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AUTHOR	Skakun, Ernest N.; And Others
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AN APPLICATION OF INFERENTIAL STATISTICS TO THE FACTORIAL INVARIANCE PROBLEM

Ernest N. Skakun
Thomas O. Maguire
A. Ralph Hakstian

University of Alberta

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Attention has been drawn to the lack of standards for evaluating the degree of goodness of fit of patterns resulting from a principal components analysis of two data sets. An empirical sampling distribution of the statistic average trace ($\underline{E}'\underline{E}$), as $\underline{E}$ is obtained in the orthogonal procrustes problem, for various orders of A matrices was developed through a Monte Carlo approach. A method is presented which can be used as a guideline in determining whether factor structures obtained from two data sets are congruent.

An Application of Inferential Statistics to the Factorial Invariance Problem

Ernest N. Skakun, Thomas O. Maguire and A. Ralph Hakstian

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The investigation of structure using factor analytic techniques has become a common practice for many educational researchers. In most studies investigators have been content with merely describing the structure of an educational phenomenon, but in the past few years it has become more apparent that inferential statements about structures are needed. For example, an investigator might apply curriculum A to one group of students and curriculum B to another, following which he may wish to make statements not only about possible differences in levels of achievement on various variables, but also about differences in the interrelationships among the variables, ie. differences in the structure of the achievement. Or, as another example, it may be useful to ask if retarded children possess the same degree of differentiated intellectual structure as normal children.

Statistically, the problem can be directly approached by looking at the differences between covariance matrices, using methods described in Anderson (1958) or Morrison (1967). However, there are situations in which significance tests on covariance matrices would be inapplicable. The most obvious situation is one in which the covariance matrix for one of the groups is not known, as for example if one tries to compare an observed structure with one found in the literature. A second situation arises when the investigator is interested in structural differences independent of differences in metric. Here, a test for differences between correlation matrices would be appropriate. Unfortunately no such test exists. Finally the investigator may want to consider only differences in structure as they are reflected in the so-called "common" portion of the test space, and not let differences in uniqueness affect inferential decisions. For these situations, a test for differences in factorial structure would be very useful.

In the psychometric literature, the problem has been seen as one of factorial invariance, and a number of steps have been taken toward its solution. Based on the nature of the data, four cases can be distinguished depending on whether the same or different variables and whether the same or different individuals have been observed in the two data sets. The most common situation, and the one of interest in the present paper is the one in which the same variables and different individuals form the basis of the analysis.

Several approaches have been taken to the problem. In one, of which Tucker (1951) is an example, coefficients of congruence have been defined as:

$$\phi_{pq} = \frac{\sum_j {}_1a_{jp}\, {}_2a_{jq}}{\sqrt{\left(\sum_j {}_1a_{jp}^2\right)\left(\sum_j {}_2a_{jq}^2\right)}}$$

where ϕ_{pq} is the coefficient of congruence between factor P in sample 1 and factor q in sample 2, and a_{jp} is the loading of variable j on factor p.

A second type of approach was used by Kaiser, Hunka and Bianchini (1971) (available in mimeo form in 1960) wherein they embedded one factor space in the other so as to optimally aline the test vectors, then used the cosine of the angles between factors to indicate the similarity between the two factors.

In a third approach, Green (1952), and more recently Cliff (1966) and Schonemann (1966) developed the orthogonal Procrustes procedure which provides an orthogonal transformation of one structure to some best fit of another. Although Cliff and Schonemann worked from different criteria; Cliff maximizing the congruence coefficients and Schonemann minimizing the sum of squared differences between the rotated matrix and the target, their two procedures yield identical results.

In most studies that make use of the orthogonal Procrustes procedure to match structures, the degree of similarity has been assessed using the magnitude of the angles through which one structure would have to be rotated in order to produce a best fit with the other (see for example Taylor and Maguire (1967)). However the structures themselves are generally rotations of principal axis solutions and are in a sense arbitrary. The "goodness" of a solution is most often a configural judgement. That is, "goodness" is measured by result, not procedure. Therefore, if two structures resulting from say principal components-varimax analyses of data for two groups, do not resemble each other initially, it may be possible to rotate one into close approximation with the other and in that case, even though the transformation matrix indicated substantial rotation was necessary, the researcher should conclude that the factor structures are similar.

The problem that still faces the researcher is in answering the inferential question of how dissimilar the matched structures should be before the implicit null hypothesis of no difference between structures in the population, should be rejected. Nessleroade and Baltes (1970), looked at the problem by considering the empirically derived sampling distrubution of congruence coefficients relating two structures based on random data sets, after one structure had been rotated to congruence to the other.

Rather than using congruence coefficients to assess the similarity, the focus of the present study was on the differences that exist between two matrices after rotation. Schonemann adopted the same emphasis in his development of the orthogonal Procrustes procedure. Given a matrix $\underline{A}_1$ (of order $\underline{n}$, variables by $\underline{r}$ components) and a target matrix $\underline{A}_2$, an orthogonal transformation matrix $\underline{T}$ (of order r by r) is applied to $\underline{A}_1$ to rotate it to a configuration as similar to $\underline{A}_2$ to possible.

In matrix form this is expressed by:

$$A_1T = A_2 + E$$

where E is the matrix of differences between the rotated A_1 and A_2. Schonemann's solution provides a T such that the trace of E'E (tr (E'E)) is a minimum. Since tr (E'E) is the criterion for the rotation used in the present study, it was decided to use tr (E'E)/nr (average trace) as the statistic describing the goodness of fit.

In the present study attention was restricted to principal components analysis as it still appears to be the most commonly used "factoring" procedure despite several decades of development of other factor models.

In summary, the purpose of the present investigation was to develop an empirical sampling distribution from one type of match procedure. Components were matched using Schonemann's (1966) procedure, and the average trace was used as the statistic for describing goodness of fit. Based on the null hypothesis that two samples are drawn at random from a population exhibiting a particular component structure, an attempt was made to develop a distribution of the average trace.

Method A population component score matrix F, of order 1000 people by 20 components was produced so that the scores on each component were approximately normally distributed with mean zero and standard deviation of one. All correlations between components were constrained to zero. This matrix of component scores served as the source for the samples used in this study. Various orthogonal component pattern matrices (A) were selected from the literature for use as population matrices. By multiplying Z = FA', we had in Z, the scores of 1000 people on n variables such that if a full component analysis were performed on the implied correlation matrix derived from the scores of $\underline{Z}$, an orthogonal variant of the matrix $\underline{A}$ would be returned. For the remaining part of this section an $\underline{A}$ matrix of order five variables by three components will be used an an example to clarify the procedure.

Two samples of 50 component scores ($\hat{\underline{F}}$) were drawn with replacement from the first three columns of $\underline{F}$. These two samples of order 50 by 3 were designated by $\hat{\underline{F}}_1$ and $\hat{\underline{F}}_2$. For each sample of 50 component scores, a product matrix was formed by $\hat{\underline{Z}} = \hat{\underline{F}}\underline{A}'$. Thus two observation matrices $\hat{\underline{Z}}_1$ and $\hat{\underline{Z}}_2$ of order 50 by 5 were produced. The procedure thus far is based on the idea that two samples of size 50 drawn from $\underline{Z} = \underline{F}\underline{A}'$ (that is the population of variable scores) is the same as drawing two samples of $\underline{F}$ and forming $\hat{\underline{Z}} = \hat{\underline{F}}\underline{A}'$

For each $\underline{Z}$ thus formed, correlations were computed among the five variables to produce a correlation matrix ($\hat{\underline{R}}$) of order 5 by 5. It is important to remember that each $\hat{\underline{R}}$ would be of rank 3. That is, neither psychometric error or uniqueness were added. This phase of the investigation was concerned only with the effects of sampling error on the average trace.

A component pattern matrix $\hat{\underline{A}}$ was calculated by taking all of the components from the $\hat{\underline{R}}$ matrix. Thus from the correlations among the variables of $\hat{\underline{Z}}_1$ an $\hat{\underline{A}}_1$ was calculated and likewise an $\hat{\underline{A}}_2$ from $\hat{\underline{Z}}_2$. Each $\hat{\underline{A}}$ was of order 5 variables by 3 components, and the two $\hat{\underline{A}}$'s represented two samples from the same population. Employing Schonemann's procedure, the two component patterns $\hat{\underline{A}}_1$ and $\hat{\underline{A}}_2$ were matched. Additional component score samples were drawn pairwise and the entire process was repeated until 1000 matches had been performed. The average trace was computed for every match so that in all, 1000 values were obtained. These 1000 values were taken as an empirical sampling distribution given the true null hypothesis that $\hat{\underline{A}}_1$ and $\hat{\underline{A}}_2$ were both estimates of a common $\underline{A}$ matrix. The entire procedure was repeated for 22 different population $\underline{A}$ matrices selected from the literature to reflect variability in order and structure. References for the $\underline{A}$ matrices used may be found in Appendix A.

Results for the Complete Component Analysis

For several of the population matrices, the procedure was repeated using samples of size 100 and 150. In such cases the sampling distribution was made up of only 500 cases.

Frequency distributions of the average trace for the different $\underline{A}$ matrices were obtained, and the 25, 50, 75, 90, 95 and 99 percentile points were determined. In addition the maximum value of the average trace was recorded. The order of the various matrices used and the percentile points for samples of 50, are shown in Table 1. Table 2 shows the effects of increased sample size on ten of the matrices. Frequency polygons were plotted for the 5 x 3, 10 x 3, and 20 x 6 matrices. These are shown in Appendix B. Frequency polygons were also plotted for some of the matrices subjected to increased sample size. These are shown in Appendix C.

Tables 1 and 2 about here

In all of the cases studied, the sampling distribution of the average trace was characterized by a large positive skewness. This was not unexpected given the composition of the statistic. In an effort to reduce the skewness, and make the distribution more symetric, a square root transformation was made on the average trace and the distribution of the transformed variable, the square root trace, determined. The mean, standard deviation, skewness and kurtosis for each transformed sampling distribution are shown in Table 3.

Table 3 about here

The characteristics of sampling distributions of the square root trace were correlated with some of the characteristics of the original pattern matrices (A), in an attempt to isolate those parameters of the A matrices that most influenced the characteristics of the sampling distribution. The parameters selected were number of variables, n, of components, r, number of elements nr , average trace (A'A), variance of the elements in A, variance of the column sum of squares of A, and the variance of the average sum of squares in columns of A. The correlations are shown in Table 4.

Table 4 about here

From Table 4 it can be seen that the number of factors was highly related to both the mean and standard deviation of the sampling distributions generated in the present study.

In an effort to determine the effect of sample size on the square root of the trace statistic, a comparison was made of the means and standard deviations of the sampling distributions of the square root trace as the sample size increased from 50 to 100 to 150. It was found that both the means and standard deviations decreased by the reciprocal of the square root of the ratio of the sample sizes.

After several attempts the means of the empirical sampling distributions were found to be fairly well approximated by:

$$\frac{1}{4}\left(\frac{r}{N}\right)^{1/2}$$

where N is the sample size.

The approximate value for the standard deviation is given by:

$$\frac{1}{(12Nr)^{1/2}}$$

The sampling distribution of the square root trace statistic was approximated by a normal distribution and the 95th and 99th percentile points were estimated using:

$$1.645\,\frac{1}{(12Nr)^{1/2}} + \frac{1}{4}\left(\frac{r}{N}\right)^{1/2}$$

and

$$2.326\,\frac{1}{(12Nr)^{1/2}} + \frac{1}{4}\left(\frac{r}{N}\right)^{1/2}$$

A comparison of the observed and approximated 95th and 99th percentile points is shown in Tables 5 and 6.

Tables 5 and 6 about here

Effects of Incomplete Components Analysis In the present study, the effects of sampling error on the square root trace statistic were investigated. In all cases, a complete components solution was used, so that neither psychometric error nor factorial uniqueness affected the data. A preliminary investigation of the effects of these influences indicates that their effects on the sampling distribution is important. For the two 20 by 6 structures, the effects of an incomplete components analysis was investigated by carrying out the sampling procedure, but using only the first two components of the $\hat{R}$ matrix. The process was then repeated for 3, 4, and 5 components. The same procedure was applied to one of the 10 by 4 structures. In each case 350 matches were carried out.

Initial results indicate that the incomplete component solution affects the critical values obtained using the normal approximation by a factor of approximately

$$\sqrt{\frac{100 - \text{percentage of variance accounted for}}{\text{number of components used} + 1}} + 1$$

This factor will be called the variance factor. A comparison of observed percentile values and those obtained by multiplying the critical values obtained using the normal approximation by the variance factor are shown in Table 7. The percentage of variance accounted for was estimated from the sample data.

Table 7 about here

Examples Three examples were selected from the literature to demonstrate the application of the technique. In two of the examples the null hypothesis should be tenable. In the third it is not.

Bechtoldt (1961) split a sample of 425 cases into two random groups. The subjects had been measured on 17 primary mental abilities. In this example, the null hypothesis is known to be true and the test should not reject. From the cor--relation matrices supplied in the article, 6 components were extracted in accordance with the procedure used there. Approximately 95%. of the variance was accounted for by the 6 components.

For a full components analysis, the critical value (at the .05 level) would be given by:

$$1.645 \times \frac{1}{(\cancel{12 \times 21 \times 6})^{1/2}} + \frac{1}{4}\left(\frac{6}{212}\right)^{1/2} = .0554$$

(12+212+6)

Since only 95% of the variance was accounted for, this value was corrected by multiplying by a factor of

$$\left(\frac{100 - 95}{6 + 1}\right)^{\frac{1}{2}} + 1 = 1.84$$

to produce a value of .102. The observed value of the square root trace statistic was .0728. Thus the null hypothesis was not rejected.

A second example was taken from Rosenbaum et. al. (1971). Semantic differential data were gathered from 33 supporters of Johnson and 33 supporters of Goldwater on 11 scales. The subjects were asked to evaluate 6 concepts on the 11 scales and the interscale correlation matrices were calculated based on the mean scale values obtained over concepts for each of the two groups. Two components were selected, accounting for approximately 50% of the variance.

For 33 subjects in each group, and a full components analysis, the critical value (p=.05) would be .145. The variance factor was 5.08. Thus the critical value becomes about .74. The observed value of the square root trace statistic was .2061 indicating a tenable null hypothesis. This conclusion agrees with the authors' conclusion which was based on subjective grounds.

Delaney (1970) measured 50 normals and 50 retarded boys on 12 Divergent production variables. Six components were extracted in each group accounting for about 75% of the variance. The observed value of the square root trace statistic was .2967. The critical value, after the variance factor has been applied is .13. The null hypothesis of similarity of structure of divergent production abilities can be rejected at the .05 level.

<u>Conclusion</u> The motivation for the present study was to provide a temporary, stop-gap answer for the problem of testing for similarity between component structures. The investigators could be accused of blind empiricism, or worse, of playing with

numbers. We plead guilty on both accounts. However, the results provided here are not an end point, but are a beginning. The sampling distribution of the average trace statistic seems to follow a chi square distribution (just as a sum of squares should), but because of the lack of independence among the elements making up the trace, it was difficult to find a way of standardizing the distribution. The distribution of the square root of the trace statistic could be reasonably approximated by a normal distribution, and appeared to behave well with changes in sample size. Admittedly, the effects of the number of roots used in the analysis are not yet well determined. Future efforts might usefully be confined to situations where error and uniqueness have been incorporated, and roots greater than one have been retained.

Two practical problems that remain even with this modest technique are the problem of different sample sizes, and differences in the variance factors of the two samples. No attempt has been made to deal with the first problem, although the harmonic mean might be an appropriate value to use. As for the second concern, the average of the two sample variance factors seems to work fairly well. Clearly, if the variance factors are greatly different, the correlation matrices themselves must be different and the null hypothesis is likely false.

References

Anderson, T.W. An introduction to multivariate statistical analysis. New York: Wiley, 1958.

Bechtoldt, H.P. An empirical study of the factor analysis stability hypothesis. Psychometrika, 1961, 26, 405-432.

Cliff, N. Orthogonal rotation to congruence. Psychometrika, 1966, 31, 33-42.

Delaney, J.O. Structure of intellectual and divergent production abilities in the lower intellectual range. Unpublished doctoral dissertation, University of Alberta, 1970.

Green, B.F. The orthogonal approximation of an oblique structure in factor analysis, Psychometrika, 1952, 17, 429-440.

Kaiser, H.F. Hunka, S.M. and Bianchini, J.C. Relating factors between studies based upon different individuals. Multivariate Behavioral Research, 1971, 6, 409-422.

Morrison, D.F. Multivariate statistical methods. New York: McGraw-Hill, 1967.

Nesselroade, J.R. and Baltes, B.P. On a dilemma of comparitive factor analysis: A study of factor matching based on random data. Educational and Psychological Measurement, 1970, 30, 935-948.

Rosenbaum, L.L. , Rosenbaum, W.B. and McGinnies. E. Semantic differential factor structure stability across subject, concept, and time differences. Multivariate Behavioral Research, 1971, 6, 1-10.

Schonemann, P.H. A generalized solution of the orthogonal Procrustes problem. Psychometrika, 1966, 31, 1-10.

Taylor, P.A. and Maguire, T.O. Perceptions of some objectives for a science curriculum. Science Education, 1967, 51, 489-493.

Tucker, L. R A method of synthesis of factor analysis studies. Personnel Research Section Report, No. 984 (1951) Washington, D.C. Department of the Army.

Appendix A

Population Pattern Matrices

Five variables - three components

1. Harman, H. Modern factor analysis, Chicago: University of Chicago Press, 1967, p. 137.

2. Hunka, S. An Investigation of Five Textbook Variables, Edmonton: University of Alberta, 1969.

3. Morrison, D.F. Multivariate statistical methods, New York: McGraw-Hill Book Company, 1967, p. 254.

4. Morrison, D.F. Multivariate statistical methods, New York: McGraw-Hill Book Company, 1967, p. 255.

Six variables - three components

5. Morrison, D.F. Multivariate statistical methods, New York: McGraw-Hill Book Company, 1967, p. 243.

6. Morrison, D.F. Multivariate statistical methods, New York: McGraw-Hill Book Company, 1967, p. 255.

Eight variables - two components

7. Harman, H. Modern factor analysis. Chicago: University of Chicago Press, 1967. p. 164.

Ten variables - four components

9. Mosychuk, H. Differential home environments and mental ability patterns, Unpublished Doctoral Dissertation. University of Alberta. 1969.

10. Noble, G. A study of Children's perceptions of intrinsic teaching machines and programmed instruction. Programmed Learning and Educational Technology, 1968, 5, 142-150.

Eleven variables - six components

11. Kraus, J. and Walker, W. A pilot study of factors in WAIS "patterns" in diffuse brain atrophy. American Journal of Mental Deficiency, 1967-68, 72, 900-904.

Twelve variables - four components

12. Hogg, J. A principal components analysis of semantic differential judgements of single colors and color pairs. The Journal of General Psychology, 1969, 80, 129-140.

13. Eyre, J. H. The prediction of bocational suitability from secondary modern school report cards. British Journal of Educational Psychology, 1966, 36, 48-50.

Twelve variables - six components

14. Lovell, K. and Gorton, A. A study of some differences between backward and normal readers of average intelligence. British Journal of Educational Psychology, 1968, 38, 240-248.

Thirteen variables - three components

15. Glass, G.V. and Maguire, T.O. Abuses of factor scores, American Educational Research Journal, 1966, 3, 297-304.

Thirteen variables - four components

16. Morrison, D.F. Multivariate statistical methods, New York: McGraw-Hill Book Company, 1967, p. 242.

Sixteen variables - three components

17. Hallworth, H.J., An analysis of C.W. Valentine's reasoning test for higher levels of intelligence. British Journal of Educational Psychology, 1963, 33, 41-46.

18. Ohnmacht, F.W. Achievement, anxiety, and creative thinking. American Educational Research Journal, 1966, 3, 131-138.

Eighteen variables - three components

19. Taylor, P. A. and Maguire, T.O. Perceptions of some objectives for a science curriculum. Science Education, 1967, 51, 489-493.

Eighteen variables - five components

20. Walberg, H.J. The structure of self-concept in prospective teachers, The Journal of Education Research, 1967, 61, 83-85.

Twenty Variables - six components

21. Hallworth, H.S. Personality ratings of adolescents: A study in a comprehensive school. British Journal of Educational Psychology, 1964, 34, 171-177.

22. Evanechko, P.O. Context and connotative meaning in grade five. Unpublished master's thesis. University of Alberta, 1968.

TABLE 1

SELECTED PERCENTILE POINTS AND MAXIMUM VALUE OF THE

CUMULATIVE FREQUENCY FOR AVERAGE tr (E'E)*

MATRIX NUMBER	ORDER	Percentile 25	50	75	90	95	99	Maximum Value
1	5x3	0015	0029	0054	0089	0119	0189	0330
2	5x3	0023	0041	0069	0104	0129	0195	0300
3	5x3	0011	0022	0038	0059	0074	0117	0225
4	5x3	0027	0046	0075	0104	0132	0202	0345
5	6x3	0021	0038	0064	0095	0116	0147	0180
6	6x3	0011	0023	0039	0060	0075	0122	0165
7	8x2	0009	0023	0049	0086	0111	0178	0420
8	8x2	0008	0018	0037	0053	0087	0131	0210
9	10x4	0041	0061	0084	0112	0133	0175	0225
10	10x4	0031	0045	0065	0087	0102	0138	0195
11	11x6	0049	0064	0082	0102	0113	0133	0210
12	12x4	0019	0030	0045	0061	0074	0112	0195
13	12x4	0026	0042	0067	0093	0113	0148	0240
14	12x4	0035	0046	0060	0074	0086	0112	0135
15	13x3	0030	0050	0070	0117	0145	0232	0315
16	13x4	0022	0034	0050	0068	0082	0118	0240
17	16x3	0030	0051	0081	0121	0145	0210	0300
18	16x3	0016	0028	0045	0069	0085	0118	0225
19	18x3	0014	0026	0044	0067	0091	0141	0240
20	18x5	0046	0063	0086	0113	0131	0161	0240
21	20x6	0045	0060	0076	0095	0110	0152	0210
22	20x6	0058	0073	0093	0114	0127	0149	0210

*Decimals have been omitted and appear before the four digit numbers.

TABLE 2

SELECTED PERCENTILE POINTS AND MAXIMUM VALUE FOR AVERAGE

tr(E'E)* FOR VARIOUS SAMPLE SIZES

Percentile	Sample Size	MATRIX 1 5x3	4 5x3	7 8x2	8 8x2	9 10x4	10 10x4	17 16x3	18 16x3	21 20x6	22 20x6
25	50	0015*	0027	0008	0009	0041	0031	0030	0016	0045	0058
	100	0005	0014	0005	0006	0019	0013	0015	0007	0021	0027
	150	0004	0008	0005	0005	0012	0073	0084	0005	0015	0019
50	50	0029	0046	0018	0023	0061	0045	0051	0028	0060	0073
	100	0012	0024	0010	0012	0028	0022	0025	0014	0028	0037
	150	0009	0017	0009	0010	0020	0015	0017	0011	0021	0024
75	50	0054	0075	0037	0049	0084	0065	0081	0046	0076	0093
	100	0019	0037	0017	0026	0041	0030	0040	0025	0039	0046
	150	0013	0027	0014	0014	0028	0024	0028	0017	0027	0030
90	50	0089	0104	0063	0086	0112	0087	0121	0069	0095	0114
	100	0030	0055	0033	0045	0055	0043	0059	0038	0049	0057
	150	0019	0040	0024	0027	0037	0029	0041	0026	0031	0040
95	50	0119	0132	0087	0111	0133	0102	0145	0085	0110	0127
	100	0039	0068	0042	0063	0064	0052	0072	0048	0056	0062
	150	0026	0045	0031	0036	0042	0036	0050	0029	0039	0043
99	50	0189	0202	0131	0178	0175	0133	0210	0118	0152	0149
	100	0060	0097	0063	0095	0086	0067	0107	0067	0072	0074
	150	0039	0066	0054	0063	0053	0045	0072	0041	0045	0049
Maxi-mum	50	0330	0345	0210	0420	0225	0210	0300	0225	0210	0210
	100	0075	0135	0110	0165	0105	0075	0120	0090	0105	0090
	150	0075	0105	0090	0090	0075	0090	0090	0045	0075	0060

*Decimals have been omitted and appear before the four digit numbers

TABLE 3

Characteristics of the Sampling Distribution of the Square Root Trace

MATRIX NUMBER	NUMBER OF VARIABLES	NUMBER OF COMPONENTS	MEAN	STANDARD DEVIATION	SKEWNESS	KURTOSIS
1	5	3	.057	.027	.914	.974
2	5	3	.067	.026	.614	.361
3	5	3	.049	.020	.869	1.283
4	5	3	.071	.025	.589	.602
5	6	3	.049	.021	.612	.310
6	6	3	.063	.024	.320	-.402
7	8	2	.045	.025	.751	.449
8	8	2	.052	.030	.893	1.011
9	10	4	.078	.021	.346	.057
10	10	4	.068	.019	.336	.301
11	11	6	.081	.014	.283	-.025
12	12	4	.056	.017	.644	.978
13	12	4	.067	.021	.49	.164
14	12	4	.067	.014	.292	.133
15	13	3	.073	.027	.711	.798
16	13	4	.061	.017	.593	1.129
17	16	3	.074	.026	.588	.316
18	16	3	.054	.021	.624	.537
19	18	3	.053	.021	.905	1.099
20	18	5	.081	.018	.456	.122
21	20	6	.078	.015	.571	.771
22	20	6	.086	.014	.294	.032

TABLE 4

Correlations Between Characteristics of the Sampling Distributions of the Square Root Trace and the Characteristics of the A Matrices

	1	2	3	4	5	6	7	8	9	10	11
1		-.402	-.689	-.520	.479	.758	.670	-.659	-.424	-.503	-.819
2			.597	.207	-.521	-.831	-.709	.671	.581	-.609	.265
3				.838	-.219	-.633	-.422	.589	.632	.416	.743
4					-.040	-.326	-.178	.346	.491	.480	.703
5						.583	.866	-.784	-.439	.326	-.395
6							.865	-.689	-.538	-.244	-.546
7								-.759	-.510	.041	-.491
8									.617	-.004	.651
9										.251	.541
10											.472
11											

Characteristics of Sampling Distribution of Square Root Trace

1. Mean
2. Standard Deviation
3. Skewness
4. Kurtosis

Characteristics of A. Matrices

5. Number of Variables
6. Number of Components
7. Number of Elements in A
8. Trace (A'A)
9. Variance of the Elements of A
10. Variance of (the Column Sum of Squares of A)
11. Variance of (the Column Sum of Squares of A ÷ n)

TABLE 5

Comparison of Observed and Approximated 95th Percentile Points for the Sampling Distribution of the Square Root Trace Statistic

		N=50		N=100		N=150	
Matrix	r	Observed	Approximated	Observed	Approximated	Observed	Approximated
1	3	.1091	.1001	.0624	.0707	.0510	.0578
2	3	.1136	.1001				
3	3	.0860	.1001				
4	3	.1149	.1001	.0825	.0707	.0671	.0578
5	3	.1077	.1001				
6	3	.0866	.1001				
7	2	.1053	.0976	.0794	.0690	.0600	.0564
8	2	.0933	.0976	.0648	.0690	.0557	.0564
9	4	.1153	.1044	.0800	.0738	.0648	.0603
10	4	.1009	.1044	.0721	.0738	.0600	.0603
11	6	.1063	.1141				
12	4	.0860	.1044				
13	4	.1063	.1044				
14	4	.0927	.1044				
15	3	.1204	.1001				
16	4	.0905	.1044				
17	3	.1204	.1001	.0848	.0707	.0707	.0578
18	3	.0922	.1001	.0693	.0707	.0539	.0578
19	3	.0954	.1001				
20	5	.1144	.1092				
21	6	.1049	.1141	.0748	.0806	.0624	.0659
22	6	.1127	.1141	.0787	.0806	.0656	.0659

TABLE 6

Comparison of Observed and Approximated 99th Percentile Points for the Sampling Distribution of the Square Root Trace Statistic

		N=50		N=100		N=150	
Matrix	r	Observed	Approximated	Observed	Approximated	Observed	Approximated
1	3	.1375	.1162	.0775	.0822	.0624	.0671
2	3	.1396	.1162				
3	3	.1082	.1162				
4	3	.1421	.1162	.0984	.0822	.0812	.0671
5	3	.1212	.1162				
6	3	.1104	.1162				
7	2	.1334	.1173	.0975	.0829	.0794	.0677
8	2	.1144	.1173	.0794	.0829	.0735	.0677
9	4	.1323	.1183	.0927	.0836	.0728	.0683
10	4	.1175	.1183	.0818	.0836	.0671	.0683
11	6	.1153	.1254				
12	4	.1058	.1183				
13	4	.1217	.1183				
14	4	.1058	.1183				
15	3	.1523	.1162				
16	4	.1086	.1183				
17	3	.1449	.1162	.1034	.0822	.0849	.0671
18	3	.1086	.1162	.0818	.0822	.0640	.0671
19	3	.1187	.1162				
20	5	.1269	.1216				
21	6	.1233	.1254	.0848	.0887	.0671	.0724
22	6	.1221	.1254	.0860	.0887	.0700	.0724

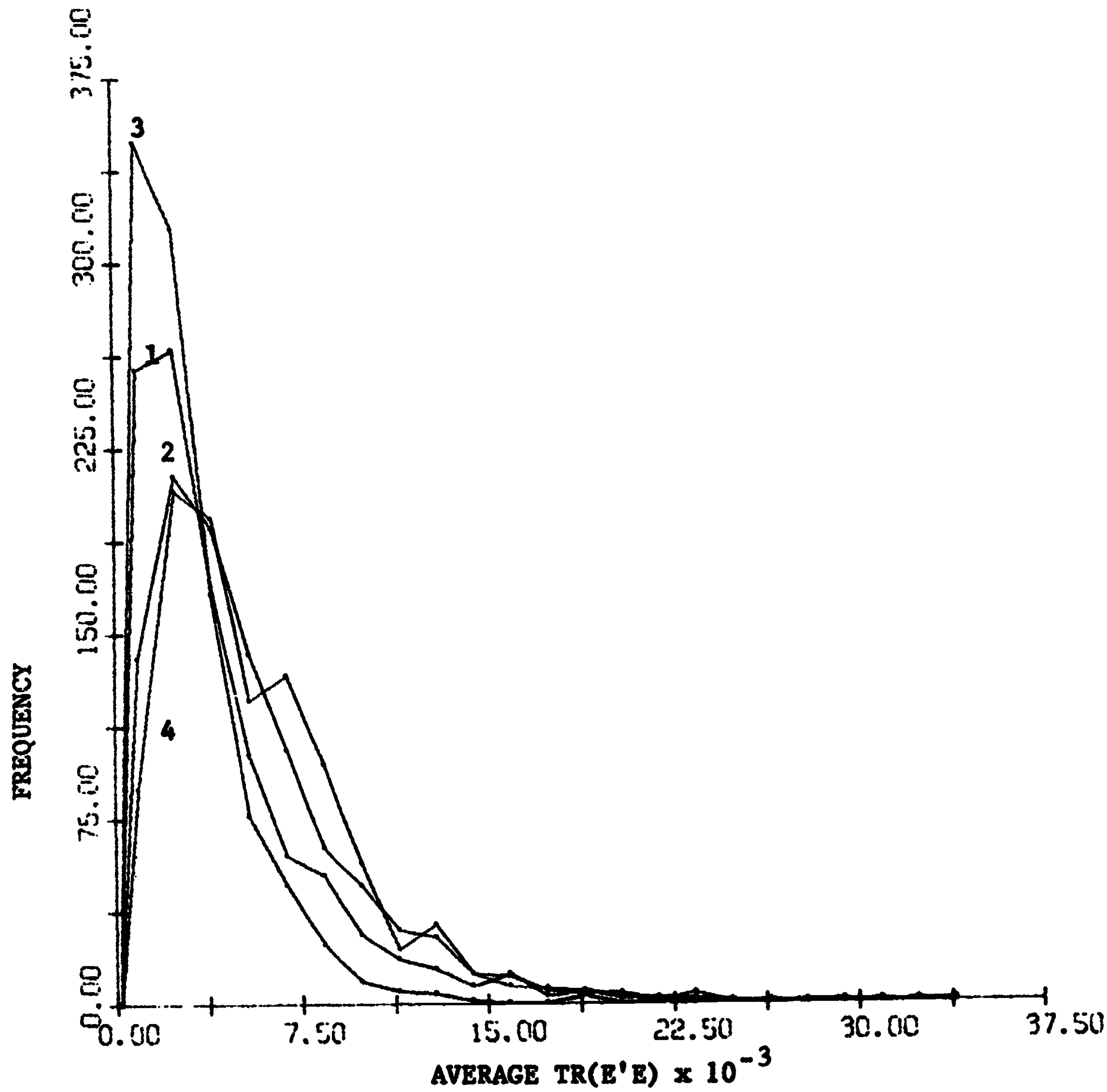
TABLE 7

Comparison of Observed 95th and 99th Percentile Points of the Square Root Trace Statistic with the Values Obtained from the Normal Approximation Multiplied by the Variance Factor (350 matches)

Matrix Number	Number of Components	% of Variance	Variance Factor	Obs 95th Percentile	Approximated 95th	Obs 99th Percentile	Approximated 99th
9	2	68	4.27	.46	.42	.54	.50
	3	84	3	.36	.30	.42	.35
	4	100	1	.11	.10	.12	.12
21	2	64	4.46	.45	.44	.48	.52
	3	80	3.24	.24	.32	.29	.38
	4	90	2.41	.19	.25	.23	.29
	5	96	1.82	.18	.19	.20	.23
	6	100	1	.10	.11	.12	.13
22	2	57	4.78	.45	.44	.51	.56
	3	72	3.65	.38	.32	.43	.42
	4	84	2.79	.32	.25	.35	.33
	5	93	2.17	.25	.19	.27	.26
	6	100	1	.10	.11	.12	.13

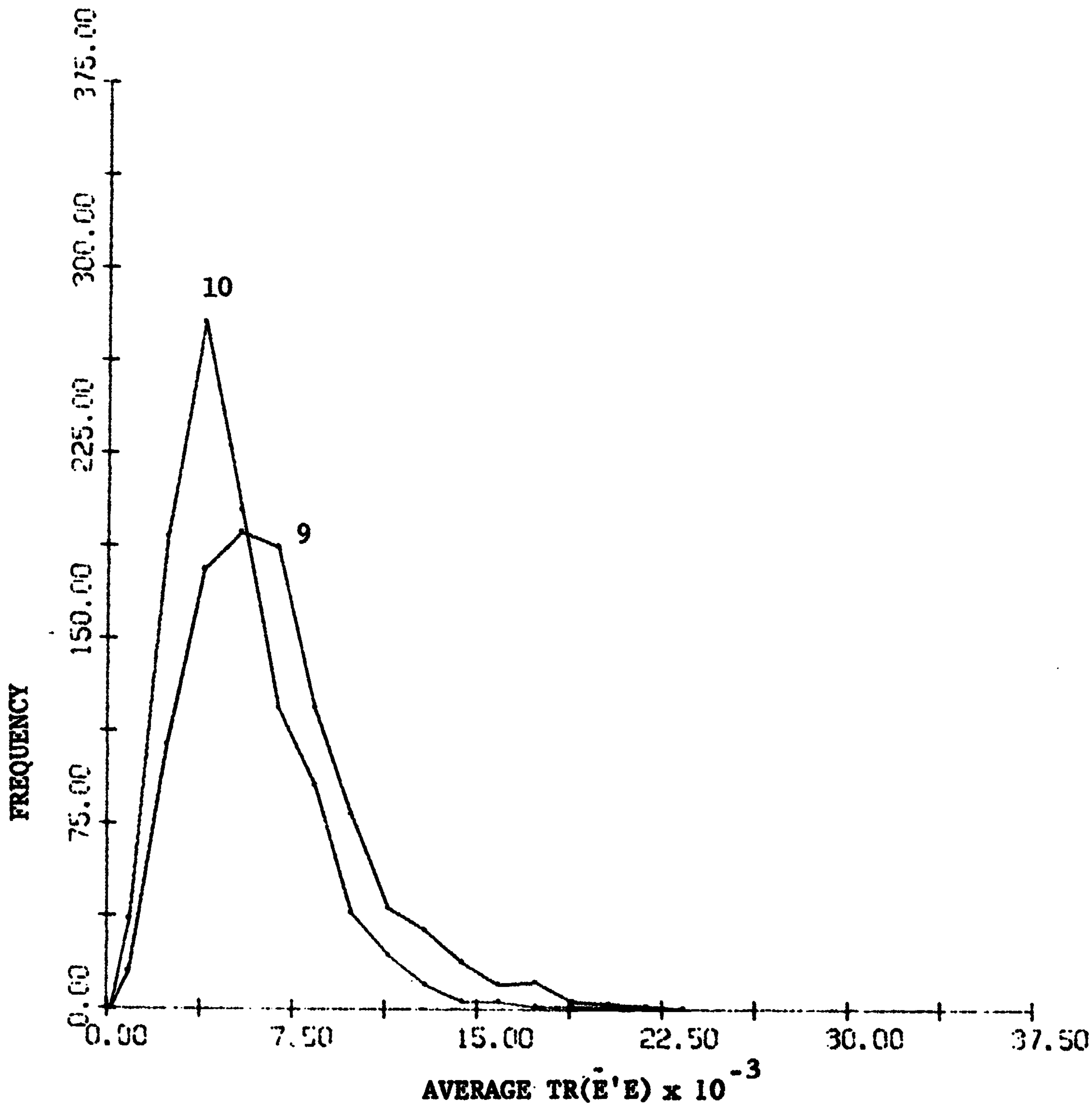
Appendix B

Frequency Polygons for Average Trace E'E



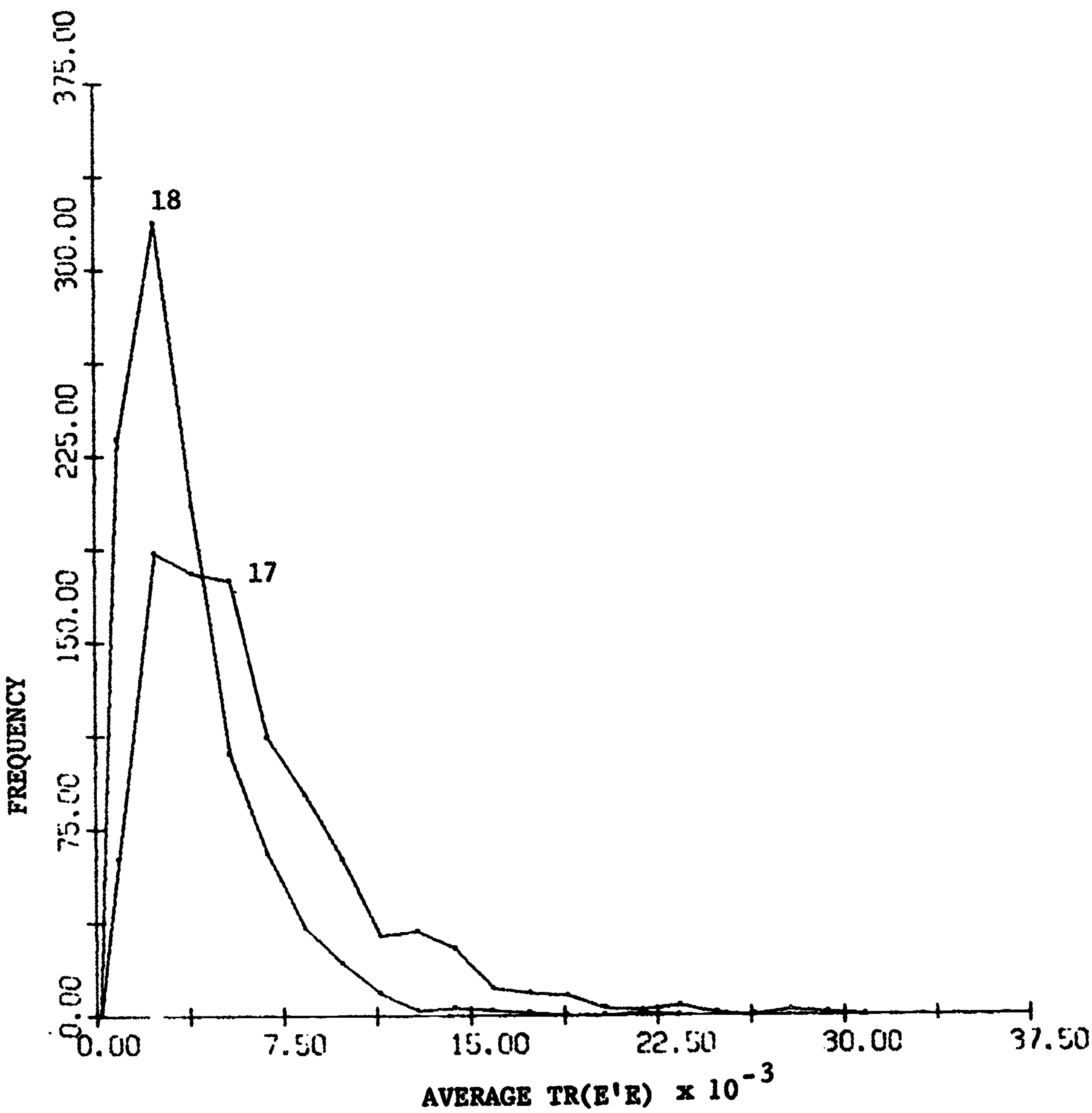
FREQUENCY POLYGONS FOR AVERAGE TRACE E'E

VALUES USING VARIOUS 5 BY 3 A MATRICES



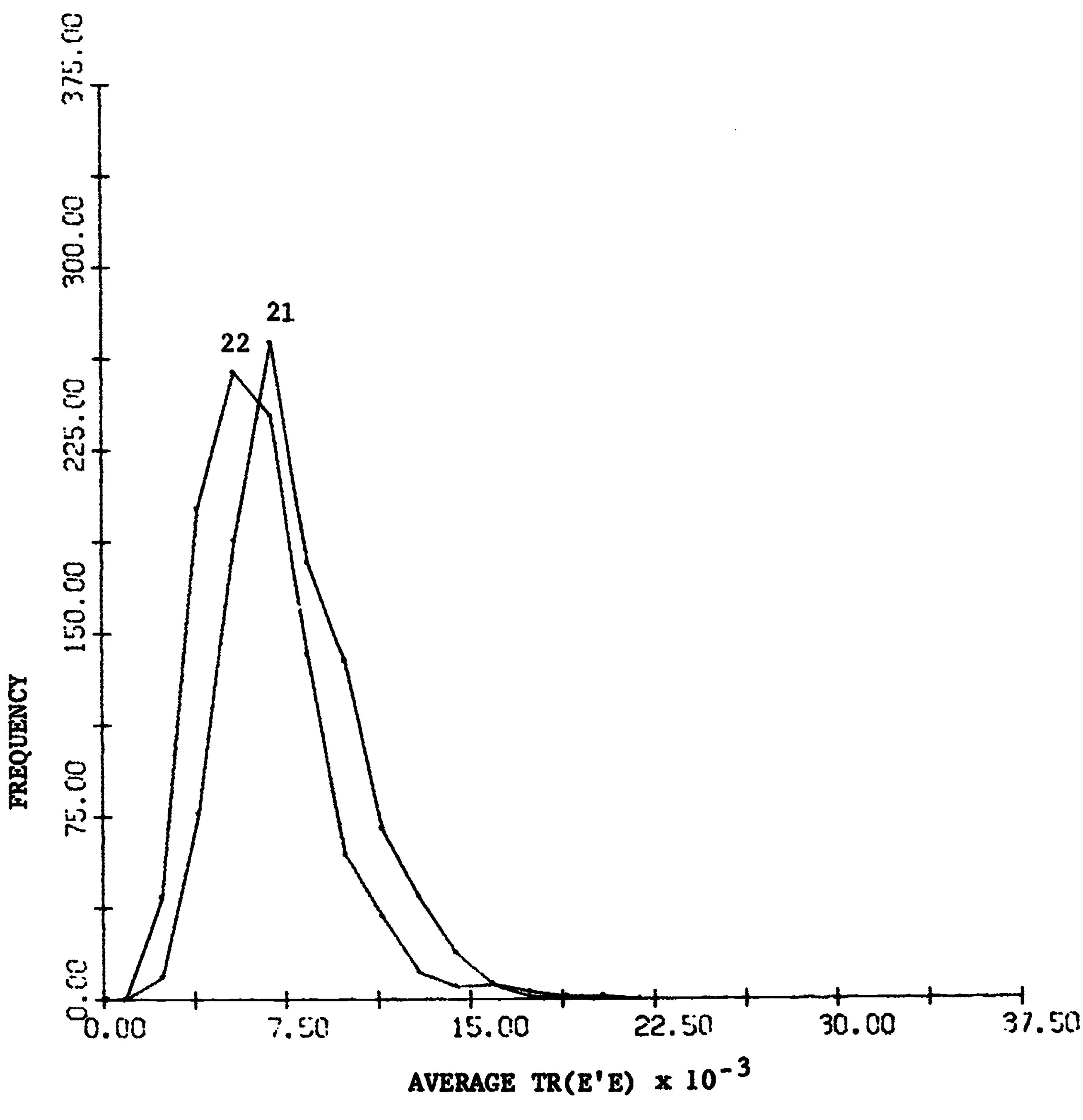
FREQUENCY POLYGONS FOR AVERAGE TRACE E'E

VALUES USING VARIOUS 10 BY 4 A MATRICES



FREQUENCY POLYGONS FOR AVERAGE TRACE E'E

VALUES USING VARIOUS 16 BY 3 A MATRICES

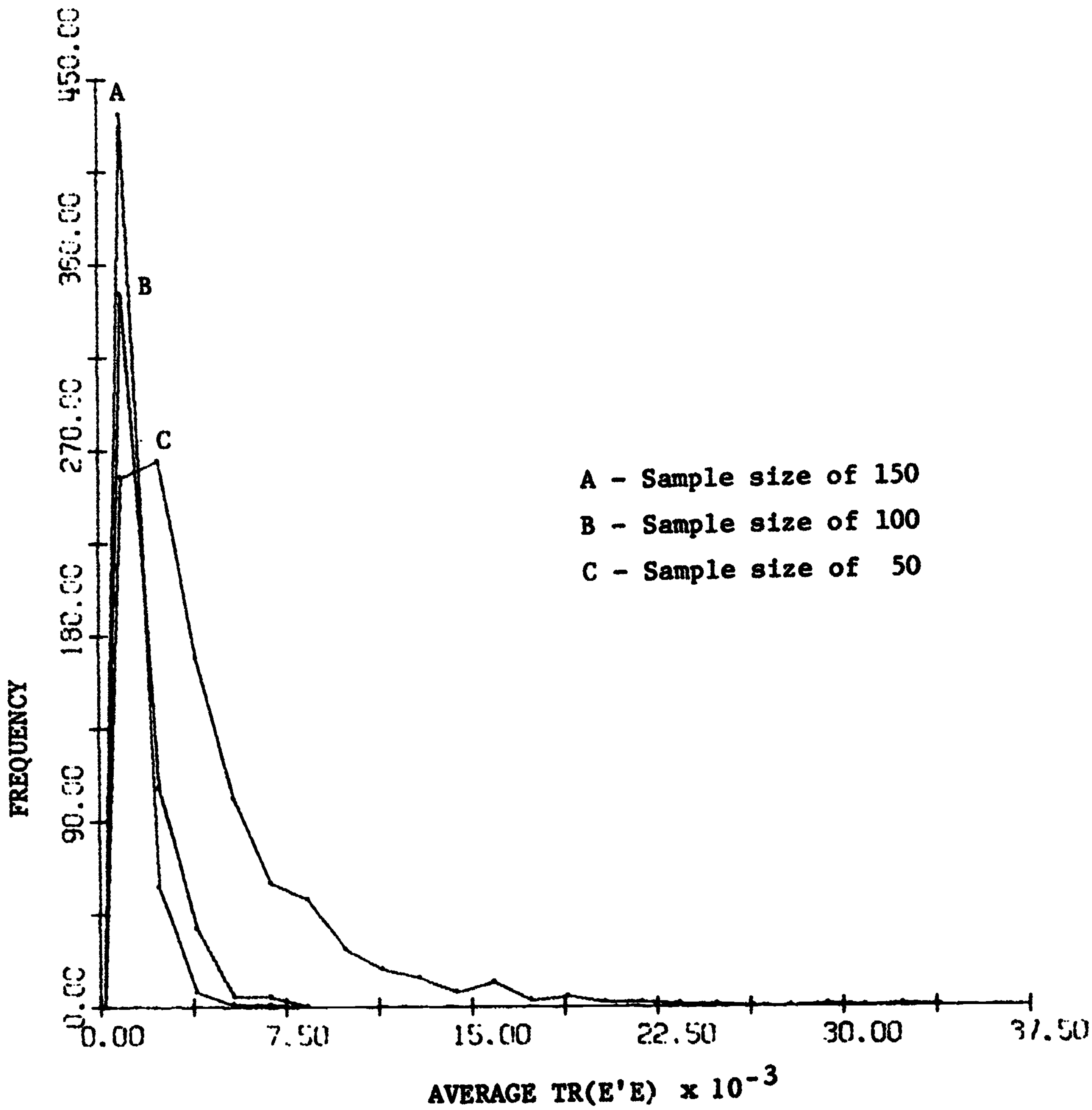


FREQUENCY POLYGONS FOR AVERAGE TRACE E'E

VALUES USING VARIOUS 20 BY 6 A MATRICES

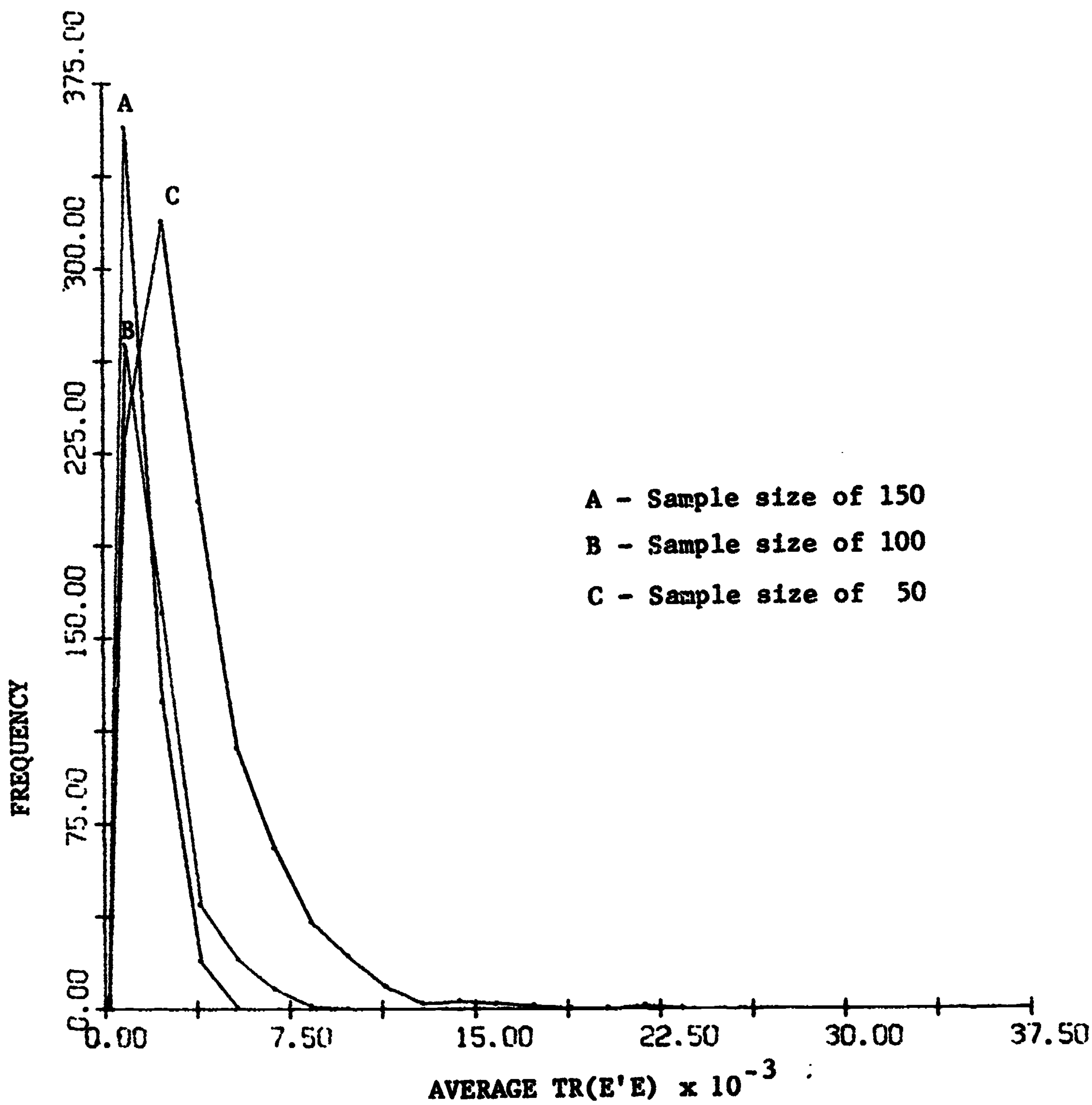
Appendix C

Frequency Polygons for Average Trace E'E
Showing Effects of Changes in Sample Size



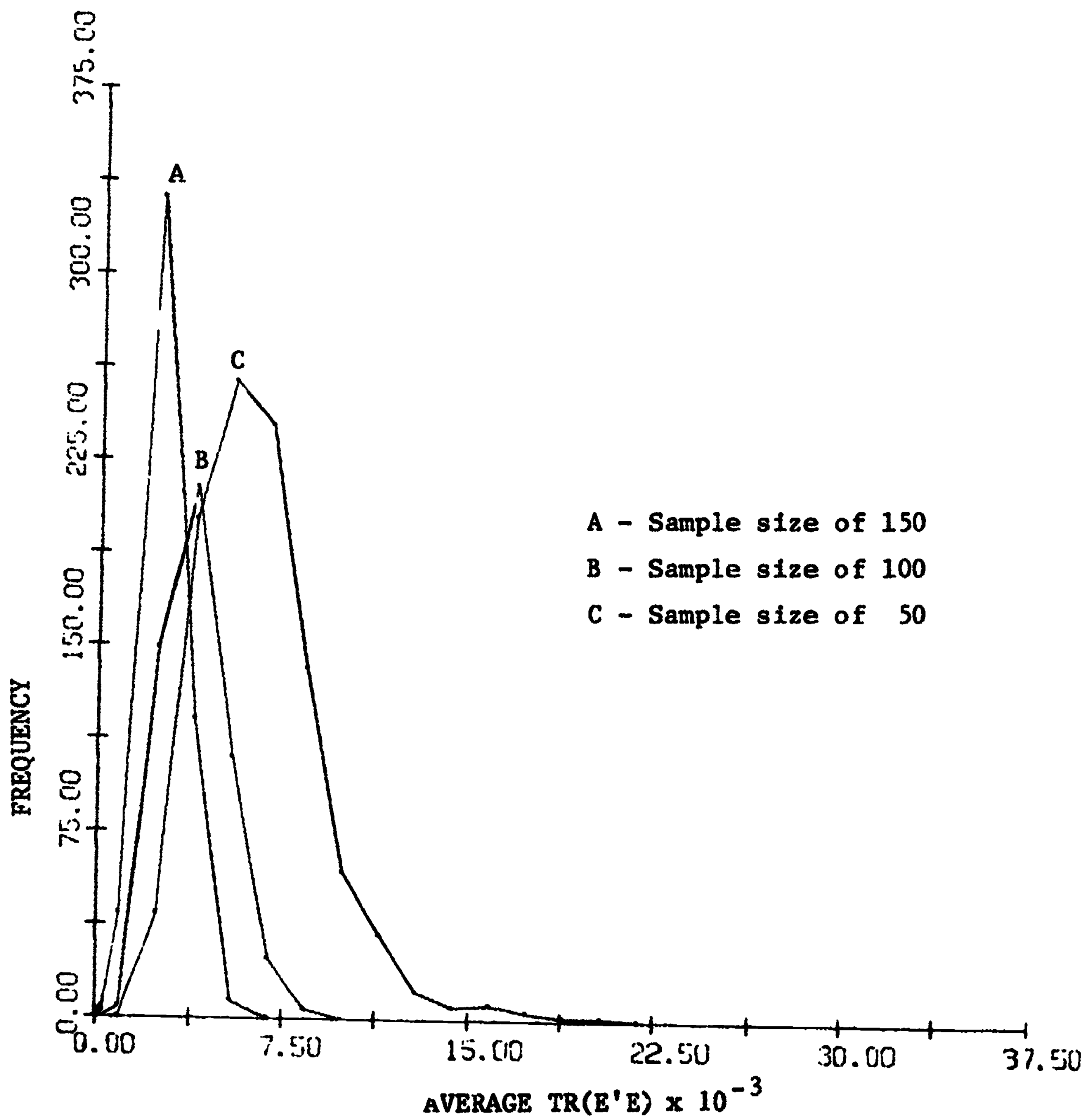
FREQUENCY POLYGONS FOR AVERAGE TR E'E VALUES FOR MATRIX NO 1 (5x3)

USING SAMPLE SIZES OF 50 100 AND 150



FREQUENCY POLYGONS FOR AVERAGE TR E'E VALUES FOR MATRIX NO. 18 (16x3)

USING SAMPLE SIZES OF 50 100 AND 150



FREQUENCY POLYGONS FOR AVERAGE TR E'E VALUES FOR MATRIX NO. 22 (20 By 6) USING SAMPLE SIZES OF 50 100 AND 150