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ABSTRACT

Box's method of approximation for the null distributions of likelihood criteria is described. It simplifies the formulas, describes a method of obtaining f, phi, and rho directly from given values, and provides two illustrations of the method. (DJ)

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RESEARCH BULLETIN

A NOTE ON BOX'S GENERAL METHOD OF APPROXIMATION FOR THE NULL DISTRIBUTIONS OF LIKELIHOOD CRITERIA

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A Note on Box's General Method of Approximation for

the Null Distributions of Likelihood Criteria[1]

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1. Introduction

Many multivariate test statistics (such as the likelihood ratio test statistic for MANOVA and the Bartlett modification of the likelihood ratio test statistic for testing the equality of covariance matrices) have null distributions whose h -th moment M_h is of the form

$$(1.1) \qquad M_h = K \left(\frac{\prod_{j=1}^{J} (y_j)^{y_j}}{\prod_{i=1}^{I} (x_i)^{x_i}} \right)^h \frac{\prod_{i=1}^{I} \Gamma(x_i(1+h)+\xi_i)}{\prod_{j=1}^{J} \Gamma(y_j(1+h)+\eta_j)} , \quad h = 0,1,2,\ldots ,$$

where K is a constant (such that $M_0 = 1$), the x's and y's are positive numbers, and

$$(1.2) \qquad \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j \quad .$$

Such statistics have a range of variation from 0 to 1, so that the moments M_h , $h = 0,1,2,\ldots$, determine the null distribution.

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For any statistic W, $0 \leq W \leq 1$, whose moments are of the form (1.1), Box (1949) has proposed an asymptotic expansion for the cumulative distribution function (c.d.f.). This expansion provides an accurate method for determining the critical constants defining rejection regions for the multivariate tests mentioned above. One form in which this expansion is often given [see Anderson (1958; p. 207)] is the following:

$$(1.3)\quad P\{-2 \log W \leq t\} = (1-\phi)P\{\chi_f^2 \leq \rho t\} + \phi P\{\chi_{f+4}^2 \leq \rho t\} + R(\delta) \quad .$$

Here, $\delta = \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_i$,

$$(1.4)\qquad f = -2[\sum_{i=1}^{I} \xi_i - \sum_{j=1}^{J} \eta_j - \tfrac{1}{2}(I-J)] \quad ,$$

$$(1.5)\qquad \phi = \frac{1}{-6\rho^2}[\sum_{i=1}^{I} x_i^{-2}B_3((1-\rho)x_i + \xi_i) - \sum_{j=1}^{J} y_j^{-2}B_3((1-\rho)y_j + \eta_j)] \quad ,$$

ρ is the solution of the equation

$$(1.6)\qquad \sum_{i=1}^{I} x_i^{-1}B_2((1-\rho)x_i + \xi_i) - \sum_{j=1}^{J} y_j^{-1}B_2((1-\rho)y_j + \eta_j) = 0 \quad ,$$

and $R(\delta)$ is a remainder term. [Note: ρ is chosen so that Anderson's ω_1 is 0.] Also $B_1(u)$, $B_2(u)$, and $B_3(u)$ are the Bernoulli polynomials of degree 1, 2, and 3 defined by:

$$(1.7)\qquad B_1(u) = u - \tfrac{1}{2} \quad , \quad B_2(u) = u^2 - u + \tfrac{1}{6} \quad , \quad B_3(u) = u^3 - \tfrac{3}{2}u^2 + \tfrac{1}{2}u \quad .$$

If $\lim_{\delta \to \infty} \delta^{-1} x_i > 0$ and $\lim_{\delta \to \infty} (1 - \rho) x_i$ exists, $i = 1,2,\ldots,I$, and if $\lim_{\delta \to \infty} \delta^{-1} y_j > 0$ and $\lim_{\delta \to \infty} (1 - \rho) y_j$ exists, $j = 1,2,\ldots,J$, then Anderson (1958, pp. 203-207) sketches an argument which shows that the remainder term $R(\delta)$ in (1.3) is $O(\delta^{-3})$ as $\delta \to \infty$. In the case of the likelihood ratio test for MANOVA, Anderson (1958) gives examples showing the high accuracy provided by the approximation (1.3).

Determination of the constants ϕ and ρ defined by (1.5) and (1.6), respectively, is usually a very cumbersome task. In Section 2, we show how this task can be considerably simplified by taking advantage of certain relationships among the Bernoulli polynomials $B_k(u)$, $k = 1,2,3$. As a result, we obtain convenient formulas for f, ρ, and ϕ for the important special case (which includes the null distributions of the likelihood ratio test statistic for MANOVA and for various special cases of the likelihood ratio test statistic for independence of sets of variates) in which $I = J$, the x_i's and y_j's are all equal, ξ_i is linear in i, and η_j is linear in j.

Let W_1 and W_2 be two statistically independent random variables, where each W_i has range of variation from 0 to 1 and has moments of the form (1.1). It follows that an asymptotic expansion (1.3) for the c.d.f. of W_i can be given, $i = 1,2$. Let f_i, ρ_i, and ϕ_i be the constants in the asymptotic expansion (1.3) for the c.d.f. of W_i, $i = 1,2$, and suppose that the values of f_1, f_2, ρ_1, ρ_2, ϕ_1, and ϕ_2 are known to us. Let $W = W_1 W_2$. It can easily be shown that $0 \leq W \leq 1$ and that the moments of W have the form (1.1). Thus, one can obtain an asymptotic expansion (1.3) for the c.d.f. of W. Although the constants f, ρ, and ϕ in this expansion can be obtained ab initio by calculating the moments of W and then using formulas (1.4) through (1.6), or the simplifications of these formulas given in Section 2, it would be more convenient to obtain the constants f, ρ, and ϕ directly from the given values of f_1, f_2, ρ_1, ρ_2, ϕ_1, and ϕ_2. Formulas for doing this are given in Section 3. Section 4 provides two illustrations of the method.

2. Simplification of the Formulas for ρ and ϕ

In this section it is shown that if the h-th moment of the random variable W, $0 \leq W \leq 1$, is given by (1.1), then the constants ρ and ϕ in Equation (1.3) can be obtained as follows:

$$(2.1) \qquad \rho = 1 - \frac{1}{f}\left[\sum_{i=1}^{I} x_i^{-1} B_2(\xi_i) - \sum_{j=1}^{J} y_j^{-1} B_2(\eta_j)\right] \quad ,$$

and

$$(2.2) \qquad \phi = -\frac{1}{6\rho^2}\left[\sum_{i=1}^{I} x_i^{-2} B_3(\xi_i) - \sum_{i=1}^{J} y_j^{-2} B_3(\eta_j) + \frac{3}{2}(1-\rho)^2 f\right] \quad ,$$

where f is given by (1.4).

To verify (2.1), note from (1.7) that

$$(2.3) \qquad B_2(w+v) = B_2(v) + 2wB_1(v) + w^2 \quad ,$$

and hence

$$(2.4)\qquad \sum_{i=1}^{L} u_i^{-1} B_2((1-\rho)u_i + v_i)$$

$$= \sum_{i=1}^{L} u_i^{-1} B_2(v_i) + 2(1-\rho) \sum_{i=1}^{L} B_1(v_i) + (1-\rho)^2 \sum_{i=1}^{L} u_i \quad .$$

Note that

$$(2.5)\qquad \sum_{i=1}^{I} B_1(\xi_i) - \sum_{j=1}^{J} B_1(\eta_j) = \sum_{i=1} \xi_i - \frac{1}{2} I - \sum_{j=1}^{J} \eta_j + \frac{1}{2} J$$

$$= -\frac{1}{2} f \quad .$$

Since $\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j$, it follows from (1.6), (2.4), and (2.5) that ρ is the solution of

$$(2.6)\qquad \sum_{i=1} x_i^{-1} B_2(\xi_i) - \sum_{j=1}^{J} y_j^{-1} B_2(\eta_j) - (1-\rho)f = 0 \quad ,$$

which is equivalent to (2.1).

To verify (2.2), note from (1.7) that

$$(2.7)\qquad B_3(w + v) = B_3(v) + 3wB_2(v) + 3w^2 B_1(v) + w^3 \quad ,$$

and hence

$$\sum_{i=1}^{L} u_i^{-2} B_3((1-\rho)u_i + v_i)$$

$$(2.8)\qquad = \sum_{i=1}^{L} u_i^{-2} B_3(v_i) + 3(1-\rho) \sum_{i=1}^{L} u_i^{-1} B_2(v_i)$$

$$+ 3(1-\rho)^2 \sum_{i=1}^{L} B_1(v_i) + (1-\rho)^3 \sum_{i=1}^{L} u_i \quad .$$

Equation (2.2) now follows from (1.5), (2.1), (2.5), and the fact that

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j \ .$$

If the h -th moment of W has the form (1.1) with $I = J \equiv L$, $x_1 = x_2 = \cdots = x_I = y_1 = y_2 = \cdots = y_J \equiv z$, $\xi_i = a + bi$, $i = 1,2,\ldots,I$, and $\eta_j = c + dj$, $j = 1,2,\ldots,J$, then from (1.4),

$$f = -2\left[\sum_{i=1}^{L} (a + bi) - \sum_{j=1}^{L} (c + dj)\right] \tag{2.9}$$

$$= -2L\left[(a - c) + \frac{1}{2}(b - d)(L + 1)\right] \ .$$

Also, from (2.1), (2.9), and (1.7),

$$\rho = 1 - \frac{1}{fz}\left[\sum_{i=1}^{L} (a + bi)^2 - \sum_{i=1}^{L} (a + bi) - \sum_{j=1}^{L} (c + dj)^2 + \sum_{j=1}^{L} (c + dj)\right]$$

(2.10)

$$= 1 - \frac{L}{fz}\left[(a^2 - c^2) + (ab - cd)(L + 1) + \frac{1}{6}(b^2 - d^2)(L + 1)(2L + 1) + \frac{f}{2L}\right] \ .$$

Finally, note from (1.7) that

$$B_3(u) = u^3 - \frac{3}{2} B_2(u) - B_1(u) - \frac{1}{4} \ .$$

From this fact, (2.1), (2.2), and (2.5), it follows that

$$-6z^2\rho^2\phi = \sum_{i=1}^{L} [(a + bi)^3 - (c + di)^3] - \frac{3}{2}(1 - \rho)zf + \frac{f}{2}$$

$$+ \frac{3}{2}(1 - \rho)^2 z^2 f \quad . \tag{2.11}$$

By a direct evaluation we have that

$$\sum_{i=1}^{L} [(a + bi)^3 - (c + di)^3]$$

$$= (a^3 - c^3)L + \frac{3}{2}(a^2 b - c^2 d)(L)(L + 1) \tag{2.12}$$

$$+ \frac{1}{2}(ab^2 - cd^2)(L)(L + 1)(2L + 1) + \frac{1}{4}(b^3 - d^3)(L^2)(L + 1)^2 \quad .$$

Thus, (2.11) and (2.12) together give us a formula for computing ϕ .

3. Approximation for the Distribution of the Product of Independent Statistics Whose Moments Are of the Form (1.1)

Let the independent random variables W_g , $0 \leq W_g \leq 1$, be independent, with h -th moment of the form

$$E(W_g)^h = K_g \left(\frac{\prod_{j=1}^{J_g} (y_{gj})^{y_{gj}}}{\prod_{i=1}^{I_g} (x_{gi})^{x_{gi}}} \right)^h \frac{\prod_{i=1}^{I_g} \Gamma(x_{gi}(1 + h) + \xi_{gi})}{\prod_{j=1}^{J_g} \Gamma(y_{gj}(1 + h) + \eta_{gj})} \quad , \quad h = 0,1,2,\ldots, \tag{3.1}$$

where K_g is a constant (such that $EW_g^0 = 1$), and $\sum_{i=1}^{I_g} x_{gi} = \sum_{j=1}^{J_g} y_{gi}$,

for $g = 1,2$. Let

$$W = W_1 W_2 \quad . \tag{3.2}$$

It then follows that $E(W)^h = E(W_1)^h E(W_2)^h$, and hence the moments of W are of the form (1.1) with $I = I_1 + I_2$, $J = J_1 + J_2$,

$$\begin{aligned} x_i &= \begin{cases} x_{1i} , \\ x_{2,i-I_1} , \end{cases} \quad \xi_i = \begin{cases} \xi_{1i} , & \text{if } i = 1,2,\ldots,I_1 , \\ \xi_{2,i-I_1} , & \text{if } i = I_1 + 1, I_2 + 1, \ldots, I , \end{cases} \\ y_j &= \begin{cases} y_{ij} , \\ y_{2,j-J_1} , \end{cases} \quad \eta_j = \begin{cases} \eta_{ij} , & \text{if } j = 1,2,\ldots,J_1 , \\ \eta_{2,j-J_1} , & \text{if } j = J_1 + 1, J_1 + 2, \ldots, J . \end{cases} \end{aligned} \tag{3.3}$$

Note also that $\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j = \delta$.

Since the moments of W_1 , W_2 , and W are all of the form (1.1), it follows from the results of Sections 1 and 2 that the c.d.f.'s of these three variables can be expanded in the form (1.3). The coefficients f_g , ρ_g , ϕ_g for the expansion of the c.d.f. of W_g , $g = 1,2$, and the coefficients f , ρ , ϕ , for the expansion of the c.d.f. of W are expressible by means of equations (2.5), (2.1), and (2.2). From these expressions and (3.3), it can be shown by some straightforward algebra that

$$f = f_1 + f_2 \quad , \tag{3.4}$$

$$\rho = \frac{f_1\rho_1 + f_2\rho_2}{f} \quad , \tag{3.5}$$

and

$$(3.6) \qquad \phi = \frac{\rho_1^2\phi_1 + \rho_2^2\phi_2}{\rho^2} + \frac{f_1 f_2}{4f\rho^2}(\rho_1 - \rho_2)^2 \quad .$$

Equations (3.4), (3.5), and (3.6) thus provide a way to compute the coefficients f , ρ , and ϕ in the expansion (1.3) of the c.d.f. of $W = W_1W_2$ in terms of the coefficients f_g , ρ_g , and ϕ_g in the expansion (1.3) of the c.d.f. of W_g , $g = 1,2$. We may generalize these results by induction and obtain the following:

Theorem 1. Let the statistically independent variables W_g , $0 \leq W_g \leq 1$, have moments of the form (3.1), $g = 1,2,\ldots,G$. Let f_g , ρ_g , ϕ_g , and $\delta_g = \sum_{i=1}^{I_g} x_{gi} = \sum_{j=1}^{J_g} y_{gj}$ be the constants in the Box expansion (1.3) of the c.d.f. of W_g , $g = 1,2,\ldots,G$. Finally, let

$$(3.7) \qquad W = \prod_{g=1}^{G} W_g \quad .$$

Then

$$(3.8) \qquad P\{-2 \log W \leq t\} = (1 - \phi)P\{\chi_f^2 \leq \rho t\} + \phi P\{\chi_{f+4}^2 \leq \rho t\} + R(\delta) \quad ,$$

where $\delta = \sum_{g=1}^{G} \delta_g$, $f = \sum_{g=1}^{G} f_g$, $\rho = \frac{1}{f}\sum_{g=1}^{G} f_g\rho_g$, and

$$\phi = \frac{1}{\rho^2}\sum_{g=1}^{G} \rho_g^2\phi_g + \frac{1}{4\rho^2 f}\sum_{g<h} f_g f_h(\rho_g - \rho_h)^2 \quad .$$

Two applications of Theorem 1 to multivariate testing problems are given in Section 4. Note that if in the Box expansions for $W_1, W_2, \ldots, W_G$, each $R_g(\delta_g)$ is $O(\delta_g^{-3})$ as $\delta_g \to \infty$, and if $\delta_1, \delta_2, \ldots, \delta_G$ are all asymptotically of the same order of magnitude (i.e., $\underline{\lim}\, \delta_g \delta_h^{-1} > 0$ as $\delta_g, \delta_h \to \infty$ for all $g \neq h$), then $R(\delta)$ in (3.8) is $O(\delta^{-3})$. In practical use of Theorem 1, the δ_g's will usually be asymptotically of the same order of magnitude. If the δ_g's are not asymptotically of the same order of magnitude, Equation (3.8) is formally correct (when δ, f, ρ, and Φ are defined as in Theorem 1), but the order of magnitude of the remainder term $R(\delta)$ in δ must be separately investigated.

4. Applications to Multivariate Hypothesis Testing Problems

Suppose we are interested in testing whether either the mean vectors and/or covariance matrices of k multivariate normal populations are identical. Suppose that an observation (p dimensional row vector) $x^{(i)}$ from the i-th population has a p-variate normal distribution with mean vector $\mu^{(i)}$ and covariance matrix $\Sigma^{(i)}$, $i = 1,2,\ldots,k$. Let $x^{(i)}$ be partitioned as $(x_1^{(i)}, x_2^{(i)})$, where $x_1^{(i)}$ is $1 \times q$, and let

$$(4.1) \qquad \mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}) \quad , \quad \Sigma^{(i)} = \begin{pmatrix} \Sigma_{11}^{(i)} & \Sigma_{12}^{(i)} \\ \Sigma_{21}^{(i)} & \Sigma_{22}^{(i)} \end{pmatrix} ,$$

be correspondingly partitioned, $i = 1,2,\ldots,k$. Suppose that we observe N_i observations from the i-th population, $i = 1,2,\ldots,k$.

We consider two tests of hypotheses. The first test compares the null hypothesis

$$(4.2) \qquad H_{mvc}: \mu^{(1)} = \mu^{(2)} = \cdots = \mu^{(k)} \quad , \quad \Sigma^{(1)} = \Sigma^{(2)} = \cdots = \Sigma^{(k)} \quad ,$$

against general alternatives. In the second test, we compare the null hypothesis H_{mvc} against the alternative:

$$(4.3) \qquad H_{m'vc'}: \mu_1^{(1)} = \mu_1^{(2)} = \cdots = \mu_1^{(k)} \quad , \quad \Sigma_{11}^{(1)} = \Sigma_{11}^{(2)} = \cdots = \Sigma_{11}^{(k)} \quad .$$

Let $\bar{x}^{(i)} = (\bar{x}_1^{(i)}, \bar{x}_2^{(i)})$ be the sample mean vector and let

$$V^{(i)} = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{pmatrix}$$

be the sample cross-product matrix from the i -th population, $i = 1,2,\ldots,k$. Then $(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(k)},V^{(1)},V^{(2)},\ldots,V^{(k)})$ is a sufficient statistic for both hypothesis testing problems. Let $N = \sum_{i=1}^{k} N_i$,

$$\bar{\bar{x}} = (\bar{\bar{x}}_1, \bar{\bar{x}}_2) = \frac{1}{N} \sum_{i=1}^{k} N_i (\bar{x}_1^{(i)}, \bar{x}_2^{(i)}) \quad ,$$

and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \sum_{i=1}^{k} N_i (\bar{x}^{(i)} - \bar{\bar{x}})' (\bar{x}^{(i)} - \bar{\bar{x}}) \quad .$$

4.1 Test of H_{mvc} Versus General Alternatives

Anderson (1958) suggests testing H_{mvc} against general alternatives by means of the test statistic

$$(4.4) \qquad W = \left(\frac{\prod_{i=1}^{k} \left| \frac{1}{n_i} V^{(i)} \right|^{\frac{1}{2}n_i}}{\left| \frac{1}{n} \sum_{i=1}^{k} V^{(i)} \right|^{\frac{1}{2}n}} \right) \left(\frac{\left| \frac{1}{n} \sum_{i=1}^{k} V^{(i)} \right|}{\left| \frac{1}{n} \left(\sum_{i=1}^{k} V^{(i)} + A \right) \right|} \right)^{\frac{1}{2}n} \equiv W_1 W_2 ,$$

where $n_i = N_i - 1$, $i = 1,2,\ldots,k$, and $n = \sum_{i=1}^{k} n_i$. The statistic W_1 is the likelihood ratio test statistic for testing $\Sigma^{(1)} = \Sigma^{(2)} = \ldots = \Sigma^{(k)}$ against general alternatives, but modified [along lines suggested by Bartlett (1937)] by everywhere replacing the sample sizes N_i by the degrees of freedom, n_i, of $V^{(i)}$. The statistic W_2 is the (n/N) -th power of the likelihood ratio test statistic for MANOVA.

As Anderson (1958) shows, the statistics W_1 and W_2 are independent when H_{mvc} holds, and also

$$(4.5) \qquad E(W_1)^h = K_1 \frac{\prod_{i=1}^{k} \prod_{s=1}^{p} \Gamma\left(\frac{1}{2} n_i(1 + h) + \frac{1}{2}(1 - s) \right)}{\prod_{t=1}^{p} \Gamma\left(\frac{1}{2} n(1 + h) + \frac{1}{2}(1 - t) \right)} .$$

Thus, the moments of W_1 under H_{mvc} are of the form (1.1), and we can use the Box expansion (1.3) for the c.d.f. of W when $N_1, N_2, \ldots, N_k$ are large and of the same order of magnitude. Although the constants f_1, ρ_1, and ϕ_1 in the expansion (1.3) of the c.d.f. of W_1 can be obtained directly from (4.5) by use of the methods of Section 2, Anderson (1958; p. 255) has already shown that

$$f_1 = \frac{1}{2}(k-1)p(p+1) \quad ,$$

$$\rho_1 = 1 - \left(\sum_{i=1}^{k} \frac{1}{n_i} - \frac{1}{n} \right) \frac{2p^2 + 3p - 1}{6(p+1)(k-1)} \quad ,$$

(4.6)

$$\phi_1 = \frac{p(p+1)[(p-1)(p+2)(\sum_{i=1}^{k} (\frac{1}{n_i})^2 - (\frac{1}{n})^2) - 6(k-1)(1-\rho_1)^2]}{48\rho_1^2} \quad .$$

[In comparing (4.6) with Anderson's results it should be noted that his q is our k .]

Anderson (1958; p. 207) gives the h -th moments of $\lambda = (W_2)^{N/n}$ under H_{mvc} as

$$(4.7) \qquad E(\lambda)^h = K_2 \frac{\prod_{s=1}^{p} \Gamma(\frac{1}{2} N(1+h) - \frac{1}{2}(k-1) - \frac{1}{2} s)}{\prod_{t=1}^{p} \Gamma(\frac{1}{2} N(1+h) - \frac{1}{2} t)} \quad ,$$

where (4.7) holds for all real h for which the gamma functions exist. Hence,

$$(4.8) \qquad E(W_2)^h = K_2 \frac{\prod_{s=1}^{p} \Gamma(\frac{1}{2} n(1+h) + \frac{1}{2} - \frac{1}{2} s)}{\prod_{t=1}^{p} \Gamma(\frac{1}{2} n(1+h) + \frac{k}{2} - \frac{1}{2} t)} \quad .$$

Thus, applying (2.9), (2.10), (2.11), and (2.12), with $L = p$, $z = n/2$, $a = 1/2$, $b = -(1/2)$, $c = k/2$, $d = -(1/2)$, we find that

$$P\{-2 \log W_2 \le t\} = (1 - \phi_2) P\{\chi^2_{f_2} \le \rho_2 t\} + \phi_2 P\{\chi^2_{f_2+4} \le \rho_2 t\} + O(n^{-3}) \quad ,$$

where

$$f_2 = p(k - 1) \quad ,$$

(4.9)
$$\rho_2 = 1 - \frac{p - k + 2}{2n} \quad ,$$

$$\phi_2 = \frac{p(k-1)[p^2 + (k-1)^2 - 5]}{48n^2(\rho_2)^2} \quad .$$

To obtain an asymptotic expansion for the c.d.f. of the test statistic W under H_{mvc} , Anderson (1958; p. 255) goes back to the h -th moments of W and applies the Box expansion method ab initio.

However, we already have the constants f_1 , f_2 , ρ_1 , ρ_2 , ϕ_1 , and ϕ_2 from the asymptotic expansions of the c.d.f.'s of W_1 and W_2 . Using Theorem 1 of Section 3, we thus conclude that the constants f , ρ , and ϕ in the Box expansion (1.3) of the c.d.f. of W are given by

(4.10)
$$f = f_1 + f_2 = \frac{1}{2}(k - 1)p(p + 3) \quad ,$$

(4.11)
$$\rho = \frac{f_1\rho_1 + f_2\rho_2}{f} = 1 - \left(\sum_{i=1}^{k} \frac{1}{n_i} - \frac{1}{n} \right) \frac{2p^2 + 3p - 1}{6(p + 3)(k - 1)} - \frac{p - k + 2}{n(p + 3)} \quad ,$$

and

$$\phi = \frac{\rho_1^2\phi_1 + \rho_2^2\phi_2}{\rho^2} + \frac{f_1 f_2}{4\rho^2 f}(\rho_1 - \rho_2)^2$$

$$(4.12) = \frac{p}{288\rho^2}[6(\sum_{i=1}^{k}\frac{1}{(n_i)^2} - \frac{1}{n^2})(p^2 - 1)(p + 2) - 36(p + 3)(k - 1)(1 - \rho)^2$$

$$- \frac{12(k - 1)}{n^2}(-2k^2 + 7k + 3pk - 2p^2 - 6p - 4)] \quad .$$

4.2 Test of H_{mvc} Versus $H_{m'vc'}$

Gleser and Olkin (1972) show that the likelihood ratio test statistic $\lambda_{mvc,m'vc'}$ for testing H_{mvc} against the alternative $H_{m'vc'}$ is

$$(4.13) \qquad \lambda_{mvc,m'vc'} = \frac{(\prod_{i=1}^{k} |\frac{1}{N_i} V_{22\cdot 1}^{(i)}|^{\frac{1}{2}N_i})(|\frac{1}{N}(\sum_{i=1}^{k} V_{11}^{(i)} + A_{11})|^{\frac{1}{2}N}}{|\frac{1}{N}(\sum_{i=1}^{k} V^{(i)} + A)|^{\frac{1}{2}N}} \quad ,$$

where

$$(4.14) \qquad V_{22\cdot 1}^{(i)} = V_{22}^{(i)} - V_{21}^{(i)}(V_{11}^{(i)})^{-1}V_{12}^{(i)} \quad , \quad i = 1,2,\ldots,k \quad .$$

However, instead of $\lambda_{mvc,m'vc'}$, let us modify the statistic by replacing N_i by $n_i = N_i - 1$, $i = 1,2,\ldots,k$, and N by $n = \sum_{i=1}^{k} n_i$, everywhere in (4.13). [There is more than one way to modify the likelihood ratio test statistic along the lines suggested by Bartlett (1937). One way is given here; another, and possibly preferable, way is considered in Gleser and Olkin (1972).] The resulting statistic is

$$(4.15)\qquad U_2 = \frac{\left(\prod_{i=1}^{k} \left|\frac{1}{n_i} V_{22\cdot 1}^{(i)}\right|^{\frac{1}{2}n_i}\right) \left|\frac{1}{n}\left(\sum_{i=1}^{k} V_{11}^{(i)} + A_{11}\right)\right|^{\frac{1}{2}n}}{\left|\frac{1}{n}\left(\sum_{i=1}^{k} V^{(i)} + A\right)\right|^{\frac{1}{2}n}} .$$

To obtain the c.d.f. of U_2 let U_1 be a similar modification of the likelihood ratio test statistic for testing hypothesis $H_{m'vc'}$ against general alternatives. Gleser and Olkin (1972) have derived the likelihood ratio test statistic. From their result, we find that

$$(4.16)\qquad U_1 = \frac{\prod_{i=1}^{k} \left|\frac{1}{n_i} V_{11}^{(i)}\right|^{\frac{1}{2}n_i}}{\left|\frac{1}{n}\left(\sum_{i=1}^{k} V_{11}^{(i)} + A_{11}\right)\right|^{\frac{1}{2}n}} .$$

Comparing the statistic W defined in (4.4) with U_1U_2, and recalling that $|V^{(i)}| = |V_{11}^{(i)}||V_{22\cdot 1}^{(i)}|$ for $i = 1,2,\ldots,k$, we see that

$$(4.17)\qquad W = U_1U_2 .$$

Since under $H_{m'vc'}$ the statistics $\bar{x}_2^{(i)} - \bar{x}_1^{(i)}(V_{11}^{(i)})^{-1}V_{12}^{(i)}$, $(V_{11}^{(i)})^{-1}V_{12}^{(i)}$, $V_{22\cdot 1}^{(i)}$, $i = 1,2,\ldots,k$, $\bar{\bar{x}}_1$, and $(\sum_{i=1}^{k} V_{11}^{(i)} + A_{11})$ are complete and sufficient, and since the distribution of U_1 is the same for all values of the parameters $\mu^{(1)},\mu^{(2)},\ldots,\mu^{(k)}$, $\Sigma^{(1)},\Sigma^{(2)},\ldots,\Sigma^{(k)}$ obeying $H_{m'vc'}$, it follows from a theorem of Basu (1955) that U_1 and U_2 are statistically independent under $H_{m'vc'}$ (and thus under H_{mvc}).

Note that U_1 has the same form as W, except that U_1 is a function only of $x^{(i)}(j)$, $j = 1,2,\ldots,N_i$; $i = 1,2,\ldots,k$. That is, U_1 is a q-dimensional version of W. The moments of W under H_{mvc} are known [they equal the product of (4.5) and (4.8)]. The moments of U_1 under H_{mvc} can be obtained from the formula for the moments of W by everywhere replacing p by q. The moments of both U_1 and W under H_{mvc} are thus of the form (1.1). Since U_1 and U_2 are independent under H_{mvc}, and since $W = U_1U_2$, the h-th moment of U_2 under H_{mvc} equals the h-th moment of W under H_{mvc} divided by the h-th moment of U_1 under H_{mvc}. Thus the moments of U_2 under H_{mvc} have the form (1.1).

From the preceding discussion and the results of Sections 1-3 (particularly Theorem 1), it follows that the c.d.f.'s of U_1, U_2, and W all have asymptotic expansions of the form (1.3). Let f, ρ, and ϕ be the coefficients in the expansion (1.3) for the c.d.f. of W; these constants are given by Equations (4.10), (4.11), and (4.12), respectively. Let f_g^*, ρ_g^*, and ϕ_g^* be the coefficients in the expansion for the c.d.f. of U_g, $g = 1,2$. Since U_1 is a q-dimensional version of W, the coefficients f_1^*, ρ_1^*, and ϕ_1^* can be obtained by substituting q for p in the formulas (4.10), (4.11), and (4.12) respectively. Finally, from Theorem 1 we know that

$$f = f_1^* + f_2^* \quad , \qquad \rho = \frac{f_1^*\rho_1^* + f_2^*\rho_2^*}{f} \quad ,$$

(4.18)

$$\phi = \frac{(\rho_1^*)^2\phi_1^* + (\rho_2^*)^2\phi_2^*}{\rho^2} + \frac{f_1^*f_2^*}{4f}(\rho_1^* - \rho_2^*)^2 \quad .$$

Solving for f_2^* , ρ_2^* , and ϕ_2^* in (4.18) yields

(4.19) $$f_2^* = f - f_1^* = \frac{1}{2}(k-1)(p-q)(p+q+3) \quad ,$$

(4.20) $$\rho_2^* = \frac{f\rho - f_1^*\rho_1^*}{f_2^*} = 1 - \left[\sum_{i=1}^{k} \frac{1}{n_i} - \frac{1}{n}\right]\left[\frac{2p^2 + 2pq + 2q^2 + 3p + 3q - 1}{6(p+q+3)(k-1)}\right] - \left[\frac{p+q-k+2}{n(p+q+3)}\right] \quad ,$$

(4.21) $$\phi_2^* = \frac{\rho^2\phi - (\rho_1^*)^2\phi_1^*}{(\rho^*)^2} - \frac{f_1^* f_2^*}{4f}\frac{(\rho_1^* - \rho_2^*)^2}{(\rho_2^*)^2}$$
$$= \frac{1}{288(\rho_2^*)^2}\{6[\sum_{i=1}^{k}\frac{1}{(n_i)^2} - \frac{1}{n^2}][(p^2-1)(p^2+2p) - (q^2-1)(q^2+2q)]$$
$$- [\frac{(12)(k-1)(p-q)}{n^2}][3(p+q)(k-2) - 2(p^2+pq+q^2)$$
$$- 2k^2 + 7k - 4] - 72f_2^*(1-\rho_2^*)^2\} \quad .$$

We conclude that

(4.22) $$P\{-2\log U_2 \le t\} = (1-\phi_2^*)P\{\chi^2_{f_2^*} \le \rho_2^* t\} + \phi_2^* P\{\chi^2_{f_2^*+4} \le \rho_2^* t\} + O(n^{-3}) \quad ,$$

where f_2^* , ρ_2^* , and ϕ_2^* are given by (4.19), (4.20), and (4.21) respectively.

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