that zero weight projects to i_o and a barely liftable object projects to i_+. Then his best estimates of the appearance project into the useable range of intensities.

It will now be assumed that all the heaviness vertices are independent of irrelevant changes in appearance in this sense: Weights differing in color, size, or texture say, but equally resisting the subject's efforts to lift, upon being lifted, have appearances projecting by such vertices to equal intensities.

This extreme position implies that all of the effects of size upon heaviness are indirect, that size affects the heaviness only by affecting the point of view. Experimental implications of this are developed in the next section, III.6.

It follows from straightness and the degeneracy of the points of view that the intensity of a weight of x pounds and size λ will be of form

$$p_\lambda[v(x)]A + [1 - p_\lambda v(x)]B$$

where A and B are points on the intensity continuum, p_λ is a projective transformation and $v(x)$ is a number valued function of weight only.

To make an initial pull the subject chooses an effort with intensity curve closest to the heaviness curve of the estimated weight. After he makes contact and has more information he adjusts the strength of contraction and completes the lifting. Finally he judges by choosing a number projecting onto the same place on the intensity continuum as the final heaviness.

This will give equation

$$qu(y) = p_\lambda v(x) \tag{2}$$

for matching number y and weight x pairs with size parameter λ. Here q is a projective transformation from the conversion of number appearances into intensities, u is a number embedding function. As noted in Section III.3 only p_λ will change when the heaviness point of view is changed.

The equation (2) may be rewritten as

$$squ(y) = sp_\lambda r^{-1}[rv(x)]$$

where r and s are any increasing projective transformations such that $rv(0) = s(i_o) = \infty$. For instance there are

$$r(x) = 1 / [\ v(0) - x\ ] \text{ and}$$

$$s(x) = 1 / [\ i_o - x\ ] .$$

Each of the p_λ must satisfy

$$p_\lambda[v(0)] = i_o \qquad \text{and}$$

$$p_\lambda[v(x_+)] = i_+ .$$

Consequently if we plot $squ(y)$ against $rv(x)$, we obtain a portion of the graph of the projective transformation $sp_\lambda r^{-1}$. But for each λ this transformation has ∞ as a fixed point and consequently is affine. Thus for some numbers a_λ and b_λ

$$sp_\lambda r^{-1}(x) = a_\lambda x + b_\lambda .$$

Thus each graph is a straight line. Furthermore, each projective transformation $sp_\lambda r^{-1}$ must take $rv(x_+)$ to the number $s(i_+)$. Since this condition is independent of λ, all of the straight lines must converge on the point of the plane corresponding to the heaviest liftable weight and its corresponding number.

In this way the Stevens-Rubin result can be interpreted geometrically.

Before concluding this section it seems advisable to call attention to the fact that the Stevens-Rubin result can be experimentally tested without curve fitting or extrapolating. There is an unusual property of the graphs of concurrent linear functions which is not affected by nonlinear transformations and which can be tested without considering the values of the function near the point of concurrence. To define this property consider the functions $f = f_\lambda$ such that $f(x)$ is the number matched to weight x of size λ. For any two such functions f and g that are strictly increasing and have overlapping domains and ranges it is possible to define a transformation of a portion of the domain of g denoted $\bar{f}g$ and defined by

$$\bar{f}g(x) = y \quad \text{if} \quad f(y) = g(x) .$$

If $f, g, h, \ldots$ can simultaneously be transformed to concurrent linear functions then the various transformations $\bar{f}g$, $\bar{f}h$, $\bar{g}h$, etc. will commute. Commutativity is a property which can be experimentally tested even when there are no data points near the point of concurrence. It is in this sense that the Stevens-Rubin result can be reformulated without curve fitting or extrapolation. For further details on functions which can be transformed to concurrent linear functions see Levine (1972, Section V).

III.6 Experiments and the Size-Weight Analysis

There will generally be several alternative interpretations of an experimental result like the Stevens-Rubin data in the theory. But each has testable experimental implications. In this section some implications of the preceding analyses are discussed.

All of the size-weight illusion was attributed to the point of view. Consequently if the point of view could be controlled, the illusion could be controlled.

The crucial part of the analysis is the selection of the heaviness point of view prior to contact with the weight. There are two ways to exploit this. A complicated way would be to incorporate estimates of heaviness made prior to contact or measures of the force exerted by the subject during his initial pull on the weight in a model and to mathematically infer the point of view. An easier and more amusing way would be to use an experimental technique developed by Nielsen (1963) to separate the feel and the sight of the lifted object.

Nielsen showed that it is possible to convince an observer that he is viewing his own body when in fact he is viewing another person's. In a study of free will he instructed each subject to move his hand along a painted line. A white glove obscured identifying marks on the hand. The line was placed in a box containing a mirror system such that the subject was actually viewing a confederate's hand. Despite the fact that the confederate repeatedly moved his hand off the line and did not respond to the compensatory motions of the subject, the illusion persisted. Eighteen of Nielsen's 20 subjects regarded the viewed hand as their own pulled by "magnets" or otherwise not completely under control.

It seems possible to adapt Nielsen's technique to obtain control of the point of view. Consider a subject lifting an object of unknown size. The subject places his hand in a poorly illuminated box. He lifts one object and views a confederate lifting another. When the object is lifted and held steady in a standard position for a full second, the illumination in the box is increased so that the subject can easily see the object in the confederate's hand.

One further manipulation is needed. In order to be certain that the subject doesn't change his point of view prior to making his judgment, he is instructed to hold his hand steady. If ne fails to obey these instructions or wishes to terminate the trial or otherwise moves his hand, the light in the box is automatically extinguished. There are two anticipated effects of this manipulation. First, if the subject changes his heaviness point of view, then, as a consequence of the hypothesized relation between strength of pull and heaviness, there should be a compensatory movement of the subject's hand. The prompt extinguishing of the light gives the subject the feedback needed to learn not to change his initial point of view. Secondly, the manipulation may protect the illusion that the subject is viewing his own hand. When he moves his hand he promptly sees an effect; when he does not move his hand, the viewed confederate's hand remains steady.

Under these conditions the theory predicts a greatly diminished size-weight illusion. The correlation between viewed size and judged weight should be insignificant or positive rather than negative.

III.7 Changes of Modulus and Changes of Point of View

In his critique of mapping theory, Krantz (1972) lists a number of generalizations about direct measurement. He argues that the alternatives to relation theory are implausible because they require mental arithmetic or complicated mental processes. This is no longer a valid criticism. Within projective theory the invariances inherent in the generalizations follow from a well-known fact of geometry: projective transformations with the same fixed points commute. Projective theory imposes projective transformations upon data; plausible (see below) psychological assumptions restrict the fixed points. Full details will be given in a separate paper (Levine, 1973c). The basic idea is sketched in an interpretation of the effects of change of modulus in this section.

In some magnitude estimation studies the experimenter instructs the observer to assign a particular number to a particular stimulus. In cross modal matching of lights to tones, the experimenter will sometimes instruct the subject to pair a particular light with a particular tone. Changing the initial pairing of number to stimulus or light to tone is called changing the modulus.

Krantz's major objection to mapping theory is that it fails to give a plausible interpretation of some invariances associated with the change of modulus. One of the central generalizations of direct measurement is this: If $f(x)$ is the average number matched to physical stimulus x in a magnitude estimation study with one modulus and $g(x)$ with another modulus, then there is a constant such that for all of the stimuli, $g(x)$ is the constant times $f(x)$. Consequently, ratios of number responses are said to be invariant. There is an analogous invariance assumed for cross modality matching. (The exponents computed from cross modality matching data are independent of modulus.)

The main purpose of this section is to develop the notion of having several points of view for a single dimension of experience. In the process it will be shown that this mapping theory quite easily accounts for changes of modulus. Even the simplest versions of the theory yield a plausible account of the invariances. No mental arithmetic or complicated ideation of any kind is required.

After the invariances are dealt with, some steps are made towards a less idealized description of the experiments. Hopefully these more realistic descriptions will lead to a specification of the conditions under which the invariances are observed and an account of some of the systematic departures from the generalizations about the experiments. The section is concluded with a list of some of the factors which may influence the selection of a point of view.

In the following discussion, only degenerate points of view are considered. The notations i_o and i_+ of Section III.5 for extreme intensities will be used.

Suppose that instructing the subject to change his modulus in a magnitude estimation study only causes him to change his number point of view. The psychophysical function and the magnitudes of the appearances of the physical stimuli are unchanged. Further suppose that each number point of view pairs i_o with the appearance of zero and i_+ with the appearance of the subject's largest number appearance. Then there is a simple geometric fact which can be related to the invariances. If p and q are projective transformations such that for two different numbers x_1 and x_2

$p(x_1) = q(x_1)$ = zero and

$p(x_2) = q(x_2)$ = infinity

times $p(x)$. Consequently for any numbers y and z such that $x_1 < y < z < x_2$ the ratios $p(y)/p(z)$ and $q(y)/q(z)$ are equal.

To avoid reference to infinity one may assume $p(x_2) = q(x_2) = M$ for some large positive M instead of $p(x_2) = q(x_2)$ = infinity. (M may be thought of as the largest number that is psychologically significant for the observer.) Then the cross ratios

$$\frac{p(y)}{p(z)} \cdot \frac{M - p(y)}{M - p(z)} \quad \text{and} \quad \frac{q(y)}{q(z)} \cdot \frac{M - q(y)}{M - q(z)}$$

are equal. If M is very large relative to $p(z)$ and $q(z)$ then the ratios $p(y)/p(z)$ and $q(y)/q(z)$ are very nearly equal. This follows from

$$\frac{M - p(y)}{M - p(z)} = 1 + \frac{p(z) - p(y)}{M - p(z)} \quad \text{and}$$

$$0 < \frac{p(z) - p(y)}{M - p(z)} < \frac{1}{M/p(z) - 1} \quad .$$

To apply these observations to magnitude estimation, recall that the interpretation of magnitude estimation as matching experiment (Section III.4) implies that the number matched to a physical stimulus is a projective transformation of the physical continuum embedding function followed by a second projective transformation relating intensities to number appearances. When the number embedding function u is linear, the invariance follows from an elementary calculation with projective transformations. (Take $u(0)$ equal to zero for interval scale u .)

The invariance of cross modality matching with changes of modulus can be interpreted similarly. The key to the argument is that the changes following the change of modulus leave the correspondence of a pair of intensities with a pair of appearances invariant. This can be interpreted as i_o and i_+ corresponding to particular appearances as above or it can be interpreted as the prolongation of a straight psychological continuum either intersecting or being parallel to the intensity line. No matter which of these alternatives is chosen, the invariances can now be interpreted in several ways, none of which require mental arithmetic, complicated ideation or other implausible p[illegible]esses.

The device of varying the point of view removes the major criticism of mapping theory by providing a number of simple processing schemes which imply the invariances. This device also seems useful in accounting for other aspects of direct matching experiments, especially experiments using a large range of physical stimuli. It seems profitable to consider the observer functioning like an automatic gain control device using inputs and a portion of its own response to adjust its characteristics in order to operate over an optimal range. This could be achieved by adjustments of the number point of view or by adjustment of the points of view relevant to the experimenter controlled events. For the present time, only number points of view will be considered.

As a special case, suppose that the observer has exactly two number points of view, labelled Lo and Hi. Both can be thought of as projecting a grid of numbers upon the intensity line. Assume that Lo projects many more of the number appearances for which the observer has number names into a neighborhood of i_o than Hi so that Lo is better suited for assigning different names to different magnitudes of weak stimuli. Lo consequently will give steeper matching functions for these weak stimuli when it is used. On any given trial the subject uses either Lo or Hi. The averaged response to stimulus x will be

$$f(x) = P_{Lo}(x)L(x) + P_{Hi}(x)H(x)$$

where P_{Lo} and P_{Hi} are the proportions of trials each is used, and L and H are conditional matching functions. If Lo is used only for very weak stimuli in the sense that $P_{Hi}(x)$ increases rapidly from zero to one, then one still predicts all of the <u>qualitative</u> properties accepted as evidence for the power law; namely, increasing, asymptotically linear, concave plots of Log f plotted against Log x .

This elaboration of projective theory has the curious implication that magnitude estimates need not be monotonic functions of physical intensity. Since L has steeper slope than H , there will be x 's such that $L(x)$ is higher than $H(x)$. In an experiment of Ross and Di Lollo (1968) observers judged very many light weights. Without warning they were given a much heavier weight to judge. Suppose the observer retained a high slope, low point of view for moderately heavy weights, but with even heavier weights changed to a high point of view. Then one would expect precisely the pattern of nonmonotonicities reported.

Another empirical consideration suggesting this elaboration of projective theory is the observation occasionally reported by experimenters having a great deal of experience with direct measurement. It is reported that in the method of free magnitude estimation (in which the subject is not assigned a modulus) it is especially easy for the subject to give reliable estimates. It is not clear in relation theory that free magnitude estimation should be preferable to any other method. In the projective mapping theory there are many possibilities. In the free method the subject selects the point of view. In the variants, the experimenter influences the selection of the point of view. The unconstrained subject is free

1. to select a point of view which gives him an opportunity to use a range of numbers of great familiarity,
2. to choose a point of view which gives good resolution over the range of intensities into which most of the stimuli project,

3. to choose a point of view which is easy to recall for successive trials,

4. to choose a point of view so that the number intensity curves will have approximately the same shape as the weight intensity curves.

The next step in developing this line of reasoning seems to be to obtain detailed individual data with deliberate manipulations to control the point of view. In the context of such experiments it may be profitable to design training procedures for giving the observer control of the point of view and statistical procedures for inferring the point of view from the fine structure of data.

III.8 On the Regression Effect

This application has been selected to illustrate arguments in which points of view which are not degenerate play an essential role. An intransitivity of matching called the regression effect and a generalization about the relation between uncertainty of judgment and intensity of stimulation called Ekman's law are considered briefly.

When an observer adjusts a bright light until it appears as intense as a loud tone, he consistently underestimates the adjusted continuum in the following sense: if light ℓ_1 is adjusted to match tone t_1 and then the tone is adjusted to t_2 matching ℓ_1, then t_2 is generally less physically intense than t_1. This phenomenon is called regression effect. It has been replicated and is valid for many pairs of continua (Stevens and Greenbaum, 1965). There is also a tendency for overestimation of faint adjusted stimuli.

In earlier mapping theories, regression was an artifact to be averaged away or made less apparent by complex experimental procedures. In the present theory magnitudes are only partially ordered, so the intransitivity of matches is not an embarrassment. Two interpretations of the phenomenon will be offered. In both cases the intransitivity follows from the manner in which the observer tries to equate magnitudes of different shape.

In the first interpretation the observer matches magnitudes of differing shape by computing an index of agreement. As a crude example, he might calculate the correlation coefficient between representative values on intensity curves and select stimuli to maximize it.

If the observer is maximizing an index of agreement between magnitudes, the intransitivities are predicted. For to find ℓ_1 he is adjusting lights to find a maximum in one set of index values. But to find t_2 he is searching for a maximum in a different set. The intransitivity occurs when the ℓ_1, t_2 pair has a higher index of agreement than the ℓ_1, t_1 pair. And since the ℓ_1, t_1 pair is included in the set of pairs available for the second match, the intransitivity is generally expected.

Some experimenters have attempted to obtain precise matches by instructing the observer to bracket. When bracketing the observer chooses a light ℓ_- just clearly less intense than tone t_1 and a light ℓ_+ just clearly more intense. Finally the observer ignores the tone and selects a light $\ell = \ell_1$ which is in some sense exactly[9] in between ℓ_- and ℓ_+.

If the shape (skewness) of the intensity curve changes appropriately then the regression effect can also be deduced from this bracketing. One set of hypotheses sufficient to assure exactly the right changes in shape is this:

1. The light and tone continua are straight.
2. The light and tone continua are parallel to the intensity continua.
3. Psychological space is two dimensional.
4. Both the brightness and the loudness points of view are radially symmetric. (Here a point of view is called radially symmetric when there is one point of view such that every rotation of psychological space leaving this point of view fixed leaves the distribution defining the point of view fixed.

To make the bracketing procedure explicit some notation is needed to express the fact that ℓ_- is just clearly less intense, ℓ_+ is just clearly more and ℓ_1 is exactly in between. Suppose the bracketing lights are selected with reference to functions

$f(x,m)$ = the measure of intensities smaller than x calculated with the magnitude of stimulus m and

$g(x,m)$ = the measure of intensities larger than x .

If brackets ℓ_-,ℓ_+ are chosen so that for small criteria $\varepsilon < \delta$ there are intensities $x < y$ such that

$$\delta = f(x,\ell_-) = g(y,\ell_+)$$

$$\varepsilon = f(x,t_1) = g(y,t_1)$$

and match $\ell = \ell_1$ such that for some $x < y$

$$f(x,\ell) = g(y,\ell)$$

$$g(x,\ell_-) = f(y,\ell_+)$$

then by elementary arguments one can deduce that there will be an apparent underestimation of very intense stimuli and an apparent overestimation of very feeble stimuli.

A further consequence of the unnecessarily strong simplifying assumptions used to deduce the regression effect is an approximation of Ekman's law (S. S. Stevens, 1966; also Ekman, 1961) on the uncertainty of judgment. For high levels of stimulation the ratio of a measure of scatter such as interquartile range to the modal intensity will be very nearly constant.

Footnotes

[1]Acknowledgments: Valuable criticism on this work has been received from Jean Claude Falmagne, Peter Freyd, Francis Irwin, Dorothea Jameson, Ann King, R. Duncan Luce, L. E. Marks, Jacob Nachmias, Klaus Riegel, Harris Savin and J. C. Stevens.

[2]See Krantz (1972) for a finer classification of the classical theories.

[3]For a lucid discussion of these constructions, see Coxeter, The Real Projective Plane. That introductory text contains much more geometry than is needed in this paper.

[4]See Hurvich (1969) for a lively account of this story. Hering's psychological ideas were rejected when introduced decades ago because they were judged to be in conflict with contemporary physiological knowledge. Now physiology has changed and Hering's color theory is widely accepted. Although Professor Hurvich does not agree in conversation or in print, the physiological realizability of Hering's psychological ideas is irrelevant in the twentieth century as it was in the nineteenth.

[5]The word "acquire" is used advisedly. The reader may choose to think of the ability as acquired in the course of the experiment, in developmental time or evolutionary time. The equations of the theory are independent of the choice.

[6]See Ross (1972) for more recent references.

[7]Instead of introducing an abstract continuum in psychological space some theorists have assumed that there are mappings from one psychological continuum to another. For example, lights of varying brightness might be

compared with sounds of varying loudness by referring each to a continuum of lines of varying length. Physical laws relating distance to a source with physical intensity are sometimes used to defend the correlation and induce quantitative relationships. This view will not be used here. In addition to the objections of Krantz (1972) there is a vulnerability to rejection in experiments using the following considerations: With procedures such as selective adaptation, the correspondence between length and perceived length as well as the relationship of appearances on a length continuum may be systematically changed. These changes should affect the crossmodal matches in predictable ways. For this reason, a new continuum is introduced.

[8]An analogy may make the criteria for an acceptable device clearer. Consider an individual with imprecise knowledge of the distribution of heights in various populations asked to make decisions of the following kind. An individual is to be selected at random from two populations. The decision maker is required to predict which population contributes the taller individual. Three critical comparisons are (1) mice with men, (2) men who have just taken their shoes off with men who have just put their shoes on, and (3) horses with men. The decision maker is predictable in (1) since the distributions do not overlap. The decision maker attempting to maximize the proportion of correct predictions is predictable in case (2) since the unknown distributions will have very nearly the same shape. Although the crucial statistic may be much larger in the last case than in the second case, the decision maker is unpredictable because he lacks the information he needs to make a rational decision. The distributions have different shapes and they overlap.

[9]Part IV of the earlier version of this paper is to be submitted for publication separately to the Journal of Mathematical Psychology. It contains an outline of some unpublished results on functional equations. Some related published papers are Levine (1970) and Levine (1972).

[10]An example of a major problem is the effect of repeated exposures to the stimuli on the observer. After many exposures the observer knows the range of stimuli that will be used. This can have complex effects on magnitude estimates. For an example of some important recent results, see Teghtsoonian (1972).

[11]In fact, almost every assertion can be proven directly from the definition by routine algebraic manipulations.

[12]Some tools for this quantification are presented in Levine (1973a) and Levine (1973b). The first paper suggests using psychophysical data to define local semigroups with essentially unique matrix representations. The entries in the matrices quantify the relations between curves. The second paper introduces a method for using fourier analysis to compute the quantifying matrices with great precision from matching functions defined on small ranges of stimuli. In collaboration with D. Saxe, a computer program to do this calculation is being prepared for public use.

[13]If the light ℓ is selected as exactly in between by bisecting an angle on the dial of an instrument used to control the lights, then the instrument can be adjusted so that by successive approximations a bias is introduced which is about the same size but in the opposite direction as the regression effect. In the analysis that follows it is assumed that the observer can learn to select the matching intermediate light without being influenced by angles on adjusting dials.

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