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ABSTRACT

Included are two curriculum guides, Section D and Section E, which continue instruction of the three guides in SE 012 723. Counting (by ones, by twos, by threes, ..., by twelves) is extended to 144 in Section D and to 1000 in Section E. The creation and manipulation of more complex equations are taught, including the notion of substitution. The relations between the four basic operations are continually stressed. The ideas of a fraction as a relation between two numbers, and as an operator, are developed, and the idea of a fraction as a number is introduced in Section E. Zero is first mentioned in Section D and is used in counting and in the discussion of place value in Section E. Throughout these two sections, Cuisenaire rods continue to be used, but more and more emphasis is placed on working without them if possible. (MM)

CURRICULUM GUIDE

PURE NUMBER

SECTION D

EDUCATION DEPARTMENT OF VICTORIA

CURRICULUM GUIDE
PURE NUMBER COURSE

Section D—The Study of Basic Mathematical Ideas in Terms of the Numbers from One to Twenty

Education Department of Victoria

4301/69

CURRICULUM GUIDE
PURE NUMBER COURSE

Section D—The Study of Basic Mathematical Ideas in Terms of the Numbers from One to Twenty

AIM

This section aims to ensure that the child is able to express, with the numbers from eleven to twenty, the understandings he previously expressed with the numbers below ten, and to develop an awareness of the pattern and order of the number system.

COMMENTS

1. In this section the child is led to realize that the concepts he has formed and expressed with numbers below ten apply equally when the numbers above ten are used. When the child realizes this he is being made aware of the internal logic and consistency of the number system.

2. This section is a natural development of Section C. The chief difference is an extension of the range and the quality of the work. Both sections aim to develop understanding of basic mathematical ideas. Similar methods are used, and the approach to testing is the same.

3. No attempt should be made to organize work so that all basic number facts and tables are covered. This coverage would be needed only if automatic response were the aim. (See comments on Aims, Section C.) The main concern here is for an understanding of basic mathematical ideas. However, automatic response, required later, can only be based on experience. In Section D, the child gains a broad experience of numbers eleven to twenty, but through substitution and planned exercises, his knowledge of numbers to ten can be considerably deepened. Similarly, when he moves to Section E, he makes a broad study of numbers to 144 and deepens his study of numbers eleven to twenty.

4. The four stages of this section are parallel, and not consecutive. Each is a continuation of its counterpart in Section C. All the skills acquired so far are used in association with one another and are not treated as separate activities.

5. As no new mathematical ideas are introduced, this section needs less time than previous ones. The actual amount of time will vary, but, for children of average ability, approximately one school term should be sufficient.

STAGE 23

AIM

To continue the study of ordinal number.

NOTES ON AIM

1. The Notes on Aim, Stage 14, Section C, outlined three aspects of counting :

i an essential skill,

ii a means of extending the child's awareness of larger numbers before he studies them in detail, and

iii a means of showing the pattern and order of the number system.

2. In Section D, counting must continue at least to 144 to prepare for the detailed work of Section E. All three aspects have received attention in previous sections and are further developed here.

3. The third aspect, iii above, is of increasing importance. The child should realize that the number system is not simply a haphazard collection of isolated facts, but is highly organized, self-consistent, complex, and inter-related. Until the child gains this view of numbers he has not understood the number system. It is the view of the number system as a logical arrangement of numbers that helps to distinguish the mathematician from the drudge who sees numbers as separate and distinct entities. One uses the logic of the system to assist his work; the other struggles with his work because he has not seen the logic.

During this stage, the child is given every opportunity to discover pattern and order. The discovery must be the child's. It is more important here than the mere knowledge. The teacher's role is, therefore, to present situations from which conclusions can be drawn.

When pattern is first seen, it is simply as a matter of interest; later there will be a probing for reason.

DEVELOPMENTAL STEPS

NOTE : These steps are not necessarily sequential.

1. Counting by Ones to One Hundred and Forty-four

This step, as well as dealing with counting for its own sake, acquaints the child superficially with numbers to be studied in detail later. He must be able to count by ones, at least to 144, forwards and backwards, and starting at any point within the range. Most of this work will be oral.

2. Order in a Series

In addition to counting to 144, the child should continue to use, wherever practicable, the idea of order in a series—e.g., 31st May, 78th birthday.

3. Figures to 144

Figures at least to 144 must be recognized and written. There is no need to discuss the techniques used; sufficient comment has been made in previous sections.

4. Words to Twenty

Children should recognize and write words to "twenty". This is not strictly a mathematical skill, and it can, with advantage, be handled in reading and spelling periods.

5. Counting to Show Pattern and Order

a **Group Counting** It was suggested in Stage 14 that children should commence group counting where they could see pattern. If some work was begun then, it should be continued; if not, it is imperative that a start be made now. Children should count by twos, by threes, by fours, etc., up to twelves. Teachers will find many useful techniques such as oral work with bead frame or counting board, recording of numbers in groups (see Stage 14, Developmental Step 2), and the oral repetition of smaller groups for rhythm—e.g., 2 4 6 8 10 12 14 16 18 . . . . (i.e., rhythmic counting by twos to discover counting by sixes).

Memorization of group counting is not yet required, though a certain amount will soon be observed. The extent to which counting by each number will be taken depends on children's ability and experience. This stage will merge with Stage 27 (Section E) with gradual extension of group counting work.

b **Number Charts** The recording of group counting leads to the building up of number charts, which provide a wide scope for study and experience, e.g.

i Final digit patterns (starting from zero):

—Counting by tens results in repetition of 0.

—Counting by fives results in repetition of 5, 0, 5, 0.

—Counting by even numbers, excepting 10, results in repetition of the digits 2, 4, 6, 8, 0, but in different order.

—Counting by threes, sevens, or nines results in a long pattern, difficult to remember at this stage.

ii Vertical and diagonal counting can lead to some interesting discoveries :

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60

In the above chart, diagonal counting from right to left reveals repetition of two final digits only. Vertical counting in each column shows another pattern.

iii Marking counting patterns on charts shows visual geometric shapes, e.g., counting by fours on the above chart reveals a triangular pattern.

c **Serial Counting** The tens chart is particularly useful for showing counting in the tens sequence, starting from any number, eg., 3, 13, 23, 33, ...

6. Doubling and Halving

Many children are capable of doing simple exercises in doubling and in halving numbers, and should be encouraged. There should, however, be no forcing of this on immature children. Experience in this activity in Section D has been found to greatly increase the quality of creative work offered by brighter children. It is very important that children should be quite free to develop their own techniques, and that they are not directed or helped by the teacher. (Refer Section E, Stage 27, for details.)

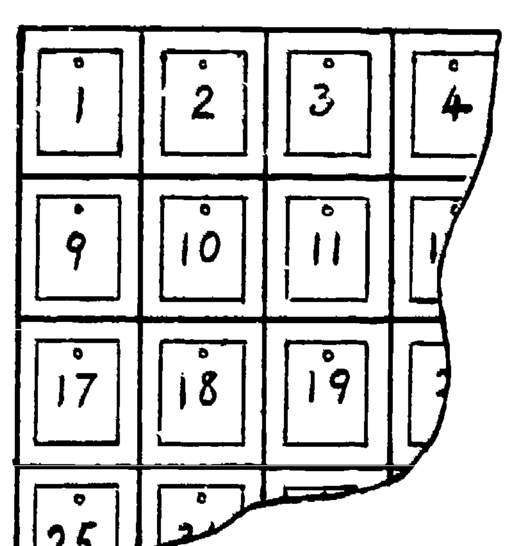
NOTES ON METHOD

1. It must be stressed that the work outlined in this stage continues side by side with the work in the following stages. It is done throughout this section.

2. The limits to the work given here are not intended to be in any way restrictive. They are minimum limits only, and the many children who are able to go beyond them should be free to do so.

3. There is no need to introduce the concept of place value at this stage. Each number may be treated as "just another number" and a detailed treatment of place value left till the next section when intensive study begins. There is, however, no strong objection to studying place value now. If this is done, it would be wise to read through the method outlined in Stage 28, where place value is treated fully.

4. Mention has been made in Developmental Step 5 of a counting board. This is an aid many teachers have found very useful. It consists of a board approximately 3 feet square marked into 144 smaller squares with a peg or hook in each, along with a set of 144 numbered tablets. For maximum efficiency the numbers should be on both sides, each side being of a different colour. The tablets required for a particular sequence are set up on the pegs, all showing the one colour. By reversing the appropriate tablets a pattern can be shown in the other colour.



TESTING

1. Counting by Ones to 144

Count—from 20 to 60
—from 94 to 108
—from 100 to 70
—from 76 to 38

What number comes after 39 ? 110 ? 137 ?

What number comes before 100 ? 63 ? 5[illegible] ?

Fill in the missing numbers : 56 57 58 — — 61
116 115 114 113 — — 110

2. Writing Figures to 144

The child writes figures dictated by the teacher.

3. Recognition and Writing of Words to " Twenty "

Write the figures for these words : fourteen, twelve, twenty (written on black-board for recognition).

Write the words for these figures : 8, 11, 14 (dictated or written on black-board).

4. Counting for Pattern and Order

Formal testing of this type of work would defeat its purpose, namely, discovery of pattern and order. The day-to-day work will show what progress a child is making.

8

STAGE 24

AIM

To continue the development of the ability to manipulate equations, using the rods.

NOTES ON AIM

1. In this stage the child uses the numbers between eleven and twenty to manipulate equations in terms of the mathematical ideas already studied.

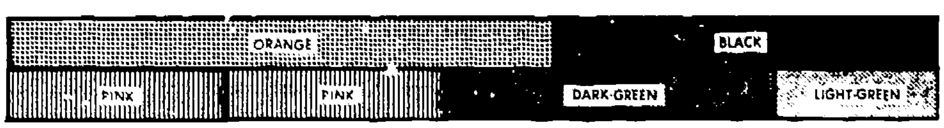
2. Broadly speaking, it corresponds to Stage 20. The larger numbers used allow for wider experience which consolidates the understanding of earlier sections.

DEVELOPMENTAL STEPS

NOTE: While the topics listed below may need separate treatment, they are not to be considered as sequential steps. Each should be read in conjunction with the others.

1 Initial Move to Numbers above 10

The child can read patterns involving all operations using numbers to 10, and can record his readings. He now applies these skills to larger numbers. The following approach is useful :— The child is asked to select a number from several suggested by the teacher, e.g., 12, 17, 20. He is then told to put out an orange rod, call it 10, and put end to end with it one other rod to make up the number he chose. Next he is asked to make a row of rods to equal this, and to read equations just as he did with smaller numbers. If he made the pattern :



he could read :

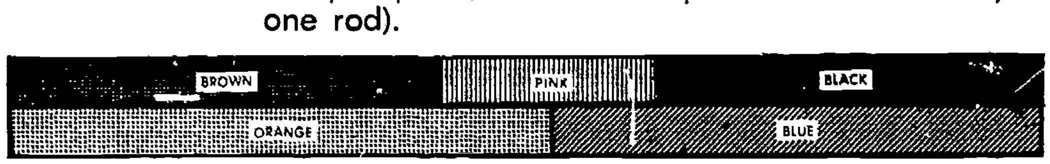
$$10 + 7 = 2 \times 4 + 6 + 3$$
$$3 + 2 \times 4 + 6 = 1$$
$$10 + 7 - (4 + 6) = \cdot + 3$$

As soon as the child can ma patterns and read them confidently, he may record his equations. Oral and written work then go on side by side.

2. Organization of Work

a In order to extend the child's experience of mathematical ideas, it is necessary to plan his work in terms of types of equations rather than specific numbers. The steps of Stage 6 outlined ideas which should be further developed. For example :

i $\mathbf{a} = \mathbf{b} + \mathbf{c} + \mathbf{d}$ (where "**a**" represents one number, not one rod).



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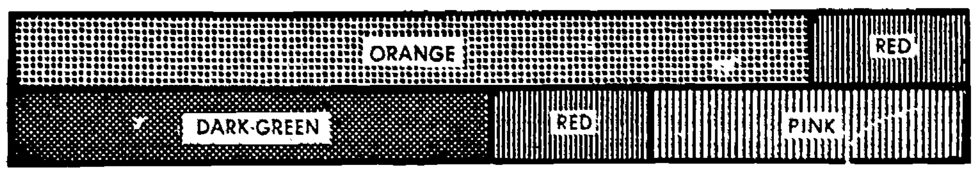
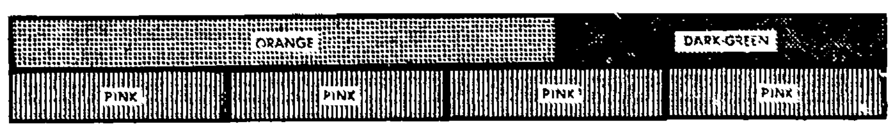
This pattern gives scope for addition and subtraction (with and without brackets), but is not useful for multiplication and division. Although it concerns the number 19, it is no more difficult than :



which could be used for 12, a smaller number.

ii **a = yb**

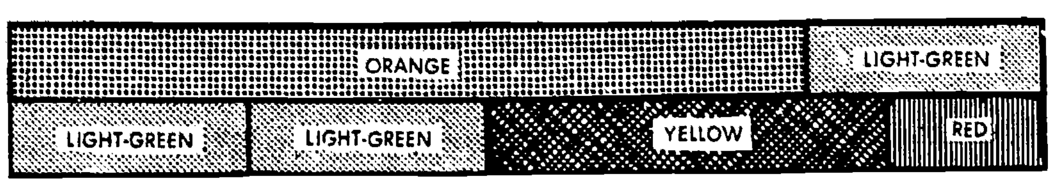
This type of pattern :



is needed to give experience of multiplication, division, and fractions. Again, the size of the original number in the pattern makes no difference to the degree of difficulty. The example above could result in the following, and many other equations :

$$4 + 4 + 4 + 4 = 16$$
$$4 \times 4 = 16$$
$$16 - 2 \times 4 = 2 \times 4$$
$$16 - (4 + 4 + 4) = 4$$
$$10 + 6 - 4 \times 4 = 0$$
$$16 \div 4 = 4$$
$$\tfrac{1}{4} \text{ of } 16 = 4$$

iii Patterns involving the ideas of both i and ii above should also be used :

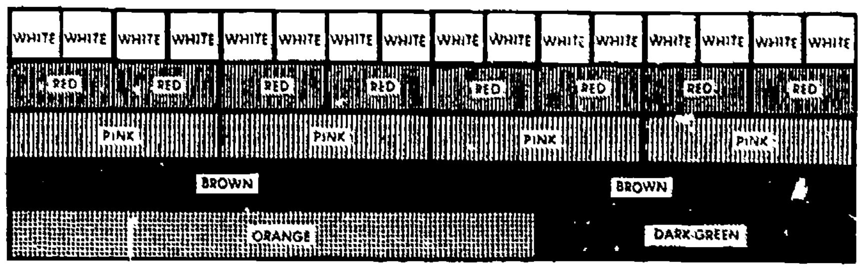


Manipulating this pattern, a child could say, or write :

$$2 \times 3 + 5 + 2 = 13$$
$$2 \times 3 + 5 + 2 - 10 = 3$$
$$10 + 3 - (5 + 2) = 2 \times 3$$
$$13 - (2 \times 3 + 5) = 2$$
$$3 = 13 - (3 + 5 + 2)$$
$$10 + 3 - (5 + 2 + 2 \times 3) = 0$$

b Mat work increases in usefulness when children have established the habit of using more than one operation in an equation. Care should be taken, however, to see that any two rows in the mat can be read as [illegible] manipulated in various ways, and not always the " base " row and one other. It is a good practice to remove the " base " row if it is being over-used.

A " multiplication " mat is very useful in presenting the idea of factors. For example, this mat shows that rows of 8, 4, 2, and 1 can be made to equal 16:



It is not suggested that children should memorize factors of numbers, but simply that they should become aware of the idea.

3. Relationships between Operations

The relationships existing between operations should be constantly in the teacher's mind, and informally brought to the child's notice.

The relation between addition and multiplication is obvious, e.g., $4 + 4 + 4 + 4 = 16$, $4 \times 4 = 16$.

Addition and subtraction relationships are seen in such examples as $8 + 5 = 13$ and $13 - 5 = 8$, both read from the one pattern.

Multiplication and division were related in Section B when the ideas were first studied, and the relationship is seen when the child reads or writes: $5 \times 3 = 15$, $15 \div 3 = 5$.

The relationship between repeated subtraction and division can be shown also, e.g., $16 - 4 - 4 - 4 - 4 = 0$, $16 \div 4 = 4$.

4. Use of Zero

When subtraction was introduced in Section B (Stage 10) examples involving " no difference " were included, opening the way for later use of " nought ", or " zero ". This should now be appearing in children's creative work, e.g., $13 - (6 + 7) = 0$. Some children freely use " 0 " in addition and multiplication, e.g., $14 + 0 = 14$, $0 + 14 = 14$, $6 \times 0 = 0$, $0 \times 6 = 0$; but definite treatment should be given for those children who need it. Division by " 0 " has no meaning.

5. Value Relations

Work as outlined in Stage 15 should continue. Any of the exercises not yet attempted should be given as soon as the children are capable of them, and those already in use must be extended.

6. Interpretation of Equations

The teacher should continue to question the child as to what he means when he makes a verbal or written statement. This is a direct continuation of the work of Section C, Stage 20, Developmental Step 4, which should be read in conjunction with this stage. Understanding is still of primary importance, and can best be assessed by the child's justification of his statements.

Experience in interpretation of the teacher's equations, following experience in reading equations from rods, lays the foundation for later, more formal work with the order of operations. For example :

i A child, asked to show with rods $9 - 2 \times 4 = 1$, will soon discover that if he starts with $9 - 2$ he is unable to proceed. This experience develops from his work in Section B, where the only numbers used were ordinal, and he encountered such work as orange minus 2 times light-green equals pink.

ii Asked to interpret with rods $6 + \frac{1}{2}$ of $8 = 10$ a child would not attempt to show 6 plus $\frac{1}{2}$. He knows he needs to show $6 + 4$ on one row, and 10 on the other.

Grading of examples is in terms of the number of operations involved.

7. Solving Uncompleted Equations

This work, begun in Stage 20, continues, using bigger numbers. Care should be taken that oral work is not neglected for excessive concentration on written work. In this stage the children may use rods. The equations should not at any time be of greater difficulty than those which children can interpret. A child should not be asked to solve $2 \times 5 + 2 \times 2 - 4 = \square$ unless he has already been asked to interpret a similar, complete equation, e.g., $2 \times 2 + 3 \times 3 - 7 = 6$.

The complexity of equations to be solved must be less than that seen in creative work. It must be remembered that the difficulty lies in the number of operations used rather than the size of numbers involved.

Less time should be spent on solving equations than on creating equations.

NOTES ON METHOD

1. This stage may be begun at the same time as the previous stage, where counting skills were the main objective. The counting skills necessary for success in this stage were, of course, taught during the previous section. (See Stage 14.)

2. It has been stressed that the work of this stage is organized in terms of the degree of complexity rather than the size of numbers. However, it is essential to see that the child has experience with each number in the range 11–20.

3. A common source of error in this stage is the over-use of written work to the detriment of oral work. This hinders a child and limits the quality and the amount of work he can cover. It is advisable to view written work as a means of recording ideas that have been thoroughly established in oral work. Writing is not used to any great extent in the actual establishing of the ideas.

4. Work without rods for the numbers studied previously continues during this stage.

5. When recorded work was first introduced (Stage 19) the sign $\div$ was used to record quotition division, e.g., " How may twos equal eight ? " was written $8 \div 2 =$. Probably, during this stage, teachers will find that children are ready for the simple vocabulary change to " divided by ". The child will then know that when he says " nine divided by three " he means " how many threes equal nine ? "

The use of " multiplied by " for the sign " $\times$ " (read in this Guide as " times ") involves more than a vocabulary change, and will be dealt with in a later section.

TESTING

The tests are basically the same as those used in Stage 20 except, of course, that larger numbers are used.

STAGE 25

AIM

To continue the development of ability to manipulate equations without using the rods.

NOTES ON AIM

1. The ability to manipulate numbers abstractly has been growing gradually and an intensive effort to speed its development was made during the last section (see Stage 21). The Notes on Method for the previous stages made it clear that, though for the numbers from eleven to twenty the child worked with the rods, he was doing some work without rods for numbers to ten. The aim of this stage is to extend that skill to the larger numbers.

2. It should be noted that there is no point in weaning from the rods a child who is not fully certain of what he is doing. On the other hand, there is no point in forcing a child who has no need for the rods to use them.

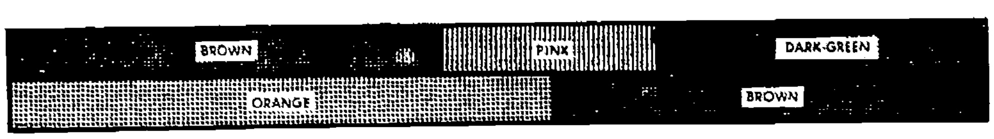
3. As mentioned in Stage 21, the ability to create and manipulate equations without using the rods at all is the culmination of this part of the course. The child who has developed this ability has developed a sound understanding of the operations.

DEVELOPMENTAL STEPS

NOTE: As the work outlined below is a simple development of that commenced earlier, the four steps may proceed side by side.

1. Mental Substitution

a A pattern is made with rods, and a simple equation is read from it, e.g.



$$8 + 4 + 6 = 18$$

A child is then asked to express one of the numbers in this equation in a different way, i.e., to substitute for it or rename it. If he substitutes for the 8, these equations could result:

$$2 \times 4 + 4 + 6 = 18$$
$$7 + 1 + 4 + 6 = 18$$
$$2 + 2 + 2 + 2 + 4 + 6 = 18$$
$$4 \times 2 + 4 + 6 = 18$$

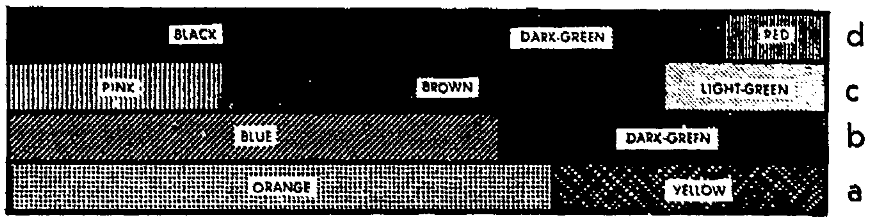
As confidence increases, children substitute for other numbers, and finally for all numbers in a given equation, with results such as :

$2 \times 4 + 3 + 1 + \frac{1}{2}$ of $12 = 10 + 8$

This step, while not fully free of the rods, is (as mentioned in Stage 21) a useful bridge to work that is completely independent of the rods.

Plenty of oral work must be done, followed by written work. A valuable group activity is to have the children work orally while the teacher records their equations on the black-board.

Mats are again useful, though they should never consist of so many rows that insufficient time is left after their construction for profitable use of the mat. For example :



From this mat, readings involving substitution could be as follows :

$3 \times 3 + 2 \times 3 = 10 + 5$	(row b = row a)
$4 + 3 + 4 + 2 + 2 = \frac{1}{2}$ of $8 + 2 \times 4 + 3$	(row d = row c)
$8 + 1 + 3 \times 2 = 7 \times 1 + \frac{1}{2}$ of $12 + \frac{1}{4}$ of 8	(row b = row d)

b It will be noted that it is easier to substitute using addition and multiplication than subtraction or division. Fractions are more readily used than quotition division. It is only as experience in number increases that a child will rename 4 as $7 - 3$. He needs a fairly wide experience before he can replace the 2 in an equation with the quotition idea $10 \div 5$ or $6 \div 3$, since this involves a change in thinking from cardinal to ordinal. It is more likely to be used when abstract work has been carried further. The teacher should not be disturbed because addition, multiplication, and fractions are used first, but must seek and encourage the subtraction and division types of substitution as soon as children are ready to handle them.

c It will readily be seen that substitution is invaluable in providing experience of number bonds for later work in automatic response. Work should be planned so that a wide experience of numbers to 10 is gained during this section. For the child who has difficulty in recalling combinations to substitute for a given number, mat work on that number will be found helpful. There should be definite experience of all number facts to 10 during Section D. The numbers 11–20 will receive similar treatment during Section E. (See Comments on Aim of Section D.)

2. Manipulating a Written Expression

a **Without Substitution** As in Stage 21, the child is given an equation and asked to rearrange it, or say it in another way, first without substitution.

e.g. $6 + 7 + 4 = 8 + 8 + 1$
$2 \times 8 + 1 = 7 + 6 + 4$
$7 + 4 + 6 - 8 = 8 + 1$
$7 + 6 + 4 - 1 = 8 + 8$
$4 + 6 + 7 - (8 + 1) = 8$
$2 \times 8 + 1 - 6 - 7 = 4$
$8 + 1 + 8 - (6 + 7 + 4) = 0$
$1 = 6 + 7 + 4 - 2 \times 8$

Ability to do this depends directly on experience in writing equations from rod patterns as required in Stages 20 and 24, and the understanding of operations.

b **With Substitution** As confidence increases in the use of substitution and manipulation separately, they should be brought together. A suitable approach is to take a simple equation, manipulate it, then substitute for the elements either one at a time or all at the one time, according to the ability of the particular child. For example,

the equation $9 + 6 = 15$ could be manipulated to give $15 - 9 = 6$. Substitution could then produce $3 \times 5 - 3 \times 3 = 2 \times 3$
or $10 + 5 - (5 + 4) = \frac{1}{3}$ of 18.

With practice, and as this section merges into Section E, there should be spontaneous use of both manipulation and substitution in the one equation; for example, from $8 + 5 = 13$ direct to $10 + 3 - 2 \times 4 = \frac{1}{2}$ of 10.

3. Creating Equations

The child is asked to create equations without reference to rods. He may be given a number as a starting point, e.g., "Write equations about the number 18". He will, of necessity, work within the limitations of the number bonds he can recall, but this will not affect the quality of his work if he understands the operations and their relation to one another. For example, if he starts with $10 + 8 = 18$, a very simple and obvious number fact, he can manipulate and substitute as follows :

$10 + 8 = 18$
$18 - 2 \times 5 = 2 \times 4$
$6 + 4 + 4 + 4 = 10 + 8$
$6 + 3 \times 4 = 10 + 8$
$4 + 2 + 3 \times 4 = 5 + 5 + \frac{1}{2}$ of 16
$3 \times 4 + 6 - (8 + 10) = 0$
$3 \times 4 + 6 - 8 - 10 = 0$
$8 = 18 - 10$

This sample does not show a great awareness of number facts about 18; such was not the intention of the exercise. It does reveal :

i ability to create equations without use of rods;

ii awareness of the relationships between addition and multiplication, and between addition and subtraction;

iii understanding of subtraction using brackets and repeated subtraction;

iv a limited use of nought; and

v knowledge of some facts about numbers smaller than 10.

4. Interpreting and Solving Equations

a The work of Stages 20 and 24 is continued, and power to attack this work abstractly should be increased. Discussion of equations provides for interpretation without rods. For example :

i $3 \times 6 + \frac{1}{2} \text{ of } 4 = 20$

Discussion of this equation should bring from children the fact that two quantities are joined or added together. They are 3×6 and $\frac{1}{2}$ of 4. Together they are equal to 20.

ii $17 - \frac{1}{4} \text{ of } 16 = 13$

In this equation children should see and be able to explain that the difference between 17 and $\frac{1}{4}$ of 16 is equal to 13. Explanations should be given in the children's own language, with as much variety as possible. They may also see that if 13 is added to $\frac{1}{4}$ of 16, 17 will result.

iii $16 + 3 = 20 - 1$

Discussion of this equation should lead to the fact that if 16 and 3 are added, the result is equal to the difference between 20 and 1.

b As experience widens, and more number bonds are known, some solving of equations without rods is possible.

It may be in two directions :

i Complete solving of simple equations.

ii Solving of certain parts of more difficult equations. For example, in $10 + 9 - 2 \times 4 = \square$, the child may know that $2 \times 4 = 8$ but need his rods to find the difference between 19 and 8.

In solving equations, use is made of substitution in the reverse direction to that seen in the renaming of a number. For example, when in solving $4 \times 2 + \frac{1}{2} \text{ of } 6 = \square$, a child simplifies this to $8 + 3 = 11$, he has substituted 8 for 4×2 and 3 for $\frac{1}{2}$ of 6.

NOTES ON METHOD

1. All four stages of Section D proceed together and there is no rigid break between work with the rods and work without them. Hence, in practice, Stages 24 and 25 merge. They are not separate and distinct. The child moves freely from one to the other, gradually gaining confidence in abstract work.

2. As always, oral work is vital. In each of the steps given above, the child should be given adequate oral work before he begins any extensive written work. Too early a move to written work and too great an emphasis on it may prohibit the development of a confident and easy mastery.

3. Some direction to children as to what form of substitution is to be used may be necessary and helpful :

a to ensure that children use all operations,

b to give experience of number bonds to 10, and

c to consolidate use of brackets with subtraction.

However, teachers should beware of over-direction. If a child is doing all the required types of work without suggestions from the teacher, his spontaneity may be lost if too much direction is given.

4. Quite often a child begins to use numbers above twenty that he has met in his counting work but not, as yet, studied intensively. This tendency, which shows the child's beginning to reach out and experiment for himself, is, of course, encouraged.

TESTING

Testing is done in terms of the exercises outlined. Ability to substitute with good variety is required, and this can be seen in daily work. Ability to manipulate equations can be assessed only by examining critically samples of the child's manipulation. Similar attitudes should prevail with creating, interpreting, and solving equations. The teacher, working closely with her group, will probably feel no need for a special test, but will be able to assess a child's progress from the work he offers. If a special test is required it must be given as an exercise in the particular field to be checked.

STAGE 26

AIM

To continue the study of fractions so that the ideas already understood may be expressed with the numbers from one to twenty.

NOTES ON AIM

The work with fractions, like the rest of the work in this section, is simply an extension of the work of Section C. It is necessary for the teacher to be familiar with Stage 22 so that he can assess the child's prc´ciency with the ideas introduced there, and extend the work to the numbers now being used.

DEVELOPMENTAL STEPS

1. A Fraction as a Relation between Two Numbers

Once this concept has been mastered (see Developmental Step 2, Stage 22), it should be applied to fractions such as :

$\frac{11}{12}$, $\frac{14}{16}$, $\frac{9}{20}$, $\frac{15}{12}$, $\frac{18}{16}$.

2. The Fraction as an Operator

The work of Stage 22 must be extended so that the child can use the fraction as an operator :

a where the denominator is the same as the number operated on, e.g., $\frac{8}{12}$ of 12, $\frac{9}{17}$ of 17, $\frac{11}{20}$ of 20;

b in any other case within the scope of numbers being used, e.g., $\frac{3}{5}$ of 15, $\frac{2}{7}$ of 14, $\frac{3}{4}$ of 16.

3. Use of Fractions in Other Work

As the child gains experience, fractions should be included in his creative work, and in uncompleted equations. It will be noticed that the child's use of fractions in creative work is limited by his number experience. For example, he will find it much easier to rename 4 as $\frac{1}{3}$ of 12 than as $\frac{1}{7}$ of 28, not because the fraction $\frac{1}{3}$ is easier to understand than $\frac{1}{7}$, but because the first involves the familiar idea, $3 \times 4 = 12$, and the second the unfamiliar idea, $7 \times 4 = 28$. It would be a simple matter to substitute $\frac{1}{7}$ of 14 for 2, because $7 \times 2 = 14$ is a fact within the child's experience.

A simple reading from this pattern would be :

$7 + 4 + 1 = 2 \times 5 + 2$

Substitution could produce :

$\frac{1}{2}$ of $14 + 2 \times 2 +$ $\frac{1}{9}$ of $9 = 2 \times 5 +$ $\frac{2}{3}$ of 3

4. Value Relations

This work continues throughout the section, and must include fractions. (Refer Section C, Stage 15, Developmental Step 2 e for suggested exercises, some of which should be possible at this level.)

If it is found necessary to illustrate halving of odd numbers with rods, it can be done simply by changing the unit. For example, to show ½ of 5, call the red rod 1. The orange rod then represents 5. Half of the orange rod is the yellow rod, which measured by the unit (red) is 2½.

NOTES ON METHOD

1. This is one of the four parallel stages of Section D. There should be no break between the fraction work of Section C and that of this section.

2. The use of the correct vocabulary was stressed in Stage 22. It is equally important in this.

3. The importance of oral work should be noted.

TESTING

The tests are the same, except for the numbers used, as those given in Stage 22.

SUMMARY OF WORK COVERED IN SECTION D

1. Ordinal Number

i Counting has been continued throughout the section:

a by ones at least to 144, forwards and backwards;

b by twos, threes, fours .... twelves, to numbers limited by the varying abilities of children. This counting has been used to introduce the pattern and order of the number system through number charts and other exercises; and

c in the tens sequence, starting from any number.

ii The idea of order in a series has been maintained (twenty-third, sixty-fifth).

iii Figures to 144 can be recognized and written.

iv Words to "twenty" can be recognized and written.

v Doubling and halving may have been commenced.

2. Basic Mathematical Ideas

The child's understanding of equality and the four basic operations has increased with his wider experience in manipulating equations. There has been an attempt to inter-relate the operations. Interpretation of equations has stressed the meanings of signs and the operations they represent, and successful solving of equations reveals the child's understanding.

3. Manipulating of Equations

Much experience has been gained in manipulating equations by:

a substituting for, or renaming, elements;

b rearranging the elements—i without substitution,
ii with substitution.

4. Abstract Work

Throughout the section, the child has continued to work both with and without rods, and he shows increasing confidence with abstract work.

5. Fractions

The understanding of a fraction as a relation between two numbers and as an operator has been consolidated and extended to larger numbers. Children use fractions in their creative work and in solving equations.

6. Value Relations

Throughout the section, exercises have been given to extend the child's ability to express, in varying ways, the relationships between rods.

7. Oral and Written Work

Although much written work has been done, oral work is still of very great importance.

8. Interpreting and Solving Equations

Children can now interpret and solve equations of increasing complexity, using rods where necessary. Emphasis is greater, however, on creating and manipulating equations than on solving them.

9. Use of Zero

A definite treatment of zero has been undertaken, and children use it confidently.

10. Preparation for Automatic Response

Although knowledge of number bonds has not been a main objective of the section, the child has gained, largely through substitution, a wide experience of numbers to 10 and their components.

11. General Notes

i The main purpose of Section D was to consolidate and extend the experience, the understandings, and the techniques of Section C. The child should now be aware that all he did with numbers to 10 he can now do with numbers to 20.

ii It is still important that knowledge gained is from the child's own experience. He is asked to explain his ideas and justify his statements, that is, to communicate verbally as well as through written symbols.

iii The numbers 11–20 have been used as a means of expressing ideas and exercising various techniques. The numbers 1–10 have been considered in more detail. At the same time, counting activities have made children aware of a much wider field that will be explored in detail in the following sections.

iv Marked differences in quality of work are evident. All children must be able to cover the basic work of each stage, but many will do work of a much greater complexity than others.

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CURRICULUM GUIDE
PURE NUMBER COURSE

Section E—The Study of Basic Mathematical Ideas in Terms of Numbers to 144

Education Department of Victoria

10251/65

CURRICULUM GUIDE
PURE NUMBER COURSE

Section E—The Study of Basic Mathematical Ideas in Terms of Numbers to 144

AIM

To extend and consolidate the work of previous sections using larger numbers.

COMMENTS

1. Section E is similar to the last in that it is devoted to the study of the mathematical ideas that have been the basis of the whole course up to this section. Thus the aim continues to be the mastery of concepts such as equality and the four operations and fractions and their inter-relation. The difference between this section and the previous ones is that larger numbers are used.

2. The introduction of larger numbers is important. While it marks no change in the basic aim, it does introduce a change in emphasis. The larger numbers widen the child's number horizon, increase his opportunities for experiment and exploration, and make his ability to handle numbers without recourse to the rods all the more imperative. Section E, therefore, sees an increased emphasis given to the abstract manipulation of numbers. This ability does not assume importance suddenly, or for the first time, during this section. What happens is that the ability, already developed to some extent in previous sections, is now given increased emphasis.

3. The suggested range of numbers has been chosen for two reasons. First, a wider range is necessary if the child's understandings are to be developed. Secondly, the range covers all the number facts to which automatic response will be required later. However, it must not be imagined that children will never use numbers greater than 144. Many will do so in their creative work, and they should be encouraged in this.

4. In Section F, the child's creative and manipulative work will be primarily with numbers to 144, as it is in this section. The difference between the two sections (as far as this aspect of the work is concerned) is not in the range of numbers used, nor in the ideas studied, but in the depth and the complexity of the treatment.

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5. Automatic response to number bonds is not a requirement of this section, but it will be of Section F. It will be achieved only if the basic experience has been obtained. In Section D the child gained a broad experience of numbers 11 to 20, and a deeper knowledge of numbers to 10. In this section, he gains a broad experience of numbers to 144, and through planned exercises (see Stage 29) deepens his study of numbers to 20. Towards the end of the section, children should have had sufficient experience of numbers to 10 to tabulate the number facts within that range and to begin working on them in preparation for automatic response.

6. The five stages proceed side by side, not consecutively. Stages 27, 29, 30, and 31 continue similar work from previous sections. Once Stage 28 (place value) has been commenced, it should be dealt with in conjunction with the work of the other stages.

7. Much more time will be needed for this section than for Section D. For children of average ability, approximately two school terms will be necessary.

STAGE 27

AIM

To extend the study of ordinal number, particularly through counting skills and exercises, to reveal the pattern and order of the number system.

NOTES ON AIM

1. Counting is a skill important in its own right. The development of this skill is continued in this section till the child is able to count confidently within the range of numbers beyond those with which he works in detail.

2. The Notes on Aim, Stage 23, Section D, stressed the importance of the realization that the number system is not a haphazard collection of numbers, but is highly organized. It would be impossible to over-emphasize this point. Mathematics has been defined as " the classification of all possible patterns ". (W. W. Sawyer—*Prelude to Mathematics*, page 12.)

The realization of order in the number system reduces greatly the amount of learning of number facts that is necessary. For example, if a child knows that $7 + 4 = 11$, and if he realizes the organization of the number system, he should be able to discover that $17 + 4 = 21$, $27 + 4 = 31$, etc.

It is worth repeating a statement made in Notes on Aim, Stage 23, namely: " It is the view of the number system as a logical arrangement of numbers that helps to distinguish the mathematician from the drudge who sees numbers as separate and distinct entities. One uses the logic of the system to assist his work; the other struggles with his work because he has not seen the logic."

3. It will be noted that some children display much greater ability than others with activities involving pattern and order. It is important that each child should develop to the best of his ability.

DEVELOPMENTAL STEPS

NOTE: These steps are not necessarily sequential.

1 **Counting in the Range from 1 to 1,000**

The counting exercises are, for the most part, oral. There is no need for the child to write the numbers he uses, or (as far as counting is concerned) to have a thorough mastery of place value. The first requirement is a knowledge of the " run of numbers ". Once the child is aware of this, he may be required to do some written counting exercises. The required counting falls into three main categories:

a **By Ones to 1,000** This is merely an extension of the work of Section D, and serves to lay out the area to be studied later in greater detail. Owing to the wide range, counting will be within specified limits. The child should be able to start anywhere between one and one thousand and count forwards or backwards to a specified point. For example: " Count by ones from 489 to 530." " Count back from 936 to 897."

b **Group Counting from Nought (Zero)**—by twos, by threes, and by all the numbers to twelves. This is a direct continuation of the work of Section D, and can probably best be studied through patterns. (See Step 2 below :—Patterns in Number Sequences.)

c **Group Counting from Any Number**—by twos, by threes, and by all the numbers to twelves. This is the activity commonly known as serial counting and is best studied through pattern work.

No specific requirement is made for memorized counting, but Section F states: " Group counting should be done abstractly at least to 144 to prepare for the later work on multiplication tables." (Course of Study, page 15.) As counting experience extends, some memorization will certainly take place and should not be overlooked.

Stress should be placed on work with both odd and even numbers.

Mixed counting is a useful exercise to consolidate counting. For example: " Count by threes to 30 and go on by fives to 50." (3 6 9 12 15 18 21 24 27 30; 35 40 45 50.) " Count by sixes to 36 and go on by fours to 48." (6 12 18 24 30 36; 40 44 48.)

2. **Patterns in Number Sequences and Use of Number Charts**

It would be impossible to list here all the activities that could be attempted. Once aware of the type of work that can be accomplished, teachers will find many more activities. As stated in Section D, investigation and discovery are of far greater importance than the knowledge of a few teacher-imposed ideas. Children should use the term " pattern " as referring to an idea that recurs.

a **" Final Digit " Patterns** This work, commenced in Section D, should be continued here. Comparisons of certain patterns can prove interesting.

e.g. i

Counting by twos	Counting by fours
2	4
4	8
6	2
8	6
0	0
2	4
4	8
6	2
8	6
0	0

A study of these patterns can show :

—Alternate numbers from the first column are taken to form the second.

—If any number in the first column is doubled, the final digit of the result is the number opposite in the second column.

Similar comparisons can be made with other " counts " as children are ready for them.

ii Counting by twos	2	4	6	8	0
Counting by fours	4	8	2	6	0
Counting by sixes	6	2	8	4	0
Counting by eights	8	6	4	2	0

If this pattern is set up on the black-board and studied by a group of children, the following discoveries may be made:—

—There is a relationship between the first and fourth columns.

—The same relationship exists between the second and third columns.

—Added horizontally each row totals 20.

—The sum of the digits in each column of the first and fourth rows is 10.

—The sum of the digits in each column of the second and third rows is 10.

Reasons for these facts will be obvious to the teacher, but may be quite obscure to the child. The important thing here is the discovery, and it should be encouraged, not forced. Children may be aware of such facts long before they seek a reason. In fact some children may see that " something happens ", but never ask why.

It is the application of final digit patterns that enables children to count with larger numbers. Once he recognizes the pattern for counting by threes (3 6 9 2 5 8 1 4 7 0) and knows he can apply it, a child can count from 781 by threes, saying 781 784 787 790 793 796 . . . .

A study of counting by nines usually results in two discoveries. (Here it is necessary to write the whole number, not only the final digit.)

9	The units decrease by one,
18	and the tens increase by
27	one.
36	
45	The sum of the digits is
54	always nine.
etc.	

b **Number Charts** Teachers should use any array of numbers that forms a profitable basis for study of pattern and order. Large number charts on black-board or paper, or a special counting board as described in Notes on Method, Stage 23, are particularly useful for group study. If brighter children show themselves ready for individual work, squared paper of suitable size could be used.

The " eights " chart is shown below. It can consist of as many horizontal rows as the teacher finds useful at a given time. Visual patterns are marked. Numbers in bold type joined by broken lines indicate diagonals revealed in counting by threes. Heavy black line shows zigzag pattern made when counting by twelves is marked.

1	2	**3**	4	5	**6**	7	8
9	10	11	**12**	13	14	**15**	16
17	**18**	19	20	**21**	22	23	**24**
25	26	**27**	28	29	**30**	31	32
33	34	35	**36**	37	38	**39**	40
41	**42**	43	44	**45**	46	47	**48**
49	50	**51**	52	53	**54**	55	56
57	58	59	**60**	61	62	**63**	64
65	**66**	67	68	**69**	70	71	**72**
73	74	**75**	76	77	**78**	79	80
81	82	83	**84**	85	86	**87**	88
89	**90**	91	92	**93**	94	95	**96**

This chart could be used to show:

i Simple counting by eights, i.e., 8 16 24 32. The final column gives a visual image of the sequence being repeated. It may, in fact, be useful to introduce some group counting by means of number charts.

ii Counting by eights, starting at any number other than 8, e.g., "Start at 4 and count by eights." (4 12 20 28 36 — — —) "Start at 7 and add on eights." (7 15 23 31 39 47 — — —)

iii Visual patterns:

—Counting by sevens, if markea on the chart, reveals a diagonal line from right to left.

—Counting by nines reveals a diagonal line from left to right.

—Counting by threes shows diagonals less direct.

—Counting by twelves results in a zigzag line.

(These same "counts" marked on different charts show different patterns.)

iv That each vertical column contains either odd or even numbers, not both.

v That the final digits in each vertical column decrease by two.

vi That sequence is repeated in each vertical column.

Children should begin to ask, "Do these things happen in other charts?" The answer of course would be, "Work with others, and find out."

c **Discovering and Continuing Pattern in a Sequence** Children are aware of pattern in their environment and arrange materials of many kinds according to pat'^rns of their own creation. The ability to recognize and repeat pattern should be applied to number.

A study of visual patterns forms a good introduction. Children can be asked to find a pattern and continue it in examples such as these:

i △ ○ ○ △ ○ ○ △ ○ ○ — — —
ii □ □ □ △ △ □ □ □ △ — — —
iii 1 2 3 1 2 3 — — — — — —
iv 4 4 8 8 4 — — — — — — —

Once children have demonstrated their ability to identify and continue a pattern, this ability should be applied to counting. Exercises need to be graded according to children's experience.

Suggested exercises:

i 4 8 12 — — —
ii 9 19 29 39 — — —
iii 6 — 18 24 30
iv 5 8 11 14 — — —
v 99 88 77 66 — — —
vi 88 86 84 — — —
vii 49 44 39 34 — — —

3. **Doubling and Halving of Numbers**

This is a most valuable exercise. Once the child has been given an understanding of "double" and "half", the procedure can be as follows :—

> The child is given a number, e.g., 2, and asked to keep doubling so that the following series is formed :

2
4
8
16
32
64
128
256
512
1024
2048
4096
8192

There is no upper limit to this exercise. If doubling was introduced as suggested in Section D, children may then have taken only a few steps, which they would gradually extend with experience. Some children may now be able to double numbers up to one million, others will drop out in the thousands. This work obviously involves an understanding of place value and an application of the ideas mastered in Stage 28.

It is most important that the child be given no formal instruction in the method of doubling. He is left to evolve his own method. There is no need for a teacher to probe to get the next number. If no child is able to give it, the exercise should be left till later. The whole purpose of the exercise is defeated if the child learns to apply a rule taught by the teacher. Such a rule may, of course, result in his doubling to even higher numbers than he could have done otherwise, but the measure of success is not necessarily the size of the number reached. The aim is to force the child to think and apply his understandings and, only in so far as he does this, is the exercise successful.

Side by side with the doubling work the child learns to halve numbers. He may, for example, begin doubling :

2
4
8
16
32
64

He is then asked to halve sixty-four and to halve each succeeding answer—-thus he gets a series such as :

$$\begin{array}{c} 64 \\ 32 \\ 16 \\ 8 \\ 4 \\ 2 \\ 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{16} \\ \frac{1}{32} \end{array}$$

When the child is asked to halve one, he has no trouble to give the answer $\frac{1}{2}$. From everyday experience he will know that the half of $\frac{1}{2}$ is $\frac{1}{4}$. He then begins to see the pattern of the numbers :

$$\begin{array}{c} 4 \\ 2 \\ 1 \\ \frac{1}{2} \\ \frac{1}{4} \end{array}$$

and, once he has seen this pattern, is in a position to halve numbers to the same extent as he can double them.

Halving is just as important as doubling, and must not be neglected. The implications of the exercise involve both doubling and halving. It is recognized that at first children may be able to double further than they can halve, because their experience is wider with whole numbers than with fractions, and because they have not yet recognized the pattern. But the teacher must be constantly on the alert for the appropriate moment to press for halving, and should then use both sides of the exercise.

It must be stressed with halving, as with doubling, that the point of the exercise is to give the child the opportunity to think and to apply the understandings he has gained earlier; the teaching of formal rules defeats this purpose.

In Section F, children are required to double and halve, beginning with any number, and to use the exercise with multiplication ideas. Some children, however, find, during work with Section E, that if they double one factor and halve the other they still have the same product. For example : $4 \times 4 = 16$, $8 \times 2 = 16$, $16 \times 1 = 16$. They apply this knowledge in their creative work.

While the majority of children will not be aware of these ideas, and there must be no forcing of them, teachers should watch for and encourage them when they occur. The use of doubling and halving greatly enriches the child's work and extends his experience.

4. Link with Value Relations

Staircases made from rods can be a useful aid to counting through value relation work. (See Stage 29.)

5. **Number Lines**

While the course makes no mention of number lines, they can be of considerable value. Teachers wishing to use them should make reference to the *Education Gazette*.

6. **Recognizing and Writing Figures to 1,000**

It is necessary for the child to recognize and write the figures from 1 to 1,000.

7. **Recognizing and Writing Words to One Hundred and ˈorty-four**

As mentioned in earlier sections, this is not strictly a mathematical skill, and can, if the teacher so desires, be handled in spelling and reading periods. The limit is simply keeping in line with the range of numbers for detailed study. The child who can write words to one hundred and forty-four should have no difficulty in writing words to nine hundred and ninety-nine, and this extension can be made at the teacher's discretion.

NOTES ON METHOD

1. It has been stressed (and because of its importance must be stressed again) that the work in this stage is mainly oral. Written work on any large scale would seriously limit the amount of work that could be covered. Thus writing in figures and words must not be allowed to dominate the work and interfere with the oral activities.

2. The variety of counting is most important. The starting-points for counting should be varied, the numbers used should change, and the child should count backwards as well as forwards.

3. There is no necessity for the child to be automatic in all of the counting work. He certainly should be able to do the simpler exercises such as counting by ones, by twos, and by threes without assistance. If, however, he still needs to refer to a number square to count by sevens, for example, there is no need to spend time drilling the sequence. The pattern and order of the number system can be seen without a complete memorization. Automatic answers are not needed until the tables are being memorized—a task that is delayed until Section F.

4. The work in this stage continues throughout the whole section side by side with that outlined for the following stages. Some of the work of this stage will only be possible when place value has been studied. (See Stage 28.)

5. It is of interest to note that the work done in serial counting is significant in laying the foundation for an important secondary school topic, "Arithmetical Progression". Likewise, doubling and halving are important for laying the foundation for another secondary school topic, "Geometrical Progression".

TESTING

1. Counting

Counting skills can be most simply tested orally by questions such as :

" Count by ones from 390 to 410."

" Count backwards by twos from 816 to 788."

" Count by threes from 21 to 45."

" Count by tens from 500 back to 440."

" Start at 19 and say the odd numbers to 47."

" What even numbers come between 240 and 256? "

2. Realization of the Pattern and Order of the Number System

Any formal attempt to test this aspect of the work would defeat its own purpose. It is best assessed by observation of everyday work. However, it is possible to obtain some idea of the child's grasp of this realization by giving him an exercise such as :

" If $8 + 4 = 12$, what must $18 + 4$ equal? "

" What must $28 + 4$, $38 + 4$ . . . . . equal? "

A child who understands the way the system is organized should quickly see what the answer is, even though these facts have not been specifically taught.

The application of a " final digit " pattern to larger numbers is another exercise which illustrates understanding of the logic of the number system.

Pattern in a sequence can be tested. For example, " When you see the pattern, complete these sequences :—

a 6 16 26 36 — — —
b 9 12 15 18 21 — — —."

3. Doubling and Halving of Numbers

The criterion here is not how far the child can go, but his understanding and his ability to double and halve numbers within the limits of his mechanical achievement.

4. Recognition and Writing of Words and Figures

Formal testing is necessary here to ensure accuracy in an essential skill. Methods of testing are straightforward. (See Stage 14, Section C; and Stage 23, Section D.)

STAGE 28

AIM

To develop an understanding of place value.

NOTE ON AIM

While place value could have been studied earlier (see Notes on Method, Stage 23) it was not necessary for successful completion of that section. However, an understanding of place value is essential for success with future work. and a definite and detailed study must now be commenced.

DEVELOPMENTAL STEPS

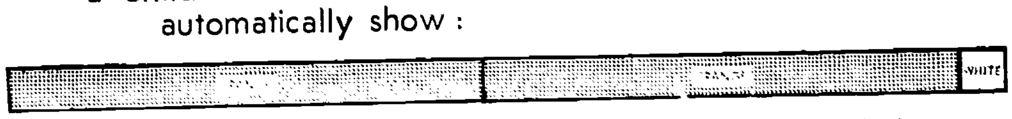
1. Awareness of Numbers to be Studied

Before any attempt is made to study place value, the child must be aware of the numbers, first through counting, then through reading and writing of the relevant numerals.

2. Introduction of the Concept of Place Value

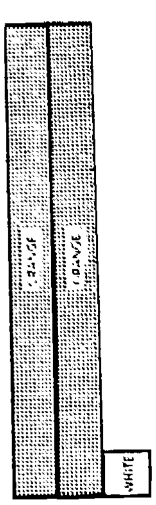
Rods can be used for this step. The following approach is suggested :

a Children are asked to show 21 with rods. Most children automatically show :

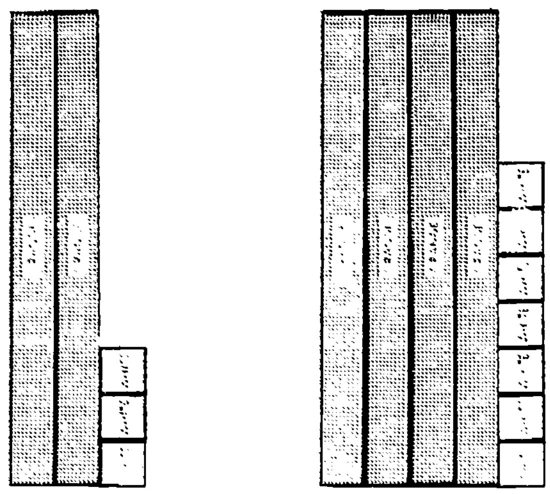


This is the representation needed, and it will be necessary to obtain it by suggestion and questioning if other rods are used. A child is then asked to write " 21 " on the chalk-board. Questioning leads to the fact that the " 2 " refers to the tens and the " 1 " to the one, or unit.

When studying the larger numbers the child may be introduced to the convention of placing the tens side by side instead of end to end—thus twenty-one can be shown as—

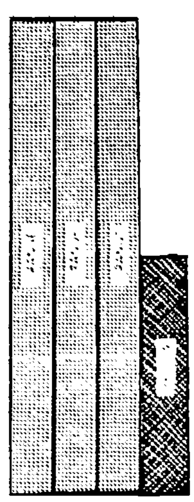


b The teacher presents numbers such as 23, 47, as shown—



The numerals are written and studied. Children see that in 23, the 2 refers to the tens and the 3 to the units; and in 47, the 4 refers to the tens and the 7 to the units. That is, in the writing of a two-digit number, the units figure is written in the right-hand place, and the tens figure in the left-hand place.

When sufficient experience has been given, the white rods can be replaced by a single rod, so that 35 would be shown as—



It would be agreed that the yellow rod represents five white rods, and " 5 " represents 5 units.

3. **Extension of the Concept (with use of rods where necessary)**

a A particular study of the numbers 11, 22, 33 . . . . . 99 will emphasize the difference in value according to place. For example, in 11, the digit " 1 " is written twice. Questions should lead children to see that the left-hand " 1 " is worth ten times as much as the right-hand " 1 ". There is one " unit " in the right-hand (or units) place, and one " ten " in the left-hand (or tens) place.

b A study of numbers where the units digit is greater than the tens digit, e.g., 26, 45, 68, etc., should be made to reinforce the idea that position affects value.

4. **Study of Numbers Involving Zero**

The teacher may introduce this idea by presenting three orange rods and saying :

" If white is one, what number am I showing? " (Thirty.)

" Write the number." (30)

" What does the 3 mean? " (3 tens.)

" Why do we need to write the 0? " (Answers should indicate that there are no separate units.)

" Would it matter if we did not write the 0? " (Yes; the number would then mean only 3 units and not 3 tens.)

Thus the child has been shown the importance of zero. The zero in 30, 40, or 50 saves confusion by showing that three, or four, or five tens are indicated. (The concept of zero as another number is studied later.)

The child should study the numbers 10 20 30 40 50 60 70 80 and 90 to ensure that the idea is grasped. Questions and instructions might proceed as follows :—

" Show me nine with your rods."

" Write it down."

" Now show me ninety."

" Write it down."

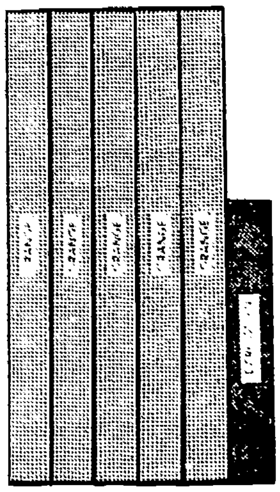
" Why did you write down the nought? "

The child should be able to say that he has written the nought to show that he means " nine tens "—not " nine units ".

5. **Consolidation of the Concept (through study of numbers without rods)**

A wide variety of exercises should be given to consolidate the concept of place value. For example :

a The teacher asks the children to pick up sets of rods such as :



and to record the numbers they represent.

b " Write the number made of 4 units."

c " Write the number made of 8 tens."

d "Write the number made of 6 tens and 3 units."

e "Write the number made of 7 units and 8 tens."

f "Which is the greater, 37 or 73?"

g "Which is the lesser, 29 or 81?"

> NOTE: The signs $>$ (is greater than) and $<$ (is less than) could be introduced to those children able to use them. There is no specific need for them in this section, but they will be found to add interest to the exercises.

h "Write these in order of size, starting with the smallest: 31 19 50."

i "Write these in order of size, starting with the largest: 26 15 79."

j "What is the 8 worth in 98? in 83? in 86?" "How do you know?"

k "Is the 3 worth more in 93 or in 37? In 32 or in 38?" "Why?"

l "Make the smallest number you can with these figures: 3 and 5."

m "Make the largest number you can, using only these figures: 1 and 9."

6. **Extended Notation**

a Understanding of place value will be deepened through the use of extended notation, which is to be used later to assist understanding of the written processes. From a single extension,

e.g.
$$26 = 20 + 6$$
$$39 = 30 + 9$$
$$70 = 70 + 0$$

the child should progress to exercises such as the following:

$24 = 20 + \square$	$9 + 60 = \square$
$38 = \square + 8$	$\square + 30 = 33$
$\square = 50 + 9$	$2 + \square = 52$
$40 + 7 = \square$	$\square = 9 + 60$
$30 + \square = 36$	$38 = \square + 30$
$\square + 2 = 82$	$52 = 2 + \square$

b Another useful form of exercise is as under:

$36 = 3$ tens $+ 6$ units
$64 = \square$ tens $+ \square$ units
$97 = 9 \square + 7 \square$
$\square$ tens $+ \square$ units $= 43$
$\square$ units $+ \square$ tens $= 75$
$\square = 6$ tens $+ 4$ units
$\square = 3$ units $+ 8$ tens

7. **Introduction of Hundreds**

a Questions and instructions could follow a pattern such as:

"Show me seventy with orange rods."
"How many rods did you use?" (Seven.)
"What does seventy mean?" (Seven tens.)
"Show me eighty with your rods."
"What does eighty mean?" (Eight tens.)
"Show me nine tens with your rods."
"What number have you?" (Ninety.)
"Write ninety in figures."
"Show me ten tens."
"Does anyone know what this number is called?"

In this way the child is shown that one hundred is not a different type of number from seventy, eighty, or ninety, but simply means ten tens in the same manner as those numbers mean seven, eight, and nine tens. And just as nine tens or ninety is written as 90, so ten tens or a hundred is written as 100.

b Once this is clear the teacher may ask questions such as:

"Write down 20 tens." (If the child is in difficulty he may use the rods. Normally he will not.)
"What number is that?"
"Write down 30 tens."
"What number is that?"
"Write down 70 tens."
"What number is that?"

8. **Study of Numbers from 100 to 999**

Once the child has mastered the previous step, the study of numbers over one hundred can commence. Already in his counting work the child should have met these numbers—this work with place value is not introducing them for the first time.

A number is written and studied. For example, children could be asked to write in figures one hundred and sixty-nine (169).

They should be able to tell, when questioned, that the "1" means one hundred, the "6" means six tens, and the "9" means nine units.

Where previously only two places were needed, a third is now used, and identified as the "hundreds" place. A numeral such as 478 can be written and discussed in the following terms:

"Which figure is in the tens place?"
"What is the 8 worth?"
"Name the figure in the hundreds place."

9. **Exercises for Consolidation**

Varied exercises should be given for experience and consolidation. For example:

a Write the number made up of 6 hundreds, 4 tens, 8 units.
b Write the number that has 7 units, 3 tens, 9 hundreds.
c Write the number made of 2 hundreds and 2 tens.

d Write the number made up of 2 units and 2 hundreds.

e Which is the greater, 263 or 236? 183 or 798? 400 or 401? 269 or 926?

f Which is the lesser, 326 or 263? 209 or 763? 111 or 999?

NOTE: As mentioned in Step 5, the signs $>$ and $<$ can profitably be used in examples e and f if the teacher finds them appropriate.

g Write these in order of size, starting with the smallest: 362 298 706.

h Write these in order of size, starting with the largest: 823 809 888.

i Is the 5 worth more in 245 or 254? In 526 or 625?

j Make the smallest number you can with these figures: 6 2 8.

k Make the largest number you can with these figures: 1 9 7.

l Make as many different numbers as you can using these figures: 3 9 8. Which is your smallest number? Which is your largest number?

10. **Extended Notation (using numbers to 999)**

This is a simple extension of Step 6 to include hundreds.

NOTES ON METHOD

1. It has been mentioned that the study of place value could have been commenced in Section D. When it is commenced, the teacher should remember that the study of place value gives added depth to the child's understanding of a number. For example, before he studies the place values contained in the number 169, he ought to have met it through counting and know how to write it.

2. The importance of giving the child sufficient variety of experience in any one step before proceeding to the next cannot be exaggerated. Unless this experience is given, understanding may be uncertain. If understanding is uncertain, the introduction of a new difficulty is dangerous.

3. It should be noted that the rods can be used in the first steps when the concept of place value is being introduced. Once it is mastered, however, the rods need not and should not be used. The numbers above one hundred should be studied without using the rods at all.

4. Some experience of place value is necessary for understanding of numbers to 144, with which this section is concerned. Therefore, this stage needs to be commenced early in the section, and to run parallel with all the other stages. In fact, all aspects of work (counting, pattern and order, value relations, creating, manipulating, interpreting and solving equations, and fractions) are being developed side by side.

TESTING

A series of activities and questions such as were outlined in Developmental Steps would be of value.

STAGE 29

AIM

To develop the ability to create and manipulate equations using the numbers from zero to one hundred and forty-four.

NOTES ON AIM

1. The child's ability to use his understanding of basic mathematical ideas to create and rearrange equations has been fostered in the last three sections. In this section it receives even more attention.

In addition, greater stress is placed on the ability to manipulate equations without using rods at all. This, too, has received attention in other sections, but it is now developed to a much greater degree.

2. As in previous sections, work is organized in terms of types of equations, but, because of the wide range of numbers involved, attention must be given to the selection of numbers for experience and study. The range is too wide for all numbers to be studied in great detail and, in any case, little needs to be known about some numbers. For example, probably all a child needs to know about 89 is its counting position (it is 9 more than 80, 1 less than 90), and it is a prime number. It is, however, useful as a vehicle through which to study mathematical ideas (see Developmental Step 5). On the other hand, there are some numbers that need to be studied because of their intrinsic value. These are numbers such as 24, 36, 48 . . . . . which are frequently used in multiplication and division tables. Children need considerable experience of these. In addition, numbers with fewer factors (such as 21, 25, 35, 42) must also be included.

3. With the above considerations in mind, the teacher should make a selection of numbers for study, remembering that some time will be available in Section F for further study of numbers to 144.

DEVELOPMENTAL STEPS

NOTE: The work of this stage is basically a continuation, using larger numbers, of that already done—that is, the oral and written creation and manipulation of equations, first from rods, but with increasing emphasis on abstract work. As children gain confidence and experience, their need for rods decreases. In fact, most children find rods an encumbrance when working with larger numbers.

In general, children start with a pattern or small mat, read and write equations from it, using the techniques of manipulation already introduced. Parallel with this is the continuation of abstract work, which gradually increases in importance till very little use of the concrete material is necessary. However, there will often be a group of weaker children who will need the help of rods for a much longer period than others.

The aspects of work that now need to be noted or developed are dealt with below. These steps, however, are not worked sequentially.

1. Types of Equations

Different types of equations must be used (both when working with rods, and in abstract work) to give experience of all operations and their relation to one another. (See Sectic · D, Stage 24, Organization of Work; and Section E, Stage 30.)

It should be remembered that the broad aim of this stage is to develop ability to create and manipulate equations, thus showing understanding of basic mathematical ideas. This understanding is best seen when a child is given one number, or an equation, and asked to create from it many different equations. If he works continually with only one type of equation, he is repeating only one type of experience and not gaining others.

The basic equations from which to work fall into three main divisions:

a **Equations Involving Addition of Different Numbers** These, when manipulated, give experience of subtraction also.

e.g.

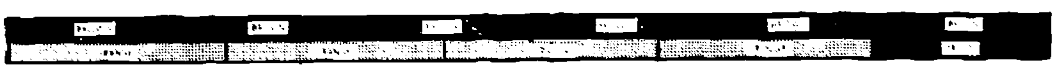
i $20 + 30 + 40 = 90$
$90 - 20 = 30 + 40$
$90 - (20 + 30) = 40$
$90 - 20 - 30 - 40 = 0$
$30 = 90 - 20 - 40$

ii $24 + 12 = 30 + 6$
$30 + 6 - 24 = 12$
$24 + 12 - 30 = 6$
$24 + 12 - (30 + 6) = 0$
$12 = 30 + 6 - 24$

Such equations are particularly useful also for experience of number bonds, because of their simplicity. An equation such as $12 + 13 + 14 = 39$ could be set, and substitution using addition required for each term on the left-hand side.

e.g. $12 + 13 + 14 = 39$
$8 + 4 + 7 + 6 + 10 + 4 = 39$
$9 + 3 + 10 + 3 + 11 + 3 = 39$

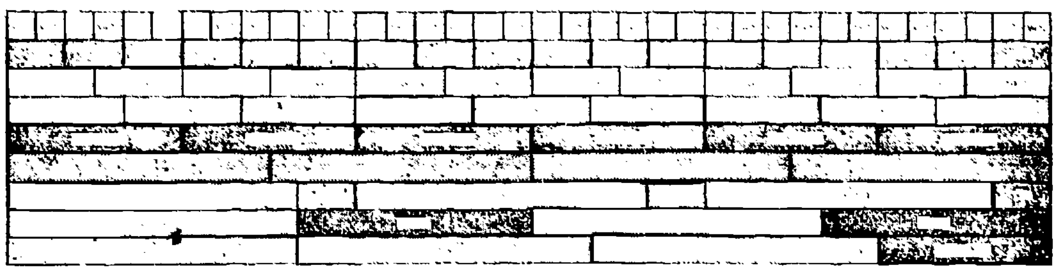
b **Equations Involving Multiplication** When rods all of one size are used to equal rods representing a given number, a most useful pattern is made, giving experience of multiplication and division, both quotition and partition. For example,



$6 \times 8 = 48$ $\frac{1}{6}$ of $48 = 8$
$48 \div 8 = 6$ $\frac{2}{6}$ of $48 = 16$
$\frac{1}{6}$ of $48 = 8$ $\frac{3}{6}$ of $48 = 24$

This idea is freely used in subsequent work without rods. The pattern could also be used for addition and subtraction.

Extension of the multiplication pattern to a mat is invaluable as background experience for later work with tables. (See Section D, Stage 24.) For example, this mat :



could result in equations such as :

$4 \times 9 = 36$

$3 \times 12 = 4 \times 9$

$6 + 6 + 6 + 6 + 6 + 6 = 3 \times 10 + 6$

$36 - 3 \times 6 = 3 \times 6$

$36 - (6 + 6 + 6) = 3 \times 6$

$\frac{1}{9}$ of $36 = 4$

$\frac{2}{9}$ of $36 = 8$

$\frac{3}{9}$ of $36 = 12$

From this mat it can also be shown that 18, 12, 9, 6, 4, 3, 2, and 1 are factors of 36; conversely, 36 is a multiple of each of these numbers. (The terms "factor" and "multiple" could now be introduced, and used by the children.)

c **Equations Combining Addition and Multiplication**

e.g. $47 = 4 \times 10 + 7$

$60 + 9 = 6 \times 10 + 3 \times 3$

2. **Substitution**

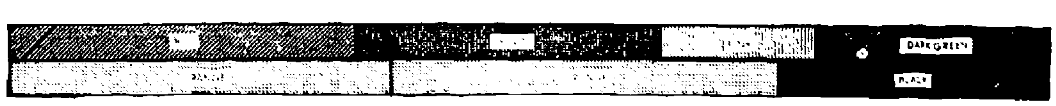
The child's experience in previous sections has given him the ability to—

a substitute numbers mentally for a rod seen in a pattern,

b substitute one numeral for another in a given equation. (A numeral is simply a name for a number. For example "6", "7 − 1", "2 × 3", "12 ÷ 2", "six", "VI", "½ of 12" are all numerals for the same number, but the simplest of these is "6".)

A great deal of experience with substitution is necessary, for it is in this exercise that the child reveals much of his understanding, consolidates his ideas, and exercises what awareness he has of number facts.

e.g. i When reading from this pattern,



the child may say—

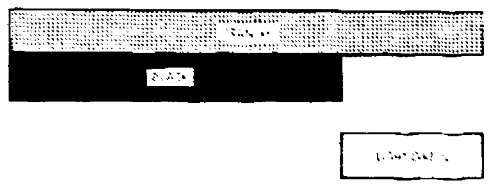
$3 \times 3 + \frac{1}{2}$ of $16 + \frac{1}{5}$ of $5 + 4 + 2 = 2 \times 10 + \frac{1}{2}$ of 14

ii Given the equation $17 + 12 = 29$, he could be expected to substitute in various ways:

$$10 + 7 + 4 \times 3 = 2 \times 10 + 9$$
$$4 \times 4 + 1 + \tfrac{1}{2} \text{ of } 24 = 4 \times 5 + 3 \times 3$$

NOTE: Substitution is affected by the child's number experience. He may rename 7 as $\frac{1}{2}$ of 14 or of 21 rather than $\frac{1}{5}$ of 35 or $\frac{1}{8}$ of 56, not because the fractions $\frac{1}{5}$ and $\frac{1}{8}$ present any greater difficulty than $\frac{1}{2}$ or $\frac{1}{3}$, but because he does not yet know that $5 \times 7 = 35$ or $8 \times 7 = 56$.

Substitution involving subtraction should now be in use. For example, $10 - 3$ or $9 - 2$ substituted for 7. If difficulty is experienced, reference to Section B, Stage 10, will be found helpful. Here the child was asked to read from a pattern such as:



"Light-green equals orange minus black."

In Section C he would have read "$3 = 10 - 7$". Attention to this type of exercise could lead to equations such as:

$$3 = 4 - 1$$
$$3 = 8 - 5$$
$$3 = 6 - 3$$

Now, any of the expressions $4 - 1$, $8 - 5$, or $6 - 3$ can replace the 3 in the equation $30 + 3 = 33$.

e.g. $3 \times 10 + 8 - 5 = 11 + 11 + 11$

$20 + 10 + 4 - 1 = 3 \times 10 + \frac{1}{2}$ of 6

Division substitution must also be sought, and similar methods could help. For example, "division" equations where the answer is 2 could be written:

$$4 \div 2 = 2$$
$$10 \div 5 = 2$$
$$6 \div 3 = 2$$
$$2 \div 1 = 2$$

Now any of the expressions $4 \div 2$, $10 \div 5$, $6 \div 3$, $2 \div 1$ can replace the 2 in the equation $20 + 2 = 22$.

e.g. $\frac{1}{2}$ of $40 + 6 \div 3 = 2 \times 10 + 2$

A further advance in substitution exercises is made when the equation given involves subtraction. For example, $24 - 6 = 18$. Substitution for 6 may produce results such as:

$$24 - (4 + 2) = 18$$
$$24 - 2 \times 3 = 18$$
$$24 - (10 - 4) = 18$$
$$24 - \tfrac{1}{2} \text{ of } 12 = 18$$
$$24 - 12 \div 2 = 18$$

Once the initial difficulty of substituting for the one number is overcome, there should be substitution for all elements of an equation.

3. **Manipulation without Substitution**

This skill has been developing from Section B, through Sections C and D, and must be continued. It is important in extending the understanding of relationships between operations. (See Stage 30—Interpretation of Equations.)

Most work has started from an "addition" equation, which, broadly speaking, has been rearranged in terms of addition and subtraction.

e.g.
$$14 + 10 + 15 = 39$$
$$15 + 14 + 10 = 39$$
$$39 = 10 + 15 + 14$$
$$39 - (14 + 10) = 15$$
$$39 - (14 + 15 + 10) = 0$$
$$14 = 39 - (15 + 10)$$
$$39 - 14 - 10 - 15 = 0$$
$$0 = 14 + 15 + 10 - 39$$

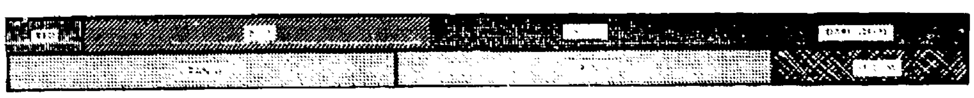
The ability to manipulate abstractly has come from the wide experience of recording equations from rod patterns. (See Stages 20 and 24.)

Simple exercises in manipulating from operations other than addition should now be introduced, in conjunction with interpretation of equations and extension of understandings of operations. Before attempting this work, however, teachers should read Stage 30, where the matter is dealt with in detail.

4. **Manipulation with Substitution**

As in Section D, rearranging and substitution should be used together, both in work with rods and in abstract work.

a Many equations could be given, either orally or written, from a pattern such as:



e.g.
$$2 + 9 + 8 + 6 = 25$$
$$20 + 5 - (\tfrac{1}{3} \text{ of } 6 + 3 \times 3 + 8) = 4 + 2$$
$$2 \times 10 + \tfrac{1}{2} \text{ of } 10 = \tfrac{8}{9} \text{ of } 9 + \tfrac{1}{2} \text{ of } 18 + \tfrac{1}{4} \text{ of } 24 + \tfrac{1}{10} \text{ of } 20.$$

b From an abstract equation such as $16 + 16 + 8 = 40$ (supplied by teacher or child) equations such as these might be given:

$$4 \times 10 - 4 \times 4 = 4 \times 4 + 2 \times 4$$
$$30 + 10 = \tfrac{1}{2} \text{ of } 16 + 2 \times 16$$
$$\tfrac{1}{3} \text{ of } 24 = \tfrac{1}{2} \text{ of } 80 - (\tfrac{16}{16} \text{ of } 16 + \tfrac{1}{2} \text{ of } 32)$$

5. **Methods of Treating Larger Numbers**

Reference has already been m[illegible] the desirability of working without rods for the larger numb[illegible] [illegible]nce a child understands the relationship between operations a[illegible] s competent in the techniques of substitution and rearrangement, all he needs is a starting-point. Two ways of obtaining this are as follows :—

a A natural breaking up of the number through extended notation, e.g., $73 = 70 + 3$. A child, working on this idea could produce—

$$73 = 7 \times 10 + 3$$

$$70 + 3 = 10 + 10 + 10 + 10 + 10 + 10 + 10 + 3$$

$$\tfrac{1}{2} \text{ of } 146 = 3 \times 10 + 4 \times 10 + \tfrac{1}{4} \text{ of } 12$$

$$70 + 3 - (\tfrac{1}{2} \text{ of } 20 + 2 \times 5 + \tfrac{1}{4} \text{ of } 40 + 20 \div 2) = 3 \times 10 + \tfrac{3}{7} \text{ of } 7$$

This is not so much a study of 73 as making use of the number to exercise mathematical ideas.

b Doubling a number already treated with rods. Children familiar with 24 from a previous study, and knowing, for example, that $3 \times 8 = 24$, may commence a study of 48 with the idea that $6 \times 8 = 48$ and work on this, producing equations such as:

$$6 \times 8 = 48$$

$$8 + 8 + 8 + 8 + 8 + 8 = 48$$

$$48 \div 8 = 6$$

$$\tfrac{1}{2} \text{ of } 16 + \tfrac{1}{3} \text{ of } 24 + \tfrac{1}{4} \text{ of } 32 + 2 \times 4 + 16 \div 2 + 8 \times 1 = 50 - 2$$

$$4 \times 10 + 8 - (8 + 8) = 4 \times 8$$

$$48 - 6 \times 8 = 0$$

$$48 - 8 - 8 - 8 - 8 - 8 - 8 = 0$$

6. Value Relations

Experience shows that individual differences are very marked in value relations work. While there should be no forcing of children beyond their capabilities, brighter children can extend their work without difficulty, and should do so.

Reference should first be made to Section C, Stage 15, which lists a graded series of exercises under the headings :

a Name Unit or Measure, Find Rods of Specified Values.

b Give Rod a Specified Value, Find Unit.

c A Combination of a and b.

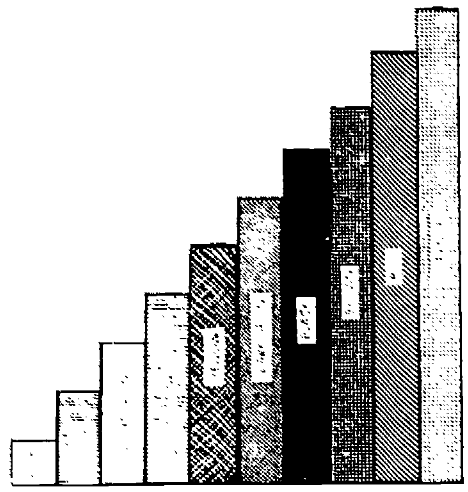
d Name Unit, Find Value of a Group of Rods.

e Use Exercises of the Above Types, Involving Fractions.

These exercises should be consolidated and extended, as suggested in Section D, Stage 24.

The new activities listed below are suggestive only, but can be used to a greater or lesser degree by most children. Note that the activities depend on children's experience with counting, fractions, and number facts, and should be kept within these bounds.

a Staircases Using "White Step"



The child can count up and down the staircase, the value of the white rod being changed to numbers within the child's counting experience.

e.g. If white is 1	:—1	2	3	4	5	6	7	8	9	10
If white is 10	:—10	20	30	40	50	60	70	80	90	100
If white is 100	:—100	200	300	400	500	600	700	800	900	1000
If white is 4	:—4	8	12	16	20	24	28	32	36	40
If white is 3	:—3	6	9	12	15	18	21	24	27	30

Questions arising from this activity might be :—

"If white is 1, what is blue?"

"If white is 5, which rod is 40?"

"If pink is 16, what is white?"

"Calling the white rod 20, which rod should we call 100?"

Children should be able to give reasons for their answers.

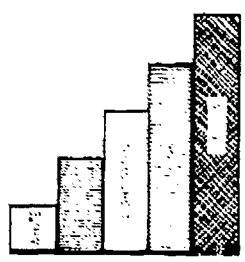
b Staircases Using Steps Other Than White

c Staircases Using Fractions

e.g. "If yellow is 1, what is white?" ($\frac{1}{5}$.

"If yellow is 1, what is red?" ($\frac{2}{5}$)

"Build a staircase using all rods from white to yellow."



"Count up the staircase". ($\frac{1}{5}$ $\frac{2}{5}$ $\frac{3}{5}$ $\frac{4}{5}$ $\frac{5}{5}$ or 1)

This exercise should be repeated, using other rods as the unit, and building the appropriate staircases.

Questions arising from this activity might be :—

" If white is $\frac{1}{6}$, what is yellow? "

" If white is $\frac{1}{8}$, which rod will be $\frac{7}{8}$? "

" If blue is 1, what is white? "

" If blue is 1, what is black? "

" If blue is 1, which rod is $\frac{5}{9}$? "

NOTES ON METHOD

1. The work of this stage should proceed parallel with that of the other stages of the section.

2. Oral work is still of great importance, and should not be neglected. Community work, where a group of children work together and the teacher records their ideas on the black-board, is of considerable value.

3. Creative work should be corrected with a view to diagnosis of errors rather than simply marking an equation right or wrong. It will be found helpful to find and follow a child's line of thought rather than to simplify an equation.

e.g. $4 \times 8 = 32$

$8 + 8 + 8 + 8 = 32$

$32 = 2 \times 8 + 2 \times 8$

$32 - (\frac{1}{2} \text{ of } 16 + \frac{1}{4} \text{ of } 32 + \frac{1}{3} \text{ of } 24) = 2 \times \frac{1}{2} \text{ of } 8$

$32 - (8 + 8 + 8) = \frac{1}{2} \text{ of } 16$

$32 - 2 \times \frac{1}{2} \text{ of } 4 - 2 \times \frac{1}{2} \text{ of } 4 - \frac{1}{3} \text{ of } 24 - \frac{1}{2} \text{ of } 8 - \frac{1}{2} \text{ of } 8 = 8$

It is obvious here that the child is thinking in terms of four eights equalling thirty-two. It is not difficult to find the four eights and the thirty-two in correct relationship in each equation, whereas it would be more difficult, and certainly much slower, to work out the last equation step by step.

4. It has been pointed out in Comments on Aim, Section D, and again in this section, that, while automatic response is not yet a requirement, the necessary background of experience must be given during these sections.

Mats for the numbers to 20 will be found useful here, particularly for facts less commonly used in substitution.

e.g. $13 + 4 = 17 \qquad 11 + 9 = 20$

$18 - 13 = 5 \qquad 9 + 7 = 16.$

By the later part of Section E, children have probably had sufficient experience of numbers to 10 to classify the addition, subtraction, multiplication, and division facts within this range, and to begin working on them for automatic response.

5. The terms " addition ", " subtraction ", " multiplication ", and " division " should be freely used by both teacher and child.

TESTING

1. As the basic aim of this stage is creation and manipulation of equations, the best test lies in observation of the child's creative work. If this reveals—

a through manipulation, an understanding of all operations,

b ability to use various types of equations, and

c ability to substitute with addition, multiplication, subtraction, division, and fractions,

there is no need for a special test. If any of these features do not appear, or appear only rarely, the child should be asked to use them. Inability or reluctance to do so reveals a need for further work.

2. The work with value relations is its own test.

STAGE 30

AIM

To extend the understandings of the basic mathematical ideas and their inter-relationship, and to further develop the ability to interpret and solve equations.

NOTES ON AIM

1. This stage is a continuation and a development of similar work in Section D. It does not introduce any new work, but rather brings together certain aspects of the child's experience.

2. Much of the work will be handled incidentally during other stages of the section. It is set out here for convenience.

3. In Section B, basic understandings of mathematical ideas were established. In Sections C and D these were consolidated through experience, both free and directed. It is important now to ensure that certain ideas are being developed in required directions.

4. It is the application of these understandings that enables the child to interpret and solve equations.

DEVELOPMENTAL STEPS

1. **Inter-relation of Operations**

In Section D, Stage 24, it was stated that "the relationships existing between operations should be constantly in the teacher's mind, and informally brought to the child's notice." Children frequently show, in their creative work, some awareness of these relationships, which should now receive more definite treatment in conjunction with manipulation of equations. (See Stage 29, Step 3.) Some of the relationships are listed below. They can be shown from rod patterns if necessary, and then be dealt with abstractly. (Note that manipularion from addition to subtraction has been freely used since Stage 21.)

a **Addition and Subtraction (an inverse relationship)**

$16 + 4 = 20$	$18 - 6 = 12$
$20 - 4 = 16$	$12 + 6 = 18$
$20 - 16 = 4$	$6 + 12 = 18$

b **Multiplication and Quotition Division (an inverse relationship)**

$5 \times 4 = 20$	$18 \div 6 = 3$
$20 \div 4 = 5$	$3 \times 6 = 18$

c **Multiplication and Partition Division**

$3 \times 5 = 15$	$\frac{1}{8}$ of $24 = 3$
$\frac{1}{3}$ of $15 = 5$	$8 \times 3 = 24$

d **Addition and Multiplication (related operations)**

$8 + 8 + 8 = 24$	$5 \times 6 = 30$
$3 \times 8 = 24$	$6 + 6 + 6 + 6 + 6 = 30$

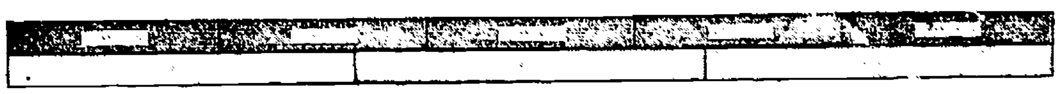
e **Successive Subtraction and Division (related operations)**

$20 - 5 - 5 - 5 - 5 = 0$	$18 \div 6 = 3$
$20 - 4 \times 5 = 0$	$18 - 3 \times 6 = 0$
$20 \div 5 = 4$	$18 - 6 - 6 - 6 = 0$

f **Quotition Division and Partition Division**

$16 \div 8 = 2$	$\frac{1}{3}$ of $18 = 6$
$\frac{1}{2}$ of $16 = 8$	$18 \div 6 = 3$

An equation of the type **xa = b** is invaluable as an illustration of the complexity of these relationships:



$6 + 6 + 6 + 6 + 6 = 30$

$5 \times 6 = 30$

$30 \div 6 = 5$

$\frac{1}{5}$ of $30 = 6$

$30 - 6 - 6 - 6 - 6 - 6 = 0$

Having studied these relationships, children should be able to take simple equations (of one operation only) and manipulate them in terms of related and inverse operations. Samples of suitable exercises are:

i Write another equation using these three numbers and subtraction:

$15 + 13 = 28$

ii Write another equation using these three numbers and addition:

$24 - 15 = 9$

iii Write another equation using these three numbers and division:

$7 \times 3 = 21$

iv Write another equation using these three numbers and multiplication:

$16 \div 8 = 2$

v Write this equation, $4 \times 6 = 24$, in this way:

$\frac{1}{\square}$ of $\square = \square$

vi Write this equation, $30 - 10 - 10 - 10 = 0$, in this way

$\square \div \square = \square$

vii Write this equation, $\frac{1}{5}$ of $20 = 4$, in this way:

$\square \div \square = \square$

2. Extension of Understanding through Interpretation of Equations

This is a continuation of the interpretative work in Stages 20, 24, and 25. The earlier work was in two directions :—

a The use of rods to illustrate an equation.

b Verbal explanation of equations created either by the child or by the teacher.

Both types of interpretation are still necessary, but emphasis on verbal explanation is greater.

It is not the intention here to cover all kinds of equations, but rather to show how the probing for understanding, established in Section B, may be applied to written and more complex work. The suggestions should be considered simply as a guide. Many other questions will occur to the teacher from observation of children's work.

Some of the complicated equations created by children are too lengthy for complete interpretation.

e.g. $35 - ($[illegible] of $6 + 10 - 4 + 14$ [illegible]

A technique useful for dealing with the more complex equations is to work from a simple equation by means of substitution to a more complicated one, and find that the same explanation holds.

e.g. $6 + 8 = 14$
$2 \times 3 + \frac{1}{2}$ of $16 = 10 + 4$

In each equation two terms are added to equal a third, first 6 and 8, and secondly 2×3, and $\frac{1}{2}$ of 16.

Examples of numeral activities are :

a $\frac{1}{2}$ of $40 + 5 \times \frac{1}{4}$ of $12 = 35$

"Which side of the equation stands for the larger number? " (Both stand for the same number.)

" How do you know? " or " What tells you this? " (The sign $=$)

" What does the plus sign mean? " (Added, or joined to.)

" What are joined? " ($\frac{1}{2}$ of 40 and $5 \times \frac{1}{4}$ of 12.)

This could be extended to the " joining " or " adding " of more than two terms :

$8 \times 2 + \frac{1}{3}$ of $12 + 18 \div 3 = 2 \times 10 + 6$

b $16 + 14 + 8 = 38$

"Which is the largest of the four numbers in this equation? " (38) Several such equations should be studied, and children brought to see that, when we add, the result (sum) is greater than any of the numbers added.

c $28 \div 7 = 4$

" What does this equation mean? " (How many sevens equal 28?—4.)

" Make the equation with rods."

" Where is the 4? " (It is the number of rods called 7.)

" Read the equation as subtraction." $(28 - 7 - 7 - 7 - 7 = 0.)$

d $(20 + 10) \div 5 = 6$

"What does this equation mean?" (How many fives equal $20 + 10$ together?—6.)

"What shows you that it does not mean you to find the number of fives that equal 10?" (The brackets.)

e $32 - (8 + 8 + 8) = \frac{1}{2}$ of 16

The child who wrote this equation could be expected to explain it by saying that the difference between 32 and 3 eights is equal to $\frac{1}{2}$ of 16, or that $\frac{1}{2}$ of 16 must be added to the 3 eights to equal 32.

3. Extension of Study of Axioms

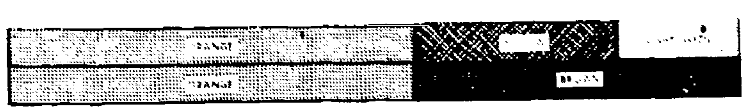
In Section B, Stage 6, the ideas of equals added to equals and to unequals were introduced. These can now be extended in preparation for future work in serial addition and subtraction, and activities leading to formal process work.

These ideas could be treated as follows: —

a If Equals Are Added to Equals, the Results Are Equal

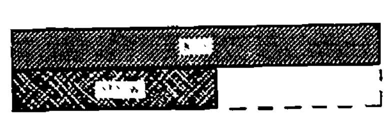
$5 + 3 = 8$

$5 + 3 + 4 = 8 + 4$

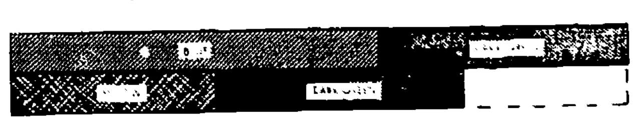


$10 + 5 + 3 = 10 + 8$

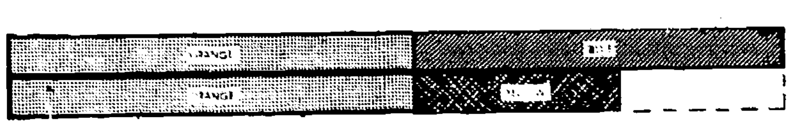
b If Equals Are Added to Unequals, the Difference Remains Constant



$9 - 5 = 4$



$9 + 6 - (5 + 6) = 4$



$10 + 9 - (10 + 5) = 4$

or $19 - 15 = 4$

Formal generalizations are not required, but the principles should be understood.

4. Solving Uncompleted Equations

The purpose of solving equations is not simply to find the correct answer, but to exercise the understanding of operations and their relationship to one another. In the absence (as yet) of a formalized rule for the order of working, the teacher must depend on the child's experience in creating, manipulating, and interpreting equations. For this reason the child must not be asked to solve an equation of a complexity beyond his experience.

It is impossible to lay down a rule for grading equations in order of difficulty, because of the difference in children's experiences. The teacher should be guided by the child's creative work, and should ensure that a complete equation can be interpreted before a similar one is presented for solving. The teacher should critically examine the thought processes involved, and make sure they are within the child's experience. Where the operations are simple, larger numbers may be used; where complex operations are involved, smaller numbers should be used at first.

Substitution (or renaming a number) is used in two directions :

i A simple equation such as $9 + 6 + 4 = \square$ is elaborated to, say, $3 \times 3 + \frac{1}{2}$ of $12 + 2 \times 2 = \square$. (This is a valuable means of complicating equations for solving, and helps the child to see the equation as the addition and/or subtraction of terms.)

ii In solving $4 \times 12 - \frac{1}{4}$ of $24 = \square$, the child substitutes 48 for 4×12 and 6 for $\frac{1}{4}$ of 24.

There should be an increasing emphasis on solving equations (either in part or wholly) without rods, but this must be governed by the child's experience of number facts.

The solving of equations should still occupy a smaller proportion of the child's time than creative and manipulative work does.

It must not be supposed that all children working in this section will be able to solve all the equations suggested below. The wide differences in ability are as evident in solving equations as in creating them. Some children will solve the simpler equations only. However, it is quite possible that other children with greater ability and a wider creative experience will be able to handle even more complicated equations than are listed.

Equations could be grouped as follows :

a **Terms Connected by Addition or Subtraction**

$\frac{3}{5}$ of $20 + 2 \times 4 = \square$
$\square + 40 \div 10 + \frac{2}{5}$ of $20 = 24$
$14 + 2 \times 5 + \square = 36$
$6 \times 6 - \frac{1}{2}$ of $12 = \square$
$\square = 3 \times 8 + \frac{2}{3}$ of 18
$28 - 2 \times \square = 20$
$32 - \square \times \square = 20$
$3 \times 9 - \square = \frac{1}{2}$ of 40

b **Terms Connected by Addition and Subtraction**

$2 \times 12 + 3 \times 4 - \frac{1}{4}$ of $16 = \square$

$16 + \frac{1}{2}$ of $24 - 2 \times \frac{1}{3}$ of $12 = \square$

$\square = 50 \div 10 + \frac{17}{100}$ of $100 - \frac{3}{7}$ of 7

$36 - 16 + \square = 24$

$\frac{4}{5}$ of $15 - \frac{1}{2}$ of $18 + 7 = \square$

$\square = 60 - 5 \times 10 + 9$

c **Subtraction with Brackets**

$6 \times 6 - (20 + 4) = \square$

$8 \times 4 - (\frac{1}{2}$ of $16 + 8 \div 2) = \square$

$\square = 24 + 6 - (18 + \frac{1}{4}$ of $8)$

$3 \times \frac{1}{2}$ of $16 - (\frac{7}{10}$ of $10 + \frac{5}{8}$ of $6) = \square$

$6 \times 6 - (12 + \square) = 16$

$36 - (10 + 6) + 5 = \square$

d **Repeated Subtraction**

$28 - 2 \times 4 - 4 \times 5 = \square$

$\frac{1}{2}$ of $100 - 30 \div 3 - \frac{7}{12}$ of $12 = \square$

$48 - 8 - 8 - 8 = 3 \times \square$

$4 \times \frac{1}{3}$ of $18 - \frac{7}{100}$ of $100 - 10 \div 2 = \square$

$50 - 2 \times 10 - \square = 20$

e **Subtraction Using the Same Numerals and Signs, with and without Brackets**

$24 - (12 + 6) = \square$

$24 - 12 + 6 = \square$

$3 \times 12 - (5 \times 2 + 12 \div 2) = \square$

$3 \times 12 - 5 \times 2 + 12 \div 2 = \square$

f **Subtraction Substitution after the Minus Sign**

$26 - 14 = \square$

$26 - (16 - 2) = \square$

$2 \times 18 - (2 \times 10 - 4) = \square$

$40 + 2 - (20 - 10) + \frac{1}{6}$ of $36 = \square$

NOTE: Omission of brackets in these examples leaves repeated subtraction.

$26 - (16 - 2) = \square$

$26 - 16 - 2 = \square$

$2 \times 18 - (2 \times 10 - 4) = \square$

$2 \times 18 - 2 \times 10 - 4 = \square$

g **Equations Involving Zero**

$18 - (6 + 6 + 6) = \square$

$3 \times \frac{1}{2}$ of $16 + 0 = \square$

$\frac{4}{5}$ of $25 - \square = 0$

$\frac{5}{9}$ of $9 - 0 = \square$

$12 + 3 \times 0 = \square$

$\square + 28 \div 7 = 4$

h **Equations Involving Multiplication and Division by One**

$5 \times 1 + 6 \times 1 + 3 \times 1 = \square$

$28 = 28 \times \square$

$\square \times 1 = 35$

$29 \div 1 = \square$

$27 \div \square = 27$

$\square \div 1 = 56$

NOTE: Examples have not been listed under the headings of " multiplication " and " division ", but these operations have been included in the above examples. The use of multiplication and division in the one equation (e.g., $24 \div 3 \times 4 = \square$ $\quad 12 \times 6 \div 3 = \square$ will be treated in a later section.

NOTES ON METHOD

1. The work of this stage is complementary to that of the other stages of the section, and should be done parallel with them

2. Oral work continues to be of considerable importance and should not be neglected.

TESTING

1. Inter-relation of Operations

The exercises outlined in Developmental Step 1 are sufficient test of understanding of the relationships existing between operations.

2. Interpretation of Equations

This can be tested only by discussions as suggested in Developmental Step 2, and by observation of the child's attack when solving equations.

3. Solving Equations

The child's ability to solve equations of various types is shown in his normal work. There is no need for a special test. If it is wished, for any reason, to set such a test, it would be in terms of examples as suggested in Developmental Step 4, provided these were within the experience of the particular child.

STAGE 31

AIM

To continue the study of fractions.

NOTES ON AIM

1. In the earlier study of fractions two main concepts have been studied—the fraction as a relation between two whole numbers and the fraction as an operator. The first task of this stage is to extend these ideas so that the child is able to express them with the numbers between twenty and one hundred and forty-four.

2. As well as extending concepts that are already understood the child is introduced to two new concepts—fractions as numbers and the equivalence of fractions

3. The particular importance of the second of these new concepts must be stressed. Almost all of the future work with fractions is dependent on a thorough understanding of the equivalence of fractions. Unless the child has mastered this concept, and is capable of working with it easily and confidently, he will be severely handicapped in his later work. Thus careful treatment of the concept in this stage is vital.

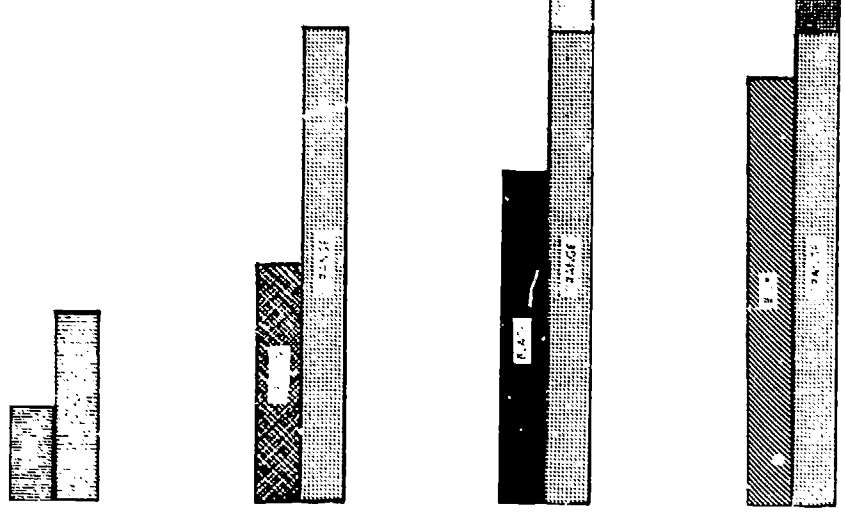
DEVELOPMENTAL STEPS

1. The Fraction as the Relation between Two Numbers

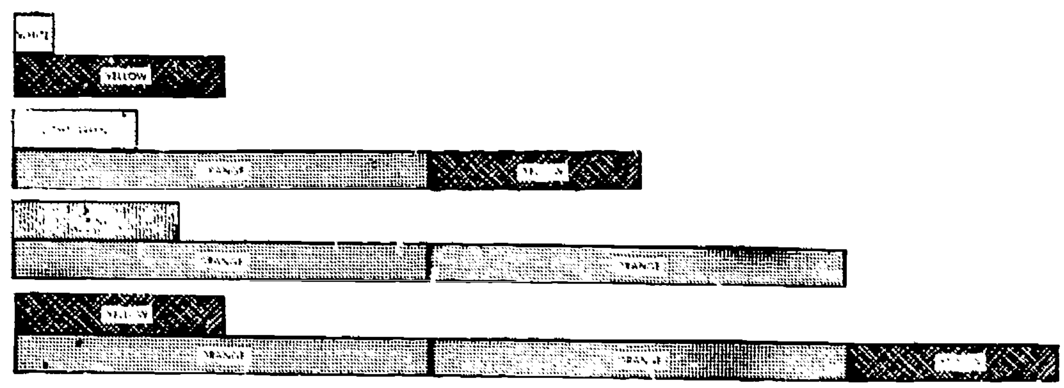
a In Sections C and D this concept was established and consolidated. The child was led to see that the relationship existing between (for example) the pink rod and the brown rod is the same as that existing between the light-green rod and the dark-green rod; that is to say, two of the smaller rods would equal the larger rod in each case.

Further experience should now be given to extend the idea. Children should be asked to show a certain relationship in a number of different ways.

e.g. i $\frac{1}{2}$ could be shown in these and other ways :



e.g. ii $\frac{1}{5}$ might be shown :



It should be seen in example i that two of the smaller rods, and in example ii that five of the smaller rods, are needed to equal the length of the rod that we now call the unit rod or " measurer ".

The child should have some experience using larger numbers. However, owing to the size of patterns needed for this it is impracticable for him to show such relationships in many ways. For example $\frac{1}{25}$ could be shown with a white rod for one row, and two orange rods and a yellow for the other, or with a red rod for one row and five orange rods for the other. To go beyond this would make a pattern too cumbersome to be useful.

b Once the child has mastered the previous step, he should be asked to show fractions such as $\frac{2}{7}$, $\frac{3}{8}$, $\frac{4}{12}$, each in various ways.

In both a and b the child should do a large number of similar exercises. The particular fractions used are not important so long as the child receives sufficient practice to ensure mastery of the idea.

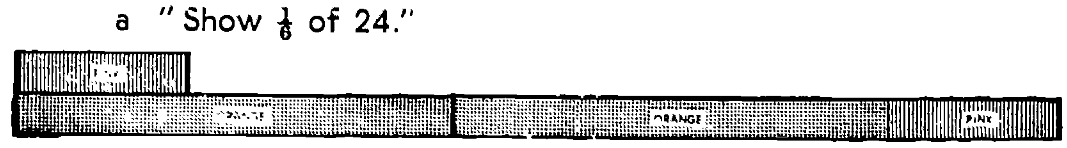
NOTE: It is useful with young children to establish a convention for the placing of rods in illustrating fractions as relationships. Any arrangement is acceptable so long as it is agreed which rod is the " measurer ". In the examples shown in a above, this is the lower rod in the horizontal pattern and the right-hand rod in the vertical pattern. This accords with the notational conventions of numerator above denominator ($\frac{3}{4}$), and the writing of a fraction as an " ordered pair ". (3, 4.)

2. The Fraction as an Operator

In Sections C and D the child used the fraction as an operator (see Stages 22 and 26). He needs to extend his experience to the numbers with which he is working now.

A typical approach is outlined below :

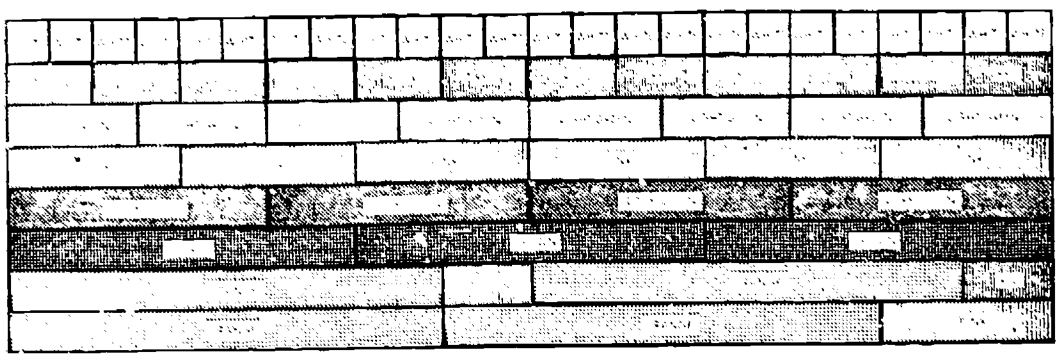
a " Show $\frac{1}{6}$ of 24."



" What does $\frac{1}{6}$ of 24 equal? " (4)

This procedure should be repeated to find $\frac{1}{3}$ of 24, $\frac{1}{8}$ of 24, $\frac{1}{2}$ of 24, $\frac{1}{4}$ of 24, $\frac{1}{12}$ of 24, $\frac{1}{24}$ of 24.

b "Make a multiplication mat for 24."



From this can be seen:

$\frac{1}{2}$ of 24 = 12	$\frac{2}{2}$ of 24 = 24		
$\frac{1}{3}$ of 24 = 8	$\frac{2}{3}$ of 24 = 16	$\frac{3}{3}$ of 24 = 24	
$\frac{1}{4}$ of 24 = 6	$\frac{2}{4}$ of 24 = 12	$\frac{3}{4}$ of 24 = 18	$\frac{4}{4}$ of 24 = 24
$\frac{1}{6}$ of 24 = 4	$\frac{2}{6}$ of 24 = 8	. . . . .	$\frac{6}{6}$ of 24 = 24
$\frac{1}{8}$ of 24 = 3	$\frac{2}{8}$ of 24 = 6	. . . . .	$\frac{8}{8}$ of 24 = 24
$\frac{1}{12}$ of 24 = 2	$\frac{2}{12}$ of 24 = 4	. . . . .	$\frac{12}{12}$ of 24 = 24
$\frac{1}{24}$ of 24 = 1	$\frac{2}{24}$ of 24 = 2	. . . . .	$\frac{24}{24}$ of 24 = 24

Children readily apply this experience to abstract work with large numbers. For example, when a child realizes that 12 fours = 48, he can apply his understanding of the relationship between multiplication and partition division and say $\frac{1}{12}$ of 48 = 4. Counting and other number experiences then enable many children to proceed to statements such as $\frac{2}{12}$ of 48 = 8, $\frac{3}{12}$ of 48 = 12, and so on.

c When children are able to work confidently with such examples as shown in a and b they should be encouraged to go on to examples such as $\frac{3}{2}$ of 24 = 36, $\frac{5}{3}$ of 24 = 40, $\frac{7}{6}$ of 24 = 28, using rods if necessary.

3. Fractions as Numbers

So far, children will have had considerable experience of the following kind of exercise:

$\frac{1}{3}$ of 6 = 2, $\frac{1}{6}$ of 24 = 4, $\frac{1}{12}$ of 48 = 4

In each of these the fraction has appeared as an "operator", operating on a counting (whole) number, and producing a counting number as a result.

Few, if any, children will have had experience of this kind of exercise:—

$\frac{1}{4}$ of 1 = $\frac{1}{4}$, $\frac{1}{8}$ of 1 = $\frac{1}{8}$, $\frac{1}{3}$ of 1 = $\frac{1}{3}$

Here the number produced by the operation is a new kind of number, namely, a fraction.

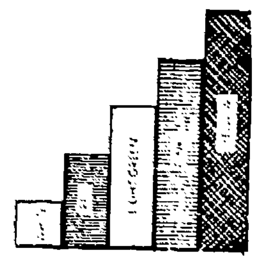
All children should now become aware of fractions as numbers, as distinct from operators or relationships.

A suggested approach is outlined below. (Note the link with the work in value relations, treated in Stage 29.)

a "If the yellow rod is called 1, what will the white rod be called?" ()

" What will the red rod be called? " ($\frac{2}{5}$)

" Build a staircase from white to yellow."



" Count up and down the staircase " ($\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$ or 1. $\frac{5}{5}$ or 1, $\frac{4}{5}$, $\frac{3}{5}$, $\frac{2}{5}$, $\frac{1}{5}$.)

This procedure should be repeated with other rods and the resulting fractions discussed.

b Staircases can be carried beyond " 1 ". For example, call black 1. Build a complete staircase from white to orange.

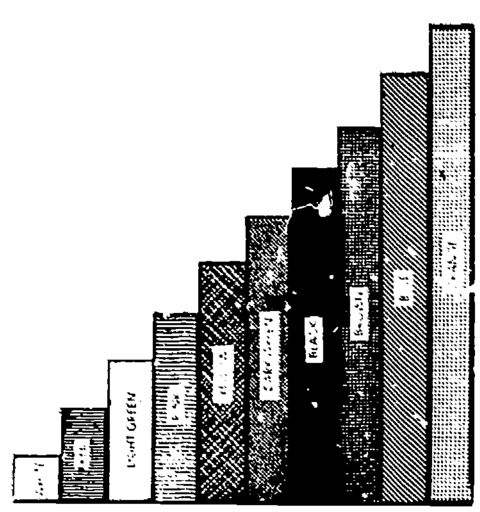
Counting would then be $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, $\frac{7}{7}$ or 1, $\frac{8}{7}$, $\frac{9}{7}$, $\frac{10}{7}$.

If it seems appropriate to read $\frac{8}{7}$ as $1\frac{1}{7}$, replacing the brown rod with a black and a white will be found helpful. The blue rod could be replaced with a black and two whites or one red to read $1\frac{2}{7}$ for $\frac{9}{7}$.

c " Call the red rod 1. What will the white rod be called? " ($\frac{1}{2}$)

" What will the light-green rod be called? " ($1\frac{1}{2}$)

" Build a staircase from white to orange."



" Count up and down the staircase." ($\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5.)

d Exercises in halving, where the children work with fractions as numbers (4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$), serve to consolidate the idea established by the preceding exercises.

NOTE: While number lines are not prescribed in the Course of Study, they are of considerable value in treating fractions as numbers.

4. **Equivalence of Fractions**

a The concept may be established by various methods, three of which are outlined below. Teachers may find others. The wider the child's experience, the sounder will be his understanding of the concept.

i *Through the Fraction as a Number (using value relations)*

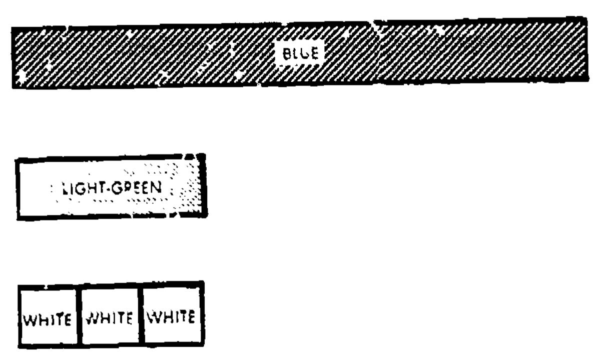
"If the blue rod is called 1, what must light-green be called?" ($\frac{1}{3}$)

"If blue is called 1, what is white?" ($\frac{1}{9}$)

"What would two white rods be called?" ($\frac{2}{9}$)

"What would three white rods be called?" ($\frac{3}{9}$)

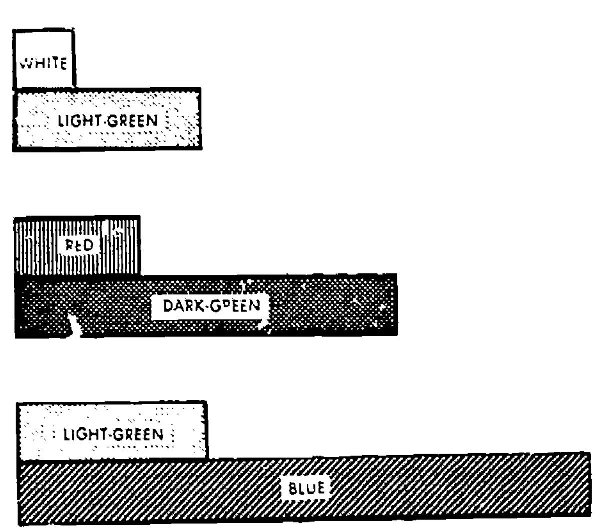
But three whites equal light-green, so $\frac{3}{9} = \frac{1}{3}$.



Using dark-green and red rods, work through the questions similar to those set out above to show $\frac{2}{6} = \frac{1}{3}$.

ii *Through the Fraction as the Relation between Two Numbers*

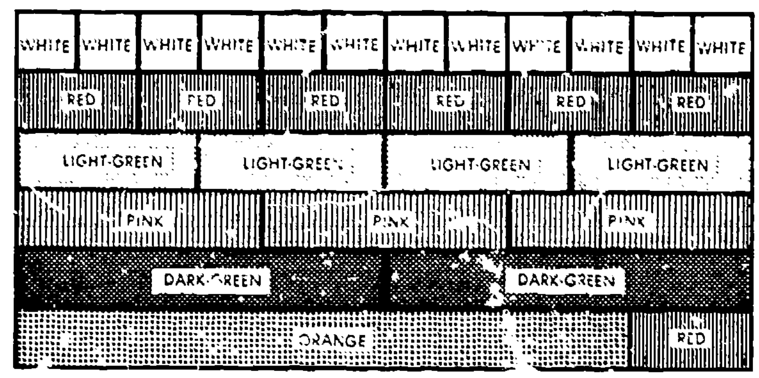
"Use pairs of rods to show $\frac{1}{3}$."
Various pairs could be used as shown below.



Questions lead children to rename $\frac{1}{3}$ as $\frac{2}{6}$ and $\frac{3}{9}$.

iii *Through the Fraction as an Operator*

" Make a multiplication mat for 12."



A study of this mat will show that :

$\frac{1}{3}$ of $12 = 4$

$\frac{2}{6}$ of $12 = 4$

$\frac{4}{12}$ of $12 = 4$ i.e. $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$

b Other fractions such as $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{4}$ should be similarly treated with rods.

e.g. $\frac{1}{2} = \frac{4}{8}$ $\frac{1}{2} = \frac{3}{6}$ $\frac{1}{2} = \frac{5}{10}$

c When children are competent with the above exercises they should find equivalents for fractions with numerators other than 1.

e.g. $\frac{2}{3} = \frac{4}{6}$ $\frac{2}{3} = \frac{6}{9}$ $\frac{3}{4} = \frac{6}{8}$ $\frac{2}{5} = \frac{4}{10}$

d Children will soon see that results can be arranged according to a pattern.

e.g. $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$

No attempt should be made to teach " rules " for finding equivalent fractions. Children should have wide experience with rods, and should see the results recorded. However, some children discover " short cuts " and put their knowledge of equivalence to use in their creative work.

NOTE: An important set of equivalents is that found for 1, e.g., $1 = \frac{5}{5}$, $1 = \frac{7}{7}$, $1 = \frac{9}{9}$. Reference to Step 3 a will show a method of obtaining these. The notation $\frac{1}{1}$ for 1 could also be shown, followed by $\frac{5}{1}$ for 5, $\frac{7}{1}$ for 7, ... The group $\frac{5}{5}$, $\frac{7}{7}$, $\frac{9}{9}$, $\frac{1}{1}$, $\frac{8}{8}$ . . . . . . are all numerals for the number whose simplest numeral is 1.

5. Use of Fractions in Other Work

The use of fractions in creative work and in the solving of equations must continue, and should increase in complexity according to the child's ability. Examples including fractions will be found in Stages 29 and 30.

NOTES ON METHOD

1. Work with fractions should be parallel with all other stages of the section. Steps 1 and 2 directly continue the work of Section D, and can therefore be dealt with from the beginning of the section. Steps 3 and 4 introduce new ideas and should not be attempted until children are sufficiently confident in the earlier use of fractions and in the appropriate value relations work.

2. The importance of the use of the rods when introducing these concepts must be stressed. At each new step the child is asked to handle the material. Gradually, as the child achieves mastery, he will work abstractly. If, however, he still needs the rods at the end of the stage, he is permitted to use them.

3. The work of each step should be oral for a considerable time. In fact some aspects of the stage require no written work at all. The advantage of oral work is that it permits a large number of examples to be covered—and the amount of experience given is a determining factor in the depth of understanding reached.

4. During the oral work it is desirable for the teacher to record the child's work on the black-board. If, for example, a group is working to discover equivalents for $\frac{1}{3}$ and the teacher records on the black-board the results of their discoveries, e.g., $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$, the work is preserved for the children to see, and the pattern in the work (the multiplication of the numerator and the denominator by the same number) is more likely to be apparent.

5. As this work is the basis for all the work with fractions in the next section, success in it is essential before that section is begun.

TESTING

1. The child's understanding of the fraction as a relationship between two numbers is demonstrated when he can :

- a show a specified fraction in various ways,
- b tell why patterns such as those used in Step 1 all represent the same fraction.

2. Competence with the fraction as an operator is most clearly shown in creative work. The child should :

- a include fractions in the numerals he uses to rename numbers when expanding equations, e.g., of 30 for 5, $\frac{7}{12}$ of 12 for 7, $\frac{3}{8}$ of 24 for 9;
- b substitute the appropriate simplest numeral for a fractional expression when solving an equation, e.g., 8 for $\frac{8}{9}$ of 9, 6 for $\frac{1}{3}$ of 18, 12 for $\frac{2}{5}$ of 30.

3. Exercises similar to those listed in Steps 3 and 4 form the tests for understanding of fractions as numbers and of the equivalence of fractions.

SUMMARY OF WORK COVERED IN SECTION E

1. **Ordinal Number**

   i Counting has been continued throughout the section :

   a by ones, in the range 0—1000, forwards and backwards;

   b by twos, by threes, by fours — — — by twelves from zero and from other numbers, with some degree of memorization.

   ii Much work has been done with number charts and number sequences to extend the child's appreciation of the pattern and order of the number system.

   iii Figures to 1000 and words to one hundred and forty-four can be recognized and written.

   iv Doubling and halving exercises are continuing.

   v A study has been made of place value to 999.

2. **Understanding of Basic Mathematical Ideas**

The child has developed his understanding of equality, the four basic operations, and simple axioms, and applied these to larger numbers. He creates equations of a greater complexity than in earlier sections. He has increased confidence in the use of the operations, and has made a study of the ways in which they are related. He has continued to manipulate equations by the use of substitution and by rearranging the elements. He shows his understanding when he interprets equations and applies this understanding to the solving of incomplete equations.

3. **Abstract Work**

The emphasis on abstract work has increased, and much less use is made of rods at the end than at the beginning of the section.

4. **Fractions**

The previously understood concepts of fractions as relationships and as operators have been applied to larger numbers. Fractions are now recognized as numbers, and children are familiar with the equivalence of fractions. Fractions are included when children are creating, manipulating, interpreting, and solving equations.

5. **Value Relations**

Ability to change the values given to the rods has been extended and applied to counting exercises and the equivalence of fractions.

6. **Zero**

Zero has been used in counting and included in the study of place value. It is used in all aspects of equation work.

7. **Preparation for Automatic Response**

Free and planned work on substitution for numbers to 20 has given children much experience of number bonds within this range. These are to be made automatic in Section F. There has been sufficient experience with number bonds to 10 for these to be tabulated.

8. **General Notes**

i The ideas studied in earlier sections are used much more confidently and mastery of the number system has increased.

ii Although much written work has been done, oral work is still of great importance, and is given a prominent place.

iii The section has seen the first exploration of the field from 0 to 144, of which the child became aware in the counting work of Section D. It will be further explored during Section F. Counting and pattern work have made the child aware of the much larger field beyond 144.

iv There continues to be a marked difference in the quality of children's work. While all children need to cover the basic work of each stage, some will do work of a much greater complexity than others.

By Authority: A. C. Brooks, Government Printer, Melbourne.