DOCUMENT RESUME

ED 068 502 TM 001 839

AUTHOR	Kristof, 'alter
TITLE	An Extension of the Reliability Concept to Vector Variables.
INSTITUTION	Educational Testing Service, Princeton, N.J.
SPONS AGENCY	National Science Foundation, Washington, D.C.
REPORT NO	RB-72-7
PUB DATE	Feb 72
NOTE	16p.
EDRS PRICE	MF-$0.65 HC-$3.29
DESCRIPTORS	Bulletins; Factor Analysis; *Mathematical Applications; *Measurement Techniques; Research Methodology; *Test Interpretation; *Test Reliability
IDENTIFIERS	Vector Variables

ABSTRACT

The coefficient of precision type of reliability originally defined for scalar variables is generalized to vector variables and named vector reliability. The new coefficient ranges from zero to one. Additional basic properties of vector reliability are derived. Vector information is defined as a simple function of vector reliability. A number of properties of vector information are demonstrated in order to justify its use as a measure of information contained in a vector variable. Two applications are appended by way of illustration. (Author)

RB-72-7

RESEARCH BULLETIN

AN EXTENSION OF THE RELIABILITY CONCEPT TO VECTOR VARIABLES

Walter Kristof

This Bulletin is a draft for interoffice circulation. Corrections and suggestions for revision are solicited. The Bulletin should not be cited as a reference without the specific permission of the author. It is automatically superseded upon formal publication of the material.

Educational Testing Service
Princeton, New Jersey
February 1972

AN EXTENSION OF THE RELIABILITY CONCEPT TO VECTOR VARIABLES

Walter Kristof

Abstract

The coefficient of precision type of reliability originally defined for scalar variables is generalized to vector variables and named vector reliability. The new coefficient ranges from zero to one. Additional basic properties of vector reliability are derived. Vector information is defined as a simple function of vector reliability. A number of properties of vector information are demonstrated in order to justify its use as a measure of information contained in a vector variable. Two applications are appended by way of illustration.

AN EXTENSION OF THE RELIABILITY CONCEPT TO VECTOR VARIABLES[1]

1. Definition and Discussion of Vector Reliability

A familiar definition of the reliability ρ_X of a scalar random variable X with true score T and error E, $X = T + E$, is given by the coefficient of precision,

$$\rho_X = \sigma_T^2/\sigma_X^2 \quad . \tag{1}$$

If ρ_X is to express an intrinsic property of variable X and if this variable represents measurement along an interval scale, then ρ_X should be independent of linear transformations of X. This is indeed the case if we introduce the rule that transformation $Y = aX + b$, $a \neq 0$ and b constants, shall imply the true score transformation $U = aT + \text{const.}$ This rule is certainly in agreement with our intuitive notion of true score.

In this note we will first extend the reliability concept to vector variables along the lines indicated recently by Conger and Lipshitz (1971). In this new presentation, the use of the covariance matrix instead of the correlation matrix leads to desirable invariance properties. Reliability is explicitly defined in terms of population characteristics. In addition, an analog to the notion of information in vector variables will be developed and presented.

Let us again employ the classical test theory model. Hence, for vector variables, $\underline{X} = \underline{T} + \underline{E}$. It will be assumed that $\underline{T}$ and $\underline{E}$ are statistically independent. As a generalization of (1) we define the reliability $\rho_{\underline{X}}$ of vector $\underline{X}$ as

$$\rho_{\underline{X}} = \frac{\mathcal{E}(\underline{T} - \underline{\mu}_T)'\underline{\Sigma}_X^{-1}(\underline{T} - \underline{\mu}_T)}{\mathcal{E}(\underline{X} - \underline{\mu}_X)'\underline{\Sigma}_X^{-1}(\underline{X} - \underline{\mu}_X)} \quad , \tag{2}$$

---

[1]Research reported in this paper has been supported by grant GB-18230 from National Science Foundation.

column vectors $\underline{X}$ and $\underline{T}$ having means $\underline{\mu}_{\underline{X}}$ and $\underline{\mu}_{\underline{T}}$, respectively. $\underline{\Sigma}_{\underline{X}}$ is the variance-covariance matrix of $\underline{X}$ and is assumed to be positive definite.

The denominator in (2) is simply the number of components of $\underline{X}$, m say. The numerator equals $\mathrm{tr}\, \underline{\Sigma}_{\underline{T}} \underline{\Sigma}_{\underline{X}}^{-1}$, $\underline{\Sigma}_{\underline{T}}$ being the variance-covariance matrix of $\underline{T}$. Hence (2) can be rewritten as

$$(3) \qquad \rho_{\underline{X}} = \frac{1}{m}\, \mathrm{tr}\, \underline{\Sigma}_{\underline{T}} \underline{\Sigma}_{\underline{X}}^{-1} .$$

Denoting the reliability of the i -th component of $\underline{X}$ by ρ_i and writing $\underline{\Sigma}_{\underline{X}} = \|\sigma_{ij}\|$, $\underline{\Sigma}_{\underline{X}}^{-1} = \|\sigma^{ij}\|$, we have also

$$(4) \qquad \rho_{\underline{X}} = 1 - \frac{1}{m} \sum_{i=1}^{m} \sigma_{ii}\sigma^{ii}(1 - \rho_i)$$

when the components of $\underline{E}$ are uncorrelated.

The following is seen to be true:

(i) If we adopt the transformation rule that a linear transformation $\underline{Y} = \underline{A}\underline{X} + \underline{B}$, $\underline{A}$ nonsingular, implies the transformation of true scores $\underline{U} = \underline{A}\underline{T} + \underline{C}$, $\underline{C}$ any constant vector, then $\rho_{\underline{X}}$ remains unchanged under transformation of $\underline{X}$ as follows from (3). Hence $\rho_{\underline{X}} = \rho_{\underline{Y}}$.

(ii) Independence of true score and error is preserved under such linear transformations of $\underline{X}$. However, initially uncorrelated errors may become correlated.

(iii) We have $0 \leq \rho_{\underline{X}} \leq 1$. For, nonnegativity of $\rho_{\underline{X}}$ follows at once from (2) since $\underline{\Sigma}_{\underline{X}}^{-1}$ is positive definite. Further, there is always a linear transformation of $\underline{X}$ that will lead to uncorrelated errors. At the same time, $\rho_{\underline{X}}$ remains unaltered according to (i). Hence, $\rho_{\underline{X}} \leq 1$ as follows from (4).

(iv) Assume that the components of $\underline{E}$ are uncorrelated and $\rho_i < 1$ for all i. Then vector reliability $\rho_{\underline{X}}$ equals the average component reliability, $\rho_{\underline{X}} = \Sigma\rho_i/m$, if and only if the components of $\underline{X}$ are uncorrelated. If the components of $\underline{X}$ are not all uncorrelated, however, then $\rho_{\underline{X}} < \Sigma\rho_i/m$. Proof:

Let us deal with the first part of the assertion first. We derive from (4) that $\rho_{\underline{X}} = \Sigma\rho_i/m$ is equivalent to

$$(5) \qquad \sum_i (1 - \sigma_{ii}\sigma^{ii})(1 - \rho_i) = 0 \quad .$$

Sufficiency of the condition follows upon noting that (5) is satisfied when $\sigma^{ii} = 1/\sigma_{ii}$. As to necessity, let $\underline{\Sigma}_{\underline{X}}$ have a canonical decomposition $\underline{\Sigma}_{\underline{X}} = \underline{P\Lambda P'}$ where $\underline{P} = \|p_{ij}\|$ is orthogonal and $\underline{\Lambda}$ is positive diagonal with λ_j in the j-th diagonal position. Then

$$\sigma_{ii} = \sum_{j=1}^{m} p_{ij}^2\lambda_j \quad , \qquad \sigma^{ii} = \sum_{j=1}^{m} p_{ij}^2/\lambda_j \quad .$$

The Cauchy-Schwarz inequality yields

$$(6) \qquad \sqrt{\sigma_{ii}\sigma^{ii}} \geq \sum_{j=1}^{m} p_{ij}^2 = 1 \quad ,$$

consequently $1 - \sigma_{ii}\sigma^{ii} \leq 0$ for all i. But $1 - \rho_i > 0$ for all i. In order to satisfy (5) it is necessary that always $1 - \sigma_{ii}\sigma^{ii} = 0$. Therefore $\sigma^{ii} = 1/\sigma_{ii}$ for all i which implies that $\underline{\Sigma}_{\underline{X}}$ is diagonal.[2]

---

[2] An explicit proof of this implication could be written down at once and has been omitted.

As to the second part of the assertion, it now follows from the above Cauchy-Schwarz inequality (6) that there must be some k for which $1 - \sigma_{kk}\sigma^{kk} < 0$. Thus

$$\sum_i (1 - \sigma_{ii}\sigma^{ii})(1 - \rho_i) < 0$$

which is equivalent to $\rho_{\underline{X}} < \Sigma\rho_i/m$. This completes the proof.

Hence we see that $\rho_{\underline{X}}$ will be at most equal to $\Sigma\rho_i/m$.

(v) $\rho_{\underline{X}} = 0$ is equivalent to $\rho_i = 0$ for all i . This fact is easily established by using (3) and considering that $\Sigma_{\underline{T}}$ is positive semidefinite and $\Sigma_{\underline{X}}^{-1}$ is positive definite.

(vi) $\rho_{\underline{X}} = 1$ is equivalent to $\rho_i = 1$ for all i provided that $\Sigma_{\underline{X}}$ remains positive definite. For, $\rho_i = 1$ for all i implies $\Sigma_{\underline{T}} = \Sigma_{\underline{X}}$, hence $\rho_{\underline{X}} = 1$ by (3). On the other hand, there is a linear transformation of $\underline{X}$ to $\underline{Y}$ that will lead to uncorrelated errors in $\underline{Y}$ and leave $\rho_{\underline{X}} = \rho_{\underline{Y}}$ according to (i). Denoting parameters which refer to $\underline{Y}$ by superscript $(\underline{Y})$ we have from (4)

$$\sum_i \sigma_{ii}^{(\underline{Y})}\sigma^{ii(\underline{Y})}(1 - \rho_i^{(\underline{Y})}) = 0 \quad .$$

Because $\sigma_{ii}^{(\underline{Y})}\sigma^{ii(\underline{Y})} > 0$ we conclude that $\rho_i^{(\underline{Y})} = 1$ for all i . Hence all components of $\underline{Y}$ are error-free. The same must then apply to components of $\underline{X}$ because these are linear combinations of the former. Therefore $\rho_i = 1$ for all i .

(vii) Statement (vi) depends crucially on $\Sigma_{\underline{X}}$ remaining positive definite. This becomes evident when we consider the following case. Let the components of $\underline{X}$ represent parallel forms of the same test with common reliability $\rho < 1$.

Assume that errors are uncorrelated. $\underline{\Sigma}_{\underline{X}}$ is positive semidefinite. We find that

$$\sigma_{ii}\sigma^{ii} = [1 + (m - 2)\rho]/[1 + (m - 2)\rho - (m - 1)\rho^2]$$

and obtain from (4) the result

$$\rho_{\underline{X}} = \rho/[1 + (m - 1)\rho] \quad . \tag{7}$$

The derivation of this formula breaks down when $\rho = 1$, i.e., when $\underline{\Sigma}_{\underline{X}}$ is not positive definite. However, we can still obtain

$$\lim_{\rho \to 1} \rho_{\underline{X}} = 1/m \quad .$$

On the other hand, if the components of $\underline{X}$ do not represent parallel (but possibly nearly parallel) forms and $\underline{\Sigma}_{\underline{X}}$ stays positive definite then $\rho_i \to 1$ implies $\rho_{\underline{X}} = 1$ according to (vi).

(viii) There is a certain relationship between $\rho_{\underline{X}}$ and the maximal reliability of a linear composite of the components of $\underline{X}$. This maximal reliability is given by the largest eigenvalue of $\underline{\Sigma}_{\underline{T}}\underline{\Sigma}_{\underline{X}}^{-1}$. For a derivation and additional references see Lord and Novick, 1968, p. 123. In contrast, $\rho_{\underline{X}}$ is the average of <u>all</u> eigenvalues of $\underline{\Sigma}_{\underline{T}}\underline{\Sigma}_{\underline{X}}^{-1}$ as follows from (3).

This concludes our enumeration of basic properties of $\rho_{\underline{X}}$.

2. Definition and Discussion of Vector Information

The reliability $\rho = \sigma_T^2/\sigma_X^2$ of a single variable X can be viewed as a measure of the amount of "true information" contained in a typical observed value of X . The scaling is such that $0 \leq \rho \leq 1$. It appears that this concept can be naturally extended to vector variables $\underline{X}$ with $m > 1$ components. The quantity $m\rho_{\underline{X}}$ exhibits features that we would require such a generalized measure to possess. Let us restrict ourselves to the nondegenerate case when $\Sigma_{\underline{X}}$ is positive definite. We find the following.

(i') $m\rho_{\underline{X}}$ remains unchanged under nonsingular linear transformations of $\underline{X}$ when the transformation rule given in (i) is adopted.

(ii') $0 \leq m\rho_{\underline{X}} \leq m$ according to (iii).

(iii') $m\rho_{\underline{X}} = 0$ precisely when $\underline{T}$ is a constant as follows from (v). Hence $\underline{X}$ may be regarded as containing only error.

(iv') $m\rho_{\underline{X}} = m$ precisely when $\underline{E}$ is a constant as follows from (vi). Hence $\underline{X}$ may be regarded as containing no error.

(v') Let the $(m_1 + m_2)$ -component vector $\underline{X}$ be partitioned into subvectors $\underline{X}_1$ and $\underline{X}_2$ with m_1 and m_2 components, respectively, $m = m_1 + m_2$. If, for general m, the quantity $m\rho_{\underline{X}}$ is to be interpreted in the sense of a measure of "information" contained in the m -component vector $\underline{X}$, then we should require the property of subadditivity,

$$(m_1 + m_2)\rho_{\underline{X}} \leq m_1\rho_{\underline{X}_1} + m_2\rho_{\underline{X}_2} \quad ,$$

when errors are not correlated across $\underline{X}_1$ and $\underline{X}_2$. The equality sign should be expected to hold when, in addition, $\underline{X}_1$ and $\underline{X}_2$ are independent of each other, i.e., when these subvectors supply independent information. We will prove that these requirements are indeed satisfied. Without loss of generality we suppose that $m_1 \geq m_2$.

Let us perform two separate linear transformations, $\underline{X}_1 \rightarrow \underline{Y}_1$ and $\underline{X}_2 \rightarrow \underline{Y}_2$, such that the partitioned variance-covariance matrix of $\underline{Y}$, where $\underline{Y}$ results from the combined transformation $\underline{X} \rightarrow \underline{Y}$, takes on the form

$$\underline{\Sigma}_{\underline{Y}} = \left[\begin{array}{c:c} \underline{I}_1 & \underline{\Gamma} \\ \hdashline \underline{\Gamma}' & \underline{I}_2 \end{array}\right] .$$

Here $\underline{I}_1$ and $\underline{I}_2$ are identity matrices of orders m_1 and m_2, respectively. Matrix $\underline{\Gamma}$ is of order $m_1 \times m_2$ and contains in its truncated diagonal the canonical correlations γ_i between $\underline{X}_1$ and $\underline{X}_2$, $i = 1,\ldots,m_2$ indicating the position, and zeroes elsewhere.

The above transformation $\underline{X} \rightarrow \underline{Y}$ carries at the same time the original true score vector $\underline{T}$ in $\underline{X} = \underline{T} + \underline{E}$ into a new true score vector $\underline{U}$ composed of subvectors $\underline{U}_1$ and $\underline{U}_2$ corresponding to $\underline{Y}_1$ and $\underline{Y}_2$. The variance-covariance matrix of $\underline{U}$ may then be partitioned accordingly,

$$\underline{\Sigma}_{\underline{U}} = \left[\begin{array}{c:c} \underline{A} & \underline{C} \\ \hdashline \underline{C}' & \underline{B} \end{array}\right]$$

with $\underline{A} = \|a_{ij}\|$, $\underline{B} = \|b_{ij}\|$, $\underline{C} = \|c_{ij}\|$. Let us partition $\underline{\Sigma}_{\underline{Y}}^{-1}$ similarly,

$$\underline{\Sigma}_{\underline{Y}}^{-1} = \left[\begin{array}{c:c} \underline{F} & \underline{G} \\ \hdashline \underline{G}' & \underline{H} \end{array}\right] .$$

Taking the special form of $\underline{\Sigma}_{\underline{Y}}$ into account we find that $\underline{F} = \|f_{ij}\|$ and $\underline{H} = \|h_{ij}\|$ are diagonal matrices with $f_{ii} = (1 - \gamma_i^2)^{-1}$ for $i \leq m_2$ and $f_{ii} = 1$ for $m_2 < i \leq m_1$, $h_{ii} = (1 - \gamma_i^2)^{-1}$ with $i \leq m_2$. The $m_1 \times m_2$ matrix $\underline{G} = \|g_{ij}\|$ contains in its truncated diagonal $g_{ii} = -\gamma_i(1 - \gamma_i^2)^{-1}$ and zeroes elsewhere.

Invoking (i') we obtain

$$(8) \qquad (m_1 + m_2)\rho_{\underline{X}} = (m_1 + m_2)\rho_{\underline{Y}} = \operatorname{tr} \underline{\Sigma}_U \underline{\Sigma}_Y^{-1}$$

$$= \operatorname{tr} \underline{AF} + 2 \operatorname{tr} \underline{CG}' + \operatorname{tr} \underline{BH} \quad .$$

The required traces are easy to evaluate due to the initial transformation $\underline{X} \to \underline{Y}$. We get

$$(9) \qquad \operatorname{tr} \underline{AF} = \sum_{i=1}^{m_1} a_{ii} f_{ii} = \sum_{i=1}^{m_1} a_{ii} + \sum_{i=1}^{m_2} \frac{\gamma_i^2}{1 - \gamma_i^2} a_{ii}$$

$$= m_1 \rho_{\underline{X}_1} + \sum_{i=1}^{m_2} \frac{\gamma_i^2}{1 - \gamma_i^2} \rho'_{1i} \quad .$$

Here we have used the fact that $\sum_{i=1}^{m_1} a_{ii} = \operatorname{tr} \underline{AI}_1$ equals $m_1 \rho_{\underline{Y}_1}$ and hence also $m_1 \rho_{\underline{X}_1}$ due to (i'). By $\rho'_{1i} = a_{ii}$ we designate the reliability of the i -th component of $\underline{Y}_1$. In complete analogy,

$$(10) \qquad \operatorname{tr} \underline{BH} = m_2 \rho_{\underline{X}_2} + \sum_{i=1}^{m_2} \frac{\gamma_i^2}{1 - \gamma_i^2} \rho'_{2i}$$

where ρ'_{2i} denotes the reliability of the i -th component of $\underline{Y}_2$. Finally,

$$(11) \qquad 2 \operatorname{tr} \underline{CG}' = -2 \sum_{i=1}^{m_2} \frac{\gamma_i}{1 - \gamma_i^2} c_{ii} = -2 \sum_{i=1}^{m_2} \frac{\gamma_i^2}{1 - \gamma_i^2} \quad .$$

In taking this step we assume that errors across $\underline{X}_1$ and $\underline{X}_2$ and so across $\underline{Y}_1$ and $\underline{Y}_2$ are uncorrelated, thus $c_{ii} = \gamma_i$. Insertion in (8) yields now

(12) $$(m_1 + m_2)\rho_{\underline{X}} = m_1\rho_{\underline{X}_1} + m_2\rho_{\underline{X}_2} - \sum_{i=1}^{m_2} \frac{\gamma_i^2}{1 - \gamma_i^2}(2 - \rho'_{1i} - \rho'_{2i}) \quad .$$

Therefore we have indeed subadditivity,

(13) $$(m_1 + m_2)\rho_{\underline{X}} \leq m_1\rho_{\underline{X}_1} + m_2\rho_{\underline{X}_2} \quad .$$

We will generally expect that $2 - \rho'_{1i} - \rho'_{2i} > 0$. Hence the equality sign in (13) will hold precisely when all canonical correlations γ_i are zero, i.e., when all correlations between components of $\underline{X}_1$ and components of $\underline{X}_2$ are zero. This completes the proof of (v').

(vi') Let the $(m_1 + m_2)$ -component vector $\underline{X}$ be partitioned as in (v'). If, for general $m = m_1 + m_2$, the quantity $m\rho_{\underline{X}}$ is to be interpreted in the sense of a measure of "information" contained in $\underline{X}$, then we should require the property

$$(m_1 + m_2)\rho_{\underline{X}} \geq \max\,(m_1\rho_{\underline{X}_1}, m_2\rho_{\underline{X}_2}) \quad .$$

We will prove this property when errors are not correlated across $\underline{X}_1$ and $\underline{X}_2$.

Upon making the same transformation of $\underline{X}_1$ and $\underline{X}_2$ as in (v') and using

(14) $$m\rho_{\underline{X}_2} = \mathrm{tr}\ \underline{\underline{BI}}_2 = \sum_{i=1}^{m_2} \rho'_{2i}$$

we obtain from (12) the difference δ ,

(15) $$\delta = (m_1 + m_2)\rho_{\underline{X}} - m_1\rho_{\underline{X}_1}$$

$$= \sum_{i=1}^{m_2} \rho'_{2i} - \sum_{i=1}^{m_2} \frac{\gamma_i^2}{1 - \gamma_i^2}(2 - \rho'_{1i} - \rho'_{2i}) \quad .$$

Since $\gamma_i^2 \leq \rho'_{1i}\rho'_{2i}$ we have further

$$(16) \qquad \delta \geq \sum_{i=1}^{m_2} \rho'_{2i} - \sum_{i=1}^{m_2} \frac{\rho'_{1i}\rho'_{2i}}{1 - \rho'_{1i}\rho'_{2i}} (2 - \rho'_{1i} - \rho'_{2i}) \quad .$$

For $0 \leq \rho'_{1i} \leq 1$ the expression

$$\frac{\rho'_{1i}\rho'_{2i}}{1 - \rho'_{1i}\rho'_{2i}} (2 - \rho'_{1i} - \rho'_{2i})$$

is a nondecreasing nonnegative function of ρ'_{1i} and tends to ρ'_{2i} when $\rho'_{1i} \to 1$. Therefore

$$(17) \qquad \delta \geq \sum_{i=1}^{m_2} \rho'_{2i} - \sum_{i=1}^{m_2} \frac{\rho'_{2i}}{1 - \rho'_{2i}} (1 - \rho'_{2i}) = 0 \quad ,$$

i.e., $(m_1 + m_2)\rho_{\underline{X}} \geq m_1\rho_{\underline{X}_1}$. An analogous derivation holds when the roles of m_1 and m_2 are interchanged. Consequently,

$$(18) \qquad (m_1 + m_2)\rho_{\underline{X}} \geq \max\,(m_1\rho_{\underline{X}_1}, m_2\rho_{\underline{X}_2})$$

as required. This completes the proof of (vi').

Properties (i') - (vi') will sufficiently justify our use of $m\rho_{\underline{X}} = \mathrm{tr}\, \underline{\Sigma}_T\underline{\Sigma}_X^{-1}$ as a measure of information contained in $\underline{X}$.

3. Illustrative Examples

Example 1: The following application[3] uses real data and will serve to illustrate the meaning of $\rho_{\underline{X}}$ and $m\rho_{\underline{X}}$. It is based on Wechsler's (1958, pp. 100-103) publication of data on the WAIS for the age group 25-34 years.

---

[3]This example is different from the one used by Conger and Lipshitz (1971).

The WAIS is composed of six verbal tests and five performance tests. The examiner will typically attempt to interpret differences in a subject's performance on the two subbatteries. We may therefore ask, for example, to what extent the two subbatteries do furnish independent information.

The WAIS contains the following subtests: 1. Information, 2. Comprehension, 3. Arithmetic, 4. Similarities, 5. Digit Span, 6. Vocabulary, 7. Digit Symbol, 8. Picture Completion, 9. Block Design, 10. Picture Arrangement, 11. Object Assembly. However, the performance test Digit Span will not be included in the analysis because the reliability of this test is not reported in Wechsler's (1958, p. 103) table for the age group considered. Hence we will be dealing with a verbal battery of six tests (tests 1-6) and a performance battery of four tests (tests 8-11) only.

The following table contains the intercorrelations of the tests, the diagonal gives the reliabilities. The first two digits after the decimal point are listed.

Test	1	2	3	4	5	6	8	9	10	11
1	91	70	66	70	53	81	67	58	62	45
2		77	49	62	40	73	56	49	57	43
3			81	55	49	59	50	51	49	37
4				85	46	74	56	52	52	39
5					66	51	39	39	47	30
6						95	61	53	62	43
8							85	62	57	54
9								83	58	61
10									60	52
11										68

The entire table can be taken as representing $\underline{\Sigma}_{\underline{T}}$ with $m = m_1 + m_2 = 10$, $m_1 = 6$, $m_2 = 4$. Matrix $\underline{\Sigma}_{\underline{X}}$ results from substituting ones for the reliabilities in the diagonal. Matrices $\underline{\Sigma}_{\underline{X}_1}$, $\underline{\Sigma}_{\underline{X}_2}$, $\underline{\Sigma}_{\underline{T}_1}$ and $\underline{\Sigma}_{\underline{T}_2}$ are obtained as portions.

We will assume independence of errors of measurement across tests. Upon inversion of $\underline{\Sigma}_{\underline{X}}$, $\underline{\Sigma}_{\underline{X}_1}$ and $\underline{\Sigma}_{\underline{X}_2}$ we use (4) and get the following vector reliabilities: $\rho_{\underline{X}} = .5445$, $\rho_{\underline{X}_1} = .6139$ and $\rho_{\underline{X}_2} = .5226$. The corresponding amounts of information become $m\rho_{\underline{X}} = 5.4445$, $m_1\rho_{\underline{X}_1} = 3.6835$ and $m_2\rho_{\underline{X}_2} = 2.0904$.

The difference $m\rho_{\underline{X}} - m_1\rho_{\underline{X}_1}$ is 1.7610. This is the portion of the information represented by the total WAIS battery that is not accounted for by the verbal subbattery and hence is added by the performance subbattery. This added amount is contributed by four tests. So, on the average, each of them contributes less than one half of the theoretically possible amount.

The information contained in the total battery comes from ten tests. Hence, on the average, each of them contributes a little more than one half of the theoretically possible amount. Figures of this sort may help in assessing a battery of tests.

<u>Example 2</u>: Suppose we wish to measure a trait by using a test whose reliability is $\rho < 1$. In order to increase accuracy we administer m parallel forms of the test with uncorrelated errors. We expect that, on the average, the sum of the m measurements will be more accurate than any single measurement. In fact, the Spearman-Brown formula of classical test theory tells us that the reliability of the sum will be $m\rho/[1 + (m - 1)\rho]$ which exceeds ρ. This is, at the same time, the amount of information contained in the sum, the sum being regarded as a new variable.

Using the theory developed in this paper we see that taking the sum is indeed optimal. We are extracting the maximum, namely, the total amount of information contained in m parallel measurements. For, according to (7), this total is $m\rho_{\underline{X}} = m\rho/[1 + (m - 1)\rho]$.

References

Conger, A. J., & Lipshitz, R. Canonical reliability for profiles and test batteries. Chapel Hill, N. C.: The L. L. Thurstone Psychometric Laboratory, University of North Carolina, 1971.

Lord, F. M., & Novick, M. R. Statistical theories of mental test scores. Reading, Mass.: Addison-Wesley, 1968.

Wechsler, D. The measurement and appraisal of adult intelligence. Fourth edition. Baltimore: The Williams & Wilkins Company, 1958.