DOCUMENT RESUME

ED 070 634 SE 015 192

TITLE [Annual Mathematics Examination, 1966-1972.]
INSTITUTION Mathematical Association of America, Lincoln, Nebr. Committee on High School Contests.
PUB DATE 72
NOTE 97p.
AVAILABLE FROM Mathematical Association of America, Committee on High School Contests, Lincoln, Nebraska

EDRS PRICE MF-$0.65 HC Not Available from EDRS.
DESCRIPTORS Achievement Tests; *Evaluation; Mathematics; *Mathematics Education; Motivation; *Secondary School Mathematics; *Tests

ABSTRACT

Sets of the Annual High School Mathematics Examination for each year from 1966 through 1972 include a copy of each test and its solution key. No mathematics beyond intermediate algebra and trigonometry is needed for solving the test problems. Prior year examinations are published as Volumes 5 and 17 of the New Mathematics Library. (DT)

[Annual Mathematics Examination, 1966-1972]

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE SOCIETY OF ACTUARIES

and

MU ALPHA THETA

M	A	A
S		A
M	A	TH

17

SEVENTEENTH ANNUAL MATHEMATICS EXAMINATION 1966

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box on the margin of your answer sheet directly under the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly under No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

THURSDAY MORNING, MARCH 10, 1966

To be filled in by the student

PRINT

last name first name middle name or initial

school number street

city county state zip code

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30

PART III

31	32	33	34	35	36	37	38	39	40

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

SEVENTEENTH ANNUAL H. S. MATHEMATICS EXAMINATION—1966

PART I (3 credits each)

1. Given that the ratio of $3x - 4$ to $y + 15$ is constant, and $y = 3$ when $x = 2$, then, when $y = 12$, x equals:

   (A) $\frac{1}{8}$ (B) $\frac{3}{7}$ (C) $\frac{7}{3}$ (D) $\frac{7}{2}$ (E) 8

2. When the base of a triangle is increased 10% and the altitude to this base is decreased 10%, the change in area is:

   (A) 1% increase (B) $\frac{1}{2}$% increase (C) 0%
   (D) $\frac{1}{2}$% decrease (E) 1% decrease

3. If the arithmetic mean of two numbers is 6 and their geometric mean is 10, then an equation with the given two numbers as roots is:

   (A) $x^2 + 12x + 100 = 0$ (B) $x^2 + 6x + 100 = 0$ (C) $x^2 - 12x - 10 = 0$
   (D) $x^2 - 12x + 100 = 0$ (E) $x^2 - 6x + 100 = 0$

4. Circle I is circumscribed about a given square and circle II is inscribed in the given square. If r is the ratio of the area of circle I to that of circle II, then r equals:

   (A) $\sqrt{2}$ (B) 2 (C) $\sqrt{3}$ (D) $2\sqrt{2}$ (E) $2\sqrt{3}$

5. The number of values of x satisfying the equation $\frac{2x^2 - 10x}{x^2 - 5x} = x - 3$ is:

   (A) zero (B) one (C) two (D) three (E) an integer greater than 3

6. AB is a diameter of a circle centered at O. C is a point on the circle such that angle BOC is 60°. If the diameter of the circle is 5 inches, the length of chord AC, expressed in inches, is:

   (A) 3 (B) $\frac{5\sqrt{2}}{2}$ (C) $\frac{5\sqrt{3}}{2}$ (D) $3\sqrt{3}$ (E) none of these

7. Let $\frac{35x - 29}{x^2 - 3x + 2} = \frac{N_1}{x - 1} + \frac{N_2}{x - 2}$ be an identity in x. The numerical value of N_1N_2 is:

   (A) -246 (B) -210 (C) -29 (D) 210 (E) 246

8. The length of the common chord of two intersecting circles is 16 feet. If the radii are 10 feet and 17 feet, a possible value for the distance between the centers of the circles, expressed in feet, is:

   (A) 27 (B) 21 (C) $\sqrt{389}$ (D) 15 (E) undetermined

9. If $x = (\log_8 2)^{(\log_2 8)}$, then $\log_3 x$ equals:

   (A) -3 (B) $-\frac{1}{3}$ (C) $\frac{1}{3}$ (D) 3 (E) 9

10. If the sum of two numbers is 1 and their product is 1, then the sum of their cubes is $(i = \sqrt{-1})$:

    (A) 2 (B) $-2 - \frac{3\sqrt{3}\,i}{4}$ (C) 0 (D) $-\frac{3\sqrt{3}\,i}{4}$ (E) -2

11. The sides of triangle BAC are in the ratio $2:3:4$. BD is the angle-bisector drawn to the shortest side AC, dividing it into segments AD and CD. If the length of AC is 10, then the length of the longer segment of AC is:

    (A) $3\frac{1}{2}$ (B) 5 (C) $5\frac{5}{7}$ (D) 6 (E) $7\frac{1}{2}$

12. The number of real values of x that satisfy the equation $(2^{6x+3})(4^{3x+6}) = 8^{4x+5}$ is:

    (A) zero (B) one (C) two (D) three (E) greater than 3

13. The number of points with positive rational coordinates selected from the set of points in the xy-plane such that $x + y \leqq 5$, is:

    (A) 9 (B) 10 (C) 14 (D) 15 (E) infinite

14. The length of rectangle ABCD is 5 inches and its width is 3 inches. Diagonal AC is divided into three equal segments by points E and F. The area of triangle BEF, expressed in square inches, is:

    (A) $\frac{3}{2}$ (B) $\frac{5}{3}$ (C) $\frac{5}{2}$ (D) $\frac{1}{3}\sqrt{34}$ (E) $\frac{1}{3}\sqrt{68}$

15. If $x - y > x$ and $x + y < y$, then

    (A) $y < x$ (B) $x < y$ (C) $x < y < 0$ (D) $x < 0, y < 0$ (E) $x < 0, y > 0$

16. If $\frac{4^x}{2^{x+y}} = 8$ and $\frac{9^{x+y}}{3^{5y}} = 243$, x and y real numbers, then xy equals:

    (A) $\frac{12}{5}$ (B) 4 (C) 6 (D) 12 (E) -4

17. The number of distinct points common to the curves $x^2 + 4y^2 = 1$ and $4x^2 + y^2 = 4$ is:

    (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

18. In a given arithmetic sequence the first term is 2, the last term is 29, and the sum of all the terms is 155. The common difference is:

    (A) 3 (B) 2 (C) $\frac{27}{19}$ (D) $\frac{13}{9}$ (E) $\frac{23}{38}$

19. Let s_1 be the sum of the first n terms of the arithmetic sequence 8, 12, $\cdots$ and let s_2 be the sum of the first n terms of the arithmetic sequence 17, 19, $\cdots$. Then $s_1 = s_2$ for:

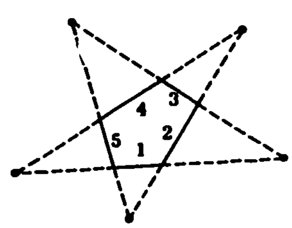
(A) no value of n (B) one value of n (C) two values of n
(D) four values of n (E) more than four values of n

20. If the proposition "a = 0" is true, the negation of the proposition "For real values of a and b, if a=0, then ab = 0" is:

(A) If $a \neq 0$, then $ab \neq 0$ (B) If $a \neq 0$, then $ab = 0$
(C) If $a = 0$, then $ab \neq 0$ (D) If $ab \neq 0$, then $a \neq 0$
(E) If $ab = 0$, then $a \neq 0$

PART II (4 credits each)

21. An "n-pointed star" is formed as follows: the sides of a convex polygon are numbered consecutively 1, 2, $\cdots$, k, $\cdots$, n, $n \geq 5$; for all n values of k, sides k and k + 2 are non-parallel, sides n + 1 and n + 2 being respectively identical with sides 1 and 2; prolong the n pairs of sides numbered k and k + 2 until they meet. (A figure is shown for the case n = 5).



Let S be the degree-sum of the interior angles at the n points of the star; then S equals:

(A) 180 (B) 360 (C) 180(n + 2) (D) 180(n − 2) (E) 180(n − 4)

22. Consider the statements: (I) $\sqrt{a^2+b^2} = 0$ (II) $\sqrt{a^2+b^2} = ab$ (III) $\sqrt{a^2+b^2} = a + b$ (IV) $\sqrt{a^2+b^2} = a - b$, where we allow a and b to be real or complex numbers. Those statements for which there exist solutions other than a = 0 and b = 0, are:

(A) (I), (II), (III), (IV) (B) (II), (III), (IV) only (C) (I), (III), (IV) only
(D) (III), (IV) only (E) (I) only

23. If x is real and $4y^2 + 4xy + x + 6 = 0$, then the complete set of values of x for which y is real, is:

(A) $x \leq -2$ or $x \geq 3$ (B) $x \leq 2$ or $x \geq 3$ (C) $x \leq -3$ or $x \geq 2$
(D) $-3 \leq x \leq 2$ (E) $-2 \leq x \leq 3$

24. If $\log_M N = \log_N M$, $M \neq N$, $MN > 0$, $M \neq 1$, $N \neq 1$, then MN equals:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 10
(E) a number greater than 2 and less than 10

25. If $F(n + 1) = \dfrac{2F(n) + 1}{2}$ and $F(1) = 2$, then $F(101)$ equals:

(A) 49 (B) 50 (C) 51 (D) 52 (E) 53

26. Let m be a positive integer and let the lines $13x + 11y = 700$ and $y = mx - 1$ intersect in a point whose coordinates are integers. Then m can be:

(A) 4 only (B) 5 only (C) 6 only (D) 7 only
(E) one of the integers 4, 5, 6, 7 and one other positive integer

27. At his usual rate a man rows 15 miles downstream in five hours less time than it takes him to return. If he doubles his usual rate, the time downstream is only one hour less than the time upstream. In miles per hour, the rate of the stream's current is:

(A) 2 (B) $\frac{5}{2}$ (C) 3 (D) $\frac{7}{2}$ (E) 4

28. Five points O, A, B, C, D are taken in order on a straight line with distances $OA = a$, $OB = b$, $OC = c$, and $OD = d$. P is a point on the line between B and C and such that $AP:PD = BP:PC$. Then OP equals:

(A) $\dfrac{b^2 - bc}{a - b + c - d}$ (B) $\dfrac{ac - bd}{a - b + c - d}$ (C) $-\dfrac{bd + ac}{a - b + c - d}$

(D) $\dfrac{bc + ad}{a + b + c + d}$ (E) $\dfrac{ac - bd}{a + b + c + d}$

29. The number of positive integers less than 1000 divisible by neither 5 nor 7, is:

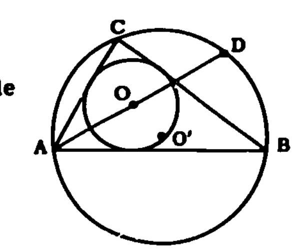
(A) 688 (B) 686 (C) 684 (D) 658 (E) 630

30. If three of the roots of $x^4 + ax^2 + bx + c = 0$ are 1, 2, and 3, then the value of $a + c$ is:

(A) 35 (B) 24 (C) -12 (D) -61 (E) -63

PART III (5 credits each)

31. Triangle ABC is inscribed in a circle with center O'. A circle with center O is inscribed in triangle ABC. AO is drawn, and extended to intersect the larger circle in D. Then we must have:



(A) $CD = BD = O'D$ (B) $AO = CO = OD$
(C) $CD = CO = BD$ (D) $CD = OD = BD$
(E) $O'B = O'C = OD$

32. Let M be the midpoint of side AB of triangle ABC. Let P be a point in AB between A and M, and let MD be drawn parallel to PC and intersecting BC at D. If the ratio of the area of triangle BPD to that of triangle ABC is designated by r, then

(A) $\frac{1}{2} < r < 1$ depending upon the position of P (B) $r = \frac{1}{2}$ independent of the position of P (C) $\frac{1}{2} \leqq r < 1$ depending upon the position of P (D) $\frac{1}{3} < r < \frac{2}{3}$ depending upon the position of P (E) $r = \frac{1}{3}$ independent of the position of P

33. If $ab \neq 0$ and $|a| \neq |b|$ the number of distinct values of x satisfying the equation $\frac{x-a}{b} + \frac{x-b}{a} = \frac{b}{x-a} + \frac{a}{x-b}$, is:

(A) zero (B) one (C) two (D) three (E) four

34. Let r be the speed in miles per hour at which a wheel, 11 feet in circumference, travels. If the time for a complete rotation of the wheel is shortened by $\frac{1}{4}$ of a second, the speed r is increased by 5 miles per hour. Then r is:

(A) 9 (B) 10 (C) $10\frac{1}{2}$ (D) 11 (E) 12

35. Let O be an interior point of triangle ABC and let $s_1 = OA + OB + OC$. If $s_2 = AB + BC + CA$, then

(A) for every triangle $s_1 > \frac{1}{2}s_2$, $s_1 \leqq s_2$ (B) for every triangle $s_1 \geqq \frac{1}{2}s_2$, $s_1 < s_2$ (C) for every triangle $s_1 > \frac{1}{2}s_2$, $s_1 < s_2$ (D) for every triangle $s_1 \geqq \frac{1}{2}s_2$, $s_1 \leqq s_2$ (E) neither (A) nor (B) nor (C) nor (D) applies to every triangle

36. Let $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ be an identity in x. If we let $s = a_0 + a_2 + a_4 + \cdots + a_{2n}$, then s equals:

(A) 2^n (B) $2^n + 1$ (C) $\frac{3^n - 1}{2}$ (D) $\frac{3^n}{2}$ (E) $\frac{3^n + 1}{2}$

37. Three men, Alpha, Beta, and Gamma, working together, do a job in 6 hours less time than Alpha alone, in 1 hour less time than Beta alone, and in one-half the time needed by Gamma when working alone. Let h be the number of hours needed by Alpha and Beta, working together, to do the job. Then h equals:

(A) $\frac{5}{2}$ (B) $\frac{3}{2}$ (C) $\frac{4}{3}$ (D) $\frac{5}{4}$ (E) $\frac{3}{4}$

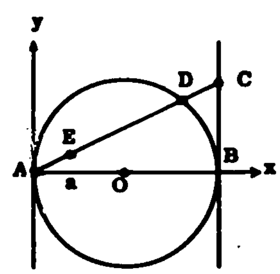
38. In triangle ABC the medians AM and CN to sides BC and AB, respectively intersect in point O. P is the midpoint of side AC and MP intersects CN in Q. If the area of triangle OMQ is n, then the area of triangle ABC is:

(A) 16n (B) 18n (C) 21n (D) 24n (E) 27n

39. In base R_1 the expanded fraction F_1 becomes $.373737\cdots$ and the expanded fraction F_2 becomes $.737373\cdots$. In base R_2 fraction F_1, when expanded, becomes $.252525\cdots$ while fraction F_2 becomes $.525252\cdots$. The sum of R_1 and R_2, each written in the base ten, is:

(A) 24 (B) 22 (C) 21 (D) 20 (E) 19

40. In this figure AB is a diameter of a circle, centered at O, with radius a. A chord AD is drawn and extended to meet the tangent to the circle at B, in point C. Point E is taken on AC so that AE = DC. If the coordinates of E are (x, y), then:



(A) $y^2 = \frac{x^3}{2a - x}$ (B) $y^2 = \frac{x^3}{2a + x}$

(C) $y^4 = \frac{x^2}{2a - x}$ (D) $x^2 = \frac{y^2}{2a - x}$

(E) $x^2 = \frac{y^2}{2a + x}$

Printed in U.S.A.

SOLUTION KEY
AND ANSWER KEY
SEVENTEENTH ANNUAL H. S. MATHEMATICS EXAMINATION

1966

Sponsored Jointly by

M A A	THE MATHEMATICAL ASSOCIATION OF AMERICA
S A	THE SOCIETY OF ACTUARIES
M A TH	MU ALPHA THETA

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions are intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

National Office Pan American College, Edinburg, Texas 78539
New York Office Polytechnic Institute of Brooklyn, Brooklyn, N. Y. 11201

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1966 examination.

1. (C) $3x - 4 = k(y + 15)$, $6 - 4 = k(3 + 15)$ $\quad \therefore k = 1/9$ $\quad \therefore 3x - 4 = 1/9\,(12 + 15) = 3$, $3x = 7$, $x = 7/3$

2. (E) $K_0 = \frac{1}{2}bh$, $K_n = \frac{1}{2}(b + .1b)(h - .1h)$ $\quad \therefore K_n = \frac{1}{2}bh + .05bh - .05bh - \frac{1}{2}(.01)\,bh = (1 - .01)K_0$. Therefore, the change is a decrease of 1% of the original area.

3. (D) Let r and s be the roots of the required equation. Since $6 = \frac{r+s}{2}$, $r + s = 12$ and since $10 = \sqrt{rs}$, $rs = 100$, so that, in the required equation, the sum of the roots is 12 and the product of the roots is 100. Hence, $x^2 - 12x + 100 = 0$.

4. (B) Let s be the length of a side of the square. The radius of circle I is $\frac{1}{2}s\sqrt{2}$ and its area $K_1 = \frac{1}{2}\pi s^2$. The radius of circle II is $\frac{1}{2}s$ and its area $K_2 = \frac{1}{4}\pi s^2$. $\quad \therefore r = \frac{K_1}{K_2} = 2$

5. (A) For $x \neq 0$, $x \neq 5$, the left side of the equation reduces to 2. $\quad \therefore 2 = x - 3$, $x = 5$, but $x = 5$ is unacceptable. The equation is, therefore, satisfied by no value of x.

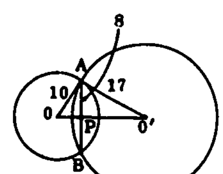
6. (C) Triangle ABC is a right triangle with hypotenuse AB = 5 inches and leg BC equal to a radius of the circle, $\frac{5}{2}$. We, therefore, have $\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2 = 25 - \frac{25}{4} = \frac{75}{4}$ $\quad \therefore AC = \frac{5\sqrt{3}}{2}$ (inches).

7. (A) $35x - 29 \equiv N_1(x - 2) + N_2(x - 1)$ is an identity in x. Letting $x = 1$ we find $N_1 = -6$ and letting $x = 2$ we find $N_2 = 41$. $\quad \therefore N_1N_2 = -246$

8. (B) From the diagram we have $\overline{OP}^2 = \overline{OA}^2 - \overline{AP}^2 = 10^2 - 8^2$, $OP = 6$

$\overline{O'P}^2 = \overline{O'A}^2 - \overline{AP}^2 = 17^2 - 8^2$, $O'P = 15$

$\therefore OO' = OP + O'P = 6 + 15 = 21$



9. (A) $\log_2 8 = 3$ and $\log_8 2 = \frac{1}{3}$ $\quad \therefore x = \left(\frac{1}{3}\right)^3$ $\quad \therefore \log_3 x = 3\log_3\frac{1}{3} = 3(0 - 1) = -3$

or

Let $y = \log_2 8 (= 3)$ $\quad \therefore 2^y = 8 (= 2^3)$ $\quad \therefore y\log_8 2 = \log_8 8 = 1$

$\therefore \log_8 2 = \frac{1}{y}$ $\quad \therefore x = \left(\frac{1}{y}\right)^y = \frac{1}{y^y} = y^{-y}$

$\therefore \log_3 x = -y\log_3 y = -3(1) = -3$

10. (E) Let the numbers be x and y. Then $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$

$\therefore x^3 + y^3 = 1^3 - 3(1)(1) = -2$

or

Consider the equation $Z^2 - Z + 1 = 0$ with roots $Z_1 = \frac{1 + i\sqrt{3}}{2}$, $Z_2 = \frac{1 - i\sqrt{3}}{2}$

The sum of these roots is 1 and their product is 1, so that we may take for our numbers the numbers Z_1 and Z_2. Therefore $Z_1^3 + Z_2^3 = \left(\frac{1 + i\sqrt{3}}{2}\right)^3 + \left(\frac{1 - i\sqrt{3}}{2}\right)^3 =$

$$\frac{1 + 3i\sqrt{3} + 3i^2 \cdot 3 + i^3 \cdot 3\sqrt{3} + 1 - 3i\sqrt{3} + 3i^2 \cdot 3 - i^3 \cdot 3\sqrt{3}}{8} = \frac{2 - 18}{8} = -2$$

11. (C) Since the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides including the given angle, taken in the proper order, we have

$\frac{10 - x}{x} = \frac{3m}{3m} = \frac{3}{4}$ $\quad \therefore 40 = 7x$, $x = 5\frac{5}{7}$, $10 - x = 4\frac{2}{7}$ $\quad \therefore$ The longer segment is $5\frac{5}{7}$.

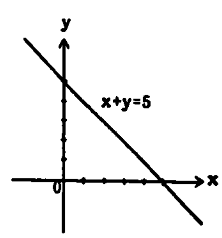
12. (E) The given equation is equivalent to $(2^{6x+3})(2^{2(3x+6)}) = 2^{3(4x+5)}$

The left side of this equation equals $2^{6x+3} \cdot 2^{6x+12} = 2^{12x+15}$ and the right side is also equal to 2^{12x+15}, so that the given equality is an identity in x. It is, therefore, satisfied by any real value of x.

13. (E) Since $x + y \leq 5$, $y \leq 5 - x$; for any rational value of x, $5 - x$ and, hence, y is rational.

or

Consider the graph of $x + y \leq 5$; it is the half-plane below the line $x + y = 5$, including the line $x + y = 5$. Since this half-plane contains an infinite set of rational points (points with rational coordinates), choice (E) is the correct one.



14. (C) The area of triangle ABC equals $\frac{1}{2} \cdot 5 \cdot 3 = \frac{15}{2}$. Since AE = EF = FC, the area of triangle BEF equals $\frac{1}{3}$ of the area of triangle ABC, that is, $\frac{1}{3} \cdot \frac{15}{2} = \frac{5}{2}$.

15. (D) Since $x - y > x$, $y < 0$ and since $x + y < y$, $x < 0$. $\therefore$ Choice (D) is correct.

16. (B) $\frac{4^x}{2^{x+y}} = \frac{2^{2x}}{2^{x+y}} = 2^{x-y} = 8 = 2^3$ $\quad \therefore\ x - y = 3 \quad \therefore x = 4, y = 1$

$\frac{9^{x+y}}{3^{5y}} = \frac{3^{2x+2y}}{3^{5y}} = 3^{2x-3y} = 243 = 3^5 \quad \therefore 2x - 3y = 5 \quad \therefore xy = 4$

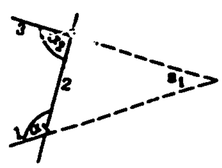
17. (C) Both curves are ellipses with the centers at the origin. The first, $\frac{x^2}{1^2} + \frac{y^2}{(\frac{1}{2})^2} = 1$, has x-intercepts +1 and −1, and y-intercepts $+\frac{1}{2}$ and $-\frac{1}{2}$. The second, $\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$, has x-intercepts +1 and −1, and y-intercepts +2 and − 2. The number of distinct points of intersection is 2.

18. (A) $s = \frac{n}{2}(a + l)$ where $a = 2$, $l = 29$, and $s = 155$ $\quad \therefore 155 = \frac{n}{2}(2 + 29) = \frac{31n}{2}$, $n = 10$.
But $l = a + (n - 1)d$; so that $29 = 2 + 9d$, $9d = 27$, $d = 3$

19. (B) $s_1 = \frac{n}{2}(16 + (n - 1)\,4)$, $s_2 = \frac{n}{2}(34 + (n - 1)\,2)$. But $s_1 = s_2$ implies $\frac{n}{2}(12 + 4n) = \frac{n}{2}(32 + 2n)$
$\therefore 12 + 4n = 32 + 2n$, $n = 10$ so that choice (B) is correct.

20. (C) To negate the proposition "p → q", where p and q themselves are propositions, we form the proposition "p and not-q". In this case p is the proposition "a = 0" and q is the proposition "ab = 0". The negation is, therefore, "a = 0 and ab ≠ 0", corresponding to "if a = 0, then ab ≠ 0".

21. (E) Let the measures of the angles at the n points be $a_1, a_2, \ldots, a_n$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the measures of the interior angles of the polygon, with $\alpha_n = \alpha_1$. We have $a_1 = 180 - (180 - \alpha_1) - (180 - \alpha_2) = \alpha_1 + \alpha_2 - 180$, $a_2 = \alpha_2 + \alpha_3 - 180, \ldots, a_n = \alpha_{n-1} + \alpha_n - 180$. Summing, we have
$a_1 + a_2 + \ldots + a_n = 2(\alpha_1 + \alpha_2 + \ldots + \alpha_n) - n \cdot 180$
$\therefore S = 2((n - 2)180) - n \cdot 180 = 180(n - 4)$



22. (A) I is satisfied when $b = a\sqrt{-1}$, II is satisfied when $b = \frac{a}{\sqrt{a^2 - 1}}$, $a \neq 1$, and III and IV are both satisfied when $b = 0$ and a is chosen arbitrarily. Therefore, (A) is the correct choice.

23. (A) We treat the equation as a quadratic equation in y for which the discriminant $D = 16x^2 - 16(x + 6) = 16(x^2 - x - 6) = 16(x - 3)(x + 2)$. For y to be real $D \geqq 0$. This inequality is satisfied when $x \leqq -2$ or $x \geqq 3$.

24. (B) Let $\log_M N = x$; then $\log_N M = \frac{1}{\log_M N} = \frac{1}{x}$. $\therefore x^2 = 1$, $x = +1$ or -1. If $x = 1$ then $M = N$, but this contradicts the given $M \neq N$. If $x = -1$, then $N = M^{-1}$ $\quad \therefore MN = 1$

or

Let $\log_M N = x = \log_N M \quad \therefore N = M^x$ and $M = N^x \quad \therefore (M^x)^x = N^x = M$
$\therefore x \cdot x = 1 \quad \therefore x = 1$ (rejected) or $x = -1 \quad \therefore N = M^{-1} \quad \therefore NM = 1$

25. (D) $2F(n + 1) = 2F(n) + 1$
$2F(n) = 2F(n - 1) + 1$
$\vdots \qquad \vdots$
$2F(2) = 2F(1) + 1$
$\therefore 2F(n + 1) = 2F(1) + n \cdot 1$
$\therefore F(n + 1) = F(1) + \frac{1}{2}n \quad \therefore F(101) = 2 + \frac{1}{2} \cdot 100 = 52$

or

$2 \sum_{1}^{100} F(n + 1) = 2F(1) + 100 = 2 \cdot 2 + 100 = 104 \quad \therefore F(101) = 52.$

26. (C) $13x + 11(mx - 1) = 700$, $x(13 + 11m) = 711$, $x = \frac{711}{13 + 11m}$, with m integral and x integral. It is easy to see that m must be even. Try 2, 4, 6 in turn. Neither 2 nor 4 produce an integral x but for m = 6, we have x = 9. Since two straight lines can intersect in at most one point (these lines do not coincide), m = 6 is the only possible value.

27. (A) Let m (miles per hour) be the man's rate in still water, and let c (miles per hour) be the rate of the current. Then

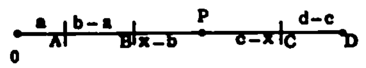
$$\frac{15}{m + c} = \frac{15}{m - c} - 5 \quad \text{and} \quad \frac{15}{2m + c} = \frac{15}{2m - c} - 1$$

$$15m - 15c = 15m + 15c - 5m^2 + 5c^2 \quad \text{and} \quad 30m - 15c = 30m + 15c - 4m^2 + c^2$$

$$\therefore -5m^2 + 5c^2 = -30c \quad \text{and} \quad -4m^2 + c^2 = -30c$$

$$\therefore m^2 - 4c^2 = 0, \quad m = 2c, \quad 3c^2 = 6c, \quad c = 2$$

28. (B) Since $\frac{AP}{PD} = \frac{BP}{PC}$, $\frac{x-a}{d-x} = \frac{x-b}{c-x}$



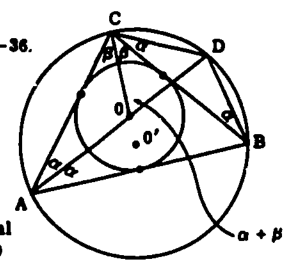
$\therefore x(a - b + c - d) = ac - bd$, $x = \frac{ac - bd}{a - b + c - d}$

29. (B) The required number n is $999 - N_1(5) - N_2(7) + N_3(35)$ where $N_1(5)$ is the number of multiples of 5, namely 199, $N_1(7)$ is the number of multiples of 7, namely 142, and $N_1(35)$ is the number of multiples of 35, namely 28. $\therefore n = 999 - 199 - 142 + 28 = 686$

30. (D) $1 + 2 + 3 + r_4 = 0$, $r_4 = -6$. Since $-a$ represents the sum of the roots taken two at a time and c represents the product of the roots, we have $-a = -2 - 3 + 6 - 6 + 12 + 18 = 25$ and $c = (1)(2)(3)(-6) = -36$ $\therefore a + c = -25 - 36 = -61$

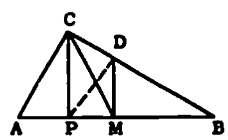
or

Solve the system $1 + a + b + c = 0$, $16 + 4a + 2b + c = 0$, $81 + 9a + 3b + c = 0$ to obtain $a = -25$, $b = 60$, $c = -36$. $\therefore a + c = -61$



31. (D) Quadrilateral ABDC is inscriptible
$\therefore \overset{\frown}{CD} = \overset{\frown}{BD}$, $\therefore CD = BD$
$\angle DCO = \alpha + \beta$, $\angle DOC = \alpha + \beta$, $\therefore OD = CD = BD$

32. (B) Since M is the midpoint of AB, the area of triangle BMC = $\frac{1}{2}$ the area of triangle BAC. But the area of $\triangle BMC$ = the area of $\triangle BMD$ + the area of $\triangle MDC$, and $\triangle MDC = \triangle MDP$ in area (they have the same base MD and equal altitudes to this base since MD ∥ PC). Therefore, in area $\triangle BMC = \triangle BMD + \triangle MDP = \triangle BPD$, $\therefore r = \frac{1}{2}$ independent of the position of P between A and M. Query; Is the theorem true when P is to the left of A?



33. (D) $a(x-a)^2(x-b) + b(x-b)^2(x-a) = ab^2(x-b) + a^2b(x-a)$
$a(x-a)[(x-a)(x-b) - ab] = -b(x-b)[(x-a)(x-b) - ab]$
$\therefore [(x-a)(x-b) - ab][ax - a^2 + bx - b^2] = 0.$
$x^2 - (a+b)x = 0$ or $(a+b)x = a^2 + b^2$ $\quad \therefore x = 0$, $x = a + b$, or $x = \frac{a^2 + b^2}{a + b}$

34. (B) $r = \frac{11}{t} \cdot \frac{1}{5280} \cdot 3600$ where t is given in seconds $\quad \therefore \frac{11}{t - \frac{1}{4}} \cdot \frac{1}{5280} \cdot 3600 = \frac{11}{t} \cdot \frac{1}{5280} \cdot 3600 + 5$
$8t^2 - 2t - 3 = (4t-3)(2t+1) = 0$, $t = \frac{3}{4}$, $r = 10$

35. (C)

$$\begin{array}{l} OA + OC > AC \\ OC + OB > BC \\ \underline{OB + OA > AB} \\ \quad 2s_1 > s_2 \\ \quad s_1 > \frac{1}{2}s_2 \end{array} \qquad \begin{array}{l} OB + OC < AB + AC \\ OC + OA < BC + AB \\ \underline{OA + OB < AC + BC} \\ \quad 2s_1 < 2s_2 \\ \quad s_1 < s_2 \end{array}$$

36. (E) $(1 - x + x^2)^n \equiv a_0 - a_1x + a_2x^2 - a_3x^3 + \ldots$ $\quad (1 + x + x^2)^n \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
$\therefore 2(a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2n}x^{2n}) \equiv (1 - x + x^2)^n + (1 + x + x^2)^n$
Let $x = 1$ $\quad \therefore 2(a_0 + a_2 + \ldots + a_{2n}) = 1^n + 3^n$ $\quad \therefore 2s = 1^n + 3^n$ $\quad \therefore s = \frac{3^n + 1}{2}$

37. (C) Let a, b, c be the number of hours needed, respectively, by Alpha, Beta, and Gamma to do the job when working alone. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a-6} = \frac{1}{b-1} = \frac{1}{c/2}$ $\quad \therefore b = a - 5$ and $c = 2a - 12$
$\therefore \frac{1}{a} + \frac{1}{a-5} + \frac{1}{2(a-6)} = \frac{1}{a-6}$ $\quad \therefore a = \frac{20}{3}$; the value $a = 3$ is rejected.
$\therefore b = \frac{5}{3}$ and $c = \frac{4}{3}$ $\quad \therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{20/3} + \frac{1}{5/3} = \frac{3}{4}$ $\quad \therefore b = \frac{4}{3}$

38. (D) In area $\triangle COM = \frac{1}{2}\triangle COB = \frac{1}{2} \cdot \frac{1}{3}\triangle ABC = \frac{1}{6}\triangle ABC \doteq \triangle CQM + n = \frac{1}{8}\triangle ABC + n$
$\therefore \frac{1}{6}\triangle ABC = \frac{1}{8}\triangle ABC + n$
$\therefore n = \frac{1}{24}\triangle ABC$ $\quad \therefore \triangle ABC = 24n$

39. (E) $F_1 = \frac{3R_1 + 7}{R_1^2 - 1} = \frac{2R_2 + 5}{R_2^2 - 1}$ and $F_2 = \frac{7R_1 + 3}{R_1^2 - 1} = \frac{5R_2 + 2}{R_2^2 - 1}$ $\quad \therefore \frac{3R_1 + 7}{2R_2 + 5} = \frac{R_1^2 - 1}{R_2^2 - 1} = \frac{7R_1 + 3}{5R_2 + 2}$
$\therefore R_2 = \frac{29R_1 + 1}{R_1 + 29}$ Knowing that R_1 and R_2 must each be integral, and that $R_1 \geqq 8$ (why?), we solve for R_2 with permissible values of R_1. For $R_1 = 8$, R_2 is not integral; for $R_1 = 9$ or 10, R_2 is not integral; for $R_1 = 11$, $R_2 = 8$; for $R_1 = 12$, R_2 is not integral; for $R_1 = 13$, $R_2 = 9$. The values $R_1 = 13$, $R_2 = 9$ do not satisfy the conditions of the problem; the values $R_1 = 11$, $R_2 = 8$ do.
$\therefore R_1 + R_2 = 19$

40. (A) $\frac{x}{2a} = \frac{AE}{AC} = \frac{CD}{AC}$ and $\overline{BC}^2 = CD \cdot CA$ and $\frac{x}{y} = \frac{2a}{BC}$ so that $\overline{BC}^2 = \frac{4a^2y^2}{x^2}$. Since $\frac{x}{2a} \cdot \overline{BC}^2 = \frac{CD}{AC} \cdot CD \cdot CA$, $\frac{x}{2a} \cdot \frac{4a^2y^2}{x^2} = \overline{CD}^2$ $\quad \therefore \frac{2ay^2}{x} = \overline{CD}^2 = \overline{AE}^2 = x^2 + y^2$ $\quad \therefore y^2(2a - x) = x^3$ $\quad \therefore y^2 = \frac{x^3}{2a - x}$

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE SOCIETY OF ACTUARIES

and

MU ALPHA THETA

M	A	A
S		A
M	A	Θ

18

EIGHTEENTH ANNUAL MATHEMATICS EXAMINATION 1967

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of this booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

THURSDAY MORNING, MARCH 9, 1967

To be filled in by the student

PRINT ____

last name   first name   middle name or initial

____

school   number   street

____

city   county   state   zip code

PART I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

PART II

21	22	23	24	25	26	27	28	29	30

PART III

31	32	33	34	35	36	37	38	39	40

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

EIGHTEENTH ANNUAL H. S. MATHEMATICS EXAMINATION—1967

PART I (3 credits each)

1. The three-digit number 2a3 is added to the number 326 to give the three-digit number 5b9. If 5b9 is divisible by 9, then a + b equals:

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 9

2. An equivalent of the expression $\left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)+\left(\frac{x^2-1}{y}\right)\left(\frac{y^2-1}{x}\right)$, $xy \neq 0$, is:

   (A) 1 (B) $2xy$ (C) $2x^2y^2+2$ (D) $2xy+\frac{2}{xy}$ (E) $\frac{2x}{y}+\frac{2y}{x}$

3. The side of an equilateral triangle is <u>s</u>. A circle is inscribed in the triangle and a square is inscribed in the circle. The area of the square is:

   (A) $\frac{s^2}{24}$ (B) $\frac{s^2}{6}$ (C) $\frac{s^2\sqrt{2}}{6}$ (D) $\frac{s^2\sqrt{3}}{6}$ (E) $\frac{s^2}{3}$

4. Given $\frac{\log a}{p}=\frac{\log b}{q}=\frac{\log c}{r}=\log x$, all logarithms to the same base and $x \neq 1$. If $\frac{b^2}{ac}=x^y$, then y is:

   (A) $\frac{q^2}{p+r}$ (B) $\frac{p+r}{2q}$ (C) $2q-p-r$ (D) $2q-pr$ (E) q^2-pr

5. A triangle is circumscribed about a circle of radius r inches. If the perimeter of the triangle is P inches and the area is K square inches, then P/K is:

   (A) independent of the value of r (B) $\sqrt{2}/r$ (C) $2/\sqrt{r}$ (D) $2/r$ (E) $r/2$

6. If $f(x)=4^x$ then $f(x+1)-f(x)$ equals:

   (A) 4 (B) $f(x)$ (C) $2f(x)$ (D) $3f(x)$ (E) $4f(x)$

7. If $\frac{a}{b}<\frac{-c}{d}$ where a,b,c,d are real numbers and $bd \neq 0$, then:

   (A) a must be negative (B) a must be positive (C) a must not be zero
   (D) a can be negative or zero, but not positive
   (E) a can be positive, negative, or zero

8. To m ounces of an m% solution of acid, x ounces of water are added to yield an (m −10)% solution. If m > 25, then x is:

   (A) $\frac{10m}{m-10}$ (B) $\frac{5m}{m-10}$ (C) $\frac{m}{m-10}$ (D) $\frac{5m}{m-20}$
   (E) not determined by the given information

9. Let K, in square units, be the area of a trapezoid such that the shorter base, the altitude, and the longer base, in that order, are in arithmetic progression. Then:

   (A) K must be an integer (B) K must be a rational fraction (C) K must be an irrational number (D) K must be an integer or a rational fraction (E) taken alone neither (A) nor (B) nor (C) nor (D) is true

10. If $\frac{a}{10^x - 1} + \frac{b}{10^x + 2} = \frac{2 \cdot 10^x + 3}{(10^x - 1)(10^x + 2)}$ is an identity for positive rational values of x, then the value of $a - b$ is:

    (A) 4/3 (B) 5/3 (C) 2 (D) 11/4 (E) 3

11. If the perimeter of rectangle ABCD is 20 inches, the least value of diagonal AC, in inches, is:

    (A) 0 (B) $\sqrt{50}$ (C) 10 (D) $\sqrt{200}$ (E) none of these

12. If the (convex) area bounded by the x-axis and the lines $y = mx + 4$, $x = 1$, and $x = 4$ is 7, then m equals:

    (A) $-1/2$ (B) $-2/3$ (C) $-3/2$ (D) -2 (E) none of these

13. A triangle ABC is to be constructed given side a (opposite angle A), angle B, and h_c, the altitude from C. If N is the number of noncongruent solutions, then N

    (A) is 1 (B) is 2 (C) must be zero (D) must be infinite (E) must be zero or infinite

14. Let $f(t) = \frac{t}{1-t}$, $t \neq 1$. If $y = f(x)$, then x can be expressed as:

    (A) $f\left(\frac{1}{y}\right)$ (B) $-f(y)$ (C) $-f(-y)$ (D) $f(-y)$ (E) $f(y)$

15. The difference in the areas of two similar triangles is 18 square feet, and the ratio of the larger area to the smaller is the square of an integer. The area of the smaller triangle, in square feet, is an integer, and one of its sides is 3 feet. The corresponding side of the larger triangle, in feet, is:

    (A) 12 (B) 9 (C) $6\sqrt{2}$ (D) 6 (E) $3\sqrt{2}$

16. Let the product (12)(15)(16), each factor written in base b, equal 3146 in base b. Let $s = 12 + 15 + 16$, each term expressed in base b. Then s, in base b, is:

    (A) 43 (B) 44 (C) 45 (D) 46 (E) 47

17. If r_1 and r_2 are the distinct real roots of $x^2 + px + 8 = 0$, then it must follow that:

(A) $|r_1 + r_2| > 4\sqrt{2}$ (B) $|r_1| > 3$ or $|r_2| > 3$ (C) $|r_1| > 2$ and $|r_2| > 2$
(D) $r_1 < 0$ and $r_2 < 0$ (E) $|r_1 + r_2| < 4\sqrt{2}$

18. If $x^2 - 5x + 6 < 0$ and $P = x^2 + 5x + 6$ then

(A) P can take any real value (B) $20 < P < 30$ (C) $0 < P < 20$
(D) $P < 0$ (E) $P > 30$

19. The area of a rectangle remains unchanged when it is made $2\frac{1}{2}$ inches longer and $\frac{2}{3}$ inch narrower, or when it is made $2\frac{1}{2}$ inches shorter and $\frac{4}{3}$ inch wider. Its area, in square inches, is:

(A) 30 (B) 80/3 (C) 24 (D) 45/2 (E) 20

20. A circle is inscribed in a square of side m, then a square is inscribed in that circle, then a circle is inscribed in the latter square, and so on. If S_n is the sum of the areas of the first n circles so inscribed, then, as n grows beyond all bounds, S_n approaches:

(A) $\frac{\pi m^2}{2}$ (B) $\frac{3\pi m^2}{8}$ (C) $\frac{\pi m^2}{3}$ (D) $\frac{\pi m^2}{4}$ (E) $\frac{\pi m^2}{8}$

PART II (4 credits each)

21. In right triangle ABC the hypotenuse AB = 5 and leg AC = 3. The bisector of angle A meets the opposite side in A_1. A second right triangle PQR is then constructed with hypotenuse $PQ = A_1B$ and leg $PR = A_1C$. If the bisector of angle P meets the opposite side in P_1, the length of PP_1 is:

(A) $\frac{3\sqrt{6}}{4}$ (B) $\frac{3\sqrt{5}}{4}$ (C) $\frac{3\sqrt{3}}{4}$ (D) $\frac{3\sqrt{2}}{2}$ (E) $\frac{15\sqrt{2}}{16}$

22. For the natural numbers, when P is divided by D, the quotient is Q and the remainder is R. When Q is divided by D′, the quotient is Q′ and the remainder is R′. Then, when P is divided by DD′, the remainder is:

(A) R + R′D (B) R′ + RD (C) RR′ (D) R (E) R′

23. If x is real and positive and grows beyond all bounds, then $\log_3(6x - 5) - \log_3(2x + 1)$ approaches:

(A) 0 (B) 1 (C) 3 (D) 4 (E) no finite number

24. The number of solution-pairs in positive integers of the equation $3x + 5y = 501$ is:

(A) 33 (B) 34 (C) 35 (D) 100 (E) none of these.

25. For every odd number $p > 1$ we have:

(A) $(p-1)^{\frac{1}{2}(p-1)} - 1$ is divisible by $p-2$

(B) $(p-1)^{\frac{1}{2}(p-1)} + 1$ is divisible by p (C) $(p-1)^{\frac{1}{2}(p-1)}$ is divisible by p

(D) $(p-1)^{\frac{1}{2}(p-1)} + 1$ is divisible by $p+1$

(E) $(p-1)^{\frac{1}{2}(p-1)} - 1$ is divisible by $p-1$

26. If one uses only the tabular information $10^3 = 1000$, $10^4 = 10,000$, $2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$, $2^{13} = 8192$, then the strongest statement one can make for $\log_{10} 2$ is that it lies between:

(A) $\frac{3}{10}$ and $\frac{4}{11}$ (B) $\frac{3}{10}$ and $\frac{4}{12}$ (C) $\frac{3}{10}$ and $\frac{4}{13}$ (D) $\frac{3}{10}$ and $\frac{40}{132}$ (E) $\frac{3}{11}$ and $\frac{40}{132}$

27. Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other, in 4 hours. At what time P.M. should the candles be lighted so that, at 4 P.M., one stub is twice the length of the other?

(A) 1:24 (B) 1:28 (C) 1:36 (D) 1:40 (E) 1:48

28. Given the two hypotheses: I Some Mems are not Ens and II No Ens are Vees. If "some" means "at least one", we can conclude that:

(A) Some Mems are not Vees (B) Some Vees are not Mems
(C) No Mem is a Vee (D) Some Mems are Vees
(E) Neither (A) nor (B) nor (C) nor (D) is deducible from the given statements.

29. AB is a diameter of a circle. Tangents AD and BC are drawn so that AC and BD intersect in a point on the circle. If AD = a and BC = b, $a \neq b$, the diameter of the circle is:

(A) $|a-b|$ (B) $\frac{1}{2}(a+b)$ (C) $\sqrt{ab}$ (D) $\frac{ab}{a+b}$ (E) $\frac{1}{2}\frac{ab}{a+b}$

30. A dealer bought n radios for d dollars, d, a positive integer. He contributed two radios to a community bazaar at half their cost. The rest he sold at a profit of \$8 on each radio sold. If the overall profit was \$72, then the least possible value of n for the given information is:

(A) 18 (B) 16 (C) 15 (D) 12 (E) 11

PART III (5 credits each)

31. Let $D = a^2 + b^2 + c^2$ where a,b are consecutive integers and c = ab. Then $\sqrt{D}$ is:

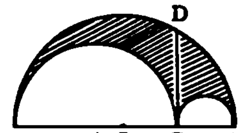
(A) always an even integer (B) sometimes an odd integer, sometimes not
(C) always an odd integer (D) sometimes rational, sometimes not
(E) always irrational

32. In quadrilateral ABCD with diagonals AC and BD, intersecting at O, BO = 4, OD = 6, AO = 8, OC = 3, and AB = 6. The length of AD is:

(A) 9 (B) 10 (C) $6\sqrt{3}$ (D) $8\sqrt{2}$ (E) $\sqrt{166}$

33. In this diagram semi-circles are constructed on diameters AB, AC, and CB, so that they are mutually tangent. If CD ⊥ AB, then the ratio of the shaded area to the area of a circle with CD as radius is:

(A) 1:2 (B) 1:3 (C) $\sqrt{3}$:7 (D) 1:4 (E) $\sqrt{2}$:6



34. Points D,E,F are taken respectively on sides AB, BC, and CA of triangle ABC so that AD:DB = BE:CE = CF:FA = 1:n. The ratio of the area of triangle DEF to that of triangle ABC is:

(A) $\frac{n^2 - n + 1}{(n + 1)^2}$ (B) $\frac{1}{(n + 1)^2}$ (C) $\frac{2n^3}{(n + 1)^3}$ (D) $\frac{n^3}{(n + 1)^3}$ (E) $\frac{n(n - 1)}{n + 1}$

35. The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. The difference between the largest and smallest roots is:

(A) 2 (B) 1 (C) 1/2 (D) 3/8 (E) 1/4

36. Given a geometric progression of five terms, each a positive integer less than 100. The sum of the five terms is 211. If S is the sum of those terms in the progression which are squares of integers, then S is:

(A) 0 (B) 91 (C) 133 (D) 195 (E) 211

37. Segments AD = 10, BE = 6, CF = 24 are drawn from the vertices of triangle ABC, each perpendicular to a straight line RS, not intersecting the triangle. Points D,E,F are the intersection points of RS with the perpendiculars. If x is the length of the perpendicular segment drawn to RS from the intersection point of the medians of the triangle, then x is:

(A) 40/3 (B) 16 (C) 56/3 (D) 80/3 (E) undetermined

38. Given a set S consisting of two undefined elements "pib" and "maa", and the four postulates: P_1: Every pib is a collection of maas, P_2: Any two distinct pibs have one and only one maa in common, P_3: Every maa belongs to two and only two pibs, P_4: There are exactly four pibs.

Consider the three theorems: T_1: There are exactly six maas, T_2: There are exactly three maas in each pib, T_3: For each maa there is exactly one other maa not in the same pib with it. The theorems which are deducible from the postulates are:

(A) T_3 only (B) T_2 and T_3 only (C) T_1 and T_2 only (D) T_1 and T_3 only (E) all

39. Given the sets of consecutive integers $\{1\}, \{2,3\}, \{4,5,6\}, \{7,8,9,10\}, \cdots$, where each set contains one more element than the preceding one, and where the first element of each succeeding set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Then S_{21} equals:

(A) 1113 (B) 4641 (C) 5082 (D) 53361 (E) none of these

40. Located inside equilateral triangle ABC is point P such that PA = 6, PB = 8, and PC = 10. To the nearest integer the area of triangle ABC is:

(A) 159 (B) 131 (C) 95 (D) 79 (E) 50

COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Pan American College, Edinburg, Texas 78539
New York Office: Polytechnic Institute of Brooklyn, Brooklyn, N.Y. 11201

SOLUTION KEY

AND ANSWER KEY

EIGHTEENTH ANNUAL H. S.

MATHEMATICS EXAMINATION

1967

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE SOCIETY OF ACTUARIES

MU ALPHA THETA

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions are intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

National Office Pan American College, Edinburg, Texas 78539
New York Office Polytechnic Institute of Brooklyn, Brooklyn, N. Y. 11201

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1967 examination.

1. (C) Since 5b9 is divisible by 9, $b = 4$ $\therefore a + 2 = 4$, $a = 2$, and $a + b = 6$.

2. (D) The given expression equals $\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + \left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)$

$$= xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy} + xy - \frac{y}{x} - \frac{x}{y} + \frac{1}{xy} = 2xy + \frac{2}{xy}.$$

or

The given expression equals $\dfrac{x^2y^2 + x^2 + y^2 + 1 + x^2y^2 - x^2 - y^2 + 1}{xy} = \dfrac{2x^2y^2 + 2}{xy} = 2xy + \dfrac{2}{xy}.$

3. (B) Let r be the radius; b, the altitude; and x, the side of the square.

$$r = \frac{1}{3}b = \frac{1}{3} \cdot \frac{1}{2} a\sqrt{3} = \frac{a\sqrt{3}}{6} \quad \therefore \text{Area (square)} = x^2 = 2r^2 = \frac{a^2}{6}.$$

4. (C) Since $\frac{\log a}{p} = \log x$, $a = x^p$. Similarly $b = x^q$ and $c = x^r$.

$$\therefore \frac{b^2}{ac} = \frac{x^{2q}}{x^p \cdot x^r} = x^{2q-p-r}. \text{ Since, also, } \frac{b^2}{ac} = x^y,\ y = 2q - p - r.$$

5. (D) Let the sides of the triangle, in inches, be a, b, and c. Then

$$K = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}rP. \quad \therefore P/K = \frac{2}{r}.$$

6. (D) $f(x + 1) - f(x) = 4^{x+1} - 4^x = 4^x(4 - 1) = 3 \cdot 4^x = 3f(x)$.

7. (E) If $bd > 0$ then $a < -\frac{bc}{d}$. When $-c > 0$, $-\frac{bc}{d} > 0$, so that a is less than a positive quantity.

$\therefore$ a may be positive, zero, or negative.

8. (A) $\dfrac{m \cdot \frac{m}{100}}{m + x} = \dfrac{m - 10}{100}$ $\therefore x = \dfrac{10m}{m - 10}$. The solution is valid for $m > 10$. It, therefore, holds for $m > 25$.

9. (E) The shorter base, the altitude, and the longer base may be represented by $a - d$, a, and $a + d$, respectively. $\therefore K = \frac{1}{2}a(a - d + a + d) = a^2$. Since a^2 may be the square of an integer, a non-square integer, the square of a rational fraction, a non-square rational fraction, or irrational, the correct choice is (E).

10. (A) $a(10^x + 2) + b(10^x - 1) = 2 \cdot 10^x + 3$ $\quad\therefore a + b = 2$ and $2a - b = 3$ $\quad\therefore a = \frac{5}{3}$ and $b = \frac{1}{3}$ $\quad\therefore a - b = \frac{4}{3}$.

11. (B) Let the dimensions be x and $10 - x$ and let the diagonal be d. Then $d^2 = x^2 + (10 - x)^2 = 2x^2 - 20x + 100 = 2(x^2 - 10x + 25) + 50 = 2(x - 5)^2 + 50$. The least value of d^2 (and, hence, of d) occurs when $x = 5$. $\therefore$ d (least) $= \sqrt{50}$.

or

The graph of $2x^2 - 20x + 100$ is a parabola with the low point (minimum) at vertex (5, 50).

$\therefore d^2$ (least) $= 50$ and d (least) $= \sqrt{50}$.

or

Of all rectangles with a fixed perimeter the one with least diagonal is the square. Since the perimeter is 20, the side of the square is 5, and, therefore, $d = \sqrt{50}$:

12. (B) The area of the trapezoid thus formed is $\frac{1}{2}(3)(m + 4 + 4m + 4) = 7$. $\quad\therefore 5m = -\frac{10}{3}$, $m = -\frac{2}{3}$.

13. (E) The given parts are not independent since $h_c = a \sin B$. There is no triangle if $h_c < a \sin B$ or $h_c > a \sin B$. When $h_c = a \sin B$, we are free to choose vertex A anywhere on BD, including right and left extensions, where D is the foot of h_c.

14. (C) Since $y = \frac{x}{1 - x}$, $y - yx = x$ and $x = \frac{y}{1 + y}$ $\quad\therefore x = -\frac{-y}{1 - (-y)} = -f(-y)$.

15. (D) Let T_1, T_2 represent the larger, smaller areas, respectively. Since the triangles are similar and 3 is a side of the smaller triangle, we have
$\frac{T_1}{T_2} = \frac{x^2}{3^2}$. $\therefore \frac{T_2 + 18}{T_2} = \frac{x^2}{3^2} = k^2$, k a positive integer $\therefore T_2 = \frac{18}{k^2 - 1}$; since T_2 is an integer, 18 must be divisible by $k^2 - 1$. Only $k^2 = 4$ is acceptable. $\therefore \frac{x^2}{3^2} = 4$ and $x = 6$.

or

Since $\frac{T_2 + 18}{T_2} = 1 + \frac{18}{T_2} = k^2$, T_2 must equal 6 and k must equal 2. But k is the ratio of corresponding sides. It follows that the side of the larger triangle corresponding to side 3 of the smaller triangle is 6.

16. (B) Let $P = (b + 2)(b + 5)(b + 6) = 3146$ (base b) $\therefore b^3 + 13b^2 + 52b + 60 = 3b^3 + b^2 + 4b + 6$.
$\therefore 0 = b^3 - 6b^2 - 24b - 27$ and $b = 9$. But $s = (b + 2) + (b + 5) + (b + 6) = 3b + 9 + 4$.
$\therefore s = 3 \cdot 9 + 9 + 4 = 4 \cdot 9 + 4 = 44$ (base b).

17. (A) Since r_1 and r_2 are real and distinct, $p^2 - 32 > 0$ $\therefore |p| > 4\sqrt{2}$. But $r_1 + r_2 = -p$
$\therefore |r_1 + r_2| = |p|$ $\therefore |r_1 + r_2| > 4\sqrt{2}$.

18. (B) Since $x^2 - 5x + 6 = (x - 3)(x - 2) < 0$, we have $2 < x < 3$. Since $P = x^2 + 5x + 6$, then $P < 3^2 + 5 \cdot 3 + 6 = 30$ and $P > 2^2 + 5 \cdot 2 + 6 = 20$, that is, $20 < P < 30$.

19. (E) If l and w represent the dimensions of the rectangle, then $\left(l + \frac{5}{2}\right)\left(w - \frac{2}{3}\right) = \left(l - \frac{5}{2}\right)\left(w + \frac{4}{3}\right) = lw$
$\therefore l = \frac{15}{2}$ and $w = \frac{8}{3}$, and the area is 20.

20. (A) The radius of the first circle is $\frac{1}{2}m$; the side of the second square is $\frac{m}{2}\sqrt{2}$; the radius of the second circle is $\frac{1}{2}\left(\frac{m\sqrt{2}}{2}\right) = \frac{m}{2\sqrt{2}}$; and so forth. If S is the limiting value of S_n, then $S = \pi\left(\frac{m}{2}\right)^2 + \pi\left(\frac{m}{2\sqrt{2}}\right)^2 + \pi\left(\frac{m}{4}\right)^2 + \ldots = \pi m^2\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) = \pi m^2\left(\frac{1}{2}\right)$.

21. (B) Let $BA_1 = x$ $\therefore \frac{x}{4 - x} = \frac{5}{3}$, $x = \frac{5}{2}$ and $4 - x = \frac{3}{2}$. $\therefore PR = 4 - x = \frac{3}{2}$ and $PQ = x = \frac{5}{2}$.
$\therefore \Delta PQR \sim \Delta BAC$, the ratio of the sides being 1:2. In ΔBAC, $\overline{AA_1}^2 = 3^2 + \left(\frac{3}{2}\right)^2 = \frac{45}{4}$
$\therefore AA_1 = \frac{3\sqrt{5}}{2}$ But $PP_1 : AA_1 = 1:2$ $\therefore PP_1 = \frac{1}{2} \cdot \frac{3\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}$.

22. (A) $P = QD + R$ and $Q = Q'D' + R'$ $\therefore P = Q'DD' + R'D + R$, which means that when P is divided by DD' the quotient is Q' and the remainder is $R + R'D$.

23. (B) $\log_3(6x - 5) - \log_3(2x + 1) = \log_3 \frac{6x - 5}{2x + 1} = \log_3 \frac{6 - \frac{5}{x}}{2 + \frac{1}{x}}$ for $x \neq 0$. As x grows beyond all bounds, the last expression approaches $\log_3 \frac{6}{2} = \log_3 3 = 1$.

24. (A) $3x = 501 - 5y$. For x to be a positive integer $\frac{501 - 5y}{3} > 0$ $\therefore 5y < 501$ and $y \leq 100$.
Also $x = 167 - y - \frac{2y}{3}$; for integral x, y must be a multiple of 3, that is, $y = 3k$. Since $y \leq 100$, $k = 1, 2, \ldots, 33$.

or

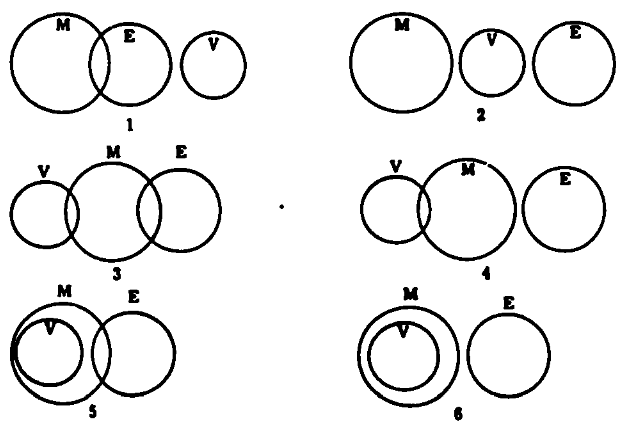
In Number Theory it is shown that if x_0, y_0 is one solution of $3x + 5y = 501$, then other solutions are $x = x_0 - \frac{5}{d}t$, $y = y_0 + \frac{3}{d}t$ where t is an integer and d is the greatest common divisor of 3 and 5, so that, in this case, $d = 1$. An obvious solution of the given equation is $x = 167$, $y = 0$. Therefore, other solutions are $x = 167 - 5t$, $y = 0 + 3t$. Since $x = 167 - 5t > 0$, $t < \frac{167}{5}$ so that $t = 1, 2, \ldots, 33$.

25. (A) Since p is odd and $p > 1$, then $\frac{1}{2}(p - 1) \geq 1$. In every case one factor of $(p - 1)^{\frac{1}{2}(p-1)} - 1$ will be $[(p - 1) - 1] = p - 2$. The other choices are either possible only for special permissible values of p or not possible for any permissible values of p.

26. (C) Since $2^{10} = 1024 > 10^3 = 1000$, $10 \log_{10} 2 > 3$ so that $\log_{10} 2 > \frac{3}{10}$. Since $2^{13} = 8192 < 10^4 = 10000$, $13 \log_{10} 2 < 4$ so that $\log_{10} 2 < \frac{4}{13}$.

27. (C) Let t represent the number of hours before 4 P.M. necessary to obtain the desired result. Then, representing the original length by L, we have $L - \frac{t}{4}L = 2\left(L - \frac{t}{3}L\right)$ or, more simply, $1 - \frac{t}{4} = 2\left(1 - \frac{t}{3}\right)$
$\therefore t = 2\frac{2}{5}$ and $4 - 2\frac{2}{5} = 1\frac{3}{5}$. Therefore, the candles should be lighted $1\frac{3}{5}$ hours after noon, that is, at 1:36 P.M.

28. (E) The given hypotheses are satisfied by any one of these six Venn diagrams:

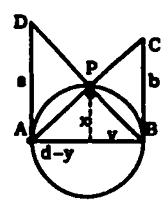


Choice (A) is contradicted by diagrams 3, 4, 5, 6. Choice (B) is contradicted by diagrams 5, 6.
Choice (C) is contradicted by diagrams 3, 4, 5, 6. Choice (D) is contradicted by diagrams 1, 2.

or

Suppose there is only one Mem and that it is not an En, and suppose there are no Vees. Then the hypotheses are satisfied but (B) and (D) are contradicted. If there are Vees and the Mem is one of them, then again the hypotheses are satisfied but (A) and (C) are contradicted.

29. (C) From similar triangles $\frac{x}{y} = \frac{a}{d}$ and $\frac{x}{d-y} = \frac{b}{d}$



$\therefore \frac{x^2}{y(d-y)} = \frac{ab}{d^2}$. But $x^2 = y(d-y)$. $\therefore 1 = \frac{ab}{d^2}$ $\therefore d = \sqrt{ab}$.

or

Let the degree measure of arc AP be m.

Then $\angle D = 90 - \frac{m}{2}$ and $\angle C = \frac{m}{2}$.

$\therefore \frac{d}{b} = \tan\frac{m}{2}$ and $\frac{d}{a} = \tan\left(90 - \frac{m}{2}\right) = \frac{1}{\tan\frac{m}{2}}$

$\therefore \tan^2\frac{m}{2} = \frac{a}{b}$ $\therefore \frac{d^2}{b^2} = \frac{a}{b}$, $d^2 = ab$, $d = \sqrt{ab}$.

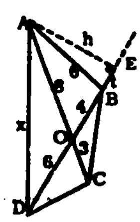
30. (D) $(n-2)\left(\frac{d}{n} + 8\right) + 2\cdot\frac{d}{2n} = 72 + d$, $8n^2 - 88n - d = 0$, $n^2 - 11n - \frac{d}{8} = 0$. Since n is a (least positive) integer, d must be such that $n^2 - 11n - \frac{d}{8}$ yields linear factors with integer coefficients. Hence $d = 96$ and $\frac{d}{8} = 12$ so that $n^2 - 11n - 12 = (n-12)(n+1) = 0$ and n (least) = 12.

or

Since $8n^2 - 88n - d = 0$, $88 = 8n - \frac{d}{n}$ whence n can be any integer > 11.

31. (C) $D = a^2 + b^2 + c^2 = a^2 + (a+1)^2 + (a(a+1))^2 = a^4 + 2a^3 + 3a^2 + 2a + 1 = (a^2 + a + 1)^2$.
$\therefore \sqrt{D} = a^2 + a + 1$, an odd integer for any integer a.

32. (E) $8^2 = h^2 + (4+t)^2$, $6^2 = h^2 + t^2$, $t = \frac{3}{2}$



$\therefore h = \frac{3\sqrt{15}}{2}$ $\therefore x^2 = h^2 + \overline{DE}^2 = \frac{135}{4} + \frac{529}{4} = 166$ $\therefore x = \sqrt{166}$.

or

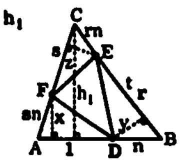
$\Delta BOC \sim \Delta AOD$ with side ratio 1 : 2 $\therefore BC = \frac{1}{2}x$

$\Delta AOB \sim \Delta DOC$ $\therefore CD = 4\frac{1}{2}$. Since ABCD is inscriptible (why?) we may use Ptolemy's Theorem.

$\therefore x\cdot\frac{x}{2} + 6\cdot 4\frac{1}{2} = (6+4)(8+3)$ $\therefore x = \sqrt{166}$.

33. (D) Shaded area $= \frac{1}{2}\left(\frac{\pi}{4}\overline{AB}^2 - \frac{\pi}{4}\overline{AC}^2 - \frac{\pi}{4}\overline{CB}^2\right)$. Since AB = AC + CB, shaded area $= \frac{1}{2}\cdot\frac{\pi}{4}\left(\overline{AC}^2 + 2AC\cdot CB + \overline{CB}^2 - \overline{AC}^2 - \overline{CB}^2\right) = \frac{\pi}{4}(AC)(CB)$. But the area of the required circle equals $\pi\overline{CD}^2$, and since $\overline{CD}^2 = (AC)(CB)$, the area of the circle equals $\pi(AC)(CB)$. Therefore, the required ratio is 1 : 4.

34. (A) Designate AD by 1, DB by n, BE by r, EC by rn, CF by s, and FA by sn. Let h_1 be the altitude from C, h_2, the altitude from A, and h_3, the altitude from B.



By similar triangles $\frac{x}{h_1} = \frac{sn}{sn+s} = \frac{n}{1+n}$ $\therefore x = \frac{n}{1+n}h_1$

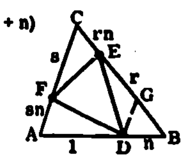
$\therefore$ area $(\Delta ADF) = \frac{1}{2}(1)(x) = \frac{1}{2}\frac{nh_1}{1+n}$. In a like manner $y = \frac{nh_2}{1+n}$ and area $(\Delta BDE) = \frac{1}{2}\frac{rnh_2}{1+n}$, and $z = \frac{nh_3}{1+n}$ and area $(\Delta CFE) = \frac{1}{2}\frac{snh_3}{1+n}$.

But the area $(\Delta ABC) = \frac{1}{3}\cdot\frac{1}{2}[h_1(1+n) + h_2(1+n)r + h_3(1+n)s] = \frac{1}{6}(1+n)(h_1 + h_2r + h_3s)$

$\therefore$ area $(\Delta DEF) = \frac{1}{6}(1+n)(h_1 + h_2r + h_3s) - \frac{1}{2}\frac{n}{1+n}(h_1 + h_2r + h_3s) = \frac{n^2-n+1}{6(1+n)}(h_1 + h_2r + h_3s)$

$\therefore$ the required ratio is $\frac{n^2-n+1}{(n+1)^2}$.

or



Draw DG ∥ AC. By similar triangles $\frac{BG}{BC} = \frac{n}{1+n}$ $\therefore BG = \frac{n}{1+n}BC = \frac{n}{1+n}\cdot r(1+n) = nr$. Also, letting K be the area of ΔABC, we have $\frac{\text{area }(\Delta DBG)}{K} = \frac{n^2}{(1+n)^2}$

$\therefore$ area $(\Delta DBG) = \frac{n^2}{(1+n)^2}K$. Since ΔBDE has base r and altitude equal to that of ΔBDG, $\frac{\text{area }(\Delta BDE)}{\text{area }(\Delta BDG)} = \frac{r}{BG} = \frac{r}{rn} = \frac{1}{n}$ $\therefore$ area $(\Delta BDE) = \frac{1}{n}\cdot\frac{n^2}{(1+n)^2}K$

$= \frac{n}{(1+n)^2}K$. In like manner $(\Delta ECF) = \frac{n}{(1+n)^2}K$ and area $(\Delta ADF) = \frac{n}{(1+n)^2}K$.

$\therefore$ area $(\Delta DEF) = K - \frac{3n}{(1+n)^2}K = \frac{n^2-n+1}{(n+1)^2}K$.

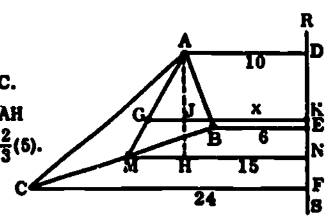
35. (B) Let the roots be $a-d$, a, and $a+d$. Then $[x-(a-d)](x-a)[x-a+d] = x^3 - 3ax^2 + x(3a^2-d^2) + (a^3-ad^2) = 0$. But $64x^3 - 144x^2 + 92x - 15 = 64\left(x^3 - \frac{9}{4}x^2 + \frac{23}{16}x - \frac{15}{64}\right) = 0$ $\therefore x^3 - \frac{9}{4}x^2 + \frac{23}{16}x - \frac{15}{64} = 0$ $= x^3 - 3ax^2 + x(3a^2-d^2) - (a^3-ad^2)$ $\therefore -3a = -\frac{9}{4}$, $a = \frac{3}{4}$ and $3a^2 - d^2 = \frac{23}{16}$, $d^2 = \frac{4}{16}$, $d = +\frac{1}{2}$ or $-\frac{1}{2}$.

$\therefore$ the roots are $\frac{5}{4}$, $\frac{3}{4}$, and $\frac{1}{4}$, and the required difference is 1.

36. (C) $S = a(1 + r + r^2 + r^3 + r^4) = 211$. If r is an integer a must equal 1 since 211 is prime. However, neither $a = 1, r = 1$ nor $a = 1, r = 2$, nor $a = 1, r = 3$ is satisfactory. $\therefore 2 < r < 3$ or $1 < r < 2$. Noting that when $r = \frac{m}{n}$, m, n integers, a must equal n^4 since 211 is prime, we try $r = \frac{3}{2}$, $a = 16$.

$\therefore s = 16\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^4\right) = 211$, and the combination $a = 16$, $r = \frac{3}{2}$ is usable. By symmetry the combination $a = 81$, $r = \frac{2}{3}$ is also usable. In either case, the odd-numbered terms are squares of integers, and their sum is $16 + 36 + 81 = 133$.

37. (A) Let M be the midpoint of CB and let G be the intersection point of the three medians. Draw MN perpendicular to RS. Then AD, x, BE, MN, and CF are parallel. MN is the median of trapezoid BEFC.



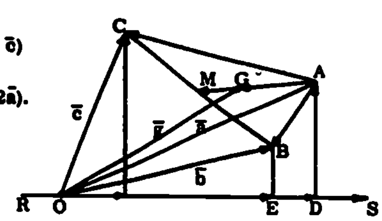
$\therefore MN = \frac{1}{2}(6 + 24) = 15$. In trapezoid ADNM draw AH ⊥ MN and let AH intersect GK in J. Then HN = 10 and MH = 5 and JK = 10 and $GJ = \frac{2}{3}(5)$.

$\therefore x = GK = \frac{10}{3} + 10 = \frac{40}{3}$.

or

Start with MN = 15 and use the Principle of Weighted Means: since the ratio AG: GM = 2: 1, we have $GK = \frac{2\cdot 15 + 1\cdot 10}{3} = \frac{40}{3}$.

or



Using vectors with O as origin, we first show that $\bar{g} = \frac{1}{3}(\bar{a} + \bar{b} + \bar{c})$ where $\bar{g} = \overline{OG}$, $\bar{a} = \overline{OA}$, $\bar{b} = \overline{OB}$, $\bar{c} = \overline{OC}$. We have

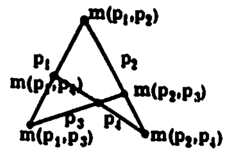
$\overrightarrow{AB} = \bar{b} - \bar{a}$, $\overrightarrow{AC} = \bar{c} - \bar{a}$, $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$ $\therefore \overrightarrow{AM} = \frac{1}{2}(\bar{b} + \bar{c} - 2\bar{a})$.

Since $\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AM}$, $\overrightarrow{AG} = \frac{1}{3}(\bar{b} + \bar{c} - 2\bar{a})$. But $\overrightarrow{AG} = \bar{g} - \bar{a}$

$\therefore \frac{1}{3}(\bar{b} + \bar{c} - 2\bar{a}) = \bar{g} - \bar{a}$ $\therefore \bar{g} = \frac{1}{3}(\bar{a} + \bar{b} + \bar{c})$.

Let unit vector i be in the direction RS and unit vector j be in the direction DA. Then $\bar{g} = ix + jy$, $\bar{a} = ix_1 + 10j$, $\bar{b} = ix_2 + 6j$, $\bar{c} = ix_3 + 24j$ $\therefore jy = \frac{1}{3}(10 + 6 + 24)j$ $\therefore y = \frac{40}{3}$.

36. (E) One method of establishing theorems for a finite geometry is to construct a model. In the one shown here the maa, $m(p_1, p_2)$, for example, is common to pib 1 and pib 2. The maa common to pib 3 and pib 4 should be labeled $m(p_3, p_4)$. In this model each of the four postulates is satisfied. Since there must be a maa for every pair of pibs, there must be a maa for each of the pairs (p_1, p_2), (p_1, p_3), (p_1, p_4), (p_2, p_3), (p_2, p_4), and (p_3, p_4), six in all. This establishes T_1. Each pib consists of exactly three maas. For example, p_3 consists of $m(p_1, p_3)$, $m(p_4, p_3)$, and $m(p_2, p_3)$. This establishes T_2. For $m(p_1, p_2)$ there is $m(p_3, p_4)$ not in p_1 or p_2, and, similarly, for each of the other maas. This establishes T_3.

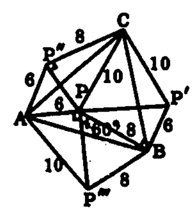


or

By P_2 and P_3 there is a one-one correspondence between the set of maas and the set of pairs of pibs. The four pibs yield six pairs (listed above) and so T_1 is true. Each pib belongs to three of the six pairs and so T_2 is true. Each pair of pibs is disjoint from one other pair (for example, the pair p_1, p_2 is disjoint from the pair p_3, p_4) and so T_3 is true.

39. (B) The first element in the nth set is 1 more than the sum of the number of elements in the preceding n − 1 sets, that is, $\{1 + 2 + \ldots + n - 1\} + 1 = \frac{(n-1)(n)}{2} + 1 = \frac{n^2 - n + 2}{2}$. Since the nth set contains n elements its last element is $\frac{n^2 - n + 2}{2} + n - 1 = \frac{n^2 + n}{2}$. Therefore, $S_n = \frac{n}{2}\left(\frac{n^2 - n + 2}{2} + \frac{n^2 + n}{2}\right) = \frac{n}{2}(n^2 + 1)$ $\therefore S_{21} = \frac{21}{2}(21^2 + 1) = 4641$.

40. (D) Rotate CP through 60° to position CP′. Draw BP′. This is equivalent to rotating △CAP into position CBP′. In a similar manner rotate △ABP into position ACP″ and △BCP into position BAP‴.



On the one hand hexagon AP‴BP′CP″ consists of △ABC and △CBP′, ACP″, BAP‴. Letting K represent area, we have, from the congruence relations,

$K(ABC) = K(CBP') + K(ACP'') + K(BAP''')$ $\therefore K(ABC) = \frac{1}{2}K$ (hexagon).

On the other hand the hexagon consists of three quadrilaterals PCP′B, PBP‴A, and PAP″C, each of which consists of the 6-8-10 right triangle and an equilateral triangle. $\therefore K(\text{hexagon}) = 3\left(\frac{1}{2}\cdot 6\cdot 8\right) + \frac{1}{4}\cdot 10^2\sqrt{3} + \frac{1}{4}\cdot 8^2\sqrt{3} + \frac{1}{4}\cdot 6^2\sqrt{3}$

$= 72 + 50\sqrt{3}$ $\therefore K(ABC) = 36 + 25\sqrt{3} \approx 79$.

or

Applying the Law of Cosines to △APB wherein ∠APB = 150°, we have $s^2 = 6^2 + 8^2 - 2\cdot 6\cdot 8\cos 150° = 100 + 48\sqrt{3}$. Therefore, $K(ABC) = \frac{s^2\sqrt{3}}{4} = 25\sqrt{3} + 36 \approx 79$.

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
and
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

19

NINETEENTH ANNUAL MATHEMATICS EXAMINATION 1968

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

***This instruction may be modified if machine-scoring is used.**

TUESDAY, MARCH 12, 1968

2E

To be filled in by the student

PRINT

last name	first name	middle initial

school	street address

city	state	zip code

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. Let P units be the increase in the circumference of a circle resulting from an increase of π units in the diameter. Then P equals:

   (A) $\frac{1}{\pi}$ (B) π (C) $\frac{\pi^2}{2}$ (D) π^2 (E) 2π

2. The real value of x such that 64^{x-1} divided by 4^{x-1} equals 256^{2x} is:

   (A) $-\frac{2}{3}$ (B) $-\frac{1}{3}$ (C) 0 (D) $\frac{1}{4}$ (E) $\frac{3}{8}$

3. A straight line passing through the point (0, 4) is perpendicular to the line $x - 3y - 7 = 0$. Its equation is:

   (A) $y + 3x - 4 = 0$ (B) $y + 3x + 4 = 0$ (C) $y - 3x - 4 = 0$
   (D) $3y + x - 12 = 0$ (E) $3y - x - 12 = 0$

4. Define an operation * for positive real numbers as $a * b = \frac{ab}{a+b}$. Then $4 * (4 * 4)$ equals:

   (A) $\frac{3}{4}$ (B) 1 (C) $\frac{4}{3}$ (D) 2 (E) $\frac{16}{3}$

5. If $f(n) = \frac{1}{3}n(n+1)(n+2)$, then $f(r) - f(r-1)$ equals:

   (A) $r(r+1)$ (B) $(r+1)(r+2)$ (C) $\frac{1}{3}r(r+1)$
   (D) $\frac{1}{3}(r+1)(r+2)$ (E) $\frac{1}{3}r(r+1)(2r+1)$

6. Let side AD of convex quadrilateral ABCD be extended through D and let side BC be extended through C, to meet in point E. Let S represent the degree-sum of angles CDE and DCE and let S′ represent the degree-sum of angles BAD and ABC. If $r = S/S'$, then:

   (A) $r = 1$ sometimes, $r > 1$ sometimes
   (B) $r = 1$ sometimes, $r < 1$ sometimes
   (C) $0 < r < 1$ (D) $r > 1$ (E) $r = 1$

7. Let O be the intersection point of medians AP and CQ of triangle ABC. If OQ is 3 inches, then OP, in inches, is:

   (A) 3 (B) $\frac{9}{2}$ (C) 6 (D) 9 (E) undetermined

8. A positive number is mistakenly divided by 6 instead of being multiplied by 6. Based on the correct answer, the error thus committed, to the nearest percent, is:

(A) 100 (B) 97 (C) 83 (D) 17 (E) 3

9. The sum of the real values of x satisfying the equality $|x+2| = 2|x-2|$ is:

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 6 (D) $6\frac{1}{3}$ (E) $6\frac{2}{3}$

10. Assume that, for a certain school, it is true that
I: Some students are not honest.
II: All fraternity members are honest.
A necessary conclusion is:

(A) Some students are fraternity members. (B) Some fraternity members are not students (C) Some students are not fraternity members (D) No fraternity member is a student (E) No student is a fraternity member.

PART II (4 credits each)

11. If an arc of $60°$ on circle I has the same length as an arc of $45°$ on circle II, the ratio of the area of circle I to that of circle II is:

(A) 16 : 9 (B) 9 : 16 (C) 4 : 3 (D) 3 : 4
(E) none of these

12. A circle passes through the vertices of a triangle with side-lengths $7\frac{1}{2}$, 10, $12\frac{1}{2}$. The radius of the circle is:

(A) $\frac{15}{4}$ (B) 5 (C) $\frac{25}{4}$ (D) $\frac{35}{4}$ (E) $\frac{15\sqrt{2}}{2}$

13. If m and n are the roots of $x^2 + mx + n = 0$, $m \neq 0$, $n \neq 0$, then the sum of the roots is:

(A) $-\frac{1}{2}$ (B) -1 (C) $\frac{1}{2}$ (D) 1 (E) undetermined

14. If x and y are non-zero numbers such that $x = 1 + \frac{1}{y}$ and $y = 1 + \frac{1}{x}$, then y equals

(A) $x - 1$ (B) $1 - x$ (C) $1 + x$ (D) $-x$ (E) x

15. Let P equal the product of any three consecutive positive odd integers. The largest integer dividing all such P is:

(A) 15 (B) 6 (C) 5 (D) 3 (E) 1

16. If x is such that $\frac{1}{x} < 2$ and $\frac{1}{x} > -3$, then:

(A) $-\frac{1}{3} < x < \frac{1}{2}$ (B) $-\frac{1}{2} < x < 3$ (C) $x > \frac{1}{2}$
(D) $x > \frac{1}{2}$ or $-\frac{1}{3} < x < 0$ (E) $x > \frac{1}{2}$ or $x < -\frac{1}{3}$

17. Let $f(n) = \frac{x_1 + x_2 + \ldots + x_n}{n}$, where n is a positive integer. If $x_k = (-1)^k$, $k = 1, 2, 3, \ldots, n$, the set of possible values of $f(n)$ is:

(A) $\{0\}$ (B) $\{\frac{1}{n}\}$ (C) $\{0, -\frac{1}{n}\}$ (D) $\{0, \frac{1}{n}\}$ (E) $\{1, \frac{1}{n}\}$

18. Side AB of triangle ABC has length 8 inches. Line DEF is drawn parallel to AB so that D is on segment AC and E is on segment BC. Line AE extended bisects angle FEC. If DE has length 5 inches, then the length of CE, in inches, is:

(A) $\frac{51}{4}$ (B) 13 (C) $\frac{53}{4}$ (D) $\frac{40}{3}$ (E) $\frac{27}{2}$

19. Let n be the number of ways that 10 dollars can be changed into dimes and quarters, with at least one of each coin being used. Then n equals:

(A) 40 (B) 38 (C) 21 (D) 20 (E) 19

20. The measures of the interior angles of a convex polygon of n sides are in arithmetic progression. If the common difference is 5° and the largest angle is 160°, then n equals:

(A) 9 (B) 10 (C) 12 (D) 16 (E) 32

PART III (5 credits each)

21. If all the operations in $S = 1! + 2! + 3! + \ldots + 99!$ are correctly performed, the units' digit in the value of S is:

(A) 9 (B) 8 (C) 5 (D) 3 (E) 0

22. A segment of length 1 is divided into four segments. Then there exists a simple quadrilateral with the four segments as sides if and only if each segment is:

(A) equal to $\frac{1}{4}$ (B) equal to or greater than $\frac{1}{8}$ and less than $\frac{1}{2}$ (C) greater than $\frac{1}{8}$ and less than $\frac{1}{2}$ (D) greater than $\frac{1}{8}$ and less than $\frac{1}{4}$ (E) less than $\frac{1}{2}$

23. If all the logarithms are real numbers, the equality $\log(x+3) + \log(x-1) = \log(x^2 - 2x - 3)$ is satisfied for:

(A) all real values of x (B) no real values of x (C) all real values of x except $x = 0$ (D) no real values of x except $x = 0$ (E) all real values of x except $x = 1$

24. A painting 18" × 24" is to be placed into a wooden frame with the longer dimension vertical. The wood at the top and bottom is twice as wide as the wood on the sides. If the frame area equals that of the painting itself, the ratio of the smaller to the larger dimension of the framed painting is:

(A) 1:3 (B) 1:2 (C) 2:3 (D) 3:4 (E) 1:1

25. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y yards, and, at a given signal, they start off in the same direction. Then the number of yards Flash must run to catch Ace is:

(A) xy (B) $\frac{y}{x+y}$ (C) $\frac{xy}{x-1}$ (D) $\frac{x+y}{x+1}$ (E) $\frac{x+y}{x-1}$

26. Let $S = 2 + 4 + 6 + \ldots + 2N$, where N is the smallest positive integer such that $S > 1{,}000{,}000$. Then the sum of the digits of N is:

(A) 27 (B) 12 (C) 6 (D) 2 (E) 1

27. Let $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1}n$, $n = 1, 2, \ldots$ Then $S_{17} + S_{33} + S_{50}$ equals:

(A) 0 (B) 1 (C) 2 (D) −1 (E) −2

28. If the arithmetic mean of a and b is double their geometric mean, with $a > b > 0$, then a possible value for the ratio $\frac{a}{b}$, to the nearest integer, is

(A) 5 (B) 8 (C) 11 (D) 14 (E) none of these

29. Given the three numbers x, $y = x^x$, $z = x^{(x^x)}$ with $.9 < x < 1.0$. Arranged in order of increasing magnitude, they are:

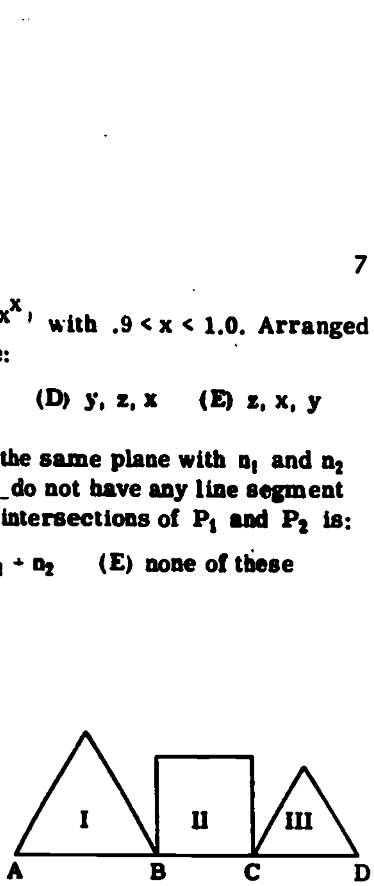
(A) x, z, y (B) x, y, z (C) y, x, z (D) y, z, x (E) z, x, y

30. Convex polygons P_1 and P_2 are drawn in the same plane with n_1 and n_2 sides, respectively, $n_1 \le n_2$. If P_1 and P_2 do not have any line segment in common, then the maximum number of intersections of P_1 and P_2 is:

(A) $2n_1$ (B) $2n_2$ (C) n_1n_2 (D) $n_1 + n_2$ (E) none of these

PART IV (6 credits each)

31. In this diagram, not drawn to scale, figures I and III are equilateral triangular regions with respective areas of $32\sqrt{3}$ and $8\sqrt{3}$, square inches. Figure II is a square region with area 32 sq. in. Let the length of segment AD be decreased by $12\frac{1}{2}$% of itself, while the lengths of AB and CD remain unchanged. The percent decrease in the area of the square is:



(A) $12\frac{1}{2}$ (B) 25 (C) 50 (D) 75 (E) $87\frac{1}{2}$

32. A and B move uniformly along two straight paths intersecting at right angles in point O. When A is at O, B is 500 yards short of O. In 2 minutes they are equidistant from O, and in 8 minutes more they are again equidistant from O. Then the ratio of A's speed to B's speed is:

(A) 4:5 (B) 5:6 (C) 2:3 (D) 5:8 (E) 1:2

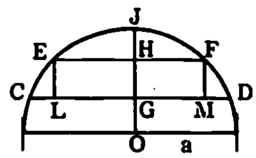
33. A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Then the middle digit is:

(A) 0 (B) 1 (C) 3 (D) 4 (E) 5

34. With 400 members voting the House of Representatives defeated a bill. A re-vote, with the same members voting, resulted in passage of the bill by twice the margin by which it was originally defeated. The number voting for the bill on the re-vote was $\frac{12}{11}$ of the number voting against it originally. How many more members voted for the bill the second time than voted for it the first time?

(A) 75 (B) 60 (C) 50 (D) 45 (E) 20

35. In this diagram the center of the circle is O, the radius is a inches, chord EF is parallel to chord CD, O, G, H, J are collinear, and G is the midpoint of CD. Let K (sq. in.) represent the area of trapezoid CDFE and let R (sq. in.) represent the area of rectangle ELMF. Then, as CD and EF are translated upward so that OG increases toward the value a, while JH always equals HG, the ratio K:R becomes arbitrarily close to:



(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}} + \frac{1}{2}$ (E) $\frac{1}{\sqrt{2}} + 1$

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

SOLUTION-ANSWER KEY

NINETEENTH ANNUAL H. S.

MATHEMATICS EXAMINATION

1968

19

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

UES-MAA-7211

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1968 examination.

1. (D) Let C, d, respectively, represent the measures of the circumference and diameter. Then $C + P = \pi(d + \pi) = \pi d + \pi^2$. Since $C = \pi d$, $P = \pi^2$.

2. (B) $64^{x-1} + 4^{x-1} = 4^{3x-3} + 4^{x-1} = 4^{2x-2}$. Since $256^{2x} = 4^{8x}$, $4^{2x-2} = 4^{8x}$ $\therefore 2x - 2 = 8x$, $x = -\frac{1}{3}$.

3. (A) Method I. $y = m_2x + b$, $4 = m_2 \cdot 0 + b$ $\therefore b = 4$. Also $m_2m_1 = \frac{1}{3}m_2 = -1$ $\therefore m_2 = -3$ $\therefore y = -3x + 4$ or $y + 3x - 4 = 0$

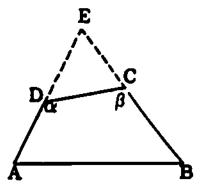
Method II. $\frac{y-4}{x-0} = -3$ $\therefore y - 4 = -3x$ or $y + 3x - 4 = 0$.

4. (C) $4 * 4 = \frac{4 \cdot 4}{4 + 4} = 2$ $\therefore 4 * (4 * 4) = \frac{4 \cdot 2}{4 + 2} = \frac{4}{3}$.

5. (A) $f(r) - f(r-1) = \frac{1}{3}r(r+1)(r+2) - \frac{1}{3}(r-1)(r)(r+1) = \frac{1}{3}r(r+1)(r+2-r+1)$
$= r(r+1)$.

6. (E) Method I. $S = \angle CDE + \angle DCE = 180° - \alpha + 180° - \beta$
$= 360° - (\alpha + \beta)$
$S' = \angle BAD + \angle ABC = 360 - (\alpha + \beta)$ $\therefore r = S/S' = 1$.

Method II. $S = \angle CDE + \angle DCE = 180° - \angle E$
$S' = \angle BAD + \angle ABC = 180° - \angle E$ $\therefore r = S/S' = 1$.

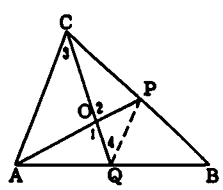


7. (E) Since $OQ = 3$, $CQ = 9$. But $OP = \frac{1}{3}AP$ and we can establish an arbitrary length for AP and, hence, for OP, as follows:
Draw $CQ = 9$ with $OQ = 3$. Through O draw OP of any length and extend PO through O to A so that $AP = 3(OP)$. Point B is then the intersection of AQ and CP and, thus, $\triangle ABC$ has AP and CQ as medians:

(1) $\frac{AO}{OP} = \frac{CO}{OQ}$, $\angle 1 = \angle 2$, $\triangle AOC \sim \triangle POQ$

(2) $\therefore \frac{AC}{QP} = \frac{CO}{OQ} = \frac{2}{1}$, $\angle 3 = \angle 4$, $QP \parallel AC$

(3) $\therefore \frac{AB}{QB} = \frac{CB}{PB} = \frac{AC}{QP} = \frac{2}{1}$ so that Q, P are midpoints.



8. (B) Let N represent the original number; the result obtained is $\frac{1}{6}N$; the correct result is 6N. The error, then, is $6N - \frac{1}{6}N$. Therefore, the per cent error is
$\frac{6N - N/6}{6N} \times 100 = \frac{3500}{36} \approx 97$.

9. (E) Since $2 + 2 \neq 2(2 - 2)$, $x \neq 2$ and since $-2 + 2 \neq -2(-2 - 2)$, $x \neq -2$.
For $x > 2$, $x + 2 = 2(x - 2)$, $x = 6$
For $-2 < x < 2$, $x + 2 = -2(x - 2)$, $x = 2/3$
For $x < -2$, $-(x + 2) = -2(x - 2)$, $x = 6$, a contradiction since $6 \not< -2$.
$\therefore$ the sum is $6 + 2/3 = 6\,2/3$

10. (C)

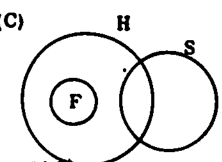


fig. 1

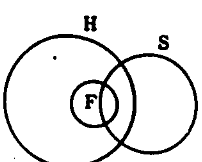


fig. 2

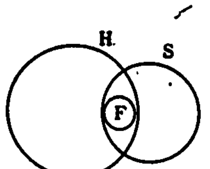


fig. 3

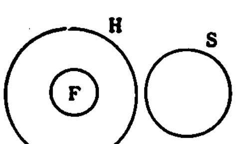


fig. 4

(A) is invalid by figures 1, 4
(B) is invalid by figure 3
(C) is valid in all cases (since some students exist who are not honest and, hence, they can't be fraternity members all of whom are honest)
(D) is invalid by figures 2, 3
(E) is invalid by figures 2, 3

11. (B) $\frac{60}{360} \times 2\pi r_1 = \frac{45}{360} \times 2\pi r_2 \therefore \frac{r_1}{r_2} = \frac{3}{4} \therefore \frac{\text{Area (I)}}{\text{Area (II)}} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

12. (C) Method I. Since $(7\frac{1}{2})^2 + 10^2 = (12\frac{1}{2})^2$, the triangle is right with hypotenuse $12\frac{1}{2}$. Therefore, $D = 2R = 12\frac{1}{2}$, $R = \frac{25}{4}$.

Method II. $R = \frac{abc}{4K} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(12\frac{1}{2})(10)(7\frac{1}{2})}{4\sqrt{15(2\frac{1}{2})(5)(7\frac{1}{2})}} = \frac{25}{4}$.

13. (B) $m + n = -m$ and $mn = n \therefore m = 1, n = -2 \therefore m + n = -1$.

14. (E) Method I. By subtraction we obtain $x - y = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$

$\therefore (x-y)\left(1 - \frac{1}{xy}\right) = 0$, $x = y$ (The result $y = \frac{1}{x}$ is rejected. Why?)

Method II. $xy = y + 1$ and $xy = x + 1 \therefore y + 1 = x + 1$, $x = y$.

Method III. Since $y = 1 + \frac{1}{x}$ and $x = 1 + \frac{1}{y}$, $y = 1 + \frac{1}{1 + \frac{1}{y}}$ and $x = 1 + \frac{1}{1 + \frac{1}{x}}$.

Therefore, $y^2 - y - 1 = 0$ and $x^2 - x - 1 = 0$. Let the roots of $z^2 - z - 1 = 0$ be r and s. Then the given equations imply that, when $x = r$, so does $y = r$ or that, when $x = s$, so does $y = s$. $\therefore y = x$.

Note. For all three methods, since $y = x \neq 0$, y cannot equal any of the other choices shown.

15. (D) Method I. Set $P = (2k-1)(2k+1)(2k+3)$. If $2k - 1 = 3m$ then $3|P$. If $2k - 1 = 3m + 1$, then $2k + 1 = 3m + 3$ and so $3|P$. If $2k - 1 = 3m - 1$, then $2k + 3 = 3m + 3$ and so $3|P$.

Method II. Set $P = (2k-1)(2k+1)(2k+3) = 8k^3 + 12k^2 - 2k - 3$.
$\therefore P = (9k^3 + 12k^2 - 3k - 3) - k^3 + k = 3Q - (k-1)(k)(k+1)$. Since the product of three consecutive integers is divisible by 3, $P = 3Q - 3R$. Hence, $3|P$.

Method III. Consider the three consecutive integers $k, k+1, k+2$; one of these is divisible by 3 (see Method II). If $k + 1$ is divisible by 3, then so is $k + 1 + 3 = k + 4$. Therefore, one of $k, k+2, k+4$ is divisible by 3. Therefore, if $P = k(k+2)(k+4)$ with k odd, P is divisible by 3.

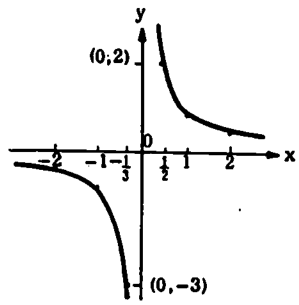
Note. For all three methods, since the greatest common factor of $1 \cdot 3 \cdot 5$ and $7 \cdot 9 \cdot 11$ is 3, no number > 3 is an exact divisor for all P.

16. (E) Method I. Let $y = \frac{1}{x}$. Then (a) For $y > 0$, when $y < 2$, $\frac{1}{y} > \frac{1}{2} \therefore x > \frac{1}{2}$.

(b) For $y < 0$, when $y > -3$, $-y < 3$, $-\frac{1}{y} > \frac{1}{3}$, $\frac{1}{y} < -3$

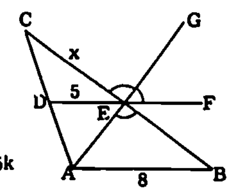
$\therefore x < -\frac{1}{3}$.

Method II. Let $y = \frac{1}{x}$ $\therefore xy = 1$; the graph is the two-branched hyperbola shown. At the point $x = \frac{1}{2}$, $y = 2$. When $y < 2$, x is to right of $\frac{1}{2}$, that is, $x > \frac{1}{2}$. At the point $x = -\frac{1}{3}$, $y = -3$. When $y > -3$, x is to the left of $-\frac{1}{3}$, that is, $x < -\frac{1}{3}$.



17. (C) Since $x_k = (-1)^k$, $x_1 = -1$, $x_2 = 1, \ldots, x_{2r-1} = -1$, $x_{2r} = 1$. Therefore, for $n = 2r$, $x_1 + x_2 + \ldots + x_{2r} = 0$ and for $n = 2r - 1$, $x_1 + x_2 + \ldots + x_{2r-1} = -1$. Therefore, $f(n) = 0$ in the former case and $f(n) = -\frac{1}{n}$ in the latter case.

18. (D) $\angle FEG = \angle CEG$. But $\angle BAE = \angle FEG$, and $\angle BEA = \angle CEG$. $\therefore \angle BAE = \angle BEA$. $\therefore BE = AB = 8$ $\therefore \frac{8 + x}{x} = \frac{8}{5}$, $x = \frac{40}{3}$.



19. (E) $25q + 10d = 1000$, $q = 40 - \frac{2d}{5}$ $\therefore 40 > \frac{2d}{5}$, $d < 100$. But $d = 5k$ $\therefore 5k < 100$ $\therefore k \le 19$ $\therefore n = 19$

20. (A) Method I. $a + (n - 1)5 = 160$, $a = 160 - (n - 1)5$
$(n - 2)180 = \frac{1}{2}n[2a + (n - 1)5] = \frac{1}{2}n[320 - (n - 1)5]$
$n^2 + 7n - 144 = 0 = (n + 16)(n - 9)$ $\therefore n = 9$

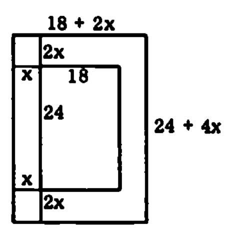
Method II. The exterior angles are 20, 25, 30, ...; their sum is 360°
$\therefore 360 = \frac{1}{2}n[40 + (n - 1)5]$ $\therefore n^2 + 7n - 144 = 0$ $\therefore n = 9$

21. (D) Each of 5!, 6!, ..., 99! ends in zero, and, in consequence, their sum ends in zero. Therefore, the units' digit of S is determined by $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$. Therefore, the units' digit of S is 3.

22. (E) Let the sides be s_1, s_2, s_3, s_4. Since it is given that $s_1 + s_2 + s_3 + s_4 = 1$, $s_1 + s_2 + s_3 = 1 - s_4$. But $s_1 + s_2 + s_3 > s_4$. Therefore, $0 < 1 - 2s_4$, $s_4 < \frac{1}{2}$. Similar reasoning applied to s_1, s_2, s_3, in turn, shows that each side is less than $\frac{1}{2}$. Conversely, a quadrilateral exists if $s_1 + s_2 + s_3 > s_4$, $s_1 + s_2 + s_4 > s_3$, $s_1 + s_3 + s_4 > s_2$, and $s_2 + s_3 + s_4 > s_1$. It is given that $s_1 + s_2 + s_3 + s_4 = 1$ and $s_1 < \frac{1}{2}$, $s_2 < \frac{1}{2}$, $s_3 < \frac{1}{2}$, $s_4 < \frac{1}{2}$. Therefore, $s_2 + s_3 + s_4 > \frac{1}{2}$ and so $s_2 + s_3 + s_4 > s_1$. In a similar manner we prove the other three cases.

23. (B) $\log(x + 3) + \log(x - 1) = \log(x^2 - 2x - 3)$ $\therefore \log(x + 3)(x - 1) = \log(x^2 - 2x - 3)$, $\therefore x^2 + 2x - 3 = x^2 - 2x - 3$, $x = 0$. But when $x = 0$, neither $\log(x - 1)$ nor $\log(x^2 - 2x - 3)$ is a real number.

24. (C) $(24 + 4x)(18 + 2x) = 2(24 \times 18)$
$\therefore x^2 + 15x - 54 = 0$, $x = 3$
$\therefore 24 + 4x = 36$, $18 + 2x = 24$, $24 : 36 = 2 : 3$



25. (C) Let z represent the number of yards Ace runs when he is caught. Let a represent Ace's speed; then xa represents Flash's speed. Then

$$\frac{y+z}{xa} = \frac{z}{a},\ z = \frac{y}{x-1}\ \therefore y + z = y + \frac{y}{x-1} = \frac{xy}{x-1}$$

26. (E) $S = 2 + 4 + 6 + \ldots + 2N = 2(1 + 2 + 3 + \ldots + N) = N(N + 1)$. Since $S > 10^6$, $N(N + 1) > 10^6$, $N \approx 10^3$. When $N = 10^3 - 1$, $S < 10^6$.
$\therefore N \geq 10^3$ $\therefore N\,(\text{smallest}) = 10^3 = 1000$; the sum of its digits is 1.

27. (B) $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1}n,\ n = 1, 2, 3, \ldots$

For n even, $S_n = 1 + 2 + 3 + \ldots + n - 2(2 + 4 + \ldots + n) = \frac{1}{2}n(n+1) - 4 \cdot \frac{1}{2} \cdot \frac{n}{2}\left(1 + \frac{n}{2}\right)$

$$\therefore S_n = \tfrac{1}{2}n(n+1) - \tfrac{1}{2}n(n+2) = -\tfrac{1}{2}n$$

For n odd, $S_n = 1 + 2 + 3 + \ldots + n - 2\big(2 + 4 + \ldots + (n-1)\big)$

$$= \frac{1}{2}n(n+1) - 4 \cdot \frac{1}{2} \cdot \frac{n-1}{2}\left(1 + \frac{n-1}{2}\right)$$

$$\therefore S_n = \tfrac{1}{2}n(n+1) - \tfrac{1}{2}(n-1)(n+1) = \tfrac{1}{2}(n+1)$$

$\therefore S_{17} = 9,\ S_{33} = 17,\ S_{50} = -25\ \ \therefore S_{17} + S_{33} + S_{50} = 1.$

28. (D) $\frac{a+b}{2} = 2\sqrt{ab},\ a + b = 4\sqrt{ab},\ a^2 + 2ab + b^2 = 16ab$

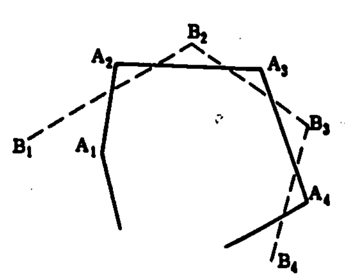
$\therefore a^2 - 14ab + b^2 = 0,\ a = \frac{14b + b\sqrt{192}}{2}\ \ \therefore \frac{a}{b} = \frac{14 + \sqrt{192}}{2} \approx \frac{14+14}{2} = 14$

Note. The value $\frac{a}{b} = \frac{14 - \sqrt{192}}{2} \approx \frac{14-14}{2} = 0$ is not listed

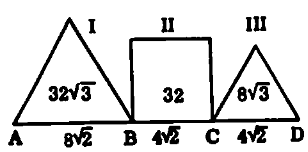
29. (A) Method I. Since $0 < x < 1$, x to any positive power is less than 1.
$\therefore \frac{x}{y} = x^{1-x} < 1$ so that $x < y$, $\frac{z}{y} = x^{y-x} < 1$ so that $z < y$, and $\frac{x}{z} = x^{1-y} < 1$ so that $x < z$. $\therefore x < z < y$.

Method II. Since $0 < x < 1$, $\log x < 0$. It follows that $\log x < x \log x$ or $\log x < \log x^x$ so that $x < x^x = y$. Since $z = x^{(x^x)} = x^y$, $\log z = y \log x$, and, since $y > x$, $y \log x < x \log x$ or $\log z < \log y$, so that $z < y$. However, since $0 < y < 1$ and $\log x < 0$, $y \log x > \log x$ or $\log z > \log x$, so that $x < z$. Finally $x < z < y$.

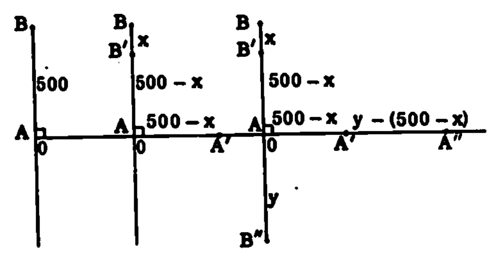
30. (A) Each side of P_1 can (1) fail to enter the boundary or interior of P_2, or (2) it can enter and terminate, or (3) it can continue on to the exterior. Therefore, at most, each side of P_1 meets two sides of P_2, so that the maximum number of intersections is $2n_1$. This maximum is attained as follows:



31. (D) $L = 16\sqrt{2}$, $L' = 16\sqrt{2} - 2\sqrt{2} = 14\sqrt{2} = 8\sqrt{2} + x + 4\sqrt{2}$
$\therefore x = 2\sqrt{2}$ $\therefore II' = (2\sqrt{2})^2 = 8$
$\therefore II - II' = 32 - 8 = 24$ (decrease), $\frac{24}{32} \times 100 = 75\%$



32. (C) Let r_A represent the speed of A, and let r_B represent the speed of B.



$\frac{x}{r_B} = 2$, $\frac{500 - x}{r_A} = 2$, $x = 2r_B$

$\therefore \frac{500 - 2r_B}{r_A} = 2$, $r_A + r_B = 250$

$\frac{500 - x + y}{r_B} = 8$, $\frac{y - 500 + x}{r_A} = 8$

$\left.\begin{array}{l} y - 500 + x = 8r_A \\ y + 500 - x = 8r_B \end{array}\right\}$ $\begin{array}{l} 2y = 8(r_A + r_B) = 8 \cdot 250 \\ y = 1000 \end{array}$

$8r_B = 1000 + 500 - 2r_B$, $r_B = 150$ $\therefore r_A = 100$ $\therefore r_A : r_B = 2:3$

33. (A) $a_1 \cdot 7^2 + a_2 \cdot 7 + a_3 = a_3 \cdot 9^2 + a_2 \cdot 9 + a_1$, $48a_1 = 80a_3 + 2a_2$, $3a_1 = 5a_3 + \frac{2a_2}{16}$

$\therefore 2a_2$ must be divisible by 16. But $a_2 \neq 8$ $\therefore a_2 = 0$

Check $3a_1 = 5a_3$ $\therefore a_1 = 5$, $a_3 = 3$

$5 \cdot 7^2 + 0 \cdot 7 + 3 = 248 = 3 \cdot 9^2 + 0 \cdot 9 + 5$

34. (B) Let y_1, n_1, respectively, represent the numbers voting for, against the bill originally, and let y_2, n_2, respectively, represent the numbers voting for, against the bill on the re-vote.

$y_1 + n_1 = 400$, $y_2 + n_2 = 400$, $y_2 = \frac{12}{11}n_1$, $y_2 - n_2 = 2(n_1 - y_1)$

$\therefore 2(y_1 + n_1) = 2(y_2 + n_2)$ and $2(-y_1 + n_1) = y_2 - n_2$

$\therefore 4n_1 = 3y_2 + n_2 = 3 \cdot \frac{12}{11}n_1 + n_2$ $\therefore n_1 = \frac{11}{8}n_2$

$\therefore y_2 = \frac{12}{11} \cdot \frac{11}{8}n_2 = \frac{3}{2}n_2$ so that $\frac{3}{2}n_2 + n_2 = 400$, $n_2 = 160$, $n_1 = 220$

$\therefore n_1 - n_2 = 60$ or, since $y_2 = 240$ and $y_1 = 180$, $y_2 - y_1 = 60$.

35. (D) $CD = 2GD = 2\sqrt{a^2 - (a - 2x)^2} = 2\sqrt{4ax - 4x^2}$

$FE = 2HF = 2\sqrt{a^2 - (a - x)^2} = 2\sqrt{2ax - x^2}$

$$\frac{K}{R} = \frac{\frac{1}{2}x\,[2\sqrt{4ax - 4x^2} + 2\sqrt{2ax - x^2}]}{x\,[2\sqrt{2ax - x^2}]} = \frac{\frac{1}{2} \cdot 2\sqrt{x}(\sqrt{4a - 4x} + \sqrt{2a - x})}{2\sqrt{x}(\sqrt{2a - x})}$$

$$= \frac{2\sqrt{a - x} + \sqrt{2a - x}}{2\sqrt{2a - x}} = \frac{\sqrt{a - x}}{\sqrt{2a - x}} + \frac{1}{2}$$

As OG increases toward the value a, x becomes arbitrarily close to zero, so that $\frac{\sqrt{a - x}}{\sqrt{2a - x}}$ becomes arbitrarily close to $\frac{\sqrt{a}}{\sqrt{2a}} = \sqrt{\frac{1}{2}}$. Therefore, as OG increases toward the value a, $\frac{K}{R}$ becomes arbitrarily close to $\sqrt{\frac{1}{2}} + \frac{1}{2} = \frac{1}{\sqrt{2}} + \frac{1}{2}$

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA

SOCIETY OF ACTUARIES

MU ALPHA THETA

and

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

20

TWENTIETH ANNUAL MATHEMATICS EXAMINATION 1969

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 11, 1969

To be filled in by the student

PRINT

last name first name middle initial

school (full name) street address

city state zip code

PART I										
	1	2	3	4	5	6	7	8	9	10
PART II										
	11	12	13	14	15	16	17	18	19	20
PART III										
	21	22	23	24	25	26	27	28	29	30
PART IV										
	31	32	33	34	35					

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = $C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. When x is added to both the numerator and the denominator of the fraction $\frac{a}{b}$, $a \neq b$, $b \neq 0$, the value of the fraction is changed to $\frac{c}{d}$. Then x equals:

   (A) $\frac{1}{c-d}$ (B) $\frac{ad-bc}{c-d}$ (C) $\frac{ad-bc}{c+d}$ (D) $\frac{bc-ad}{c-d}$ (E) $\frac{bc-ad}{c+d}$

2. If an item is sold for x dollars, there is a loss of 15% based on the cost. If, however, the same item is sold for y dollars, there is a profit of 15% based on the cost. The ratio $y:x$ is:

   (A) 23 : 17 (B) 17y : 23 (C) 23x : 17 (D) dependent upon the cost (E) none of these.

3. If N, written in base 2, is 11000, the integer immediately preceding N, written in base 2, is:

   (A) 10001 (B) 10010 (C) 10011 (D) 10110 (E) 10111

4. Let a binary operation * on ordered pairs of integers be defined as $(a,b) * (c,d) = (a-c, b+d)$. Then, if $(3,2) * (0,0)$ and $(x,y) * (3,2)$ represent identical pairs, x equals:

   (A) -3 (B) 0 (C) 2 (D) 3 (E) 6

5. If a number N, $N \neq 0$, diminished by four times its reciprocal, equals a given real constant R, then, for this given R, the sum of all such possible values of N is:

   (A) $\frac{1}{R}$ (B) R (C) 4 (D) $\frac{1}{4}$ (E) $-R$

6. The area of the ring between two concentric circles is $12\frac{1}{2}\pi$ square inches. The length of a chord of the larger circle tangent to the smaller circle, in inches, is:

   (A) $\frac{5}{\sqrt{2}}$ (B) 5 (C) $5\sqrt{2}$ (D) 10 (E) $10\sqrt{2}$

7. If the points $(1,y_1)$ and $(-1,y_2)$ lie on the graph of $y = ax^2 + bx + c$, and $y_1 - y_2 = -6$, then b equals:

   (A) -3 (B) 0 (C) 3 (D) $\sqrt{ac}$ (E) $\frac{a+c}{2}$

8. Triangle ABC is inscribed in a circle. The measures of the non-overlapping minor arcs AB, BC, and CA are, respectively, $x + 75°$, $2x + 25°$, $3x - 22°$. Then one interior angle of the triangle, in degrees, is:

(A) $57\frac{1}{2}$ (B) 59 (C) 60 (D) 61 (E) 122

9. The arithmetic mean (ordinary average) of the fifty-two successive positive integers beginning with 2, is:

(A) 27 (B) $27\frac{1}{4}$ (C) $27\frac{1}{2}$ (D) 28 (E) $28\frac{1}{2}$

10. The number of points equidistant from a circle and two parallel tangents to the circle, is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) infinite

PART II (4 credits each)

11. Given points $P(-1,-2)$ and $Q(4,2)$ in the xy-plane; point $R(1,m)$ is taken so that $PR + RQ$ is a minimum. Then m equals:

(A) $-\frac{3}{5}$ (B) $-\frac{2}{5}$ (C) $-\frac{1}{5}$ (D) $\frac{1}{5}$ (E) either $-\frac{1}{5}$ or $\frac{1}{5}$.

12. Let $F = \frac{6x^2 + 16x + 3m}{6}$ be the square of an expression which is linear in x. Then m has a particular value between:

(A) 3 and 4 (B) 4 and 5 (C) 5 and 6 (D) -4 and -3
(E) -6 and -5

13. A circle with radius r is contained within the region bounded by a circle with radius R. The area bounded by the larger circle is $\frac{a}{b}$ times the area of the region outside the smaller circle and inside the larger circle. Then $R:r$ equals:

(A) $\sqrt{a}:\sqrt{b}$ (B) $\sqrt{a}:\sqrt{a-b}$ (C) $\sqrt{b}:\sqrt{a-b}$ (D) $a:\sqrt{a-b}$
(E) $b:\sqrt{a-b}$

14. The complete set of x-values satisfying the inequality $\frac{x^2-4}{x^2-1} > 0$ is the set of all x such that:

(A) $x > 2$ or $x < -2$ or $-1 < x < 1$ (B) $x > 2$ or $x < -2$
(C) $x > 1$ or $x < -2$ (D) $x > 1$ or $x < -1$
(E) x is any real number except 1 or -1

15. In a circle with center at O and radius r, chord AB is drawn with length equal to r (units). From O a perpendicular to AB meets AB at M. From M a perpendicular to OA meets OA at D. In terms of r the area of triangle MDA, in appropriate square units, is:

(A) $\frac{3r^2}{16}$ (B) $\frac{\pi r^2}{16}$ (C) $\frac{\pi r^2\sqrt{2}}{8}$ (D) $\frac{r^2\sqrt{3}}{32}$ (E) $\frac{r^2\sqrt{6}}{48}$

16. When $(a-b)^n$, $n \geq 2$, $ab \neq 0$, is expanded by the binomial theorem, it is found that, when $a = kb$, where k is a positive integer, the sum of the second and third terms is zero. Then n equals:

(A) $\frac{1}{2}k(k-1)$ (B) $\frac{1}{2}k(k+1)$ (C) $2k-1$ (D) $2k$ (E) $2k+1$

17. The equation $2^{2x} - 8 \cdot 2^x + 12 = 0$ is satisfied by:

(A) $\log 3$ (B) $\frac{1}{2}\log 6$ (C) $1 + \log \frac{3}{2}$ (D) $1 + \frac{\log 3}{\log 2}$
(E) none of these

18. The number of points common to the graphs of $(x-y+2)(3x+y-4) = 0$ and $(x+y-2)(2x-5y+7) = 0$ is:

(A) 2 (B) 4 (C) 6 (D) 16 (E) infinite

19. The number of distinct ordered pairs (x, y) where x and y have positive integral values satisfying the equation $x^4y^4 - 10x^2y^2 + 9 = 0$ is:

(A) 0 (B) 3 (C) 4 (D) 12 (E) infinite

20. Let P equal the product of 3,659,893,456,789,325,678 and 342,973,489,379,256. The number of digits in P is:

(A) 36 (B) 35 (C) 34 (D) 33 (E) 32

PART III (5 credits each)

21. If the graph of $x^2 + y^2 = m$ is tangent to the graph of $x + y = \sqrt{2m}$, then:

(A) m must equal $\frac{1}{2}$ (B) m must equal $\frac{1}{\sqrt{2}}$ (C) m must equal $\sqrt{2}$
(D) m must equal 2 (E) m may be any non-negative real number

22. Let K be the measure of the area bounded by the x-axis, the line $x = 8$, and the curve defined by $f = \{(x, y) \mid y = x$ when $0 \leq x \leq 5$, $y = 2x - 5$ when $5 \leq x \leq 8\}$. Then K is:

(A) [illegible] (B) 36.4 (C) 36.5 (D) 44
(E) less than 44 but arbitrarily close to it.

23. For $n \geq$ [illegible] the number of prime numbers greater than $n! + 1$ and less than $n! + n$, is: [$n! = 1 \cdot 2 \cdots (n-1) \cdot n$; thus: $3! = 1 \cdot 2 \cdot 3 = 6$; $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$]

(A) 0 (B) 1 (C) $\frac{n}{2}$ for n even, $\frac{n+1}{2}$ for n odd
(D) $n - 1$ (E) n

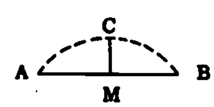
24. When the natural numbers P and P', with $P > P'$, are divided by the natural number D, the remainders are R and R', respectively. When PP' and RR' are divided by D, the remainders are r and r', respectively. Then:

(A) $r > r'$ always (B) $r < r'$ always
(C) $r > r'$ sometimes and $r < r'$ sometimes
(D) $r > r'$ sometimes and $r = r'$ sometimes (E) $r = r'$ always

25. If it is known that $\log_2 a + \log_2 b \leq 6$ and that ab is a maximum, then the least value that can be taken on by $a + b$ is:

(A) $2\sqrt{6}$ (B) 6 (C) $8\sqrt{2}$ (D) 16 (E) none of these.

26. A parabolic arch has a height of 16 inches and a span of 40 inches. The height, in inches, of the arch at a point 5 inches from the center M, is:



M center AB span
MC height C vertex

(A) 1 (B) 15 (C) $15\frac{1}{3}$ (D) $15\frac{1}{2}$ (E) $15\frac{3}{4}$

27. A particle moves so that its speed for the second and subsequent miles varies inversely as the integral number of miles already traveled. For each subsequent mile the speed is constant. If the second mile is traversed in 2 hours, then the time, in hours, needed to traverse the nth mile is:

(A) $\frac{2}{n-1}$ (B) $\frac{n-1}{2}$ (C) $\frac{2}{n}$ (D) 2n (E) $2(n-1)$

28. Let n be the number of points P interior to the region bounded by a circle with radius 1, such that the sum of the squares of the distances from P to the endpoints of a given diameter is 3. Then n is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) infinite

29. If $x = t^{\frac{1}{t-1}}$ and $y = t^{\frac{t}{t-1}}$, $t > 0$, $t \neq 1$, a relation between x and y is:

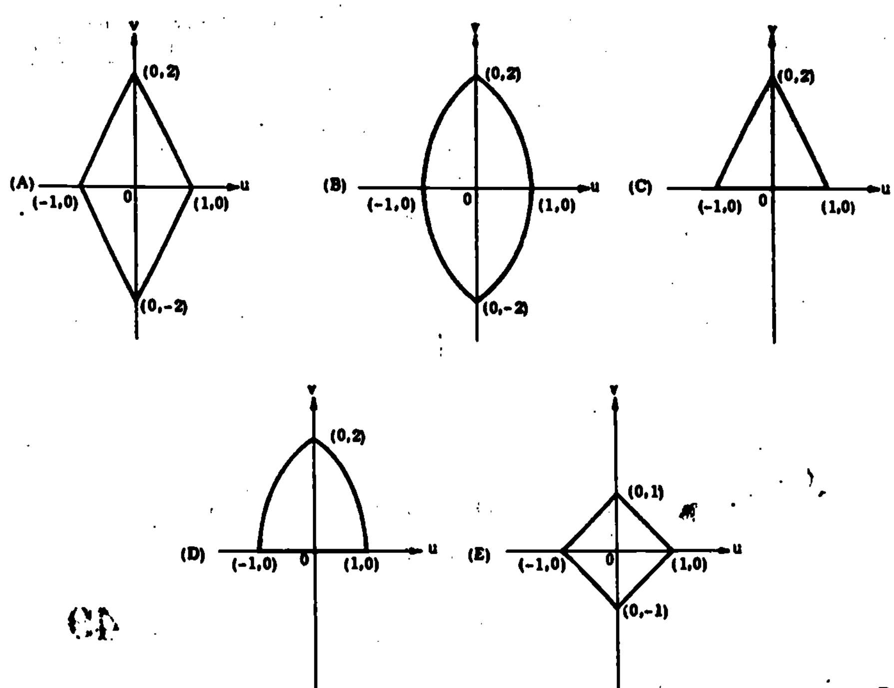
(A) $y^x = x^{\frac{1}{y}}$ (B) $y^{\frac{1}{x}} = x^y$ (C) $y^x = x^y$ (D) $x^x = y^y$
(E) none of these

30. Let P be a point of hypotenuse AB (or its extension) of isosceles right triangle ABC. Let $s = AP^2 + PB^2$. Then:

(A) $s < 2CP^2$ for a finite number of positions of P
(B) $s < 2CP^2$ for an infinite number of positions of P
(C) $s = 2CP^2$ only if P is the midpoint of AB or an endpoint of AB
(D) $s = 2CP^2$ always (E) $s > 2\ CP^2$ if P is a trisection point of AB

PART IV (6 points each)

31. Let OABC be a unit square in the xy-plane with O(0,0), A(1,0), B(1,1) and C(0,1). Let $u = x^2 - y^2$ and $v = 2xy$ be a transformation of the xy-plane into the uv-plane. The transform (or image) of the square is:



32. Let a sequence $\{u_n\}$ be defined by the relation $u_n + 1 - u_n = 3 + 4(n - 1)$, $n = 1, 2, 3, \ldots$, and $u_1 = 5$. If u_n is expressed as a polynomial in n, the algebraic sum of its coefficients is:

(A) 3 (B) 4 (C) 5 (D) 6 (E) 11

33. Let S_n and T_n be the respective sums of the first n terms of two arithmetic series. If $S_n : T_n = (7n + 1) : (4n + 27)$ for all n, the ratio of the eleventh term of the first series to the eleventh term of the second series, is:

(A) 4 : 3 (B) 3 : 2 (C) 7 : 4 (D) 78 : 71 (E) undetermined

34. The remainder R obtained by dividing x^{100} by $x^2 - 3x + 2$ is a polynomial of degree less than 2. Then R may be written as:

(A) $2^{100} - 1$ (B) $2^{100}(x - 1) - (x - 2)$ (C) $2^{100}(x - 3)$
(D) $x(2^{100} - 1) + 2(2^{99} - 1)$ (E) $2^{100}(x + 1) - (x + 2)$

35. Let $L(m)$ be the x - coordinate of the left end point of the intersection of the graphs of $y = x^2 - 6$ and $y = m$ where $-6 < m < 6$. Let $r = \dfrac{L(-m) - L(m)}{m}$. Then, as m is made arbitrarily close to zero, the value of r is:

(A) arbitrarily close to zero (B) arbitrarily close to $\frac{1}{\sqrt{6}}$
(C) arbitrarily close to $\frac{2}{\sqrt{6}}$ (D) arbitrarily large
(E) undetermined

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

Printed in U.S.A. 325 M-69

SOLUTION-ANSWER KEY

TWENTIETH ANNUAL H. S.

MATHEMATICS EXAMINATION

1969

20

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1969 examination.

1. (B) $\frac{a+x}{b+x} = \frac{c}{d}$ $\therefore ad + xd = bc + xc$ $\therefore ad - bc = x(c-d)$ $\therefore x = \frac{ad-bc}{c-d}$

Comment. If $\frac{c}{d} = \frac{b}{a}$, $x = -(a+b)$.

2. (A) Let C represent the cost (in dollars). Then $x = C - .15C = .85C$, and $y = C + .15C = 1.15C$. Therefore, $y : x = (1.15C) : (.85C) = 23 : 17$.

3. (E) Method I. $N - 1 = 11\,000_2 - 1 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 0 - 1$
$\therefore N - 1 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + (2-1) = 10\,111_2$

Method II. $N = 11\,000_2 = 24_{10}$ and $24_{10} - 1 = 23_{10}$
Since $23_{10} = 10\,111_2$, $N - 1 = 10\,111_2$

4. (E) By definition $(3, 2) * (0, 0) = (3-0, 2-0) = (3, 2)$ and $(x, y) * (3, 2) = (x-3, y-2)$ $\therefore x - 3 = 3 - 0$ and $y - 2 = 2 - 0$. $\therefore x = 6$ (and $y = 4$).

5. (B) $N - 4 \cdot \frac{1}{N} = R$ $\therefore N^2 - RN - 4 = 0$. Let the values of N satisfying this equation be N_1 and N_2. Therefore, $N_1 + N_2 = R$. For example, if $R = 3$, then N_1, N_2 are $4, -1$ and the sum of 4 and -1 equals 3.

6. (C) Let R be the larger radius, let r be the smaller radius, and let L (inches) be the length of the chord. Then $R^2 - r^2 = \left(\frac{L}{2}\right)^2$. Since $\pi R^2 - \pi r^2 = \frac{25\pi}{2}$, $R^2 - r^2 = \frac{25}{2}$.
$\therefore \left(\frac{L}{2}\right)^2 = \frac{25}{2}$ and $L = \sqrt{50} = 5\sqrt{2}$.

7. (A) $y_1 = a + b + c$ and $y_2 = a - b + c$ $\therefore y_1 - y_2 = 2b$. Since $y_1 - y_2 = -6$, $2b = -6$, $b = -3$.

8. (D) $x + 75 + 2x + 25 + 3x - 22 = 360$ $\therefore x = 47$, $\widehat{AB} = 122°$, $\widehat{BC} = 119°$, $\widehat{CA} = 119°$.

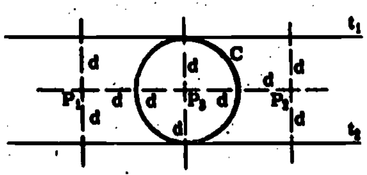
The interior angles of $\triangle ABC$ are, in degrees, 61, $59\frac{1}{2}$, $59\frac{1}{2}$.

9. (C) Method I. We have an arithmetic sequence with the first term $a = 2$, the common difference $d = 1$, and the last term $l = a + (n-1)d = 2 + (52-1)(1) = 53$.
Since $S_{52} = \frac{52}{2}(2 + 53)$, $AM. = \frac{52/2(2+53)}{52} = \frac{55}{2} = 27\frac{1}{2}$.

Method II. Designate the terms of the arithmetic sequence by $u_1, u_2, \ldots, u_n$.
Then $A.M. = \frac{1}{2}(u_1 + u_n)$; since $\frac{1}{2}(u_1 + u_n) = \frac{1}{2}(a + a + (n-1)d =$
$\frac{1}{2} \cdot \frac{n(2a + (n-1)d)}{n} = \frac{1}{2} \cdot \frac{S_n}{n}$. $\therefore A.M. = \frac{1}{2}(55) = 27\frac{1}{2}$.

Comment. Generally, $A.M. = \frac{u_i + u_{n+1-i}}{2}$, $i = 1, 2, \ldots, n$.

10. (C) Points P_1, P_2, and P_3 are each d distance from tangent lines t_1 and t_2 and circle C.



11. (B) Method I. Since PR + RQ is a minimum, points P_1, R_1 and Q are collinear. Therefore, $\frac{-2-m}{-1-1} = \frac{-2-2}{-1-4}$ so that $m = -\frac{2}{5}$.

Method II. Since PR + RQ is a minimum, points P_1, R_1 and Q are collinear. Therefore, PR + RQ = PQ so that $\sqrt{(1-(-1))^2 + (m+2)^2} + \sqrt{(4-1)^2 + (2-m)^2}$
$= \sqrt{(4-(-1))^2 + (2+2)^2}$

$\therefore \sqrt{4 + (m+2)^2} + \sqrt{9 + (2-m)^2} = \sqrt{25 + 16} = \sqrt{41}$

$\therefore \sqrt{4 + m^2 + 4m + 4} - \sqrt{41} = -\sqrt{9 + 4 - 4m + m^2}$

$\therefore 4 + m^2 + 4m + 4 - 2\sqrt{41}\,\sqrt{m^2 + 4m + 8} + 41 = 9 + 4 - 4m + m^2$

$\therefore 8m + 36 = 2\sqrt{41}\,\sqrt{m^2 + 4m + 8},\ 4m + 18 = \sqrt{41}\,\sqrt{m^2 + 4m + 8}$

$\therefore 16m^2 + 144m + 324 = 41m^2 + 164m + 328$

$\therefore 25m^2 + 20m + 4 = 0,\ 5m + 2 = 0,\ m = -\frac{2}{5}$

12. (A) Method I. $F = x^2 + \frac{8x}{3} + \frac{m}{2} = (x + r)^2 = x^2 + 2rx + r^2$

$\therefore 2r = \frac{8}{3}$ and $r^2 = \frac{m}{2}$. $\therefore r = \frac{4}{3}$ and $m = \frac{32}{9} = 3\frac{5}{9}$ and $3 < 3\frac{5}{9} < 4$.

Comment. When $m = \frac{32}{9}$, $F = (x + \frac{4}{3})^2$.

Method II. By completing the square we find that the trinomial $x^2 + \frac{8x}{3} + \frac{16}{9} = \left(x + \frac{4}{3}\right)^2$. Therefore, $x^2 + \frac{8x}{3} + \frac{16}{9} = x^2 + \frac{8x}{3} + \frac{m}{2}$, so that $\frac{m}{2} = \frac{16}{9}$ and $m = \frac{32}{9}$.

13. (B) $\pi R^2 = \frac{a}{b}(\pi R^2 - \pi r^2)$ $\therefore r^2\frac{a}{b} = R^2(\frac{a}{b} - 1)$ so that $\frac{R^2}{r^2} = \frac{a/b}{a/b - 1} = \frac{a}{a - b}$ and, hence, $R : r = \sqrt{a} : \sqrt{a - b}$.

14. (A) Method I. Since $\frac{x^2 - 4}{x^2 - 1} > 0$, $(x^2 - 1 > 0) \Rightarrow (x^2 - 4 > 0)$. Therefore, all real values of x such that $x > 2$ or $x < -2$ satisfy the inequality. Also $(x^2 - 1 < 0) \Rightarrow (x^2 - 4 < 0)$. Therefore, all real values of x such that $-1 < x < 1$ satisfy the inequality.

Method II. Since $\frac{x^2 - 4}{x^2 - 1} > 0$, $\frac{x^2 - 1}{x^2 - 1} - \frac{3}{x^2 - 1} > 0$. Therefore, $1 > \frac{3}{x^2 - 1}$.
This latter inequality is satisfied by all x such that $x^2 - 1 > 3$, that is, by $x > 2$ or $x < -2$, and by all x such that $x^2 - 1 < 0$, that is, by $-1 < x < 1$.

15. (D) Since AB = r, the measure of angle A is 60 (degrees) and $AM = \frac{1}{2}r$. In right triangle MDA, since $AM = \frac{1}{2}r$ and the measure of angle AMD is 30 (degrees), $AD = \frac{1}{4}r$ and $MD = \frac{1}{4}r\sqrt{3}$.
Therefore, the area of $\triangle MDA = \frac{1}{2}(AD)(MD) = \frac{1}{2} \cdot \frac{r}{4} \cdot \frac{r\sqrt{3}}{4} = \frac{r^2\sqrt{3}}{32}$.

16. (E) $(a - b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 - \ldots$

$\therefore -n(kb)^{n-1}b + \frac{n(n-1)}{2}(kb)^{n-2}b^2 = 0$

$\therefore -k^{n-1} + \frac{n-1}{2}k^{n-2} = 0.$ $\therefore n - 1 = \frac{2k^{n-1}}{k^{n-2}}$ $\therefore n = 2k + 1$.

17. (D) $2^{2x} - 8 \cdot 2^x + 12 = (2^x - 6)(2^x - 2) = 0$ $\therefore 2^x - 6 = 0,\ 2^x = 6$.

$\therefore x \log 2 = \log 6 = \log 2 + \log 3$ $\therefore x = 1 + \frac{\log 3}{\log 2}$.

Comment. The equation is also satisfied by $x = 1$.

18. (B) Each of the graphs is a pair of non-parallel straight lines. Each line of the first pair intersects the second pair in two distinct points, giving a total of four points.

Comment. The intersection points are $(0, 2)$, $(-1, 1)$, $(1, 1)$, $(\frac{13}{11}, \frac{23}{11})$.

19. (B) $x^4y^4 - 10x^2y^2 + 9 = (x^2y^2 - 1)(x^2y^2 - 9) = 0$ $\therefore x^2y^2 = 1$ or $x^2y^2 = 9$. $\therefore xy = +1$ or -1, $xy = +3$ or -3. Ordered pairs of positive integral values satisfying these equations are $(1, 1)$, $(1, 3)$, $(3, 1)$.

20. (C) Let N_1 represent the first factor of P and let N_2 represent the second factor. Then $(4)(10)^{18} > N_1 > (3)(10)^{18}$ and $(\frac{1}{2})(10)^{15} > N_2 > (\frac{1}{3})(10)^{15}$. Therefore, $(2)(10)^{33} > N_1 N_2 > (1)(10)^{33}$ so that the number of digits in $P(= N_1 N_2)$ is 34.

21. (E) Method I. Let OA be the x-intercept of the straight line $x + y = \sqrt{2m}$ and let OB be its y- intercept. Then $OA = OB = \sqrt{2m}$ so that AOB is a 45°-45°-90° triangle.

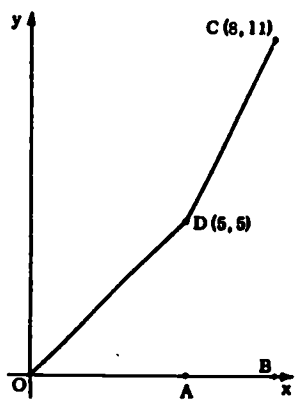
Let the point of tangency be labeled C. Since the graph of $x^2 + y^2 = m$ is a circle, OC, the radius r of the circle is perpendicular to AB; that is, OC is the median to hypotenuse AB. Therefore, $OC = r = \frac{1}{2} \cdot 2\sqrt{m}$ $\sqrt{m}$, and this is the same value of the radius given by the equation $x^2 + y^2 = m = (\sqrt{m})^2$. Consequently, m may be any non-negative real number.

Method II. Using the distance formula from a line to a point, we have $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ where $Ax + By + C = 0$ is the given line and (x_1, y_1) is the given point. The given line is $x + y - \sqrt{2m} = 0$ and the given point is (0, 0). Therefore, the required $d = \frac{|0 + 0 - \sqrt{2m}|}{\sqrt{1^2 + 1^2}} = \sqrt{m}$. But $d = r$, the radius of the circle $x^2 + y^2 = m = (\sqrt{m})^2$. Hence, m may be any non-negative real number.

Method III. Let (x, y) be the intersection point of the graphs of $x^2 + y^2 = m$ and $x + y = \sqrt{2m}$. This pair of equations yields $xy = \frac{m}{2}$ and $x + y = \sqrt{2m}$.

Thus x and y may be taken as the roots of $t^2 - \sqrt{2m}\ t + \frac{m}{2} = 0$. For tangency the discriminant of this last equation must equal zero. Therefore $2m - 4 \cdot \frac{m}{2} = 0$, an identity in m. Hence, m may be any non-negative real number.

22. (C) The total area consists of the triangular region OAD and the trapezoidal region ABCD.
$\therefore K = \frac{1}{2}(5)(5) + \frac{1}{2}(3)(5 + 11) = 36.5$



23. (A) Consider the integer $n! + k$ where $1 < k < n$. Since n! contains each of the factors 1, 2, 3, . . . , n, it contains the factor k. Since $n! + k$ can be written as the product of two factors, one of which is k, it is composite. Hence, there are no primes between $n! + 1$ and $n! + n$.

Comment. The conclusion holds for $n \geqq 1$.

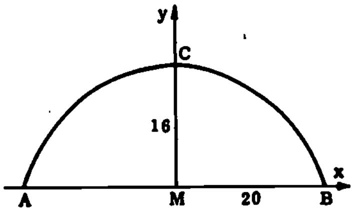
24. (E) We may write $P = Q_1D + R$ where $0 < R < D$, and $P' = Q_2D + R'$ where $0 \leqq R' < D$. Therefore, $PP' = (Q_1D + R)(Q_2D + R') = Q_1Q_2D^2 + Q_2DR + Q_1DR' + RR'$. But $RR' = Q_4D + r'$. Therefore, $PP' = Q_1Q_2D^2 + Q_2DR + Q_1DR' + Q_4D + r' = D(Q_1Q_2D + Q_2R + Q_1R' + Q_4) + r'$. But $PP' = Q_5D + r$. Therefore $Q_5 = Q_1Q_2D + Q_2R + Q_1R' + Q_4$ and $r = r'$.

25. (D) Since $\log_2 a + \log_2 b \leqq 6$, $\log_2 ab \leqq 6$. $\therefore ab \leqq 2^6 = 64$. Since ab is a maximum, ab = 64. Therefore, the least value that can be taken on by a + b is 8 + 8 = 16. [If P = ab,

the least value of $a + b$ is $\sqrt{P} + \sqrt{P} = 2\sqrt{P}$. One way of proving this theorem is to consider the minimum value of $f = a^2 - aS + P$ where $S = a + b$.

Hint. $S = a + b = a + \frac{P}{a}$.]

26. (B) Choose axes so that the x-axis coincides with the span AB and the y-axis coincides with the height MC. We may then write the equation for the parabola as $y = ax^2 + 16$. Since $0 = a \cdot 20^2 + 16$, $a = -\frac{1}{25}$, so that $y = -\frac{x^2}{25} + 16$. When $x = 5$, $y = -1 + 16 = 15$.



27. (E) Let v_n be the speed for the n-th mile. Then $v_n = \frac{k}{n-1}$, $n \geq 2$. $\therefore v_2 = \frac{k}{2-1} = k$.

Since $T = \frac{D}{R}$, $2 = \frac{1}{v_2} = \frac{1}{k}$ so that $k = \frac{1}{2}$. Therefore, $v_n = \frac{1}{2(n-1)}$. $\therefore T_n = \frac{1}{v_n} = 2(n-1)$.

28. (E) Method I. $AP^2 - (r-x)^2 = OP^2 - x^2$, $BP^2 - (r+x)^2 = OP^2 - x^2$.
$\therefore AP^2 + BP^2 - 2r^2 - 2x^2 = 2OP^2 - 2x^2$. $\therefore 3 = 2 + 2OP^2$.
$\therefore OP = \frac{1}{\sqrt{2}}$, that is P may be any point of a circle with radius $\frac{1}{\sqrt{2}}$.

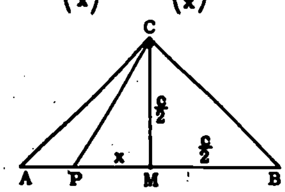
Method II. Since $AP^2 + BP^2 = 3 < 4 = AB^2$, angle APB > 90° so that P is interior to the circle. Therefore, there are many points P satisfying the given conditions such that $0 < AP \leq \sqrt{\frac{1}{2}}$.

29. (C) Method I. $x = t^{\frac{1}{t-1}}$, $y = t^{\frac{t}{t-1}} = t^{1+\frac{1}{t-1}} = t \cdot t^{\frac{1}{t-1}}$ $\therefore y = tx$. Also $y = t^{\left(\frac{1}{t-1}\right)t} = x^t$
$\therefore y^x = (x^t)^x = x^{tx}$ $\therefore y^x = x^y$.

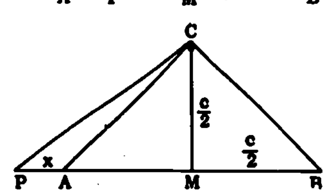
Method II. $x = t^{\frac{1}{t-1}}$, $y = t^{\frac{t}{t-1}}$ $\therefore \frac{y}{x} = t^{\frac{t}{t-1} - \frac{1}{t-1}} = t$ $\therefore x = \left(\frac{y}{x}\right)^{\frac{1}{y/x-1}} = \left(\frac{y}{x}\right)^{\frac{x}{y-x}}$
$\therefore x^{y-x} = \frac{y^x}{x^x}$ $\therefore x^y = y^x$

30. (D) Method I. (see fig. 1) Let M be the midpoint of hypotenuse c. Then $S = AP^2 + PB^2 = \left(\frac{c}{2} - x\right)^2 + \left(\frac{c}{2} + x\right)^2 = \frac{c^2}{2} = 2x^2$. But $x^2 = CP^2 - \left(\frac{c}{2}\right)^2$

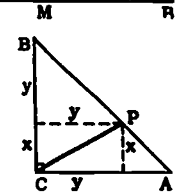
$\therefore S = \frac{c^2}{2} + 2CP^2 - \frac{c^2}{2} = 2CP^2$



(see fig. 2) $S = AP^2 + PB^2 = x^2 + (c+x)^2 = c^2 + 2cx + 2x^2$. But $CP^2 = \left(x + \frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2 = x^2 + cx + \frac{c^2}{2}$. $\therefore S = 2CP^2$

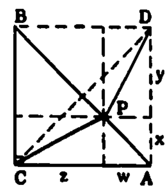


Method II. (for P interior to AB, fig. 3)
$S = AP^2 + PB^2$, $AP^2 = x^2 + x^2 = 2x^2$, $PB^2 = y^2 + y^2 = 2y^2$. $\therefore S = 2x^2 + 2y^2 = 2(x^2 + y^2) = 2CP^2$

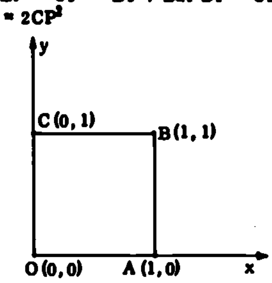


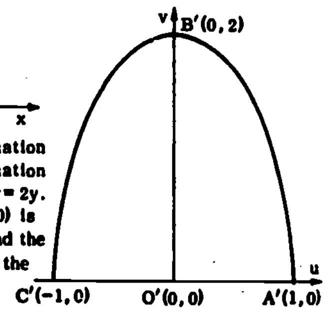
Method III. (for P interior to AB, fig. 4)

Complete ΔACB to square ACBD. Draw PD and draw perpendiculars from P to AC, CB, BD, DA. $AP^2 = x^2 + w^2$, $BP^2 = y^2 + z^2$, $CP^2 = x^2 + z^2$, $DP^2 = y^2 + w^2$. $\therefore AP^2 + BP^2 = x^2 + y^2 + z^2 + w^2$, $CP^2 + DP^2 = x^2 + y^2 + z^2 + w^2$
$\therefore S = AP^2 + BP^2 = CP^2 + DP^2$. But DP = CP (Why?) $\therefore S = 2CP^2$



31. (D) The image of A(1, 0) in the xy-plane is A′ (1, 0) in the uv-plane since $u = 1^2 - 0^2 = 1$ and $v = 2(1)(0) = 0$. The image of B(1, 1) is B′ (0, 2), the image of C (0, 1) is C′(−1, 0), and the image of O (0, 0) is O′(0, 0).





The image of the straight line AB whose xy− equation is $x = 1$ is the parabolic arc A′B′ whose uv− equation is $v^2 = 4(1 - u)$, a parabola, since $u = 1 - y^2$ and $v = 2y$. The image of the straight line OA (equation $y = 0$) is the straight line O′A′ (equations $v = 0$, $u = x^2$), and the image of the straight line OC (equation $x = 0$) is the straight line O′C′ (equations $v = 0$, $u = -y^2$).

32. (C)
$$\left.\begin{aligned} u_{n+1} - u_n &= 3 + 4(n-1) \\ u_n - u_{n-1} &= 3 + 4(n-2) \\ &\ \,\vdots \\ u_3 - u_2 &= 3 + 4(2-1) \\ u_2 - u_1 &= 3 \end{aligned}\right\}$$
By "telescopic" addition we obtain $u_{n+1} - u_1 = 3n + 4 \cdot \frac{1}{2}(n-1)(n-1+1)$. Since $u_1 = 5$, $u_{n+1} = 2n^2 + n + 5$. Therefore, $u_n = 2(n-1)^2 + (n-1) + 5 = 2n^2 - 3n + 6$.

Since $2 - 3 + 6 = 5$, the correct answer is (C).

33. (A) Let a_1 and d_1 be the first term and the common difference, respectively, of the first series, and let a_2 and d_2 be the first term and the common difference, respectively, of the second series.

Then $\frac{Sn}{Tn} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$.

Let u_{11} and v_{11}, respectively, be the eleventh terms of the two series whose sums are Sn and Tn. Then

$\frac{u_{11}}{v_{11}} = \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$. For $n = 21$, $\frac{Sn}{Tn} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$.

Therefore, $\frac{u_{11}}{v_{11}} = \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$.

34. (B) $x^{100} = Q(x)(x^2 + 3x + 2) + R(x) = Q(x)(x-2)(x-1) + R(x)$ where $R(x) = ax + b$.
$\therefore R(1) = 1^{100} = 1 = a + b$ and $R(2) = 2^{100} = 2a + b$
$\therefore a = 2^{100} - 1$, $b = 2 - 2^{100}$. $\therefore R(x) = x(2^{100} - 1) + 2 - 2^{100} = x \cdot 2^{100} - x + 2 - 2^{100}$.
$\therefore R(x) = 2^{100}(x-1) - (x-2)$.

35. (B) $r = \frac{L(-m) - L(m)}{m} = \frac{-\sqrt{6-m} + \sqrt{6+m}}{m}$

$\therefore r = \frac{-\sqrt{6-m} + \sqrt{6+m}}{m} \cdot \frac{-\sqrt{6-m} - \sqrt{6+m}}{-\sqrt{6-m} - \sqrt{6+m}} = \frac{-2m}{-m(\sqrt{6-m} + \sqrt{6+m})}$

$\therefore r = \frac{2}{\sqrt{6-m} + \sqrt{6+m}}$. As $m \to 0$, $r \to \frac{2}{\sqrt{6} + \sqrt{6}} = \frac{1}{\sqrt{6}}$.

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA

SOCIETY OF ACTUARIES

MU ALPHA THETA

and

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

21

TWENTY FIRST ANNUAL MATHEMATICS EXAMINATION 1970

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 10, 1970

To be filled in by the student

PRINT

last name first name middle initial

school (full name) street address

city state zip code

PART I										
	1	2	3	4	5	6	7	8	9	10

PART II										
	11	12	13	14	15	16	17	18	19	20

PART III										
	21	22	23	24	25	26	27	28	29	30

PART IV					
	31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE = C $- \frac{1}{4}$W		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is

   (a) $\sqrt{2}+\sqrt{3}$ (B) $\frac{1}{2}(7+3\sqrt{5})$ (C) $1+2\sqrt{3}$ (D) 3 (E) $3+2\sqrt{2}$

2. A square and a circle have equal perimeters. The ratio of the area of the circle to the area of the square is

   (A) $4/\pi$ (B) $\pi/\sqrt{2}$ (C) $4/1$ (D) $\sqrt{2}/\pi$ (E) $\pi/4$

3. If $x = 1 + 2^p$ and $y = 1 + 2^{-p}$, then y in terms of x is

   (A) $\frac{x+1}{x-1}$ (B) $\frac{x+2}{x-1}$ (C) $\frac{x}{x-1}$ (D) $2-x$ (E) $\frac{x-1}{x}$

4. Let S be the set of all numbers which are the sum of the squares of three consecutive integers. Then we can say that

   (A) No member of S is divisible by 2
   (B) No member of S is divisible by 3 but some member is divisible by 11
   (C) No member of S is divisible by 3 or by 5
   (D) No member of S is divisible by 3 or by 7
   (E) None of these

5. If $f(x) = \frac{x^4+x^2}{x+1}$, then $f(i)$, where $i = \sqrt{-1}$, is equal to

   (A) $1+i$ (B) 1 (C) -1 (D) 0 (E) $-1-i$

6. The smallest value of $x^2 + 8x$ for real values of x is

   (A) -16.25 (B) -16 (C) -15 (D) -8 (E) None of these

7. Inside square ABCD with side s, quarter-circle arcs with radii s and centers at A and B are drawn. These arcs intersect at a point X inside the square. How far is X from side CD?

   (A) $\frac{1}{2}s(\sqrt{3}+4)$ (B) $\frac{1}{2}s\sqrt{3}$ (C) $\frac{1}{2}s(1+\sqrt{3})$ (D) $\frac{1}{2}s(\sqrt{3}-1)$ (E) $\frac{1}{2}s(2-\sqrt{3})$

8. If $a = \log_8 225$ and $b = \log_2 15$, then

   (A) $a = \frac{1}{2}b$ (B) $a = 2b/3$ (c) $a = b$ (D) $b = \frac{1}{2}a$ (E) $a = 3b/2$

9. Points P and Q are on line segment AB, and both points are on the same side of the midpoint of AB. Point P divides AB in the ratio 2:3 and Q divides AB in the ratio 3:4. If PQ = 2, then the length of segment AB is

(A) 12 (B) 28 (C) 70 (D) 75 (E) 105

10. Let F = .48181··· be an infinite repeating decimal with the digits 8 and 1 repeating. When F is written as a fraction in lowest terms, the denominator exceeds the numerator by

(A) 13 (B) 14 (C) 29 (D) 57 (E) 126

PART II (4 credits each)

11. If two factors of $2x^3 - hx + k$ are $x + 2$ and $x - 1$, the value of $|2h - 3k|$ is

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

12. A circle with radius r is tangent to sides AB, AD, and CD of rectangle ABCD and passes through the midpoint of diagonal AC. The area of the rectangle, in terms of r, is

(A) $4r^2$ (B) $6r^2$ (C) $8r^2$ (D) $12r^2$ (E) $20r^2$

13. Given the binary operation * defined by $a*b = a^b$ for all positive numbers a and b. Then for all positive a, b, c, n, we have

(A) $a*b = b*a$ (B) $a*(b*c) = (a*b)*c$ (C) $(a*b^n) = (a*n)*b$
(D) $(a*b)^n = a*(bn)$ (E) None of these

14. Consider $x^2 + px + q = 0$ where p and q are positive numbers. If the roots of this equation differ by 1, then p equals

(A) $\sqrt{4q+1}$ (B) $q - 1$ (C) $-\sqrt{4q+1}$ (D) $q + 1$ (E) $\sqrt{4q-1}$

15. Lines in the xy-plane are drawn through the point (3, 4) and the trisection points of the line segment joining the points (−4, 5) and (5, −1). One of these lines has the equation

(A) $3x - 2y - 1 = 0$ (B) $4x - 5y + 8 = 0$ (C) $5x + 2y - 23 = 0$
(D) $x + 7y - 31 = 0$ (E) $x - 4y + 13 = 0$

16. If F(n) is a function such that $F(1) = F(2) = F(3) = 1$, and such that $F(n+1) = \frac{F(n) \cdot F(n-1) + 1}{F(n-2)}$ for $n \geqq 3$, then $F(6)$ is equal to

(A) 2 (B) 3 (C) 7 (D) 11 (E) 26

17. If $r > 0$, then for all p and q such that $pq \neq 0$ and $pr > qr$, we have

(A) $-p > -q$ (B) $-p > q$ (C) $1 > -q/p$ (D) $1 < q/p$ (E) None of these

18. $\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$ is equal to

(A) 2 (B) $2\sqrt{3}$ (C) $4\sqrt{2}$ (D) $\sqrt{6}$ (E) $2\sqrt{2}$

19. The sum of an infinite geometric series with common ratio r such that $|r| < 1$, is 15, and the sum of the squares of the terms of this series is 45. The first term of the series is

(A) 12 (B) 10 (C) 5 (D) 3 (E) 2

20. Lines HK and BC lie in a plane. M is the midpoint of line segment BC, and BH and CK are perpendicular to HK. Then we

(A) always have MH = MK (B) always have MH > BK
(C) sometimes have MH = MK but not always
(D) always have MH > MB (E) always have BH < BC

PART III (5 credits each)

21. On an auto trip, the distance read from the instrument panel was 450 miles. With snow tires on for the return trip over the same route, the reading was 440 miles. Find, to the nearest hundredth of an inch, the increase in radius of the wheels if the original radius was 15 inches.

(A) .33 (B) .34 (C) .35 (D) .38 (E) .66

22. If the sum of the first 3n positive integers is 150 more than the sum of the first n positive integers, then the sum of the first 4n positive integers is

(A) 300 (B) 350 (C) 400 (D) 450 (E) 600

23. The number 10! (10 is written in base 10), when written in the base 12 system, ends with exactly k zeros. The value of k is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 2, then the area of the hexagon is

(A) 2 (B) 3 (C) 4 (D) 6 (E) 12

25. For every real number x, let $[x]$ be the greatest integer which is less than or equal to x. If the postal rate for first class mail is six cents for every ounce or portion thereof, then the cost in cents of first-class postage on a letter weighing W ounces is always

(A) $6W$ (B) $6[W]$ (C) $6([W] - 1)$ (D) $6([W] + 1)$ (E) $-6[-W]$

26. The number of distinct points in the xy-plane common to the graphs of $(x + y - 5)(2x - 3y + 5) = 0$ and $(x - y + 1)(3x + 2y - 12) = 0$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) infinite

27. In a triangle, the area is numerically equal to the perimeter. What is the radius of the inscribed circle?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

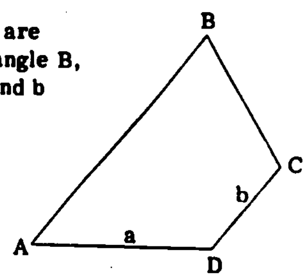
28. In triangle ABC, the median from vertex A is perpendicular to the median from vertex B. If the lengths of sides AC and BC are 6 and 7 reapectively, then the length of side AB is

(A) $\sqrt{17}$ (B) 4 (C) $4\frac{1}{2}$ (D) $2\sqrt{5}$ (E) $4\frac{1}{4}$

29. It is now between 10:00 and 11:00 o'clock, and six minutes from now, the minute hand of a watch will be exactly opposite the place where the hour hand was three minutes ago. What is the exact time now?

(A) $10:05\frac{5}{11}$ (B) $10:07\frac{1}{2}$ (C) 10:10 (D) 10:15 (E) $10:17\frac{1}{2}$

30. In the accompanying figure, segments AB and CD are parallel, the measure of angle D is twice that of angle B, and the measures of segments AD and CD are a and b respectively. Then the measure of AB is equal to



(A) $\frac{1}{2}a + 2b$ (B) $\frac{3}{2}b + \frac{3}{4}a$ (C) $2a - b$
(D) $4b - \frac{1}{2}a$ (E) $a + b$

PART IV (6 credits each)

31. If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?

(A) 2/5 (B) 1/5 (C) 1/6 (D) 1/11 (E) 1/15

32. A and B travel around a circular track at uniform speeds in opposite directions, starting from diametrically opposite points. If they start at the same time, meet first after B has travelled 100 yards, and meet a second time 60 yards before A completes one lap, then the circumference of the track in yards is

(A) 400 (B) 440 (C) 480 (D) 560 (E) 880

33. Find the sum of the digits of all the numerals in the sequence 1, 2, 3, 4, ···, 10000.

(A) 180,001 (B) 154,756 (C) 45,001 (D) 154,755 (E) 270,001

34. The greatest integer that will divide 13,511, 13,903 and 14,589 and leave the same remainder is

(A) 28 (B) 49 (C) 98 (D) an odd multiple of 7 greater than 49
(E) an even multiple of 7 greater than 98

35. A retiring employee receives an annual pension proportional to the square root of the number of years of his service. Had he served a years more, his pension would have been p dollars greater, whereas, had he served b years more ($b \neq a$), his pension would have been q dollars greater than the original annual pension. Find his annual pension in terms of a, b, p, and q.

(A) $\frac{p^2-q^2}{2(a-b)}$ (B) $\frac{(p-q)^2}{2\sqrt{ab}}$ (C) $\frac{ap^2-bq^2}{2(ap-bq)}$ (D) $\frac{aq^2-bp^2}{2(bp-aq)}$ (E) $\sqrt{(a-b)(p-q)}$

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

Printed in U.S.A.

350M-70

SOLUTION-ANSWER KEY

TWENTY FIRST ANNUAL H. S.

MATHEMATICS EXAMINATION

1970

21

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA

SOCIETY OF ACTUARIES

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

Note: The letter following the problem number designates the correct choice of the five listed for the problem in the 1970 examination.

1. (E) If x denotes the given expression, then $x^2 = 1 + \sqrt{2}$ and $x^4 = 3 + 2\sqrt{2}$.

2. (A) Let the common perimeter be p. The area of the circle is $\frac{1}{4}p^2/\pi$ and the area of the square is $p^2/16$, so the ratio is $4/\pi$.

3. (C) $x - 1 = 2^p,\ y - 1 = 2^{-p} \therefore (x - 1)(y - 1) = 1 \therefore y = x/(x - 1)$.

4. (B) Let the consecutive integers be $(n - 1)$, n, $(n + 1)$. Then the sum of their squares is $3n^2 + 2$ which is never divisible by 3 but is 77 when n = 5 and is then divisible by 11 as required in (B). Choices (A), (C), and (D) are eliminated by taking the middle integer n equal to 0, 4, and 2 respectively.

5. (D) Since i^2 and i^4 are -1 and 1, $f(i) = (1 - 1)/(1 + i) = 0/(1 + i) = 0$.

6. (B) $x^2 + 8x = (x + 4)^2 - 16$ which is least (-16) when $(x + 4)^2 = 0$ or when $x = -4$.

7. (E) Triangle ABX is equilateral with altitude $\frac{1}{2}s\sqrt{3}$. The required distance is $s - \frac{1}{2}s\sqrt{3}$ or $\frac{1}{2}s(2 - \sqrt{3})$.

8. (B) $225 = 8^a = 2^{3a} \therefore 15 = 2^{3a/2}$. Also $15 = 2^b \therefore 3a/2 = b \therefore a = 2b/3$.

9. (C) $AP = \frac{2}{5}AB$ and $AQ = \frac{3}{7}AB \therefore PQ = \left(\frac{3}{7} - \frac{2}{5}\right)AB = AB/35 = 2 \therefore AB = 70$.

10. (D) $F = .4 + .081 + .00081 + \cdots = .4 + .081(1 + .01 + .0001 + \cdots)$

$$= .4 + .081 \times \frac{1}{1 - .01} = (4/10) + (81/1{,}000)(1/.99) = 53/110.$$

Denominator $-$ Numerator $= 110 - 53 = 57$.

11. (E) Write $f(x) = 2x^3 - hx + k$. Then $f(-2) = -16 + 2h + k = 0$ and $f(1) = 2 - h + k = 0$. Hence $h = 6$, $k = 4$ and $|2h - 3k| = 0$.

12. (C) The length of side AD is 2r as are the distances from the midpoint of the diagonal AC to AD and BC. Hence the length of AB is 4r and the area of the rectangle is $(2r)(4r) = 8r^2$.

13. (D) $(a * b)^n = a^{bn}$ and $a * (bn) = a^{bn}$.

14. (A) The roots are $\frac{1}{2}(-p + \sqrt{p^2 - 4q})$ and $\frac{1}{2}(-p - \sqrt{p^2 - 4q})$. The difference is $\sqrt{p^2 - 4q} = 1$. Hence $p^2 = 4q + 1$ and $p = \sqrt{4q + 1}$.

15. (E) The trisection points are $(-1,3)$ and $(2,1)$. The slopes of lines joining these points to the point $(3,4)$ are $\frac{1}{4}$ and 3. Only the line of (E) has slope $\frac{1}{4}$ and none of the lines has slope 3.

16. (C) Substitution yields $F(4) = 2$, $F(5) = 3$, and finally $F(6) = 7$.

17. (E) Choices (A) and (B) are both false when p and q are 2 and 1 respectively, and choices (C) and (D) are both false when p and q are 1 and -2 respectively. Hence none of these holds for all values of p and q.

18. (A) The required difference is 2 because it is positive and its square is 4.

19. (C) Let a be the first term of the infinite series. Using the formula for the sum, we have $a/(1 - r) = 15$. Also $a^2/(1 - r^2) = 45$. Dividing gives $a/(1 + r) = 3$. $\therefore a = 3 + 3r$ and $a = 15 - 15r \therefore a = 5$.

20. (A) Regardless of how the diagram is drawn, M lies on the perpendicular bisector of HK so that MH = MK always.

21. (B) The distances read are inversely proportional to the radii of the tires so that

$450 \times 15 = 440(15 + d)$ where d is the increase. Hence the increase $d = 15/44 = .34$ inches.

[illegible] (A) Let S_m denote the sum of the first m positive integers. Using the formula for the sum of an arithmetic progression, $S_{3n} - S_n = 3n(3n + 1)/2 - n(n + 1)/2 = 4n^2 + n = 150 \therefore 4n^2 + n - 150 = 0 \therefore (n - 6)(4n + 25) = 0 \therefore n = 6 \therefore S_{4n} = S_{24} = 12(24 + 1) = 300$.

23. (D) The number $10! = 7 \times 5^2 \times 12^4$ when written in the base 12 system, ends with exactly 4 zeros.

24. [illegible] If the side of the triangle is 2s and hence its area is $s^2\sqrt{3} = 2$, then the side and area of the hexagon are s and $3s^2\sqrt{3}/2 = 3$ respectively.

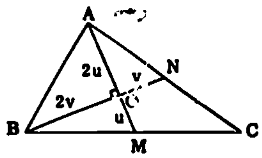
25. (E) If the number of ounces is $W = w + p$ where w is a non-negative integer and $0 < p \leq 1$, then the postage is

$$6(w + 1) = -6(-w - 1) = -6[-w - 1] = -6[-w - p] = -6[-(w + p)]$$
$$= -6[-W] \text{ cents}$$

26. (B) The two lines which are the graph of the first equation intersect at the point (2,3) and so do the lines which are the graph of the second equation which are distinct from those of the first. Hence the two graphs have only the one point (2,3) in common.

27. (A) If r is the radius and p the perimeter, then the area of the triangle is $\frac{1}{2}pr = p$. Therefore $r = 2$.

28. (A) Let the medians be AM and BN which intersect at O and let AO and BO have lengths 2u and 2v so that OM and ON have lengths u and v respectively. From right triangles AON and BOM, we obtain



$$4u^2 + v^2 = (6/2)^2 = 36/4$$
$$u^2 + 4v^2 = (7/2)^2 = 49/4$$

Four-fifths of the sum of these two equations gives $4u^2 + 4v^2 = 17$ which is the square of the hypotenuse AB of right triangle AOB. Hence $AB = \sqrt{17}$.

29. (D) Let x be the number of minutes after 10 o'clock now. Then equating vertical angles (measured in minute spaces) formed by a line through 6 and 12 and one in the "exactly opposite" directions, we have $x + 6 = 20 + (x - 3)/12$ so that $x = 15$. Therefore the time is 10:15 now.

30. (E) Let the bisector of angle D be drawn and intersect AB at P. Then PDCB is a parallelogram and PAD an isosceles triangle. The measures of AP = a, PB = b, and hence $AB = a + b$.

31. (B) To total 43, five digits must be one 7 and four 9's or two 6's and three 9's. There are five and ten or a total of 15 such numbers three of which are divisible by 11. Hence the probability is 3/15 or 1/5. Divisibility by 11 requires that the sum of the first, third, and fifth digits minus the sum of second and fourth be divisible by 11; only 98,989 and 97,999 and 99,979 satisfy this requirement.

32. (C) Let 2C be the circumference. The ratio of the distances travelled by A and B remains constant because their speeds are uniform. Therefore $(C - 100)/100 = (2C - 60)/(C + 60)$ so that $C^2 = 240C$, $C = 240$ and the circumference is 480 yards.

33. (A) Omit 10,000 for the moment. In the sequence 0000, 0001, 0002, 0003, $\cdots$, 9999 each digit appears the same number of times. There are (10,000)(4) digits in all, each digit appearing 4,000 times. The sum of the digits is $4,000(0 + 1 + 2 + \cdots + 9) = 4,000(45) = 180,000$. Now add the 1 in 10,000 to get 180,001.

34. (C) A number that dividea two numbers and leaves the same remainder, divides their difference exactly. We seek the greatest integer that will divide the difference $(13{,}903 - 13{,}511) = 392 = 7^2 \cdot 2^3$ and the difference $(14{,}589 - 13{,}903) = 686 = 7^3 \cdot 2$. The required common factor is $7^2 \cdot 2$ or 98.

35. (D) Let X be the amount of the annual pension and y the number of years of service. Th(·.ith constant of proportionality k,

$$X = k\sqrt{y} \qquad X + p = k\sqrt{y + a} \qquad X + q = k\sqrt{y + b}$$
$$X^2 = k^2 y \qquad (X + p)^2 = k^2(y + a) \qquad (X + q)^2 = k^2(y + b)$$

Replacing k^2y by X^2 in the last two equations and simplifying,

$$2pX + p^2 = k^2a \qquad \text{and} \qquad 2qX + q^2 = k^2b$$

Hence

$$\frac{2pX + p^2}{2qX + q^2} = \frac{a}{b} \qquad \therefore\ X = \frac{aq^2 - bp^2}{2(bp - aq)}.$$

The student should verify that $bp - aq$ can not be zero.

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA

SOCIETY OF ACTUARIES

MU ALPHA THETA

and

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

TWENTY SECOND ANNUAL MATHEMATICS EXAMINATION

1971

22

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 9, 1971

To be filled in by the student

PRINT ____

last name first name middle initial

____

school (full name) street address

____

city state zip code

PART I

1	2	3	4	5	6	7	8	9	10

PART II

11	12	13	14	15	16	17	18	19	20

PART III

21	22	23	24	25	26	27	28	29	30

PART IV

31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE $= C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. The number of digits in the number $N = 2^{12} \times 5^8$ is

   (A) 9 (B) 10 (C) 11 (D) 12 (E) 20

2. If b men take c days to lay f bricks, then the number of days it will take c men working at the same rate to lay b bricks, is

   (A) fb^2 (B) b/f^2 (C) f^2/b (D) b^2/f (E) f/b^2

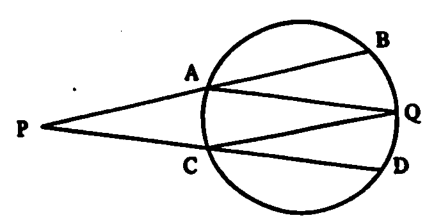
3. If the point $(x,-4)$ lies on the straight line joining the points $(0,8)$ and $(-4,0)$ in the xy-plane, then x is equal to

   (A) -2 (B) 2 (C) -8 (D) 6 (E) -6

4. After simple interest for two months at 5% per annum was credited, a Boy Scout Troop had a total of $255.31 in the Council Treasury. The interest credited was a number of dollars plus the following number of cents

   (A) 11 (B) 12 (C) 13 (D) 21 (E) 31

5. Points A,B,Q,D, and C lie on the circle shown and the measures of arcs $\widehat{BQ}$ and $\widehat{QD}$ are 42° and 38° respectively. The sum of the measures of angles P and Q is



   (A) 80° (B) 62° (C) 40° (D) 46°
   (E) None of these

6. Let * be a symbol denoting the binary operation on the set S of all non-zero real numbers as follows: For any two numbers a and b of S, $a * b = 2ab$. Then the one of the following statements which is not true, is

   (A) * is commutative over S
   (B) * is associative over S
   (C) ½ is an identity element for * in S
   (D) Every element of S has an inverse for *
   (E) $\frac{1}{2a}$ is an inverse for * of the element a of S

7. $2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k}$ is equal to

   (A) 2^{-2k} (B) $2^{-(2k-1)}$ (C) $-2^{-(2k+1)}$ (D) 0 (E) 2

8. The solution set of $6x^2 + 5x < 4$ is the set of all values of x such that

   (A) $-2 < x < 1$ (B) $-\frac{4}{3} < x < \frac{1}{2}$ (C) $-\frac{1}{2} < x < \frac{4}{3}$ (D) $x < \frac{1}{2}$ or $x > -\frac{4}{3}$
   (E) $x < -\frac{4}{3}$ or $x > \frac{1}{2}$

9. An uncrossed belt is fitted without slack around two circular pulleys with radii of 14 inches and 4 inches. If the distance between the points of contact of the belt with the pulleys is 24 inches, then the distance between the centers of the pulleys in inches is

   (A) 24 (B) $2\sqrt{119}$ (C) 25 (D) 26 (E) $4\sqrt{35}$

10. Each of a group of 50 girls is blonde or brunette and is blue or brown eyed. If 14 are blue-eyed blondes, 31 are brunettes, and 18 are brown-eyed, then the number of brown-eyed brunettes is

    (A) 5 (B) 7 (C) 9 (D) 11 (E) 13

PART II (4 credits each)

11. The numeral 47 in base a represents the same number as 74 in base b. Assuming that both bases are positive integers, the least possible value for a + b written as a Roman numeral, is

    (A) XIII (B) XV (C) XXI (D) XXIV (E) XVI

12. For each integer $N > 1$, there is a mathematical system in which two or more integers are defined to be congruent if they leave the same non-negative remainder when divided by N. If 69, 90, and 125 are congruent in one such system, then in that same system, 81 is congruent to

    (A) 3 (B) 4 (C) 5 (D) 7 (E) 8

13. If $(1.0025)^{10}$ is evaluated correct to 5 decimal places, then the digit in the fifth decimal place is

    (A) 0 (B) 1 (C) 2 (D) 5 (E) 8

14. The number $(2^{48} - 1)$ is exactly divisible by two numbers between 60 and 70. These numbers are

    (A) 61, 63 (B) 61, 65 (C) 63, 65 (D) 63, 67 (E) 67, 69

15. An aquarium on a level table has rectangular faces and is 10 inches wide and 8 inches high. When it was tilted, the water in it just covered an $8'' \times 10''$ end but only three-fourths of the rectangular bottom. The depth of the water when the bottom was again made level was

(A) $2\frac{1}{2}''$ (B) $3''$ (C) $3\frac{1}{4}''$ (D) $3\frac{1}{2}''$ (E) $4''$

16. After finding the average of 35 scores, a student carelessly included the average with the 35 scores and found the average of these 36 numbers. The ratio of the second average to the true average was

(A) 1 : 1 (B) 35 : 36 (C) 36 : 35 (D) 2 : 1 (E) None of these

17. A circular disk is divided by $2n$ equally spaced radii ($n > 0$) and one secant line. The maximum number of non-overlapping areas into which the disk can be divided is

(A) $2n + 1$ (B) $2n + 2$ (C) $3n - 1$ (D) $3n$ (E) $3n + 1$

18. The current in a river is flowing steadily at 3 miles per hour. A motor boat which travels at a constant rate in still water goes downstream 4 miles and then returns to its starting point. The trip takes one hour, excluding the time spent in turning the boat around. The ratio of the downstream to the upstream rate is .

(A) 4 : 3 (B) 3 : 2 (C) 5 : 3 (D) 2 : 1 (E) 5 : 2

19. If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 1$ exactly once, then the value of m^2 is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$ (E) $\frac{5}{6}$

20. The sum of the squares of the roots of the equation $x^2 + 2hx = 3$ is 10. The absolute value of h is equal to

(A) -1 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 2 (E) None of these

PART III (5 credits each)

21. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$, then the sum $x + y + z$ is equal to

(A) 50 (B) 58 (C) 89 (D) 111 (E) 1296

22. If w is one of the imaginary roots of the equation $x^3 = 1$, then the product $(1 - w + w^2)(1 + w - w^2)$ is equal to

(A) 4 (B) w (C) 2 (D) w^2 (E) 1

23. Teams A and B are playing a series of games. If the odds for either team to win any game are even and Team A must win two or Team B three games to win the series, then the odds favoring Team A to win the series are

(A) 11 to 5 (B) 5 to 2 (C) 8 to 3 (D) 3 to 2 (E) 13 to 6

24. Pascal's triangle is an array of positive integers (See figure), in which the first row is 1, the second row is two 1's, each row begins and ends with 1, and the kth number in any row when it is not 1, is the sum of the kth and (k – 1)th numbers in the immediately preceding row. The quotient of the number of numbers in the first n rows which are not 1's and the number of 1's is

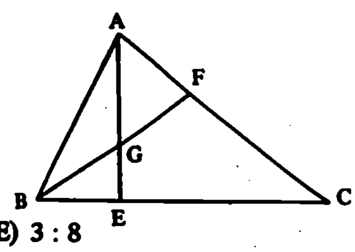
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
etc.

(A) $\frac{n^2 - n}{2n - 1}$ (B) $\frac{n^2 - n}{4n - 2}$ (C) $\frac{n^2 - 2n}{2n - 1}$ (D) $\frac{n^2 - 3n + 2}{4n - 2}$ (E) None of these

25. A teen age boy wrote his own age after his father's. From this new four place number he subtracted the absolute value of the difference of their ages to get 4,289. The sum of their ages was

(A) 48 (B) 52 (C) 56 (D) 59 (E) 64

26. In triangle ABC, point F divides side AC in the ratio 1 : 2. Let E be the point of intersection of side BC and AG where G is the midpoint of BF. Then point E divides side BC in the ratio



(A) 1 : 4 (B) 1 : 3 (C) 2 : 5 (D) 4 : 11 (E) 3 : 8

27. A box contains chips, each of which is red, white, or blue. The number of blue chips is at least half the number of white chips, and at most one third the number of red chips. The number which are white or blue is at least 55. The minimum number of red chips is

(A) 24 (B) 33 (C) 45 (D) 54 (E) 57

28. Nine lines parallel to the base of a triangle divide the other sides each into 10 equal segments and the area into 10 distinct parts. If the area of the largest of these parts is 38, then the area of the original triangle is

(A) 180 (B) 190 (C) 200 (D) 210 (E) 240

29. Given the progression $10^{\frac{1}{11}}, 10^{\frac{2}{11}}, 10^{\frac{3}{11}}, 10^{\frac{4}{11}}, \cdots, 10^{\frac{n}{11}}$. The least positive integer n such that the product of the first n terms of the progression exceeds 100,000 is

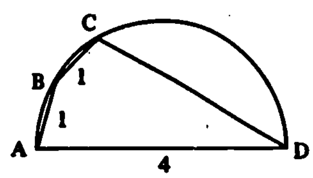
(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

30. Given the linear fractional transformation of x into $f_1(x) = \frac{2x-1}{x+1}$. Define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, \cdots$. Assuming that $f_{35}(x) = f_5(x)$, it follows that $f_{28}(x)$ is equal to

(A) x (B) $\frac{1}{x}$ (C) $\frac{x-1}{x}$ (D) $\frac{1}{1-x}$ (E) None of these

PART IV (6 credits each)

31. Quadrilateral ABCD is inscribed in a circle with side AD, a diameter of length 4. If sides AB and BC each have length 1, then side CD has length



(A) $\frac{7}{2}$ (B) $\frac{5\sqrt{2}}{2}$ (C) $\sqrt{11}$ (D) $\sqrt{13}$ (E) $2\sqrt{3}$

32. If $s = (1 + 2^{-\frac{1}{32}})(1 + 2^{-\frac{1}{16}})(1 + 2^{-\frac{1}{8}})(1 + 2^{-\frac{1}{4}})(1 + 2^{-\frac{1}{2}})$, then s is equal to

(A) $\frac{1}{2}(1 - 2^{-\frac{1}{32}})^{-1}$ (B) $(1 - 2^{-\frac{1}{32}})^{-1}$ (C) $1 - 2^{-\frac{1}{32}}$ (D) $\frac{1}{2}(1 - 2^{-\frac{1}{32}})$ (E) $\frac{1}{2}$

33. If P is the product of n quantities in Geometric Progression, S their sum, and S′ the sum of their reciprocals, then P in terms of S, S′, and n is

(A) $(SS')^{\frac{1}{2}n}$ (B) $(S/S')^{\frac{1}{2}n}$ (C) $(SS')^{n-2}$ (D) $(S/S')^{n}$ (E) $(S'/S)^{\frac{1}{2}(n-1)}$

34. An ordinary clock in a factory is running slow so that the minute hand passes the hour hand at the usual dial positions (12 o'clock, etc.) but only every 69 minutes. At time and one-half for overtime, the extra pay to which a \$4.00 per hour worker should be entitled after working a normal 8 hour day by that slow running clock, is

(A) \$2.30 (B) \$2.60 (C) \$2.80 (D) \$3.00 (E) \$3.30

35. Each circle in an infinite sequence with decreasing radii is tangent externally to the one following it and to both sides of a given right angle. The ratio of the area of the first circle to the sum of areas of all other circles in the sequence, is

(A) $(4+3\sqrt{2}):4$ (B) $9\sqrt{2}:2$ (C) $(16+12\sqrt{2}):1$ (D) $(2+2\sqrt{2}):1$
(E) $(3+2\sqrt{2}):1$

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Univ. of Nebraska, Lincoln, Nebr. 68508
Omaha Office: Univ. of Nebraska at Omaha, Omaha, Nebr. 68101
New York Office: Fred F. Kuhn, 119 Fifth Avenue, New York, N.Y. 10003

Printed in U.S.A. 360M-71

SOLUTION-ANSWER KEY

TWENTY SECOND ANNUAL H. S.
MATHEMATICS EXAMINATION

1971

22

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

UES-MAA-7212

Note: The letter following the problem number designates the correct choice of the five listed for the problem in the 1971 examination.

1. (B) Rearranging the factors, $N = 2^4(2^8 \times 5^8) = 16 \times 10^8 = 1{,}600{,}000{,}000$ which is a ten digit number.

2. (D) If x, y, and z represent the number of men, days, and bricks respectively, and k is the constant of proportionality, then $z = kxy$ and the given data yields $k = f/(bc)$. With this k, the number of days $y = bcz/(fx) = bcb/(fc) = b^2/f$.

3. (E) Equate the reciprocals of the slopes of the segments joining $(x,-4)$ with $(0,8)$ and $(-4,0)$ with $(0,8)$ to get $-x/12 = 4/8$. Hence $x = -6$.

4. (A) The principal at the beginning of the 2 months was $P = \$255.31/(1 + .05/6) = \$253.20$ so that the interest credited was \$2.11 and the number of cents was 11.

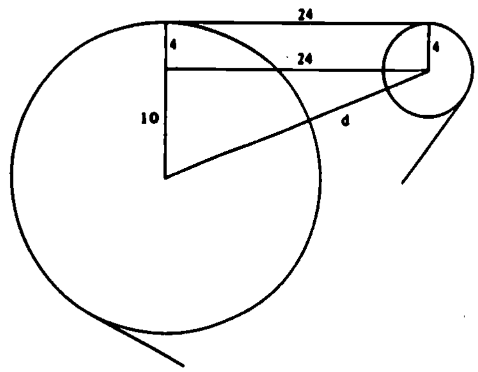
5. (C) $\measuredangle P + \measuredangle Q = 360^\circ - (\measuredangle PAQ + \measuredangle PCQ)$
$= 360^\circ - (180^\circ - 21^\circ) - (180^\circ - 19^\circ) = 40^\circ$.

6. (E) $a * \frac{1}{2a} = 1$ which is not the identity element as required.

7. (C) The given expression $= 2^{-2k}(2^{-1} - 2 + 1) = -2^{-2k}/2 = -2^{-(2k+1)}$

8. (B) The given inequality is equivalent to $(3x + 4)(2x - 1) < 0$ which requires $0 < 3x + 4$ and $2x - 1 < 0 \therefore -\frac{4}{3} < x < \frac{1}{2}$.

9. (D) A right triangle with legs 24 and 10 inches is formed by the line of centers, a radius, and a line parallel to the belt (See figure). $\therefore$ the required length d satisfies $d^2 = 24^2 + 10^2 = 26^2$, $d = 26$ inches.



10. (E) There are $50 - 31 = 19$ blondes and $\therefore 19 - 14 = 5$ brown eyed blondes. $\therefore 18 - 5 = 13$ are brown eyed brunettes.

11. (D) Both a and b must be greater than 7 and $4a + 7 = 7b + 4 \therefore 7b - 4a = 3 \therefore (a,b) = (15,9) \therefore a + b = 24 =$ XXIV.

12. (B) The difference of any two congruent integers must be divisible by N. Since $90 - 69 = 21$ and $125 - 90 = 35$, $N = 7$. Since $81 - 4 = 77$, 81 is congruent to 4.

13. (E) The binomial expansion gives $(1.0025)^{10} = (1 + .0025)^{10} = 1 + 10(.0025) + 45(.0025)^2 + 120(.0025)^3 + 210(.0025)^4 + \cdots = 1.025 + .00028125 + .000001875 +$ terms less than $10^{-8} = 1.02528$ accurate to 5 decimal places.

14. (C) Two factors are 63 and 65 because $2^{48} - 1 = (2^{24} - 1)(2^{24} + 1) = (2^{12} - 1)(2^{12} + 1)(2^{24} + 1) = (2^6 - 1)(2^6 + 1)(2^{12} + 1)(2^{24} + 1) = 63 \times 65 \cdot (2^{12} + 1)(2^{24} + 1)$.

15. (B) Let u and h be the length of the bottom and depth of water when the bottom is level. Then the volume V of the water is $V = 10hu = 10 \cdot \frac{1}{2} \cdot 8 \cdot \frac{3}{4}u$, $h = 3''$.

16. (A) Let the 35 scores be denoted by $x_1, x_2, x_3, \cdots, x_{35}$ and their average by $\bar{x}$. Then $35\bar{x} = x_1 + x_2 + x_3 \cdots + x_{35}$. The average of the 36 numbers is $(x_1 + x_2 + x_3 + \cdots + x_{35} + \bar{x})/36 = (35\bar{x} + \bar{x})/36 = 36\bar{x}/36 = \bar{x}$. Hence the ratio is 1 : 1.

17. (E) A secant can cut across $(n + 1)$, but no more, of the 2n equal sectors, dividing each into two parts. The total number of distinct areas is then $2n + (n + 1) = 3n + 1$.

18. (D) Let v = the boat's speed in still water in miles per hour. Then $1 = \frac{4}{v+3} + \frac{4}{v-3}$
$\therefore v^2 - 9 = 4(v - 3) + 4(v + 3) \therefore v^2 - 8v - 9 = 0, (v - 9)(v + 1) = 0 \therefore v = 9$
$\frac{\text{Speed downstream}}{\text{Speed upstream}} = \frac{v+3}{v-3} = \frac{12}{6} = \frac{2}{1}$. The ratio is 2 : 1.

19. (C) Exactly one intersection requires the roots of the quadratic $x^2 + 4(mx + 1)^2 = 1$ to be equal and hence the discriminant to be zero. i.e. $(8m)^2 - 4(4m^2 + 1) \cdot 3 = 0$. Hence $m^2 = \frac{3}{4}$.

20. (E) If the roots are r and s, then $r + s = -2h$ and $rs = -3$. Hence $(r + s)^2 = r^2 + s^2 + 2rs = 10 - 6 = 4h^2 \therefore |h| = 1$.

21. (C) Since the antilog of 0 is 1 regardless of base, $\log_3(\log_4 x) = \log_4(\log_2 y) = \log_2(\log_3 z) = 1$ and hence $\log_4 x = 3$, $\log_2 y = 4$ and $\log_3 z = 2$. $\therefore x + y + z = 4^3 + 2^4 + 3^2 = 89$.

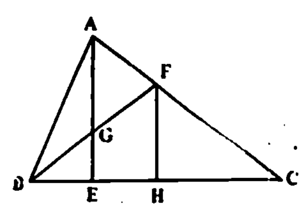
22. (A) Factoring the given equation $x^3 - 1 = 0$ gives $(x - 1)(x^2 + x + 1) = 0$. The imaginary root w satisfies $w^2 + w + 1 = 0$. Hence $1 + w^2 = -w$, $1 + w = -w^2$ $\therefore (1 - w + w^2)(1 + w - w^2) = (-2w)(-2w^2) = 4w^3 = 4$ because $w^3 = 1$.

23. (A) The four sequences of wins (each followed by its probability) for Team A to lose the series are BBB(1/8), ABBB(1/16), BABB(1/16), BBAB(1/16). The total probability for Team A to lose is 5/16 so that, to win is 11/16. The odds favoring Team A to win are 11 to 5.

24. (D) In the first n rows, there are $(2n - 1)$ 1's and $\frac{1}{2}(n - 2)(n - 1) = \frac{1}{2}(n^2 - 3n + 2)$ other numbers. The quotient is $\frac{n^2 - 3n + 2}{4n - 2}$.

25. (D) Let b and f denote the boy's and father's age respectively. Then $13 \leq b \leq 19$. Also $99f + 2b = 4289$. Equating the remainders in the division of both members of this equation by 9, (casting out nines), $2b = 32$, $b = 16$. Now $99f = 4289 - 32 = 4257$ and $f = 43$. $\therefore f + b = 59$.

26. (B) Draw FH parallel to AE. Then BE = EH because BG = GF. Also 2EH = HC because 2AF = FC. ∴ 3BE = EH + HC = EC. ∴ E divides BC in the ratio 1 : 3.



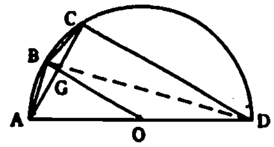
27. (E) Let w, b, and r denote the numbers of white, blue, and red chips respectively. Then $w \leq 2b$ and $3b \leq r$. Now $w + b \geq 55 \therefore 2b + b = 3b \geq 55 \therefore b \geq 18\frac{1}{3} \therefore 19 \leq b \therefore 57 \leq 3b \leq r$. The minimum number of red chips is 57.

28. (C) Let b and h denote the length of the base and altitude respectively. Since 38 = $\frac{1}{2}(b + .9b)(.1h)$, the area $\frac{1}{2}bh = 200$.

29. (E) Since $100{,}000 = 10^5$, the sum of the first n exponents must exceed 5. i.e. $n(n + 1)/22 > 5 \therefore n(n + 1) > 110 \therefore n = 11$.

30. (D) If $g(x)$ is the inverse of the transformation $f_1(x)$, then $g(f_{n+1}(x)) = f_n(x)$. Since $f_{35}(x) = f_5(x)$, successive application of this formula yields $f_{31}(x) = f_1(x) \therefore f_{30}(x) = g(f_1(x)) = x \therefore f_{29}(x) = g(x) \therefore f_{28}(x) = g(g(x))$. But $g(x) = \frac{x+1}{2-x} \therefore f_{28}(x) = g(g(x)) = \frac{1}{1-x}$.

31. (A) Draw radius OB which is perpendicular to and bisects chord AC at G. Side CD is parallel to and has length twice that of segment GO. Similar right triangles BGA and ABD yield



$$\frac{BG}{1} = \frac{1}{AD} = \frac{1}{4} \therefore GO = BO - BG = 2 - \frac{1}{4} = \frac{7}{4}. \quad \therefore CD = 2\,GO = \frac{7}{2}.$$

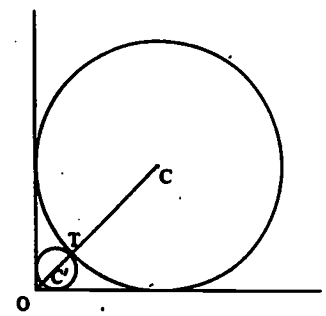
32. (A) Let $x = 2^{-\frac{1}{32}}$. Then $s = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})$
Now $(1 - x)s = 1 - x^{32} = \frac{1}{2}. \therefore s = \frac{1}{2}(1 - x)^{-1} = \frac{1}{2}(1 - 2^{-\frac{1}{32}})^{-1}$

33. (B) Let the progression be $a, ar, ar^2, \cdots ar^{n-1}$, then $P = a^n r^{\frac{1}{2}(n-1)n}$, $S = a\frac{1-r^n}{1-r}$,

$$S' = \frac{1}{a} \cdot \frac{1-r^{-n}}{1-r^{-1}} = \frac{r^{-(n-1)}}{a} \cdot \frac{1-r^n}{1-r} \therefore S/S' = a^2r^{(n-1)} \therefore (S/S')^{\frac{1}{2}n} = a^n r^{\frac{1}{2}(n-1)n} = P.$$

34. (B) 12 hours by the slow clock = 69 X 11 minutes = 12 hrs. + 39 min. ∴ 8 hours by the slow clock = 8 hrs. + 26 min. Overtime of 26 min. @ \$6 per hr. or 10¢ per min. gives \$2.60.

35. (C) Let O denote the vertex of the right angle, C and C′ the centers, r and r′ ($r > r'$) the radii of any two consecutive circles. If T is the point of contact of the circles, then $OT = OC' + r' = (\sqrt{2} + 1)r'$ and $OT = OC - r = (\sqrt{2} - 1)r$. Equating these expressions for OT yields the ratio of consecutive radii $r'/r = (\sqrt{2} - 1)/(\sqrt{2} + 1) = (\sqrt{2} - 1)^2$. If r is the radius of the first circle in the sequence, then πr^2 is its area, and the sum of the areas of all the other circles, which form a geometric series, is



$$\pi r^2 [(\sqrt{2} - 1)^4 + (\sqrt{2} - 1)^8 + \ldots] = \frac{\pi(\sqrt{2} - 1)^4 r^2}{1 - (\sqrt{2} - 1)^4} = \frac{\pi r^2}{(\sqrt{2} + 1)^4 - 1}$$

The quotient of areas $= (\sqrt{2} + 1)^4 - 1 = 16 + 12\sqrt{2}$ and the required ratio is $(16 + 12\sqrt{2}) : 1$.

85

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA

SOCIETY OF ACTUARIES

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

and

CASUALTY ACTUARIAL SOCIETY

23

TWENTY THIRD ANNUAL MATHEMATICS EXAMINATION 1972

INSTRUCTIONS

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off this cover along the dotted line inside. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you write the information required in the first three lines.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*This instruction may be modified if machine-scoring is used.

TUESDAY, MARCH 14, 1972

M.A.A. COMMITTEE ON HIGH SCHOOL CONTESTS

National Office: Univ. of Nebraska, Lincoln, Nebr. 68508
Omaha Office: Univ. of Nebraska at Omaha, Omaha, Nebr. 68101
New York Office: Fred F. Kuhn, 270 Madison Avenue, New York, N.Y. 10016

To be filled in by the student

PRINT ____

last name   first name   middle initial

____

school (full name)   street address

____

city   state   zip code

PART I										
	1	2	3	4	5	6	7	8	9	10

PART II										
	11	12	13	14	15	16	17	18	19	20

PART III										
	21	22	23	24	25	26	27	28	29	30

PART IV					
	31	32	33	34	35

Not to be filled in by the student

Part	Value	Points Correct, C	Points Wrong, W (do not include omissions)
I	3 points each	3 times =	3 times =
II	4 points each	4 times =	4 times =
III	5 points each	5 times =	5 times =
IV	6 points each	6 times =	6 times =
	TOTALS	C =	W =
	SCORE $= C - \frac{1}{4}W$		
		Use for computation	Write score above (2 dec. places)

PART I (3 credits each)

1. The lengths in inches of the three sides of each of four triangles I, II, III, and IV are as follows:

I 3, 4, and 5 — III 7, 24, and 25
II 4, $7\frac{1}{2}$, and $8\frac{1}{2}$ — IV $3\frac{1}{2}$, $4\frac{1}{2}$, and $5\frac{1}{2}$

Of these four given triangles, the only right triangles are

(A) I and II (B) I and III (C) I and IV
(D) I, II, and III (E) I, II, and IV

2. If a dealer could get his goods for 8% less while keeping his selling price fixed, his profit, based on cost, would be increased to $(x + 10)\%$ from his present profit of $x\%$ which is

(A) 12% (B) 15% (C) 30% (D) 50% (E) 75%

3. If $x = \frac{1 - i\sqrt{3}}{2}$ where $i = \sqrt{-1}$, then $\frac{1}{x^2 - x}$ is equal to

(A) -2 (B) -1 (C) $1 + i\sqrt{3}$ (D) 1 (E) 2

4. The number of solutions to $\{1,2\} \subseteq X \subseteq \{1,2,3,4,5\}$ where X is a subset of $\{1,2,3,4,5\}$ is

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of these

5. From among $2^{1/2}$, $3^{1/3}$, $8^{1/8}$, $9^{1/9}$ those which have the greatest and the next to the greatest values in that order, are

(A) $3^{1/3}, 2^{1/2}$ (B) $3^{1/3}, 8^{1/8}$ (C) $3^{1/3}, 9^{1/9}$ (D) $8^{1/8}, 9^{1/9}$
(E) None of these

6. If $3^{2x} + 9 = 10(3^x)$, then the value of $(x^2 + 1)$ is

(A) 1 only (B) 5 only (C) 1 or 5 (D) 2 (E) 10

7. If $yz:zx:xy = 1:2:3$, then $\frac{x}{yz} : \frac{y}{zx}$ is equal to

(A) 3:2 (B) 1:2 (C) 1:4 (D) 2:1 (E) 4:1

8. If $|x - \log y| = x + \log y$ where x and log y are real, then

(A) $x = 0$ (B) $y = 1$ (C) $x = 0$ and $y = 1$
(D) $x(y-1) = 0$ (E) None of these

9. Ann and Sue bought identical boxes of stationery. Ann used hers to write 1-sheet letters and Sue used hers to write 3-sheet letters. Ann used all the envelopes and had 50 sheets of paper left, while Sue used all of the sheets of paper and had 50 envelopes left. The number of sheets of paper in each box was

(A) 150 (B) 125 (C) 120 (D) 100 (E) 80

10. For x real, the inequality $1 \le |x - 2| \le 7$ is equivalent to

(A) $x \le 1$ or $x \ge 3$ (B) $1 \le x \le 3$ (C) $-5 \le x \le 9$
(D) $-5 \le x \le 1$ or $3 \le x \le 9$ (E) $-6 \le x \le 1$ or $3 \le x \le 10$

Part II (4 credits each)

11. The value(s) of y for which the following pair of equations

$$x^2 + y^2 - 16 = 0 \text{ and } x^2 - 3y + 12 = 0$$

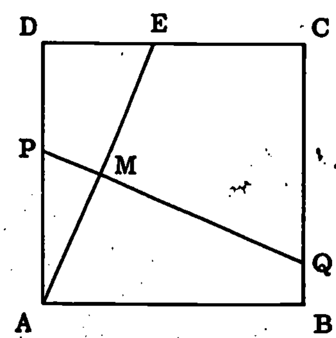
may have a real common solution, are

(A) 4 only (B) $-7, 4$ (C) $0, 4$ (D) no y (E) all y

12. The number of cubic feet in the volume of a cube is the same as the number of square inches in its surface area. The length of the edge expressed as a number of feet is

(A) 6 (B) 864 (C) 1728 (D) 6×1728 (E) 2304

13. Inside square ABCD (See figure) with sides of length 12 inches, segment AE is drawn where E is the point on DC which is 5 inches from D. The perpendicular bisector of AE is drawn and intersects AE, AD, and BC at points M, P, and Q respectively. The ratio of segment PM to MQ is



(A) 5:12 (B) 5:13 (C) 5:19
(D) 1:4 (E) 5:21

14. A triangle has angles of 30° and 45°. If the side opposite the 45° angle has length 8, then the side opposite the 30° angle has length

(A) 4 (B) $4\sqrt{2}$ (C) $4\sqrt{3}$ (D) $4\sqrt{6}$ (E) 6

15. A contractor estimated that one of his two bricklayers would take 9 hours to build a certain wall and the other 10 hours. However, he knew from experience that when they worked together, their combined output fell by 10 bricks per hour. Being in a hurry, he put both men on the job and found that it took exactly 5 hours to build the wall. The number of bricks in the wall was

(A) 500 (B) 550 (C) 900 (D) 950 (E) 960

16. There are two positive numbers that may be inserted between 3 and 9 such that the first three are in geometric progression while the last three are in arithmetic progression. The sum of those two positive numbers is

(A) $13\frac{1}{2}$ (B) $11\frac{1}{4}$ (C) $10\frac{1}{2}$ (D) 10 (E) $9\frac{1}{2}$

17. A piece of string is cut in two at a point selected at random. The probability that the longer piece is at least x times as large as the shorter piece is

(A) $\frac{1}{2}$ (B) $\frac{2}{x}$ (C) $\frac{1}{x+1}$ (D) $\frac{1}{x}$ (E) $\frac{2}{x+1}$

18. Let ABCD be a trapezoid with the measure of base AB twice that of base DC, and let E be the point of intersection of the diagonals. If the measure of diagonal AC is 11, then that of segment EC is equal to

(A) $3\frac{2}{3}$ (B) $3\frac{3}{4}$ (C) 4 (D) $3\frac{1}{2}$ (E) 3

19. The sum of the first n terms of the sequence

$1, (1+2), (1+2+2^2), \ldots (1+2+2^2+\ldots+2^{n-1})$

in terms of n is

(A) 2^n (B) $2^n - n$ (C) $2^{n+1} - n$

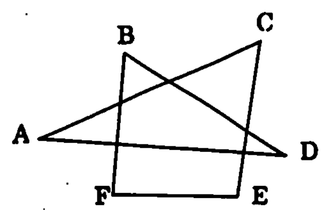
(D) $2^{n+1} - n - 2$ (E) $n \cdot 2^n$

20. If $\tan x = \frac{2ab}{a^2 - b^2}$ where $a > b > 0$ and $0° < x < 90°$, then $\sin x$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{\sqrt{a^2 - b^2}}{2a}$ (D) $\frac{\sqrt{a^2 - b^2}}{2ab}$ (E) $\frac{2ab}{a^2 + b^2}$

PART III (5 credits each)

21. If the sum of the measures in degrees of angles A, B, C, D, E, and F in the figure to the right is 90n, then n is equal to

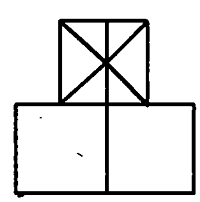


(A) 2 (B) 3 (C) 4
(D) 5 (E) 6

22. If $a \pm bi$ $(b \neq 0)$ are imaginary roots of the equation $x^3 + qx + r = 0$ where a, b, q, and r are real numbers, then q in terms of a and b is

(A) $a^2 + b^2$ (B) $2a^2 - b^2$ (C) $b^2 - a^2$
(D) $b^2 - 2a^2$ (E) $b^2 - 3a^2$

23. The radius of the smallest circle containing the symmetric figure composed of the 3 unit squares shown at the right is



(A) $\sqrt{2}$ (B) $\sqrt{1.25}$ (C) 1.25
(D) $\frac{5\sqrt{17}}{16}$ (E) None of these

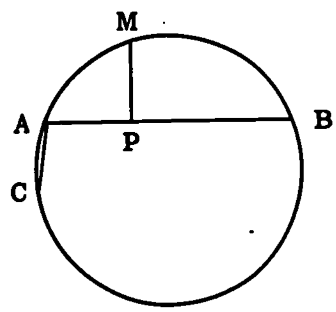
24. A man walked a certain distance at a constant rate. If he had gone $\frac{1}{2}$ mile per hour faster, he would have walked the distance in four-fifths of the time; if he had gone $\frac{1}{2}$ mile per hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. The distance in miles he walked was

(A) $13\frac{1}{2}$ (B) 15 (C) $17\frac{1}{2}$ (D) 20 (E) 25

25. Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. The diameter of this circle has length

(A) 62 (B) 63 (C) 65 (D) 66 (E) 69

26. In the circle to the right, M is the mid-point of arc CAB and segment MP is perpendicular to chord AB at P. If the measure of chord AC is x and that of segment AP is (x + 1), then segment PB has measure equal to



(A) $3x + 2$ (B) $3x + 1$
(C) $2x + 3$ (D) $2x + 2$
(E) $2x + 1$

27. If the area of ΔABC is 64 square inches and the geometric mean (mean proportional) between sides AB and AC is 12 inches, then sin A is equal to

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{8}{9}$ (E) $\frac{15}{17}$

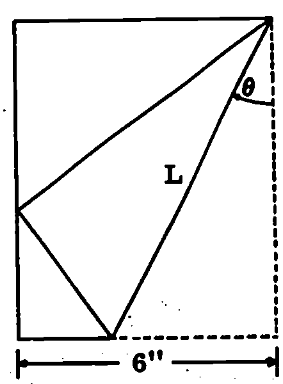
28. A circular disc with diameter D is placed on an 8×8 checkerboard with width D so that the centers coincide. The number of checkerboard squares which are completely covered by the disc is

(A) 48 (B) 44 (C) 40 (D) 36 (E) 32

29. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ for $-1 < x < 1$, then $f\left(\frac{3x+x^3}{1+3x^2}\right)$ in terms of $f(x)$ is

(A) $-f(x)$ (B) $2f(x)$ (C) $3f(x)$ (D) $[f(x)]^2$
(E) $[f(x)]^3 - f(x)$

30. A rectangular piece of paper 6 inches wide is folded as in the diagram so that one corner touches the opposite side. The length in inches of the crease L in terms of angle θ is



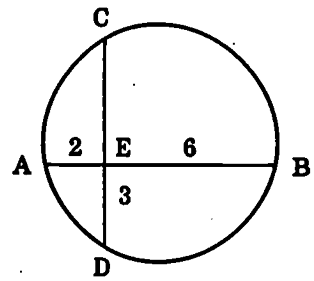
(A) $3 \sec^2 \theta \csc \theta$
(B) $6 \sin \theta \sec \theta$
(C) $3 \sec \theta \csc \theta$
(D) $6 \sec \theta \csc^2 \theta$
(E) None of these

PART IV (6 credits each)

31. When the number 2^{1000} is divided by 13, the remainder in the division is

(A) 1 (B) 2 (C) 3 (D) 7 (E) 11

32. Chords AB and CD in the circle to the right intersect at E and are perpendicular to each other. If segments AE, EB, and ED have measures 2, 6, and 3 respectively, then the length of the diameter of the circle is



(A) $4\sqrt{5}$ (B) $\sqrt{65}$ (C) $2\sqrt{17}$
(D) $3\sqrt{7}$ (E) $6\sqrt{2}$

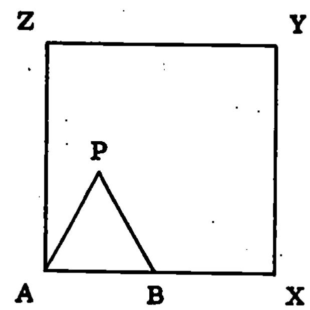
33. The minimum value of the quotient of a (base ten) number of three different nonzero digits divided by the sum of its digits is

(A) 9.7 (B) 10.1 (C) 10.5 (D) 10.9 (E) 20.5

34. Three times Dick's age plus Tom's age equals twice Harry's age. Double the cube of Harry's age is equal to three times the cube of Dick's age added to the cube of Tom's age. Their respective ages are relatively prime to each other. The sum of the squares of their ages is

(A) 42 (B) 46 (C) 122 (D) 290 (E) 326

35. Equilateral triangle ABP (See figure) with side AB of length 2 inches is placed inside square AXYZ with side of length 4 inches so that B is on side AX. The triangle is rotated clockwise about B, then P, and so on along the sides of the square until P returns to its original position. The length of the path in inches traversed by vertex P is equal to



(A) $20\pi/3$ (B) $32\pi/3$ (C) 12π
(D) $40\pi/3$ (E) 15π

SOLUTION-ANSWER KEY

TWENTY THIRD ANNUAL H. S.

MATHEMATICS EXAMINATION

1972

23

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra and trigonometry is needed to solve the problems posed.

UES-MAA-7305

e8

Note: The letter following the problem number designates the correct choice of the five listed for the problem in the 1972 examination.

1. (D) The Pythagorean Theorem applies to show that IV is not but I, II, and III are right triangles.

2. (B) Let C be the present cost, hence .92C the discounted cost, and x the present profit, in percentage, then equating the fixed proceeds of sale, gives

$C(1 + .01x) = .92C[1 + .01(x + 10)]$, $.08(.01x) = (.92)(1.1) - 1 = .012$, $x = 15$.

3. (B) $x^2 - x = \frac{1}{4}(1 - i\sqrt{3})^2 - \frac{1}{2}(1 - i\sqrt{3}) = \frac{1}{4}(-2 - 2i\sqrt{3}) - \frac{1}{2}(1 - i\sqrt{3}) = -1$ and the reciprocal requested is also -1.

4. (D) The union of $\{1,2\}$ with each of the 8 distinct subsets of $\{3,4,5\}$ results in a different solution X and there are no other solutions.
<u>Remark</u>: In making the count, one need only enumerate the 8 subsets of $\{3,4,5\}$ but it is not difficult to write down all 8 solutions. This is left to the student.

5. (A) First note that $(2^{1/2})^6 = 8 < (3^{1/3})^6 = 9$ so that $2^{1/2} < 3^{1/3}$. Also $(2^{1/2})^{18} = 2^9 = 512 > (9^{1/9})^{18} = 81$ so that $2^{1/2} > 9^{1/9}$, and $2^{1/2} > 8^{1/8} = 2^{3/8}$. Hence $3^{1/3}$, $2^{1/2}$ have the greatest and next to the greatest values in that order.

6. (C) Let $y = 3^x$ and the given equation is equivalent to $y^2 - 10y + 9 = 0$ or $(y - 9)(y - 1) = 0$. Hence $y = 3^x = 9$ or 1, $x = 2$ or 0 so that $x^2 + 1 = 5$ or 1.

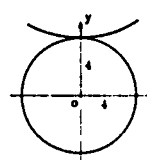
7. (E) The ratio $\frac{x}{yz} : \frac{y}{zx} = \frac{zx}{yz} : \frac{yz}{zx} = 2 : \frac{1}{2} = 4 : 1$ because it is given that $yz : zx = 1 : 2$ and hence $2yz = zx$ so that $\frac{zx}{yz} = 2$ and $\frac{yz}{zx} = \frac{1}{2}$.

8. (D) If $(x - \log y)$ is nonnegative, then the given equation requires that $x - \log y = x + \log y$ so that $-\log y = \log y = 0$ and $y = 1$. On the other hand, if $(x - \log y)$ is negative $-(x - \log y) = x + \log y$ so that $2x = 0$. We can write $x(y - 1) = 0$ to say that $x = 0$ or $y = 1$ or both.

9. (A) Let x and y denote the number of sheets of paper and of envelopes respectively in each box. Then $x - y = 50$ and $y - \frac{1}{3}x = 50$ give $\frac{2}{3}x = 100$, $x = 150$ sheets of paper in each box.

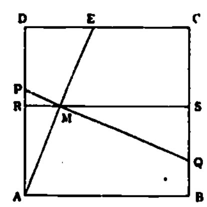
10. (D) First when $x - 2 \geq 0$, than $1 \leq x - 2 \leq 7$, $3 \leq x \leq 9$. Again when $x - 2 \leq 0$, then $1 \leq 2 - x \leq 7$, $-1 \leq -x \leq 5$ gives $-5 \leq x \leq 1$ as the other possibility as stated in choice (D).

11. (A) Graphing the circle of radius 4 about the origin and the parabola with y – intercept 4, it becomes apparent that they intersect only when $y = 4$. Alternately, we may subtract the second from the first equation to get $y^2 + 3y - 28 = 0$, $y = -7$ or 4. For $y = 4$, $x = 0$, but there is no real x for $y = -7$.



12. (B) Let an edge be f feet and hence 12f inches long. Then $f^3 = 6(12f)^2$ so that $f = 6(12)^2 = 6 \times 144 = 864$ ft.

13. (C) Let R and S be the intersections with sides AD and BC of the line through M parallel to AB (See figure). Then $RM = \frac{1}{2}DE = 2\frac{1}{2}$ inches and hence MS has length $9\frac{1}{2}$ inches. Since PMR and SMQ are similar right triangles $PM : MQ = RM : MS = 2\frac{1}{2} : 9\frac{1}{2} = 5 : 19$.



14. (B) Let s denote the required side. Then the Law of Sines gives

$$\frac{s}{\sin 30^\circ} = \frac{8}{\sin 45^\circ}, \; s = \frac{8 \sin 30^\circ}{\sin 45^\circ} = \frac{8(\frac{1}{2})}{\frac{1}{2}\sqrt{2}} = 4\sqrt{2}.$$

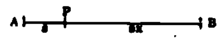
Alternately, the altitude between sides s and 8 has length 4 and is one leg of an isosceles right triangle with hypothenuse $s = 4\sqrt{2}$.

15. (C) If x is the number of bricks in the wall, then this reduced number of bricks for 5 hours of work together, is

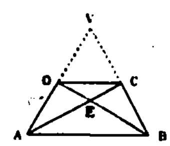
$$5\left(\frac{x}{9} + \frac{x}{10} - 10\right) = x, \; x = 900 \text{ bricks.}$$

16. (B) Let 3, x, y be the first three numbers. Then $\frac{x}{3} = \frac{y}{x}$, $x^2 = 3y$. The last three numbers are x, y, 9 so that $y - x = 9 - y$, $x + 9 = 2y$. Eliminate y getting $2x^2 - 3x - 27 = 0$. Factoring, $(x + 3)(2x - 9) = 0$. $x = 4\frac{1}{2}$, $y = \frac{x^2}{3} = 6\frac{3}{4}$. $\therefore x + y = 11\frac{1}{4}$.

17. (E) Let AB represent the string (See figure) and let P be the point on it such that $AP:PB = 1:x$. If AP has length s inches, then the length of PB is sx. The probability that the cut lie on AP is $\frac{s}{s + sx} = \frac{1}{1 + x}$. Since the cut is equally likely to lie within the same distance from the other end B of the string, the probability of either is $\frac{2}{x + 1}$ of choice (E).



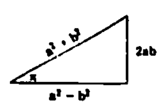
18. (A) Let sides AD and BC of the quadrilateral intersect at V. Then diagonals AC and BD are medians from A and B of triangle ABV intersecting at E which divides AC in the ratio 2 : 1 so that the length of EC is $\frac{11}{3}$ or $3\frac{2}{3}$ units.



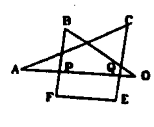
19. (D) The k^{th} term of the given sequence is equal to $2^k - 1$ so that the sum of the first n terms may be written as

$$(2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^n - 1)$$
$$= (2 + 2^2 + 2^3 + \cdots + 2^n) - (1 + 1 + 1 + \cdots \text{ to n terms})$$
$$= (2^{n+1} - 2) - n = 2^{n+1} - n - 2.$$

20. (E) Angle x may be taken as the acute angle opposite the side of length 2ab in a right triangle (See figure) whose other leg then has length $(a^2 - b^2)$. The hypothenuse is then the square root of $(2ab)^2 + (a^2 - b^2)^2 = a^4 + 2a^2b^2 + b^4$ or $a^2 + b^2$. Hence $\sin x = 2ab/(a^2 + b^2)$ by the definition of sine.



21. (C) Let P and Q denote the intersections of AD with BF and CE respectively. Then 3 sums of angles in degrees are



$$\angle F + \angle FPD + \angle EQA + \angle E = 360°$$
$$\angle B + (180° - \angle FPD) + \angle D = 180°$$
$$\angle A + (180° - \angle EQA) + \angle C = 180°$$

Adding these equations member by member gives

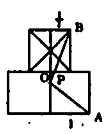
$$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + 360° = 720°$$

and the required sum is $360° = 90n°$ so that $n = 4$ as in choice (C).

22. (E) Let the third root of the given equation be s. The sum of the roots is zero, $2a + s = 0$, $s = -2a$. The sum of the products of the roots taken two at a time is q,

$$(a + bi)(a - bi) + (a + bi)s + (a - bi)s = q$$
$$(a^2 + b^2) + a(-2a) + bi(-2a) + a(-2a) - bi(-2a) = b^2 - 3a^2 = q \text{ which is choice (E).}$$

23. (D) Let P be the center of the circumscribing circle. (See figure). We must have $\overline{AP}^2 = \overline{PB}^2$ so that $(1 - \overline{OP})^2 + 1^2 = (1 + \overline{OP})^2 + (\frac{1}{2})^2$ and $-2\overline{OP} + 1 = 2\overline{OP} + \frac{1}{4}$, $OP = \frac{3}{16}$. Hence $AP^2 = (1 - OP)^2 + 1^2 = \left(\frac{13}{16}\right)^2 + 1 = \frac{169 + 256}{256} = \frac{425}{256} = \frac{25(17)}{256}$ and $AP = \frac{5\sqrt{17}}{16}$.



24. (B) Let D, R, T denote the distance (miles), rate (miles per hour), time (hours) respectively. Then

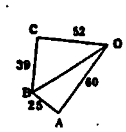
$$D = RT, \quad D = \tfrac{4}{5}(R + \tfrac{1}{2})T, \quad D = (R - \tfrac{1}{2})(T + 2\tfrac{1}{2}).$$

The first and second equations give $\frac{1}{5}D = \frac{2}{5}T$. $\therefore D = 2T$. $\therefore 2T = RT$ and $R = 2$. Replacing T by $\frac{1}{2}D$ and R by 2 in the third equation gives $D = \frac{3}{2}(\frac{1}{2}D + 2\frac{1}{2})$, $D = 15$ miles.

25. (C) Since angles A and C are supplementary, (See figure) one suspects that each may be a right angle. This turns out to be the case with diagonal BD of length 65 as the common hypothenuse and the diameter of the circumscribing circle.

or

The same result may be obtained using the Law of Cosines on triangles ABD and CBD. Thus



$$\overline{BD}^2 = 39^2 + 52^2 - 2 \times 39 \times 52 \cos C$$
$$\overline{BD}^2 = 25^2 + 60^2 - 2 \times 25 \times 60 \cos A$$

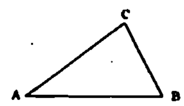
Since C is the supplement of A, replacing cos C by its equal (−cos A) and subtracting, gives

$0 = 0 + (2x39x52+2x25x60) \cos A$. $\therefore \cos A = 0$

and A and its supplement C are both right angles. The common hypothenuse BD has length 65 and is the diameter of the circumscribing circle as before.

26. (E) Draw NQ perpendicular to AB at Q where chord MN has measure x. Since arcs BN and AM are equal, PQMN is a rectangle and PQ has measure x. Hence PB = PQ + QB has measure $x + (x+1) = 2x + 1$.

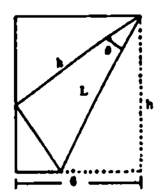
27. (D) The area of ΔABC (See figure) is $64 = \frac{1}{2}AB \cdot AC \sin A$.
Now $AB \cdot AC = 144$. Hence $\sin A = \frac{2 \times 64}{144} = \frac{8}{9}$.



28. (E) All border checkerboard squares are not entirely covered by the disc. Only the 4 corner squares of the 6 × 6 "checkerboard" of the 36 remaining interior squares are not entirely covered. Hence 36 − 4 = 32 squares are entirely covered by the disc.

29. (C) $f\left(\frac{3x+x^3}{1+3x^2}\right) = \log \frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}} = \log \frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3} = \log \frac{(1+x)^3}{(1-x)^3} = \log\left(\frac{1+x}{1-x}\right)^3 = 3 \log \frac{1+x}{1-x} = 3f(x)$.

30. (A) Let h denote the length of the sheet. Then $h = \frac{6}{\cos(90° - 2\theta)} = \frac{6}{\sin 2\theta}$
$= 3 \sec \theta \csc \theta$.
Also $L = h \sec \theta = 3\sec^2\theta \csc \theta$.



31. (C) The remainder in the division of 2^{12} by 13 is 1 because that of $2^6 = 64$ is −1. Hence $2^{1000} = 2^{996} \times 2^4 = (2^{12})^{83} \times 16$ leaves a remainder of $1^{83} \times 3 = 3$ when divided by 13.

or

Using congruences, $2^6 \equiv -1 \mod 13$. Hence $2^{1000} \equiv 2^{996} \times 2^4 \equiv (-1)^{166} \times 16 \equiv 1 \times 3 \equiv 3 \mod 13$.

32. (B) If AC and BD be drawn, then inscribed angles B and C subtend the same arc AD and hence right triangles ACE and DBE are similar. Hence CE : AE = BE : DE and the measure of segment CE is 4. The center of the circle at the intersection of the perpendicular bisectors of chords AB and CD is 4 units to the right of and $\frac{1}{2}$ unit above point A. Hence $(\text{Radius})^2 = (\frac{1}{2}\text{ Diameter})^2 = 4^2 + (\frac{1}{2})^2 = (\frac{1}{2}\sqrt{65})^2$. $\therefore$ The length of the diameter is $\sqrt{65}$.

33. (C) Let the units, tens, and hundreds digits be denoted by U, T, and H respectively so that the quotient to be minimized is $\frac{U + 10T + 100H}{U + T + H} = 1 + \frac{9(T + 11H)}{U + T + H}$ which is least when U = 9 regardless of the values of T and H. Now $\frac{T + 11H}{T + H + 9} = 1 + \frac{10H - 9}{T + H + 9}$ which is least when T = 8 and H = 1. Hence the number is 189 and the minimum quotient $\frac{189}{18} = 10.5$.

34. (A) Let T, D, and H denote the ages of Tom, Dick, and Harry respectively. Then $3D + T = 2H$ and $2H^3 = 3D^3 + T^3$ or equivalently $2(H - D) = D + T$ and $2(H^3 - D^3) = D^3 + T^3$. Since $D + T \neq 0$, $\frac{2(H^3 - D^3)}{2(H - D)} = \frac{D^3 + T^3}{D + T}$ or $H^2 + HD + \cancel{D^2} = \cancel{D^2} - DT + T^2$. Hence $T^2 - H^2 = D(T + H)$ so that $T - H = D$. Eliminating T from the last and very first equation gives H = 4D so that D = 1 and H = 4 because H and D are relatively prime integers. Also T = H + D = 5 and the required sum of squares is $T^2 + D^2 + H^2 = 5^2 + 1^2 + 4^2 = 42$.

35. (D) The point P returns to its original position after 24 = 8 × 3 moves. In 8 of these moves, the rotation is about vertex P with no path traversed by P. In the other 16 moves, 8 are about a mid-point of a side of the square with the radius to P sweeping through 120° or $\frac{2}{3}\pi$ radians, and 8 are about a corner of the square with the radius to P sweeping through 30° or $\frac{1}{6}\pi$ radians. The total angle swept through by the radius to P is $8(\frac{2}{3}\pi + \frac{1}{6}\pi) = \frac{20}{3}\pi$ radians. Hence the length of the path traversed by P is $\frac{40}{3}\pi$ inches. The general solution is:

$\frac{8}{3}\pi(2+5K)$ or $\frac{40}{3}\pi(1+K)$, $K = 0, 1, 2, \ldots$

How about a Career with Mathematics?

The Occupational Outlook Handbook (OOH) of the U.S. Government's Bureau of Labor Statistics states, "Mathematics is both a profession and a tool essential for many kinds of work. The expression of ideas in mathematical language provides a framework within which these ideas can be understood." Have you ever thought of some of the ways in which Mathematics can help you earn a living? For example:

As a Teacher

Do you like mathematics? Do you like to share your enthusiasm for mathematics with others? Do you like to help people? Do you like school? If your answer is "yes" to these questions, then you would undoubtedly enjoy teaching mathematics as a profession. Since a knowledge of mathematics is so essential in this growing technological society, there is a need for mathematics teachers at all levels including; elementary school, junior high school or middle school, senior high school, junior college, college, and graduate school. You can choose the age of student with which you would like to work as well as the level of mathematics that you would like to teach. Mathematics teachers find it exciting to see others learn mathematical concepts and challenging to discover new ways of teaching concepts. Also, good mathematics teachers find additional excitement and challenge, since they continue to learn new mathematics throughout their careers. Mathematics teaching is not without financial reward. Salaries of mathematics teachers have been steadily increasing and there is reason to believe that they will continue to increase. *Consider entering this profession of excitement, challenge, and reward.*

Time (Sept. 28, 1970; p. 38) reports teacher "shortages still exist in mathematics and science."

As a Statistician

A power company trying to supply enough electrical power for peak periods of demand, a pharmaceutical chemist trying to determine the effectiveness of a new drug, and an expert in educational methods trying to decide whether a new approach to teaching reading is better than older methods are concerned with apparently different problems; but each is vitally in need of statistical tools which can be used with the scientific method to make decisions. A statistician is trained to analyze such problems, to design experiments whose results may yield some answers to the problems, and to interpret the results when they are obtained; he may also be an expert in another field such as communication, agriculture, psychology, economics, sociology, engineering, medicine, genetics, etc. A statistician needs a background in undergraduate mathematics as well as in the special fields of statistics. *If you are mathematically oriented and interested in the decision-making aspects of research problems of all kinds, consider statistics as a career.*

As an Actuary

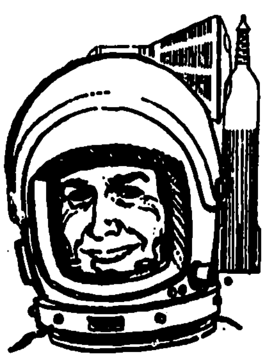
The actuary is an executive professionally trained in the science of mathematical probabilities. He uses his skills on behalf of people to design insurance plans that will help keep the family stable financially if the head of the family becomes disabled or dies. He also designs pension programs to meet financial needs in retirement years. Protection against financial disaster which might follow an accident or fire also calls for the skills of the actuary. The actuary is a businessman deeply involved in all aspects of the insurance business. A large percentage of actuaries are employed by insurance companies, although other areas of employment include consulting firms, state and local governments and academic institutions. Admission to the profession requires completion of a series of examinations—the earlier examinations being based on college mathematics. Salaries offered are substantial, and extensive training programs allow the student to obtain professional status while working in a stimulating job. *If you enjoy mathematics and wish a business career, consider the actuarial profession.*

In a letter dated September 1, 1970 announcing the Actuarial Examinations, the Presidents of the Casualty Actuarial Society and the Actuarial Society state in part "the demand for actuaries continues to outrun the supply even though increasing numbers qualify for professional status each year. It appears that the shortage of actuaries will continue for a number of years."

As an EDP Specialist

Electronic Data Processing (EDP) has been made today's fastest growing industry by a seemingly infinite series of advances in computer development. New and more powerful computers are being used to attack an increasing variety of problems. Successful EDP activity, however, depends on the imaginative application of human intelligence, without which even the most powerful computers are useless. The demand for highly trained EDP specialists who can provide guidance grows steadily each year. To be successful in EDP work, one must have ability to think logically and to analyze thoroughly a wide variety of problems. Solutions frequently involve complex calculations and the use of sophisticated mathematical techniques. A solid background in mathematics provides an excellent foundation for entering this field. *If you like mathematics, enjoy the challenge of solving complicated* *problems, and would like to be part of a new and dynamic field, consider a career as an EDP specialist.*

As a Worker in Applied Mathematics



Mathematicians are in great demand in the challenging fields of radio chemistry, rocket propulsion, nuclear physics and space exploration. Research in these fields is conducted by the Federal Government, by industries, by universities and by foundations. There are other fields too, and just as exciting, in which mathematics is needed to do research. There has been an unusually rapid increase in the use of advanced mathematical techniques and concepts in economics, psychology, sociology, political science, in business management and in biology and medical research. In addition to calculus, linear algebra, and probability, entirely new branches of mathematics (such as the theory of games, linear and dynamic programming)and new research journals (such as JOURNAL OF MATHEMATICAL PSYCHOLOGY and MANAGEMENT SCIENCE) now serve this new group of applied mathematicians. Typical problems that use advanced mathematics involve competitive economic systems, learning theory in psychology, inventory control and production scheduling in business management, cell growth and the spread of disease in biology and medicine. Or perhaps your future lies with one of the many kinds of engineering–aeronautical, chemical, civil, electrical, industrial management, mechanical, metallurgical, nuclear, petroleum, quality control, sanitary, and so forth. You can see they cover a wide range of activities. Mathematics as you know is a basic tool of the engineer. *If you enjoy mathematics and wish to make it your lifework, why not investigate more fully one of the above exciting fields?*

In the Summer 1969 issue of the Occupational Outlook Quarterly (pp. 25-27), Michael F. Crowley reports "Scientists and Engineers a Fast Paced Employment Expansion", an article based on (NSF-68-30) for sale by the U.S. Government Printing Office; Price 70¢.

Interested participants in the Annual Examination and their teachers may gain additional Career information from the OOH which gives more extended (but still brief) descriptions of about 700 occupations, each followed by the places of employment, the training needed, and the future outlook which is based on five assumptions stated for the first time in the 1970-1971 biennial revision (p. 11). The OOH is probably the nation's foremost source of information on Careers. Bulletin (NSF-68-30) states the meaning of Scientist including Mathematician carefully in Appendix C (p. 54), and Appendix D (p. 55) gives the Basic Sources of Employment Data on Scientists and Engineers, 1950-1966.

Somewhat more complete information for Mathematicians may be obtained from *Professional Opportunities in Mathematics* (POM) published by the Mathematical Association of America. The Eighth Edition (1971) gives a large number of references to further reading and the names and locations of many non-academic employers of mathematicians. It is available for $.35 from the MAA Washington Office. The address is noted below. In contrast to POM, the MAA has for distribution *You'll Need Math* (YNM) addressed principally to junior high school students and potential high school math dropouts. YNM contains 14 "comic strip type" cartoons under nine of which are very brief, but accurate, descriptions of jobs. YNM has four lists on page 5 naming 75 jobs in which mathematics ranging from "some High School math" to "some College math" is needed. The brochure ends on page 15 with the prediction, "Half the jobs you'll see 10 years from now do not exist today."

Specifically we suggest that by including three or four years of mathematics in high school, and one or two years in college, you may keep open the doors, otherwise closed, to many opportunities for your future.

This material has been made available to the student participants by the following named Sponsors of the Annual High School Mathematics Examination:

The Mathematical Association of America
1225 Connecticut Avenue NW, Washington, D.C. 20036

Society of Actuaries
208 South LaSalle Street, Chicago, Illinois 60604

Mu Alpha Theta
University of Oklahoma, Norman, Oklahoma 73069

National Council of Teachers of Mathematics
1201 Sixteenth Street NW, Washington, D.C. 20036

Casualty Actuarial Society
200 East 42nd St., New York, N.Y. 10017

ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION
JOINTLY SPONSORED BY THE
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES • MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

IMPORTANT INFORMATION FOR PARTICIPANTS

Item 1. Definitions and Symbols and their Meanings:

$=$ the same as	$\leq$ less than or equal to
$\neq$ different from	$\geq$ greater than or equal to
$<$ less than	$\lvert k\rvert$ $+k$ if $k \geq 0$, $-k$ if $k < 0$
$>$ greater than	

XY may mean line XY or the length of segment XY, according to context

f a function; f(x) is sometimes used in the sense of f

f(a) the unique value of f when the (independent) variable assumes the permissible value a, e.g. $f(a) = a - 2\sqrt{a}$

$\sum$ continued sum, e.g. $\sum_{k=1}^{n} C_k = C_1 + C_2 + \dots + C_n$

$\prod$ continued product, e.g. $\prod_{k=1}^{n} C_k = C_1C_2\dots C_n$

n! $n(n-1)(n-2)\dots(2)(1)$ for n, a natural number.

Item 2. Below are the essential examination instructions to the student:

1. Do not open this booklet until told to do so by your teacher.
2. You will have an answer sheet on which you are to indicate the correct answer to each question.
3. This is a multiple choice test. Each question is followed by five answers, marked A, B, C, D, E. For each question decide upon the correct answer; then *write the capital letter that precedes the correct answer in the box of your answer sheet directly above the number of the question. For example: In question No. 3 suppose that the correct answer is preceded by the letter C; you write the capital letter C in the box directly above No. 3. Fill in the answers as you find them.
4. If unable to solve a problem leave the corresponding answer-box blank. *Avoid random guessing since there is a penalty for wrong answers.*
5. Use pencil. Scratch paper, graph paper, ruler, compasses and eraser are permitted.
6. *When your teacher gives the signal tear off the cover of the examination booklet along the dotted line and turn the cover over. Page 2 is your answer sheet.
7. Keep the questions covered with the answer sheet while you fill in your name and the name of your school on it.
8. When your teacher gives the signal begin working the problems. You have 80 minutes working time for the test.

*The instruction may be modified if machine-scoring is used.