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ABSTRACT

This booklet briefly traces the history of algebra up through the present day modern algebra. The use of proofs, matrices, properties of groups, and linear equations and computers are discussed. (DT)

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20TH CENTURY ALGEBRA IN HIGH SCHOOL

RICHARD V. ANDREE

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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CONTENTS

INTRODUCTION

SOME OF the earliest algebraic problems on record are "puzzle problems" that presumably were solved by algebraic reasoning expressed in words, without the use of mathematical symbols. Many puzzles are still solved in this fashion. Diophantus of Alexandria (probably late third century A.D.) is often credited with the introduction of the algebraic abbreviations that led eventually to the "ninth-grade algebra" of the first half of the twentieth century. Much of this was in reality nothing but arithmetical juggling with numbers replaced by letters, combined with the solution of quadratic equations and of systems of linear equations. About the middle of the nineteenth century the more abstract and vastly more general nature of algebra became apparent to mathematical scholars, and the modern algebraic concepts evolved—concepts now finding their way into the elementary and secondary school curricula.

In general, modern mathematics is the study of basic structure. This includes the structures of number systems, geometries, topologies, and calculi. Mathematicians study the abstract structure of problems in engineering, physics, chemistry, and logic. The basic structure of an economy, a political system, or a language is also part of modern mathematical investigation. Many different disciplines apply theory and theorems developed by mathematicians. However, the study of abstract structures is the mathematician's primary field of research.

ALGEBRA
THE STUDY OF FINITE STRUCTURES

ONCE it was popular to separate mathematics into divisions, such as algebra, analysis, geometry, and statistics. Such divisions still persist in the general vocabulary and in some school curricula, but the boundaries have merged and coalesced. In general, *algebra deals with finite structures,* while analysis is the study of continuous structures. Algebra includes the theory of equations, number theory, matrix theory, algebraic systems (group, ring, field, and so on), and many other branches. Typical branches of analysis are calculus and differential equations. Since the book you are reading, the table on which it rests, and indeed the entire Galaxy are each composed of only a *finite* number of particles, it may seem that little of practical importance is lost by restricting one's attention to finite structures. That conclusion, however, would be incorrect. Although analysis assumes a continuous (infinite) structure, its elegant theory has many applications to finite, real-life situations. Indeed, it is only the existence of the modern computer, born in the last half of the twentieth century, that has made the extensive use of finite (algebraic) methods feasible substitutes for the continuous methods of classical analysis. In today's world, many difficult problems of analysis are being solved by using algebraic techniques.

Algebra from the Ancient World to Today

Originally algebra (that is, the study of finite structures) arose in problems relating to numbers. Babylonian tablets (circa 3100 B.C.) contain algebraic problems. The Rhind (Ahmes) Papyrus, a copy (circa 1700 B.C.) of an earlier manuscript, contains abstract problems and puzzles such as "A number and a quarter of that number together give fifteen. What is the number?" Practical problems such as the equitable distribution of wages to laborers, the calculation of the amount of grain needed to produce bread and beer for a given number of people, the number of bricks needed to construct a building ramp of given dimensions, and other area and volume problems are also included. (The reader interested in further details about ancient Egyptian, Babylonian, and Greek history will find no more delightful source than the book *Science Awakening*, written by one of the world's outstanding algebraists, B. L. van der Waerden.[1])

About 2,500 years later (circa A.D. 800) a more favorable notation permitted the serious study of algebraic equations. In the sixteenth century, the solution of the cubic equation by Italian mathematicians led gradually to semimodern notation and to the development of the basic structural theory of polynomial equations. The fundamental theorem of elementary algebra (that a polynomial equation of degree *n* has at least one root) was not proved until almost 1800 by K. F. Gauss.

By 1900 the more general and abstract nature of modern algebra was apparent to specialists. However, even though almost every university had a course devoted essentially to the theory of equations (see Chrystal's *Algebra*[2] or Bocher's *Algebra*[3]), the modern abstract algebra was seldom taught. By 1920, invariant theory, matrix theory, group theory, and field theory were studied in most graduate programs (see Dickson's *Algebraic Theories*[4]), but undergraduate programs in abstract algebra did not begin to appear regularly until about 1940. By 1960 almost every college and university worthy of the name provided undergraduate courses in modern abstract algebra. Of even greater importance is the effort of large groups of competent mathematicians to make modern algebra available at the secondary school and elementary school levels. The Twenty-third Yearbook of the National Council of Teachers of Mathematics, *Insights into Modern Mathematics*, published in 1957, presents an excellent overview of the mathematical preparation expected of a good high school teacher. (It might be well to take this yearbook off your library shelf and devote at least thirty minutes a week to its uninterrupted study. This practice will pay unexpected dividends.) The work of the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, the University of Oklahoma Mathematics Service Committee,

[1] *Science Awakening*, trans. Arnold Dresden (Groningen, Netherlands: Erven P. Noordhoff, 1954).

[2] George Chrystal, *Algebra: An Elementary Text-Book for the Higher Classes of Secondary Schools and Colleges* (New York: The Macmillan Co., 1886).

[3] M. Bocher, *Introduction to Higher Algebra* (New York: The Macmillan Co., 1907).

[4] L. E. Dickson, *Modern Algebraic Theories* (Boston: Benjamin H. Sanborne Co., 1926).

the University of Maryland Mathematics Project, the Ball State Teachers College Experimental Program, the Minnemath (Minnesota Mathematics Project) group, and many others have provided the foundation for thousands of high school students' study of Boolean algebra, matrices, groups, fields, and rings, as well as systems of linear equations and the "laws of algebra." More good structural algebra is being taught in many high schools today than was available in the finest graduate schools a hundred years ago.

Although algebra is one of the oldest branches of mathematics, more algebraic theory was developed in the fifty years from 1900 to 1950 than in the 5,000 years between 3100 B.C. and A.D. 1900. Furthermore, more algebraic (finite) mathematical theory was published in the fifteen years from 1950 to 1965 than in the previous, highly productive fifty-year period. Abstract algebra is an interesting, growing branch of mathematics.

Long ago Euclid realized that both algebraic and geometric theorems required proofs based on postulates (axioms) and previously proved theorems. If you peruse the books of Euclid, you will find that they contain algebra, particularly number theory, with proofs set up in much the same manner as that currently used in high school geometry. Examine, for example, the theorem that

$$N = 2^{n-1} \cdot (2^n - 1)$$

is a perfect number if $(2^n - 1)$ is prime, given in Proposition 36, Book 9 of the *Elements*. The rigorous approach to geometry flourished and grew, but the concept of algebra as a coherent whole to be developed from a set of basic premises using deductive logic seems to have been more or less lost until it was revived in the current century.[5]

The Use of Proofs in Algebra

It is now common practice in better high schools to derive the rules of algebra from a few premises just as geometric theorems are proved. Interestingly enough, the proofs of algebraic theorems turn out to be neater and simpler than the geometric proofs usually studied in tenth-

[5] Some of the most important recent contributions to modern abstract algebra were published under the pseudonym "Nicholas Bourbaki." During the 1940's, mathematicians began to notice that an unknown French mathematician named Nicholas Bourbaki was producing really excellent mathematics at a prolific rate and covering a wide variety of topics. By 1950 most mathematicians were aware that "Nicholas Bourbaki" was actually a pseudonym (pilfered from an obscure Napoleonic general) used by a coalition of some of the best French mathematicians. The actual members of the coalition change, but the production of high-quality mathematics continues. Some careless future historian may well credit Bourbaki with being the greatest mathematician of the twentieth century.

grade geometry. Hence, they provide an earlier introduction to the techniques of mathematical proof.

A simple example is in order. Consider the axioms for equality. (Did you realize that the notion of equality was based on axioms?) Originally the concept of equality meant identity. Two things were equal if and only if they were identical (exactly the same). Gradually it became clear that there were various types of equality. The two "drumsticks" of a chicken are not identical (at best, they are mirror images of one another); but for the purpose of eating, they may well be equal. The fractions $\frac{21}{42}, \frac{1}{2}, \frac{13}{26}, \frac{1749}{3498}$, and $\frac{i(2 + 3i)}{-6 + 4i}$ are not identical, but they are said to be equal. In one sense, two triangles are "equal" if they are similar triangles (i.e., equal shape). In another sense, two triangles may be equal if they have the same area, even though they differ in shape. In still another sense (congruence), two triangles are thought of as equal only if they are both similar and equal in area. (Note, however, that they still may not be identical, since one may be the mirror image of the other.)

If the concept of identity is to be generalized to the concept of equality (or equivalence, if you prefer), we must analyze the basic structure of the concept and see what common properties these various brands of equality have.

Let us introduce the symbol $\mathrel{E}$ to represent equality or equivalence. We shall choose S, a *specific set of elements*. The set S may be quite general, but it must be specified; and changing the set may change the answer to whether or not a specific comparison is an equivalence relation. A relation $\mathrel{E}$ will not be considered to be an equivalence relation for a set of elements S unless the following postulates are satisfied *for all elements of* S:

1. (Determinative) For any two elements a, b of the set S, either $a \mathrel{E} b$, or $a \not\mathrel{E} b$, but *not both* (where $\not\mathrel{E}$ means "is not equivalent to").
2. (Reflexive) For each a in S, $a \mathrel{E} a$.
3. (Symmetric) If $a \mathrel{E} b$, then $b \mathrel{E} a$.
4. (Transitive) If $a \mathrel{E} b$ and if $b \mathrel{E} c$, then $a \mathrel{E} c$.

These axioms are valid not only for equality between numbers and algebraic expressions but also for (1) equality of shape (similarity) of geometric objects and for (2) relations such as "has the same color of eyes as" or "has the same parents as" for the set of all people, or "has the same number of sides as" for the set of all polygons (the latter is really just equality of number).

If we consider the concept of equality for a set of numbers (or other objects for which addition is defined), we may desire the following additional axioms:

5. (a) If $a = b$, then $a + x = b + x$ for all x in set S.
   (b) If $a = b$, then $x + a = x + b$ for all x in set S.
   (Note that since we have not postulated that $b + x$ and $x + b$ are equal, we need both forms. In many real-life situations, $b + x$ and $x + b$ are *not* equal. More will be said about this later on.)

From this very simple basis we can prove the following powerful theorem:

THEOREM. If equals are added to equals, the results are equal; or, in symbols: If $a = b$ and $c = d$, then $a + c = b + d$.

Given: Axioms 1, 2, 3, 4, 5(a), 5(b); $a = b$, and $c = d$.
To prove: $a + c = b + d$.

PROOF

1. $a = b$.	1. Given.
2. $a + c = b + c$.	2. Step 1 and Axiom 5(a) with $x = c$.
3. $c = d$.	3. Given.
4. $b + c = b + d$.	4. Step 3 and Axiom 5(b) with $x = b$.
5. $a + c = b + d$.	5. Step 2, Step 4, and Axiom 4.

Note that each step of the proof has been justified by a statement or axiom from the "given hypothesis" in exactly the same manner as is customary in geometry.

This is the type of "elementary algebra" being taught in many U.S. schools today. (It is being taught earlier than the ninth grade in some schools. For example, in 1964 some eighth-graders in a Norman, Oklahoma, school proved that

$$(^-a) \cdot (^-b) = {^+}(a \cdot b)$$

and that

$$(^-a) \cdot (b) = {^-}(a \cdot b)$$

as homework exercises in their mathematics course. What is more important, they *created the proof themselves* rather than merely memorizing a proof given in the text. Thousands of junior high school students throughout the United States are having similar experiences.) It is essential that academic high schools, colleges, and universities build courses on *this* foundation rather than on the foundation common in the 1940's and before. Some of the "old bricks" are no longer there and should not be expected. It is neither necessary nor desirable for a modern high school or college teacher to "reteach" so as to supply these missing old-fashioned foundation bricks. You should expect a foundation for a steel building to be different from that for an adobe hut. Learn what modern algebra is all about; then use it.

Changes in Basic Concepts: Matrices

The current axiomatic approach to the "rules of algebra" is not the only important change in the algebra being taught today. The basic concepts themselves have changed. The inclusion of introductory matrix theory (born during the last century) in the high school curriculum was extremely rare before 1950, but it gives promise of becoming standard practice before too many years. A discussion of the commutative law of multiplication, $a \cdot b = b \cdot a$, seems rather pointless to many students until they discover that there exist mathematical systems in which this "law" is *not* valid. Matrices provide such an example. Matrix theory is one of the most powerful tools of modern applied mathematics, since matrices well represent many complex interconnections evidenced in nature.

There are many algebras in which $a \cdot b$ and $b \cdot a$ are not the same thing. Portions of modern physics, chemistry, psychology, and statistics are based on such noncommutative systems. The following simple experiment provides a physical example in which $a \cdot b$ and $b \cdot a$ are not identical.

> Place two closed books flat on the table in front of you with their faces upward and their spines (bound edges) on the left. (This is the normal position in which a book might lie before it is opened.) The books will remain closed throughout the experiment.
>
> Rotate the first book through 90° about its bottom edge. (It will now be standing upright on the table.) Now rotate the same book through 90° about its spine. Leave the book in this position.
>
> Rotate the second book through 90° about its spine. (If the book were released at this point, it would fall open in reading position.) Now rotate it through 90° about its bottom edge.

Note that the two books are not in the same final position. Each book has been rotated through 90° about its bottom edge and 90° about its spine; but the order was not the same, and the results are different. It is possible to use matrix theory to forecast the result of these operations and also of much more complicated rotations in three-dimensional, four-dimensional, or higher-dimensional space.

A matrix is a rectangular array of numbers (elements) for which multiplication is defined in a special way. A matrix should not be confused with a determinant, which is a single number or value associated with a square array. The matrix is the array itself. Two matrices are said to be equal if, and only if, the elements in corresponding position are equal. For example, if the elements are ordinary integers,

$$\begin{bmatrix} 7 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5+2 & 2 \\ 3 & 5 \end{bmatrix}, \text{ but } \begin{bmatrix} 7 & 2 \\ 3 & 5 \end{bmatrix} \neq \begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix}.$$

The matrices

$$\begin{bmatrix} 3 & 9 \\ -2 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} -3 & 3 \\ 4 & 2 \end{bmatrix}$$

are unequal if the elements are integers, but they are equal if the elements are the integers mod 6. Thus, to discuss matrices, it is necessary first to consider the set from which the elements of the matrix are to be selected. Equality of matrices depends upon the equivalence relation used in the set from which the elements of the matrix are selected.

If

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } N = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

are two matrices, then their product $M \cdot N$ is defined as follows:

$$M \cdot N = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}.$$

The element in the first (horizontal) row and second (vertical) column of the product $M \cdot N$ is a sum of elements, each of which is the product of an element from the first row of M multiplied by a corresponding element from the second column of N.

Thus,

$$\begin{bmatrix} a & b \\ * & * \end{bmatrix} \cdot \begin{bmatrix} * & x \\ * & z \end{bmatrix} = \begin{bmatrix} * & ax + bz \\ * & * \end{bmatrix}.$$

In a similar fashion, the element in row R and column C of the product $M \cdot N$ is the sum of the products of the elements of the Rth row of M multiplied by the corresponding elements of the Cth column of N.

$$\text{If } A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 4 \\ 5 & 8 \end{bmatrix},$$

$$\begin{aligned}\text{then } A \cdot B &= \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & 4 \\ 5 & 8 \end{bmatrix} \\ &= \begin{bmatrix} (1)(7) + (-1)(5) & (1)(4) + (-1)(8) \\ (3)(7) + (2)(5) & (3)(4) + (2)(8) \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 \\ 31 & 38 \end{bmatrix}.\end{aligned}$$

However,

$$\begin{aligned}B \cdot A &= \begin{bmatrix} 7 & 4 \\ 5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} (7)(1) + (4)(3) & (7)(-1) + (4)(2) \\ (5)(1) + (8)(3) & (5)(-1) + (8)(2) \end{bmatrix} \\ &= \begin{bmatrix} 19 & 1 \\ 29 & 11 \end{bmatrix}.\end{aligned}$$

Thus, in the system of 2×2 matrices, $A \cdot B$ and $B \cdot A$ are not necessarily the same.

The reader may check his understanding of matrix multiplication by showing that

$$\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & 4 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} -9 & -38 \\ 19 & 24 \end{bmatrix}.$$

Matrices containing thousands of elements are used in many practical applications. At the Naval Ordnance Testing Station, matrices are used in computations involving rocket and projectile flight. Matrices are used in modern economic theory. The branch of psychology known as factor analysis applies matrix methods. Systems of thirty-five (or more) equations in thirty-five (or more) unknowns, which arise in industrial research, may be neatly solved by using matrix methods. Modern vibration analysis uses matrices of more than 1,000 rows and 1,000 columns. In heat-transfer problems in nuclear reactors, even larger matrices are common. Competent biologists and geneticists find matrix methods helpful in the study of the complex interrelations of heredity and genetics. Large laboratories and oil refineries often ask universities to recommend graduates who are facile in the use of matrices.

The widespread utility of matrices stems from their unusual method of multiplication, in which each element of the product matrix is obtained through the interaction of several elements of the original matrices.

The Pauli matrices, used in the study of electron spin in quantum mechanics, have an interesting arithmetic. If $i^2 = -1$, the Pauli matrices ar is follows:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

$$F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

The Pauli matrices form a closed set under matrix multiplication; that is, the product of two or more Pauli matrices is again a Pauli matrix.

$$C \cdot D = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = G.$$

$$F \cdot A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = C.$$

$$B \cdot B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = D.$$

Try it yourself and see—there are only sixty-four cases, and fourteen of them are "obvious."

Changes in Basic Concepts: Groups

We now turn our attention to one of the simplest mathematical systems of modern algebra, the *group*. To have a group we need first of all a set of elements, G, and an equals (equivalence) relation. We also need an operation such as multiplication with which we can combine elements of the set G.

An operation, $\cdot$, is said to be *well defined* with respect to an equivalence relation, $=$, if, when a and b are replaced by equivalent elements, an equivalent result is obtained. This means that $x = x'$ and $y = y'$ imply $x \cdot y = x' \cdot y'$. Since each element is equivalent to itself by the reflexive property (Equality Postulate 2), no replacement need be apparent, as in

$$\begin{array}{rcr} \frac{1}{2} & \rightarrow & \frac{2}{4} \\ +\frac{1}{4} & \rightarrow & +\frac{1}{4} \\ \hline & & \frac{3}{4} \end{array}$$

where $\frac{2}{4}$ is equivalent to $\frac{1}{2}$ and $\frac{1}{4}$ is equivalent to $\frac{1}{4}$. The distributive property is actually used here to obtain

$$\left(2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4}\right) = (2 + 1) \cdot \frac{1}{4} = \frac{3}{4}.$$

The reader should note the difference between an operation's being well-defined and the theorem "If equals are added to equals, the results are equal." An operation's being well defined with respect to an equivalence relation implies the substitution property with respect to the operation, whereas the theorem discusses only the results of the operation for addition.

A *group* consists of a set of elements, G, having an equals (equivalence) relation, =, and a well-defined operation, $\cdot$, such that the following postulates are satisfied:

1. *Closure:* If $a, b \varepsilon G$, then $a \cdot b \varepsilon G$.
   (The symbol ε means "is an element of.")
2. *Associative Law:* If $a, b, c \varepsilon G$, then
   $$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$
3. *Existence of an Identity (Unity):* There exists an element $u \varepsilon G$ such that for each $b \varepsilon G$,
   $$u \cdot b = b \cdot u = b.$$
4. *Existence of an Inverse:* For each $b \varepsilon G$, there exists an element $b^* \varepsilon G$ such that
   $$b \cdot b^* = b^* \cdot b = u,$$
   for the u of Postulate 3.

It may seem that with so simple a set of postulates, no important theory could develop. However, *much of the power of modern mathe-*

matics stems from group theory. E. P. Wigner, the physical chemist who was director of research at the Oak Ridge National Laboratory and the Metallurgical (Atomic) Laboratory at the University of Chicago, is said to have presented the essentials of the thermodynamics of heat transfer to his staff of physicists on *one sheet of paper* by using group theory.

Consider some actual examples.

Example 1

The integers (positive, negative, and zero) form a group under the operation of addition.

1. *Closure:* The sum of two integers is an integer.
2. *Associative Law:* The integers are associative under addition.
3. *Existence of an Identity (Unity):* $u = 0$, since
$$b + 0 = 0 + b = b$$
for all integers b.
4. *Existence of an Inverse:* The inverse of b is $b^* = {}^-b$.
(Be sure you understand this example before continuing. What is the inverse of $^-11$?)

Example 2

The integers do not form a group under multiplication. The first three postulates are satisfied ($u = 1$, in this case); but Postulate 4 is not, since the number 5 does not have an inverse in the system of integers. (Actually, no integers other than ± 1 have multiplicative inverses in the system of integers, but a single counterexample is sufficient.)

Example 3

The rational numbers, with zero excluded, form a group under the operation of multiplication.

1. *Closure:* The product of two rational numbers is a rational number.
2. *Associative Law:* The rational numbers are associative.
3. *Existence of an Identity (Unity):* $u = 1$, since
$$b \cdot 1 = 1 \cdot b = b$$
for all rational numbers b.
4. *Existence of an Inverse:* The inverse of b ($\neq 0$) is $\frac{1}{b}$, which is rational if b is rational.

In the above examples, the elements of the group are numbers, and the equals relationship is ordinary equality of numbers. However, a group may be more general than this.

Example 4

Let the elements of a group be substitutions of one set of letters for another. For example,

$$\text{Let } a = \begin{cases} \text{for } A \text{ substitute } B \\ \text{for } B \text{ substitute } C \\ \text{for } C \text{ substitute } A \end{cases} = \begin{cases} A \rightarrow B \\ B \rightarrow C \\ C \rightarrow A \end{cases}$$

so that the substitution a carries (A, B, C) into (B, C, A), and if substitution a is applied to $3A^2 - 4BC + 2A$, it becomes $3B^2 - 4CA + 2B$.

If

$$b = \begin{cases} A \rightarrow C \\ B \rightarrow B \\ C \rightarrow A \end{cases}$$

is another substitution, then b carries (A, B, C) into (C, B, A). Also note that b carries (B, C, A) into (B, A, C).

If the substitution a is followed by the substitution b, we obtain the following:

$$(A, B, C) \underset{a}{\overset{\text{Using}}{\text{becomes}}} (B, C, A) \underset{b}{\overset{\text{Using}}{\text{becomes}}} (B, A, C).$$

However, there is a substitution that will carry (A, B, C) into (B, A, C) directly, namely,

$$c = \begin{cases} A \rightarrow B \\ B \rightarrow A \\ C \rightarrow C. \end{cases}$$

We write $a \cdot b = c$ to express this.

There are six possible substitutions of the three letters A, B, C, including the identity substitution

$$i = \begin{cases} A \rightarrow A \\ B \rightarrow B \\ C \rightarrow C. \end{cases}$$

Let these six substitutions be elements of a mathematical system, G, in which the operation $\cdot$ is the following of one substitution by another substitution, and the equals relation is identity of result; i.e., $a \cdot b$ means "first do 'a,' then do 'b.'" Thus, $a \cdot b = c$ for the substitutions given above.

When a group is being defined, it is essential that the equals relation be specified and that it satisfy the postulates for an equals relation given earlier. (See p. 6.)

It is usual to represent such substitutions (often called permutations) by a notation

$$a = \begin{pmatrix} A\ B\ C \\ B\ C\ A \end{pmatrix},$$

where the notation implies that each letter in the top row is replaced by the letter just below it.

The six possible substitutions are as follows:

$$a = \begin{cases} A \to B \\ B \to C \\ C \to A \end{cases} = \begin{pmatrix} A\ B\ C \\ B\ C\ A \end{pmatrix}; \quad b = \begin{cases} A \to C \\ B \to B \\ C \to A \end{cases} = \begin{pmatrix} A\ B\ C \\ C\ B\ A \end{pmatrix};$$

$$c = \begin{cases} A \to B \\ B \to A \\ C \to C \end{cases} = \begin{pmatrix} A\ B\ C \\ B\ A\ C \end{pmatrix}; \quad d = \begin{cases} A \to C \\ B \to A \\ C \to B \end{cases} = \begin{pmatrix} A\ B\ C \\ C\ A\ B \end{pmatrix};$$

$$e = \begin{cases} A \to A \\ B \to C \\ C \to B \end{cases} = \begin{pmatrix} A\ B\ C \\ A\ C\ B \end{pmatrix}; \quad i = \begin{cases} A \to A \\ B \to B \\ C \to C \end{cases} = \begin{pmatrix} A\ B\ C \\ A\ B\ C \end{pmatrix}.$$

Thus,

$$d \cdot e = \begin{pmatrix} A\ B\ C \\ C\ A\ B \end{pmatrix} \cdot \begin{pmatrix} A\ B\ C \\ A\ C\ B \end{pmatrix} = \begin{pmatrix} A\ B\ C \\ B\ A\ C \end{pmatrix} = c,$$

while

$$e \cdot d = \begin{pmatrix} A\ B\ C \\ A\ C\ B \end{pmatrix} \cdot \begin{pmatrix} A\ B\ C \\ C\ A\ B \end{pmatrix} = \begin{pmatrix} A\ B\ C \\ C\ B\ A \end{pmatrix} = b.$$

Note that $d \cdot e \neq e \cdot d$. Hence this system of substitutions is not commutative. It need not be, to satisfy the group postulates.

The group postulates are to be satisfied by the system of substitution just discussed.

1. *Closure:* The "product" of two substitutions in this list is also a substitution in the list. You are asked below to complete the "multiplication" table for this group, and hence prove closure.
2. *Associative Law:* Examine a typical letter, under the transformation $[(a \cdot b) \cdot c]$ and under $[a \cdot (b \cdot c)]$.

$$a = \begin{pmatrix} \ldots x \ldots\ldots \\ \ldots y \ldots\ldots \end{pmatrix}$$

$$b = \begin{pmatrix} \ldots\ldots y \ldots\ldots \\ \ldots\ldots z \ldots\ldots \end{pmatrix}$$

$$c = \begin{pmatrix} \ldots\ldots z \ldots \\ \ldots\ldots w \ldots \end{pmatrix}$$

Then $(a \cdot b)$ takes x into z, while $(b \cdot c)$ takes y into w.

$$(a \cdot b) = \begin{pmatrix} \ldots x \ldots\ldots \\ \ldots z \ldots\ldots \end{pmatrix}, \text{ while } (b \cdot c) = \begin{pmatrix} \ldots\ldots y \ldots \\ \ldots\ldots w \ldots \end{pmatrix}.$$

Thus,

$$[(a \cdot b) \cdot c] = \begin{pmatrix} \ldots x \ldots\ldots \\ \ldots z \ldots\ldots \end{pmatrix} \cdot \begin{pmatrix} \ldots\ldots z \ldots \\ \ldots\ldots w \ldots \end{pmatrix} = \begin{pmatrix} \ldots x \ldots\ldots \\ \ldots w \ldots\ldots \end{pmatrix},$$

while

$$[a \cdot (b \cdot c)] = \begin{pmatrix} \ldots x \ldots\ldots \\ \ldots y \ldots\ldots \end{pmatrix} \cdot \begin{pmatrix} \ldots\ldots y \ldots \\ \ldots\ldots w \ldots \end{pmatrix} = \begin{pmatrix} \ldots x \ldots\ldots \\ \ldots w \ldots\ldots \end{pmatrix}.$$

Thus not only these substitutions but every set of substitutions is associative.

3. *Existence of an Identity (Unity):* $u = i$, as already noted.
4. *Existence of an Inverse:* By forming the products $x \cdot x^* = x^* \cdot x = i$, show that each $x \in G$ has the inverse x^* indicated below.

$$\begin{array}{ll} a^* = d & d^* = a \\ b^* = b & e^* = e \\ c^* = c & i^* = i \end{array}$$

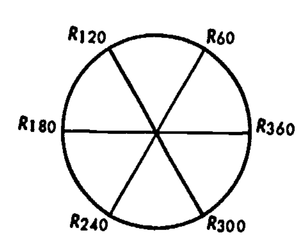
PARTIAL MULTIPLICATION TABLE FOR EXAMPLE 4

	i	*a*	*b*	*c*	*d*	*e*
i	*i*	*c*	*b*	*c*	*d*	*e*
a	*a*				*i*	
b	*b*		*i*			
c	*c*		*a*	*i*		
d	*d*	*i*			*a*	*c*
e	*e*				*b*	*i*

Fill in the entries not already supplied above. Filling in blanks is an important part of "reading" mathematics.

Example 5

Let the elements of a group be rotations of a plane figure through integral multiples of 60°. Let the operation be "following one such rotation by another," and let the equals relation be "identity of position." Rotation through $-120°$ results in the same position as rotation through $+240°$. Rotation through $(k \cdot 60° \pm n \cdot 360°)$ results in the same position as rotation through $k \cdot 60°$ for each integer n. Hence, the elements of this group may be represented as R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, R_{360}. Note that each of these group elements, R_x, represents an entire equivalence class $\{R \pm n \cdot 360°\}$,



where R_x denotes a rotation through x degrees in the counterclockwise direction. Thus, $R_{120} \cdot R_{180} = R_{300}$, and $R_{240} \cdot R_{180} = R_{60}$. Check the group postulates for this system.

1. *Closure:* If one of the above rotations is followed by another rotation, the resulting position could have been obtained as a single rotation.
2. *Associative Law:* True.
3. *Existence of an Identity (Unity):* $u = R_{360}$.
4. *Existence of an Inverse:* The inverse of R_x is $R_x^{-1} = R_{360^\circ - x}$.

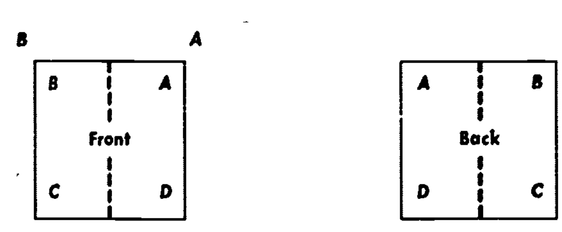
It is instructive to form the multiplication tables for the group in Example 5. Here, as well as in Example 4, you should fill in the entries not supplied.

PARTIAL MULTIPLICATION TABLE FOR EXAMPLE 5

	R_{360}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}
R_{360}	R_{360}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}
R_{60}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	R_{360}
R_{120}	R_{120}	R_{180}	R_{240}	R_{300}		
R_{180}	R_{180}					
R_{240}	R_{240}				R_{120}	
R_{300}	R_{300}		R_{60}			R_{240}

Example 6

Cut out a two-inch square of cardboard or paper. Label the vertices A, B, C, D as shown below.



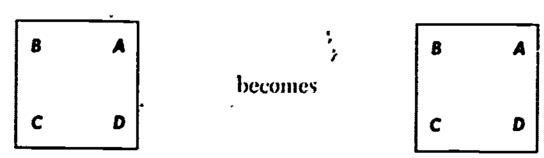
Place a letter A on the back face of the square in the same vertex as the letter A on the front. Vertex A is now uniquely identified. Repeat for vertices B, C, and D.

The elements of this group are certain movements of the square. The operation, $\cdot$, consists of following one movement by another. Equality is identity of vertex positions.

The permitted movements (elements) are as follows:

R_{180}: A rotation (in the plane) of the square through 180° about its center.

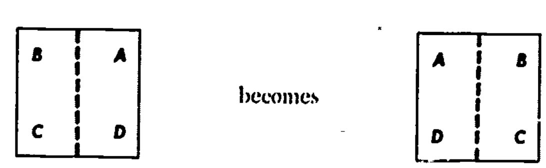
R_{360}: A rotation (in the plane) of the square through 360°.



This is the identity element, since the positions of the vertices are unchanged.

H: A flip (in three-space) about a horizontal line through the center of the square.

V: A flip (in three-space) about a vertical line through the center of the square.



Then $(H \cdot V)$ is that element that results when H (horizontal flip) is followed by V (vertical flip). Actually, experiment will show that

$$(H \cdot V) = R_{180},$$

that

$$(R_{180} \cdot V) = H,$$

and that

$$(H \cdot H) = R_{360}.$$

You should construct a model and carry out the operations just suggested as well as supply the remainder of the entries in the multiplication table of this group.

PARTIAL MULTIPLICATION TABLE FOR EXAMPLE 6

	R_{360}	R_{180}	H	V
R_{360}	R_{360}	R_{180}	H	V
R_{180}				
H			R_{360}	R_{180}
V				

After the table is completed, it will be possible to see that these four elements do form a group, as shown below:

1. *Closure:* Only the elements R_{180}, R_{360}, H, and V are needed to complete the table.
2. *Associative Law:* The four given elements are substitutions for the corner letters. Substitutions are associative, by the discussion of Example 4.

$$R_{180} = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}, \quad R_{260} = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix},$$

$$H = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}, \quad V = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}.$$

3. *Existence of an Identity (Unity):* $u = R_{360}$.
4. *Existence of an Inverse:* Since $u = R_{360}$ appears once, and only once, in each row and column of the "multiplication" table, it follows that

$$x \cdot a = R_{360} \quad \text{and} \quad b \cdot x = R_{360}$$

are solvable for all a, b in $G = (R_{360}, R_{180}, V, H)$. Examination shows that if $x \cdot y = R_{360}$, then $y \cdot x = R_{360}$.

As was emphasized before, the serious mathematics teacher must realize that one learns mathematics by working problems, not merely by reading. I have suggested that you complete the partial multiplication tables given in Examples 4, 5, and 6. It would also be instructive to prove that the set of Pauli matrices discussed on page 11 forms a group under the operation of matrix multiplication, and that the multiplication table given below is *not* the multiplication table of a group.

(Not a group!)

	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	a_3	a_4	a_5
a_2	a_2	a_1	a_4	a_5	a_3
a_3	a_3	a_5	a_1	a_2	a_4
a_4	a_4	a_3	a_5	a_1	a_2
a_5	a_5	a_4	a_2	a_3	a_1

Actually, even the simple set of group postulates given here is redundant (i.e., parts of some postulates can be proved from the remaining postulates). You may, for example, wish to prove that in Postulate 3 (there exists an element $u \varepsilon G$ such that for each $b \varepsilon G$, $u \cdot b = b \cdot u = b$), the equation

$$u \cdot b = b \cdot u = b$$

may be replaced by the equation

$$b \cdot u = b$$

and that one can then derive the remaining assertion,

$$u \cdot b = b,$$

as a theorem.

Linear Equations and Computers

Not only are new areas of mathematics being explored, but very ancient areas are flowering in fashions that would have been unbelievable even twenty years ago. Ancient Babylonian tablets discuss problems that, in modern notation, require the solution of systems of linear equations such as

$$3x + y = 19,$$
$$2x - y = 1.$$

Modern schoolboys still solve similar systems of two linear equations in two unknowns and even systems of three or four equations in three or four unknowns, but until recently only the hardy ventured beyond this. Today's computers make it possible to solve, as routine problems, systems of 700 equations in 700 unknowns, which arise in modern economic theory. Systems of 1,728 equations in 1,728 unknowns arise daily in vibration theory and flutter analysis in modern rocket and jet design. Much larger linear systems arise in the heat-transfer problems of atomic physics. Algebraic problems that were beyond consideration ten years ago are now solved routinely by the use of modern computers. Entire new vistas open up.

With the new vistas, new problems appear. Although a great deal is known about continuous variables (analysis) and much has been learned about certain finite algebraic systems such as groups, fields, rings, and integral domains, *no one has yet completely studied the basic structure of the arithmetic used in any major computer now in operation!* Modern computers are amazing arithmetical engines, but they violate many basic postulates of high school algebra. For example, high school algebra assumes that for all a, b, c,

$(a + b) + c = a + (b + c)$	Associative Addition
$(a \cdot b) \cdot c = a \cdot (b \quad c$	Associative Multiplication
$s \cdot (a + b) = s \cdot a + s \cdot b$	Distributive Property

It also assumes (or proves) that

$$\text{if } a \neq 0 \text{ and } a \cdot x = a \cdot y, \text{ then } x = y$$

and

$$\text{if } a + x = a + y, \text{ then } x = y$$

and that

$$\text{if } a \cdot b = 0, \text{ then either } a = 0, \text{ or } b = 0, \text{ or both.}$$

However, *none* of these rules is valid in computer arithmetic. In spite of this, computers provide the majority of the arithmetic answers needed in today's engineering and science. Mathematicians must study the basic structure of computer algebras if science is to make reasonable use of this vital new tool.

CONCLUSION

Abstract algebra has many other vital problems of even more interest to mathematicians than the ever-changing structure of computer arithmetic. Today's economics, psychology, business administration, physics, chemistry, and engineering each lean heavily on mathematics. Each brings new problems for the mathematician's study. Modern algebra provides the language and the tools of mathematics, much as mathematics provides the language and tools of today's science and engineering. It is not surprising that most colleges and universities have demands for ten times as many students of abstract algebra as are presently graduated.

Algebra has a long and interesting history from ancient times to the present, but only the current century has seen the rapid growth of abstract algebra. To any high school teacher wishing to know more about recent developments, I would say: First study the Twenty-third Yearbook of the NCTM, *Insights into Modern Mathematics;* then consult the 512.8 (Dewey decimal) or QA 266 (Library of Congress) area in your library. Continuous study is needed. Tomorrow's history is being molded in the minds of active secondary school students today by teachers who are still learning about abstract algebra.

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