DOCUMENT RESUME

ED 062 397 TM 001 336

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TITLE The Relation of the Method of Reciprocal Averages to Guttman's Internal Consistency Scaling Model.

PUB DATE Apr 72

NOTE 19p.; Paper presented at the annual meeting of the American Educational Research Association (Chicago, Illinois, April 1972)

EDRS PRICE MF-$0.65 HC-$3.29

DESCRIPTORS Analysis of Variance; Correlation; *Evaluation Methods; *Internal Scaling; *Item Analysis; Mathematical Applications; *Mathematical Models; Psychometrics; Scores; Statistical Analysis

IDENTIFIERS *Guttmans Internal Consistency Scaling Model

ABSTRACT

A scaling technique known as the Method of Reciprocal Averages has been in use since the early 1930's. This technique yields a set of item response weights for a psychological inventory which maximizes the internal consistency of the inventory for a group of subjects. Although the technique has been used for many years, its mathematical foundations have not been made explicit. In the present paper, it is shown that the informal data processing procedures of this technique actually solve the set of linear equations yielded by Guttman's Least Squares Model for internal consistency scaling. The Method of Reciprocal Averages can be implemented as a simple extension to existing item analysis computer programs. (Author/CK)

THE RELATION OF THE METHOD OF RECIPROCAL AVERAGES TO GUTTMAN'S INTERNAL CONSISTENCY SCALING MODEL

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Paper Presented at the
Annual Meeting of the American Educational Association
Chicago, Illinois
April 3-7, 1972

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Abstract

A scaling technique known as the Method of Reciprocal Averages has been in use since the early 1930's. This technique yields a set of item response weights for a psychological inventory which maximizes the internal consistency of the inventory for a group of subjects. Although the technique has been used for many years, its mathematical foundations have not been made explicit. In the present paper it is shown that the informal data processing procedures of this technique actually solve the set of linear equations yielded by Guttman's Least Squares Model for internal consistency scaling. The constraint imposed by Guttman to insure that the solution yields a nonextraneous set of weights is also met. From a computational point of view the Method of Reciprocal Averages has an advantage over the principal components approach employed by Guttman's solution as it does not require the calculation of an item response category co-occurrence matrix. In addition, The Method of Reciprocal Averages can be implemented as a simple extension to existing item analysis computer programs.

THE RELATION OF THE METHOD OF RECIPROCAL AVERAGES TO GUTTMAN'S INTERNAL CONSISTENCY SCALING MODEL

Over the years a common practice among psychologists has been the creation of ad hoc computational procedures yielding various scores, indices, loadings, etc. which aid in the understanding of data. Such procedures often were developed within the context of a particular study to meet some practical need of the researcher. In some cases, the general usefulness of the procedure led to the derivation of an underlying mathematical rationale and what was once an ad hoc procedure developed into standard psychometric technique. A good example of this process is factor analysis where Spearman's early procedures were developed by later workers into a mathematically sophisticated major area of psychometrics. The field of psychological scaling is also one in which many ad hoc procedures have been developed and some of these have become established techniques. One of these ad hoc scaling procedures that has been used by researchers for many years (Klausmeier, Quilling & Wardrop, 1968; Mitzel & Hoyt, 1954; Moiser, 1942) and currently implemented in a widely distributed computer program (Baker, 1960; Baker & Martin, 1969) is the Method of Reciprocal Averages. This scaling procedure yields a set of item response weights for a psychological inventory that maximizes the internal consistency index of the inventory for a group of subjects. Despite its use over a period of many years, the procedure remains an ad hoc one in that an explicit mathematical model for the Method of Reciprocal Averages has not appeared in the literature. However, a general mathematical model for internal consistency scaling has been provided by Guttman (1941). It is the purpose of the present paper to demonstrate that although the Method of Reciprocal Averages preceeded Guttman's (1941) work, it is actually a particular implementation of that model.

The Method of Reciprocal Averages has its origin in a scaling procedure partially described by Richardson and Kuder (1933). The procedure in the article was not named, but it became well known to psychometricians of the era as both Horst (1935) and Guttman (1941) attributed the Method of Reciprocal Averages to M. Richardson, citing the 1933 article. A detailed description of the data processing and computational procedures of the Method of Reciprocal Averages were not avilable until they were presented by Moiser (1946). Attempts to provide a formal mathematical model for this scaling procedure were also spread over a considerable period of years. Guttman (1941) provided a general model for internal consistency scaling based upon a least squares approach. Mosteller (1949) developed a scaling technique in which only the positive response to an item was weighted and a set of equations were solved for the item response weights that maximized the internal consistency index. The equations solved under this approach were essentially the same as those due to Guttman (1941). In an unpublished paper, Hoyt and Collier (1953) showed empirically that in the single response situation, the Method of Reciprocal Averages and Mosteller's techniques yielded the same item response weights. Suggesting for this case at least, that a connection exists between the Method of Reciprocal Averages and Guttman's (1941) least squares model. In addition, Guttman (1941) mentioned he felt that his model and the Method of Reciprocal Averages were related, but he did not pursue the issue. On the surface, it is not obvious that the Method of Reciprocal Averages is a particular implementation of the solution of the equations yielded by Guttman's least squares approach. The former involves rather informal data processing procedures whereas the latter is based upon a complex mathematical approach. In order to fully develop the relationship between the two, the existing bases of both are presented below and then the relationship is shown.

The Method of Reciprocal Averages

Moiser's (1946) procedures were designed for implementation on punched card equipment and are reformulated here to agree with the computer program due to Baker and Martin (1969). Fundamentally, the situation is one involving a population of N subjects who respond to a universe of m items, where: each item has two or more possible item response categories, each subject can select only one item response category per item, but must respond to all items in the universe of items. Thus, there will be m items having a total of r possible item response categories. The basic assumption is that a single variable underlies the items in the universe. The goal, then, is to obtain a set of item response category weights X_j $(j=1,2,...,r)$ which will maximize the internal consistency index of the instrument for the population of subjects.

The Method of Reciprocal Averages is an iterative procedure in which an a priori set of item response weights is used to obtain a score for each subject, then the scores in conjunction with the subject's item response choices are used to derive a new set of item response weights. The derived weights are then used to obtain a new score for each individual and the iterative process is continued until a convergence criterion is met. The final set of item response weights will be those which maximize the internal consistency index for the group of subjects on the given instrument. In the following paragraphs the procedural steps are presented and a notational scheme for representing the variables involved is developed.

Step A

The investigator assigns an a priori (though not necessarily distinct) weight to each of the r possible item response categories in the instrument. These weights are usually integer numbers ranging in value from unity to some arbitrary upper limit. Let X_k (k=1,2,3,...,r) denote an arbitrary set of item response weights, the a priori weights at this stage.

Step B

The a priori weights are used as a scoring key and a total score for each subject is obtained. Let $\varepsilon_{ik} = 1$ if the i-th subject (i=1,2,...,N) chooses the k-th item response category $\varepsilon_{ik} = 0$, otherwise. A subject's total score T_i is given by the sum of the item weights corresponding to his choices.

(1) $$T_i = \sum_{k=1}^{r} \varepsilon_{ik} X_k$$

Step C

The mean score for all subjects choosing a given item response is computed for each item response category (the mean item response score). Let $\varepsilon_{ij} = 1$ if the i-th subject responds to the j-th $(j=1,2,3,\ldots,r)$ item response category and 0 otherwise (and alternative specification for ε_{ik}). Let $\varepsilon_{ij}\varepsilon_{ik} = 1$ if the i-th subject selects both item response categories j and k. The total score for a person choosing a specific item response category, say j ($\varepsilon_{ij} = 1$) is given by:

$$(2) \qquad \tilde{T}_i = \sum_{k=1}^{r} \varepsilon_{ij}\varepsilon_{ik}X_k$$

Note that ε_{ij} merely designates the item response category of interest, whereas ε_{ik} specifies the subject's response choice to all items including the one of interest. Now, summing over all subjects, the sum of the total scores for all persons choosing response category j is

$$(3) \qquad \sum_{i=1}^{N} \tilde{T}_i = \sum_{i=1}^{N} \left[\sum_{k=1}^{r} \varepsilon_{ij}\varepsilon_{ik}X_k\right]$$

and the number of subjects choosing item response j is given by

$$(4) \qquad \sum_{i=1}^{N} \varepsilon_{ij} = n_j$$

The mean item response score is

$$(5) \qquad M_j = \frac{\sum_{i=1}^{N} \tilde{T}_i}{n_j} = \frac{\sum_{i=1}^{N} \left[\sum_{k=1}^{r} \varepsilon_{ij}\varepsilon_{ik}X_k\right]}{\sum_{i=1}^{N} \varepsilon_{ij}}$$

Step D

The mean item response scores are now used to assign the derived weights X_j to the r item response categories. The frequency distribution of the M_j, is divided into L equal area intervals and an integer weight assigned to each interval. Then, the interval into which a given M_j falls is determined and the derived item response weight X_j is the integer number corresponding to this interval. Thus, the derived weight X_j is proprotional to the mean item response score based upon the weights from the previous iterations.

Step E

The criterion used to determine whether the iterative procedures should be terminated is the difference between the Hoyt ANOVA reliability index (Hoyt, 1941), on two successive iterations. If the positive difference is sufficiently small, the most recent set of derived weights are considered to be the "optimum" set. If not, X_k's are replaced by the X_j's and steps B-E are repeated.

This disarmingly simple procedure results in a set of weights with very useful psychometric properties. According to Moiser (1946) there are:

> "(1) The reliability of each item and the internal consistency of the weighted inventory are maximized. (2) The correlation between the item and the total score is maximized and the product moment correlation coefficient becomes identical with the correlation ratio. (3) The relative variance of the distribution of scores (coefficient of variation) is maximized. (4) The relative variance of item scores within a single case is minimized. (5) The correlation between an item and total score is proportional to the standard deviation of the item weights for that item. (6) Questions which bear no relation to the total-score variable are automatically weighted so that they exert no effect in the scoring."

Guttman's Least Squares Model

Guttman (1941) established a situation identical to that described above for the Method of Reciprocal Averages involving a population of N subjects and a universe of m items. He wanted to obtain a set of item response weights which maximized the ratio of the variance between people to the total variance, i.e, maximize a correlation ratio. A symbology for this situation can be developed by letting X be a diagonal matrix of item response weights $(X_1, X_2, X_3, \ldots, X_r)$ and letting E by the matrix of ε_{ik} as defined above. Now the matrix B is given by:

$$(7) \qquad B = XE$$

where the r rows of B correspond to item response categories, and the N columns correspond to subjects. Summing across the rows of B within a given column one obtains the sum of the weights, i.e., the total scores, for a given subject (i). The arithmetic mean of these weights is given by

$$a_i = \frac{\sum_{k=1}^{r} \varepsilon_{ik} X_k}{m}$$, where m (the number of items) is the same for all individuals. Note that $ma_i = \sum_{k=1}^{r} \varepsilon_{ik} X_k$ is merely a subject's total score. The grand mean of all the non-zero elements of the matrix B is given by:

(8) $$\alpha = \frac{1}{mN}\left[\sum_{i=1}^{N}\sum_{k=1}^{r} \epsilon_{ik}X_k\right] = \frac{1}{mN}\sum_{i=1}^{N} ma_i = \frac{1}{N}\sum_{i=1}^{N} a_i.$$

The variance between people is

(9) $$R = \frac{1}{N}\sum_{i=1}^{N}\left[ma_i - \sum_{i=1}^{N}\frac{ma_i}{N}\right] = \frac{\sum_{i=1}^{N} m^2a_i^2 - m^2N\alpha^2}{N}.$$

The total variance is

(10) $$W = \frac{1}{mN}\sum_{i}^{N}\sum_{k}^{r}(\epsilon_{ik}\,{}_k - \alpha)^2 = \frac{1}{mN}\sum_{k=1}^{r} n_kX_k^2 - \alpha^2$$

where $\sum_{i=1}^{N} \epsilon_{ik} = n_k$.

The correlation ratio to be maximized is:

(11) $$\eta_x^2 = \frac{R}{W} = \frac{m^2\sum_{i=1}^{N} a_i^2 - m^2N\alpha^2}{\frac{1}{m}\sum_{k=1}^{r} n_kX_k^2 - N\alpha^2},$$

Now because the variances of weights X_k in a given column of B are unaffected by a shift in the origin of measurement of the X_k the correlation ratio will be invariant to such a shift. A simplification can then be achieved by letting

(12) $$\alpha = \frac{1}{mN}\sum_{k=1}^{r} n_kX_k = 0.$$

Substituting (12) in (11), the correlation ratio to be maximized becomes:

$$(13)\qquad \eta_x^2 = \frac{m^2 \sum_{i=1}^{N} a_i}{\frac{1}{m} \sum_{k=1}^{r} n_k X_k^2} = \frac{\sum_{i=1}^{N} \left\{ \sum_{k=1}^{r} \epsilon_{ik} X_k \right\}^2}{\frac{1}{m} \sum_{k=1}^{r} n_k X_k^2} .$$

The reader is referred to Guttman (1941) and Torgerson (1958, p. 338) for mathematical details of this maximization process. The net result is a system of linear equations which are solved iteratively for the derived item response weights. Guttman (1941) recognized that these equations were identically those solved by Hotelling (1933) for principal components. Thus, he succeeded in showing that internal consistency scaling is merely a special case of the more general principal components model. Intuitively, one would anticipate that the single variable underlying the items would be reflected in the first principal component, but, Guttman (1941) shows that the first principal component yields an extraneous solution consisting of a vector of weights all equal to unity, which maximizes the correlation ratio. Thus, the desired weights correspond to the second principal component. It appears to the present author that this artifact results from having set $\alpha = 0$ prior to maximizing the correlation ratio. In order to obtain a nonextraneous set of weights, a constraint must be introduced that forces the derived weights to be orthogonal to the extraneous weights. Guttman (1941) used the constraint that the sum of the $n_k X_k$ across the response categories of each item must be equal to zero. Mosteller (1949) used the less stringent constraint that the sum of the $n_k X_k$ across all item response categories must be equal to zero which is equivalent to requiring the mean score over all subjects to be zero.

Torgerson (1958) used this latter constraint when presenting the derivation of the Guttman model.

The linear equations (Torgerson, 1958) resulting from the maximization of the correlation ratio are:

$$
(14) \quad
\begin{array}{ccccc}
n_{11}X_1 + n_{12}X_2 & = \dots + & n_{1r}X_r & = & mn^2 n_{11}X_1 \\
n_{22}X_1 + n_{22}X_2 & = \dots + & n_{2r}X_r & = & mn^2 n_{22}X_2 \\
. \quad . & \dots & . & & . \\
. \quad . & \dots & . & & . \\
n_{r1}X_1 + n_{22}X_2 & = \dots + & n_{rr}X_r & = & mn^2 n_{rr}X_r
\end{array}
$$

subject to the constraint

$$n_{11}X_1 + n_{22}X_2 + \dots + n_{rr}X_r = 0.$$

Note that the weights on the left correspond to X_k and on the right to X_j. In a more compact notation these equations are:

$$(15) \qquad \sum_{k=1}^{r} n_{jk}X_k = mn^2 n_{jj}X_j; \text{ for each } j;$$

Subject to the constraint

$$(16) \qquad \sum_{k=1}^{r} n_{kk}X_k = 0.$$

These equations are solved iteratively and one must first create the co-occurrence matrix for the item response categories, a $\underline{r}$ x $\underline{r}$ matrix whose cells are the n_{jk}. Then a set of r-1 a priori weights $(X_1, X_2, \dots X_{r-1})$ are selected and the constraint (16) solved for the value of X_r. These weights are then substituted for the X_k on the left side of (15) and each row can be solved for the value of X_j. The term $m\eta^2$ appears in each row but can be ignored as it is a constant of proportionality which is the same for all X_j. The X_j's are then rescaled so that they can be compared with the previous set of weights. If they are the same, the iterative process is terminated. If not, the derived weights X_j become the X_k's and the process is repeated.

Guttman (1941) showed that the identical results are obtained if one starts out to obtain scores which maximize the ratio of the variance between categories to the total variance. He also showed that if the correlation between the scores and the weights is maximized, the same results are obtained and that the square of the correlation ratio is equal to the square of the product-moment correlation. Thus, the numerous properties attributed to the Method of Reciprocal Averages by Moiser (1946) stem from these relationships.

Relating the Method of Reciprocal Averages to The Guttman Model

Inspection of the equations (14) to be solved for the weights readily reveals that the sum of the $n_{jk}X_k$ terms on the left hand side in a given equation is equal to the sum of the total scores of all persons choosing item response j. On the right hand side, the n_{jj} term is the number of subjects choosing item response j. The Guttman equations (14) can be rewritten as:

$$(17) \qquad \frac{\sum_{k=1}^{r} n_{jk}X_k}{n_{jj}} = m\eta^2 X_j \ .$$

Now, because the numerator of this equation and equation (5) are identical it is clear that the derived weights X_j are proportional to the mean item response scores, a point made by Guttman. Under the Method of Reciprocal Averages the derived weights were also proportional to the mean item response scores. The difference between the solutions being the nature of this proportionality. In the former, $m\eta^2$ is the constant of proportionality which is the same for all weights. In the latter, the integer weight represents a point on the frequency distribution of the mean item response scores, hence, is proportional to them. In Richardson and Kuder (1933) the mean item response scores were used as the derived weights. The use of integer weights was introduced by Moiser (1946) in order to simplify the computational procedures when accounting machines were used. In practice this substitution appears to have negligible effect upon the maximization of the internal consistency index.

Under the Method of Reciprocal Averages, the investigator is free to choose a priori weights and need not be concerned directly with meeting the constraint imposed upon the equations by Guttman (1941). In that the constraint does not appear explicitly in the Method of Reciprocal Averages, it must be met implicitly. The constraint (16) can be expressed in terms of the ϵ_{ik} as follows:

$$(18) \qquad \sum_{k=1}^{r} n_k X_k = \sum_{k=1}^{r} \left(\sum_{i=1}^{N} \epsilon_{ik} \right) X_k = 0.$$

Now at the end of each iteration:

$X_j = \dfrac{\sum_{i=1}^{N} \epsilon_{ij} T_i}{n_j}$, then substituting X_j for X_k and j for k in (18) one gets

$$(19) \qquad \sum_{j=1}^{r} n_j \sum_{i=1}^{N} \frac{\varepsilon_{ij} T_i}{n_j} = m \sum_{i=1}^{N} T_i = 0 \; ;$$

Substituting from (1) for T_i, equation (19) becomes

$$\sum_{i=1}^{N} \sum_{k=1}^{r} \varepsilon_{ik} X_k = \sum_{k=1}^{r} \sum_{i=1}^{N} \varepsilon_{ik} X_k = 0, \text{ but } \sum_{i=1}^{N} \varepsilon_{ik} = n_k$$

and one obtains

$$(20) \qquad \sum_{k=1}^{r} n_k X_k = 0$$

which is identically the constraint given by Mosteller (1949) and Torgerson (1958). Thus, using the mean item response scores as the item response weights will meet the constraint and the weights corresponding to the second principal component will be obtained. It should be noted that the constraint is not immediately satisfied under the Method of Reciprocal Averages as the initial a priori weights do not meet the constraint, however, the derived weights will meet the constraint.

From the above it can be seen that the Method of Reciprocal Averages actually implements a solution of Guttman's equations. The technique employs direct computation of the mean item response scores rather than obtaining them as the solution to a set of equations. Yet, the derived weights satisfy the constraint that insures the derived weights and the extraneous set of weights are orthogonal.

Computational Considerations

Although the mathematical model due to Guttman (1941) shows that internal consistency scaling can be performed using the computational procedures of principal components analysis, there are certain computational disadvantages even when digital computers are employed. The principal components approach requires calculation of the co-occurrence matrix for the item-response categories and the size of this matrix depends on the total number of item response categories. In that most measuring instruments contain many item response categories a large co-occurrence matrix results. Methods are available for handling large matrices of this type but they are expensive in terms of computer memory and execution time. Thus, due to storage requirements, Guttman's procedure is limited to a modest number of item-response categories even when computers are employed. In contrast, the computational procedures in the Method of Reciprocal Averages do not require a co-occurrence matrix, and the size of the instrument to be analyzed is limited only by the length of the vector of item response weights. In the current computer program (Baker and Martin, 1969) the length of this vector is set arbitrarily at 1800. The amount of computer time used is a function primarily of the number of subjects rather than of the size of the instrument.

A significant feature of the Method of Reciprocal Averages as a scaling technique is that it can be appended to an existing item analysis program such as was done by Baker and Martin (1969). The mean item response score is part of the item-criterion correlation calculation for both the biserial and point biserial correlations commonly used as item discrimination indices. The calculation of an index of internal consistency is routinely part of an item analysis program. Therefore to implement the Method of Reciprocal Averages one merely adds a simple weight assignment subroutine and a convergence test to an item analysis program.

In present implementations of the Method of Reciprocal Averages, the mean item response scores are not used as the derived weights, rather, the frequency distribution of the mean item response scores is divided into a number of regions and an integer number associated with each region serves as the weight rather than the obtained mean item response score. The effect of this substitution appears to be minor and its simplicity from a computer programming point of view outweights other considerations.

The convergence criterion for the iterative procedures used by Guttman (1941), Moiser (1946) and Mosteller (1949), was that two successive sets of weights did not differ. The weight by weight comparison can be avoided if one recalls that maximizing internal consistency is the goal of the scaling procedure. Thus, Baker and Martin (1969) used the difference between two successive values of Hoyt's ANOVA index of internal consistency as the convergence variable. When this difference is less than some arbitrarily small value, the iterative procedure is terminated. Such a convergence criterion is more in keeping with the basic rationale of the scaling procedure.

Summary

The Method of Reciprocal Averages first appeared as an informal computational procedure in the work of Richardson and Kuder (1933) and has been used by psychologists and others since that time. But, since its inception, a deficiency of the technique has been the lack of an explicit mathematical model. Guttman (1941) had derived a model for internal consistency scaling that was equivalent to the principal components model due to Hotelling (1933). In that the Method of Reciprocal Averages also maximizes the internal consistency index of an instrument for a given population, a connection between the two should exist. The present paper used the work of Hoyt and Collier (1953), Moiser (1946), Mosteller (1949), and Torgerson (1958) to show that the Method of Reciprocal Averages actually solves the equations yielded by Guttman's Least Squares approach under the appropriate constraint.

The Method of Reciprocal Averages has considerable appeal as a scaling technique on computational grounds because it does not require the computation of an item response co-occurrence matrix. Hence it can cope with very large instruments at a reasonable cost as the computer time used is a function of the number of subjects rather than the size of the instrument. In addition, the computational procedures are such that they can be implemented as simple extensions to existing item analysis programs.

In practice, the Method of Reciprocal Averages has proved to be an extremely useful scaling procedure. Clarification of its underlying mathematical model means that it can be employed with confidence in a wide range of psychological, educational and behavioral science research.

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