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ABSTRACT

These recommendations cover the graduate course work for a doctorate in mathematics. A strong background is assumed of courses such as those outlined in "A General Curriculum in Mathematics for Colleges" and "Preparation for Graduate Study in Mathematics", some of which are reprinted in an appendix. This document then gives detailed outlines for three one-semester courses in analysis, three in topology, two in algebra, and three in applications. There is also a discussion of apprenticeship in teaching. (MM)

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MATHEMATICAL ASSOCIATION OF AMERICA

A BEGINNING GRADUATE PROGRAM IN MATHEMATICS FOR PROSPECTIVE TEACHERS OF UNDERGRADUATES

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS
1969

SE 013 602

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A BEGINNING GRADUATE PROGRAM IN MATHEMATICS

For Prospective Teachers of Undergraduates

Report of the
Graduate Task Force

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

February, 1969

The Committee on the Undergraduate Program in Mathematics is a committee of the Mathematical Association of America charged with making recommendations for the improvement of college and university mathematics curricula at all levels and in all educational areas. Financial support for CUPM has been provided by the National Science Foundation.

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INTRODUCTION

More than half of the college and university mathematics teachers in the United States do not hold a PhD in one of the mathematical sciences.* Thus, the bulk of undergraduate teaching of mathematics in the country is being done by men and women whose graduate training, for one reason or another, has broken off short of the doctorate. There is no convincing evidence that this situation will soon change: the rising output of PhD's in mathematics is probably more than offset by rapidly rising college enrollments and by increasing demands on mathematics as a service discipline.

Those concerned with the preparation of college teachers are therefore faced with a basic problem: What is the best way to arrange the early part of the graduate program in mathematics to provide background for effective college teaching? This problem is further complicated because the research potential of most students is still untested when they begin graduate work; thus, it is not possible to separate those who will complete a PhD from those who will not. Accordingly, the choice of topics for the first year or two of graduate study must permit students to progress unretarded toward the PhD. This booklet explores one solution to this problem.

We contend that all graduate students of mathematics should be treated as future teachers, first, because most of them do in fact go into teaching, and second, because virtually all professional mathematicians are engaged to some extent in the communication of mathematics. Hence, graduate programs aimed at producing better teachers may be expected to benefit everyone.

In 1965, The Committee on the Undergraduate Program in Mathematics (CUPM), created an ad hoc Committee on the Qualifications of College Teachers, whose report, QUALIFICATIONS

* According to the 1967 document, Aspects of Undergraduate Training in the Mathematical Sciences: Report of the Survey Committee, Conference Board of the Mathematical Sciences, Washington, D.C., p.32, about 3,500 mathematics teachers in universities hold a PhD degree in the mathematical sciences, while 1,200 do not. In four-year colleges, however, only 1,600 hold such a degree, while 4,400 do not.

FOR A COLLEGE FACULTY IN MATHEMATICS, was issued in 1967. In particular, this qualifications report outlines a graduate program ("first graduate component") that provides both a reasonable first segment of a PhD program and adequate background for teaching the lower division courses* described in the 1965 CUPM publication, A GENERAL CURRICULUM IN MATHEMATICS FOR COLLEGES (GCMC). The qualifications report goes on to outline additional training ("advanced graduate component") needed to teach the full GCMC program.

In 1967, a task force, with membership drawn from CUPM and its Panel on College Teacher Preparation, was given the assignment of preparing a more detailed description of the first graduate component. A description appears in the pages that follow.

Clearly, this set of recommendations is not the only possible solution to the problem stated above, but we believe that it forms a sound basic program which each university can adapt to local conditions. The time for its completion will depend upon the student's ability and preparation, but in most cases one to two years beyond the bachelor's degree should be adequate. Satisfactory completion will ensure that the student has sound academic qualifications for teaching lower division courses in calculus, linear algebra, probability, and advanced multivariable calculus.

This program can be adapted readily to the special needs of teachers of university parallel curricula in two-year colleges. A separate report on the qualifications and graduate preparation of teachers in two-year colleges is forthcoming.

Besides serving the purposes already described, appropriate parts of the course of study we recommend would constitute an excellent sabbatical program for established teachers who wish to improve their acquaintance with modern approaches to mathematics.

The recommended program is discussed in detail in the following section. Here we mention some of its characteristic features and reasons for them. Two considerations figure prominently in the selection of topics for courses: first, the relative importance of the topic

* These are the GCMC courses in calculus, linear algebra, elementary probability and advanced multivariable calculus; according to the qualifications report, those who complete this first graduate component will also have the technical qualifications needed to teach some of the upper division GCMC courses.

in all of mathematics; and second, its relevance to teaching the lower division courses described in the GCMC report.

In analysis, to follow a year of undergraduate real analysis and a semester of undergraduate complex analysis (like the GCMC courses* 11, 12, 13), the program includes a semester of measure and integration followed by a semester of functional analysis. The course in measure and integration is obviously relevant to the GCMC courses in calculus and probability. We believe that the course in functional analysis is more important at this stage than a second course in complex analysis, since functional analysis will further develop the methods of linear algebra, the concept of uniform convergence, and various other topics in analysis. Moreover, functional analysis provides an immediate application of the course in measure and integration.

In topology we recommend a sequence which, in addition to the usual material in basic topology, includes an introduction to manifold theory and differential forms, to provide the prospective teacher with a deeper understanding of multivariable calculus.

Since lower division mathematics needs to be illustrated liberally with uses of the subject, college teachers must command a broad knowledge of the applications of mathematics. Also, many will be called upon to teach elementary probability and statistics. Hence, we recommend that a course of study for college teachers include two or three semesters of work at the advanced undergraduate or beginning graduate level chosen from courses in probability, statistics, differential equations, numerical analysis or subjects in applied mathematics. Moreover, all courses in the program should give attention to the relevance of their subjects to undergraduate mathematics and related disciplines.

In algebra, we believe that a year-long advanced undergraduate course such as that described in the CUPM publication, PREPARATION FOR GRADUATE STUDY IN MATHEMATICS,**is essential, but that further study of algebra is less important in the preparation of teachers of lower division mathematics than the suggested work in analysis, topology and applied mathematics; therefore, graduate level algebra has been treated as an elective. Likewise, we do not advocate a special requirement in geometry, partly because a considerable amount of geometry in

* Outlines of the advanced undergraduate courses in analysis are included in the appendix on pp. A1-7.

** An outline of the advanced undergraduate course in algebra is included in the appendix on pp. A8-12.

various forms is distributed throughout other recommended courses, and partly because advanced training in geometry does not seem essential either as background for lower division teaching or as general preparation for further graduate study. Nevertheless, geometrical points of view should be stressed in courses whenever they are appropriate.

It is certain that many teachers of lower division mathematics will, in the very near future, be called upon to use the computer to some extent in their courses. However, in view of the rapid developments in computer science and the many non-mathematical factors involved, any explicit recommendations on the role of computing in this program must be regarded as tentative at this time. We do expect that students completing this program will have acquired at least a basic knowledge of computers, perhaps by way of an introductory undergraduate course in computer science such as Course B1 outlined in recent curriculum recommendations from the Association for Computing Machinery (see pp. 13-17 of the Appendix).

Apart from formal course work, we feel that a meaningful apprenticeship in teaching is an essential aspect of the student's preparation, and activities to provide such an apprenticeship should form an integral part of beginning graduate work.

A master's degree would suitably recognize completion of the first graduate component. However, in place of a master's thesis, we strongly recommend the substitution of a comprehensive examination. Foreign language requirements are not discussed here because we believe that they are irrelevant for a student whose graduate training stops at the first graduate component; however, the student who hopes to earn a PhD should be advised that a reading knowledge of foreign languages is likely to be essential in his subsequent work.

It must be understood that the student who has completed this or any other program will not be, by that reason alone, a complete teacher or mathematician for the rest of his career; sustained intellectual and professional growth is essential to continued competence as a teacher and as a mathematician. For this reason, we urge a graduate faculty to make vigorous efforts to involve the students seriously, as participants rather than observers, in the mathematics they are studying. It is important for the student who stops short of the PhD, even more than for the one who will complete it, that course work of the first two years of graduate study emphasize fundamentals and basic understanding. This applies especially to the prospective college teacher who must be able to relate his graduate work to the material he will be teaching later. Courses which involve the student in doing mathematics as distinct from hearing about mathematics would seem to be particularly valuable. Depth of understanding on the part of the student is to be preferred to superficial exposure to mathematical terms. Our course outlines should be understood in this context.

Finally, we emphasize that it has not been our objective to design a separate track in graduate mathematics. This program is intended to prepare the student as an effective and well-informed teacher of lower division mathematics; but, at the same time, we believe it moves him toward the PhD at a satisfactory rate.

PROGRAM DESCRIPTION

The "first graduate component," as described in the qualifications report, is a program of graduate study built upon strong undergraduate preparation in mathematics. Because the undergraduate preparation of graduate students varies widely, it is useful to describe the first graduate component in terms of the combined undergraduate and graduate preparation of the candidate. It is likely that many students will have to complete in graduate school some properly undergraduate work, and therefore, for such students, up to two years of post-baccalaureate study may be required to complete this program.

We assume that every student has already completed lower division courses equivalent to the GCMC courses listed below, including a basic course in computer science. Each course is a one semester, three or four-credit course.

1, 2, 4 Introductory Calculus, Mathematical Analysis

These courses include the differential and integral calculus of the elementary functions with the associated analytic geometry as well as the usual techniques of the single-variable calculus. There is some treatment of limits, series, multivariable calculus and differential equations.

2P Probability

This is an introductory course in probability and statistical inference using some calculus.

3 Linear Algebra

Systems of linear equations, vector spaces, linear transformations, matrices, quadratic forms, and geometrical applications are topics in this basic course.

5 Advanced Multivariable Calculus

This course deals with vector analysis, Stokes' and Green's theorems, and includes an introduction to partial differential equations.

Introduction to Computer Science

This course deals with algorithms, programs, and computers. Among the suggested topics are basic programming, programming and computing systems, debugging, computer languages, and applications. (A detailed outline is provided on pp. A13-17.)

In addition, he will have studied some, but probably not all, of the following upper division courses:

7B Probability* (Prerequisite: 2P)

This second course in probability covers moments of distributions, joint density functions, random variables, Markov chains, and stochastic processes.

7A Statistics* (Prerequisite: 2P or 7B)

This is a course dealing with statistical inference and estimation, regression, analysis of variance, design of experiments, decision theory, and testing of hypotheses.

8 Numerical Analysis

Machine oriented, this course deals with numerical solutions of differential equations, numerical methods in linear algebra, and error propagation and stability.

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* The GCMC course in upper division probability and statistics, Mathematics 7, is a one-semester course defined in a flexible way. In particular, the two suggested course outlines on pages 55-58 of the GCMC report describe essentially distinct courses, one of which might properly be described as "statistics," the other as "probability." We denote these alternative versions by 7A and 7B, respectively.

9 Geometry

This is a course dealing with a single geometric theory from a modern axiomatic point of view.

10 Applied Mathematics

Illustrated in this course are the principles and basic styles of thought in solving real world problems by mathematical methods. Model building in a particular discipline is emphasized.

The next five courses form the core of preparation for graduate study. Outlines are provided on pp. A1-12. The first three are reproduced from the GCMC and the last two are reprinted from the CUPM publication, PREPARATION FOR GRADUATE STUDY IN MATHEMATICS.

11,12 Introductory Real Variable Theory

13 Complex Analysis

D, E Abstract Algebra

Graduate courses which are especially appropriate for the first graduate component, and for which suggested course descriptions are given starting on page 13 of this booklet are:

P. Measure and Integration
Q. Functional Analysis
R. Complex Analysis
S. Topology
T. Homology and Multivariable Integration
U. Topology and Geometry of Manifolds
V. Galois and Field Theory
W. Ring Theory and Multilinear Algebra
X. Advanced Ordinary Differential Equations with Applications
Y. Problem-Oriented Numerical Analysis
Z. Seminar in Applications

Courses 11, 12, 13, D, E, P, Q, S and T should be in every student's program. Because of the great importance of applied mathematics, every program of study should include at least one year of applied work, of which the following four sequences are examples: 7B-7A, X-10, X-Y, X-Z.

Each student should, if possible, include a third course from among the courses 7A, 7B, 10, X, Y, Z in his program. Students who plan to continue into the advanced graduate component and to specialize in some area of pure mathematics are advised to take as many as possible of the courses R, U, V, W. Other students may substitute electives in geometry, logic, foundations, number theory or other subjects.

For the sake of convenience, we have stated our recommendations in terms of semester courses. However, we believe that courses at the graduate level are best thought of as year courses. The material outlined for pairs of related courses can, of course, be rearranged within the year to suit local conditions.

Since beginning graduate programs ordinarily include year courses in analysis, topology, and algebra, our recommendations depart from the norm only in ways intended to enhance the ability of the student to teach lower division mathematics.

Effective exposition is a skill of major importance to any prospective mathematician, whether he expects his principal professional emphasis to be research or teaching. However, it is unrealistic to assume that a beginning graduate student is qualified to teach well, even in introductory undergraduate courses. Therefore, we propose that he be required to complete an apprenticeship in teaching under the thoughtful direction of experienced members of the faculty. Suggestions for such a program are discussed on pp. 54-56.

To complete the program, we suggest that every student be required to take a comprehensive examination designed specifically to test the breadth and depth of the candidate's understanding of mathematics relevant to the undergraduate curriculum. Whenever feasible, the examination should be scheduled so that students have several weeks devoted exclusively to preparation for it. We believe that a truly comprehensive examination is a more appropriate requirement than the traditional master's thesis, principally because preparation for such an examination demands that the student regard his subject as a whole rather than a collection of parts.

In summary, the first graduate component, as described here, consists of the following work:

(1) Completion (if necessary) of a strong undergraduate major program which includes these upper division courses: three semesters of real and complex analysis, a full year of abstract algebra.*

(2a) A year of graduate topology, including differential forms.

(2b) A year of graduate analysis: measure and integration and functional analysis.

(2c) A year of work at the advanced undergraduate or beginning graduate level, emphasizing the applications of mathematics: such as a year of probability and statistics; or a semester of differential equations followed by a semester of numerical analysis, a seminar in applications, or a "model building" course.

(3) A year or more of work focussed on problems of teaching undergraduates.

For a student whose undergraduate preparation does not meet the standards described in (1) and (2c), completion of the first graduate component may require two years of study beyond his bachelor's degree. For example, if his undergraduate preparation in algebra and in analysis is weak, his program for the first graduate component might be as follows:

<u>First Year</u>

Analysis (GCMC 11)	Analysis (GCMC 12)
Complex Analysis (GCMC 13)	Applied Mathematics (GCMC 10)
Algebra D	Algebra E
Apprenticeship in Teaching	Apprenticeship in Teaching

* Even though a student may have studied undergraduate courses bearing the same titles as those outlined in the Appendix, it should not automatically be assumed either that he has been exposed to, or that he understands, the material required. The mathematics department should determine, by testing or otherwise, the actual extent of each student's preparation.

Second Year

Analysis P Topology S Probability(GCMC 7B) and	Analysis Q Topology T Statistics (GCMC 7A)
OR	
Differential Equations X and	Numerical Analysis Y or Applications Seminar Z
Apprenticeship in Teaching	Apprenticeship in Teaching Comprehensive Examination

Most students will have completed some of the undergraduate courses in this program and thus will be able to substitute electives for some of the subjects listed. A student who has a very strong undergraduate major in mathematics will be able to complete the program in one year, for example, by taking the second year of the preceding schedule.

Graduate departments are urged to give careful attention to the proper placement of entering graduate students and to continue to advise them regarding course selections.

ANALYSIS

The following section includes suggested outlines for three one-semester graduate courses in analysis:

P. Measure and Integration (two suggested outlines are offered)

Q. Functional Analysis

R. Complex Analysis

Each student should include courses P and Q in his program of study.

P. Measure and Integration

This course provides an introduction to and essential background for Course Q, can be used in Course R, and is naturally useful in more advanced courses in real analysis. We present two outlines, which represent different approaches and a somewhat different selection of material. If presented in the right spirit, a course in Measure and Integration provides insights into the material of lower division courses that the student will have to teach.

First Outline

1. The limitations of the Riemann integral. Examples of a series that fails to be integrable term by term only because its sum is not integrable; of a differentiable function with a non-integrable derivative. Limitations of integration in general: there is no countably additive, translation-invariant integral for all characteristic functions of sets (the usual construction of a non-measurable set will serve).

2. Lebesgue integration on the line. Outer measure; definition of measurable sets by means of outer measure. Measurability of sets of measure 0, of intersections and unions, of Borel sets. Countable additivity. Application: the Steinhaus theorem on the set of distances of a set of positive measure. Measurable functions, Borel measurability, measurability of continuous functions. Egoroff's theorem. Definition of the integral of a bounded measurable function as the common value of $\inf \int \psi(x)dx$ for simple majorants ψ of f and $\sup \int \varphi(x)dx$ for simple minorants φ. Riemann integrable functions are Lebesgue integrable. Bounded convergence and applications (necessary and sufficient condition for Riemann

integrability; $\log 2 = 1 - 1/2 + 1/3 + \ldots$). Integrability of non-negative functions, Fatou's lemma, monotone convergence, integrability of general functions. A non-negative function with zero integral is zero almost everywhere.

3. L^p spaces, with emphasis on L^2; motivation from orthogonal series. Schwarz inequality; with little extra effort one gets (via convex functions) the Hölder, Minkowski and Jensen inequalities. L^∞ as a formal limit of L^p via $(\int f^p)^{1/p} \to$ ess sup f as $p \to \infty$. Parseval's theorem, Riesz-Fischer theorem. Rademacher functions; proof that almost all numbers are normal. Convergence of $\Sigma \pm \frac{1}{n}$ and other series with random signs. Proof (by Bernstein polynomials or otherwise) that continuous functions on an interval are uniformly approximable by polynomials. Hence, continuous functions are dense in L^p.

4. Differentiation and integration. Proof that an indefinite integral is differentiable almost everywhere and its derivative is the integrand; the Lebesgue set. Equivalence of the properties of absolute continuity and of being an integral.

5. Lebesgue-Stieltjes integral with respect to a function of bounded variation. A rapid survey pointing out what changes have to be made in the previous development. Applications in probability, at least enough to show how to treat discrete and continuous cases simultaneously. Riesz representation for continuous linear func-

tionals on C[a,b].

6. General measure spaces. Definition of the integral and convergence theorems in the general setting; specialization to n-dimensional Euclidean space. Fubini's theorem. Application to convolutions and to such matters as gamma-function integrals and $\int e^{-x^2} dx$. The one-dimensional integral as the integral of the characteristic function of the ordinate set.

7. (If time permits) Complex measures. Decompositions. Radon-Nikodým theorem.

Second Outline

1. Lebesgue integration on the line. F. Riesz's step function approach. Definition of the integral for simple step functions and extension to the class of functions which are limits almost everywhere of monotone sequences of simple step functions. Definition of a summable function and fundamental properties of the integral. Extension to complex-valued functions. The basic convergence theorems including monotone, bounded and dominated convergence theorems. Fatou's lemma and convergence in measure. Illustrations and applications: (a) justifications, by bounded convergence, of term by term integration of series leading to formulas such as $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$; (b) use of dominated convergence to perform operations such as $\frac{\partial}{\partial\alpha}\int f(x,\alpha)dx = \int \frac{\partial}{\partial\alpha} f(x,\alpha)dx$; (c) proof of the analogue of Fatou's theorem for series: $\underline{\lim}_{i}\, \Sigma\, a_{in} \geq \Sigma\, \underline{\lim}_{i}\, a_{in}$ when $a_{in} \geq 0$. Comparison of the Riemann and Lebesgue integral.

2. Measure and absolute continuity. Measurable functions and measurable sets. Properties of measurable sets. Egoroff's theorem. Cantor's function and the relationship between Lebesgue and Borel sets. Non-measurable sets. Proof that the integral of a summable function is a countably additive set function. Almost everywhere differentiability of monotone functions. Review of basic properties of functions of bounded variation. Absolutely continuous

functions. Fundamental theorem concerning differentiation of the integral of a summable function. Proof that an absolutely continuous function on an interval is of bounded variation and that its total variation is equal to the L^1 - norm of its derivative. Helly's theorem on compactness of families of normalized functions of bounded variation.

3. L^p spaces and orthogonal expansions. Convex functions and the inequalities of Hölder and Minkowski. Proof that the L^p spaces are complete. Theorem: If $\{f_i\}$ is a sequence of measurable functions and $f_i \to f$ pointwise almost everywhere and if $\lim_i \int |f_i| = \int |f|$, then $\lim_i \int |f_i - f| = 0$. Lusin's theorem: f measurable and finite almost everywhere and $\delta > 0$ implies there exists a continuous function φ such that $\varphi = f$ except on a set of measure less than δ. Hence continuous functions are dense in L^p, $1 \leq p < \infty$. Representation of continuous linear functionals on L^p. Orthonormal systems in $L^p(a,b)$. Bessel's inequality, Parseval's inequality and the Riesz-Fischer theorem. Proof that the Cesàro means of the Fourier series of a function f in $L^p(0, 2\pi)$ converge to f in the L^p-norm ($1 \leq p < \infty$) and uniformly provided that f is periodic and continuous. From this latter fact deduce the Weierstrass theorem on polynomial approximation of continuous functions on an interval. The trigonometric functions form a complete orthonormal system in $L^2 (0, 2\pi)$.

4. Integration on product spaces. Integration and measure theory in R^n. Theorems of Fubini and Tonelli. Applications to nonlinear change of variable in multiple integrals.

5. Convolution (optional). If f is summable, then f(t-x) is a measurable function on the plane. By Fubini's theorem, f and g in L^1 implies that $f*g(t) = \int f(t-x)\, g(x)dx$, the convolution of f with g, is finite for almost all t, f*g is in L^1 and $\|f*g\| \leq \|f\| \ \|g\|$. In fact, for $p > 1$, $q > 1$ and $0 < 1/r = 1/p + 1/q - 1$, $L^p * L^q \subset L^r$ and $\|f*g\|_r \leq \|f\|_p \|g\|_q$. (Actually, $L^1 * L^p = L^p$ for $1 \leq p < \infty$.) Also, if $1 < p < \infty$ and $p' = p/(p-1)$ and $f \in L^p$, $g \in L^{p'}$, then f*g is bounded. L^1 under convolution is an algebra without unit. Proof that $\widehat{f*g} = \hat{f}\hat{g}$, where $\wedge$ denotes the Fourier transform. Riemann-Lebesgue lemma.

6. General measure theory (optional). Set functions and introduction of abstract measure spaces. Definition of the integral and rapid review of standard theorems. Total variation of measures, regularity properties of Borel measures. Identification of Borel measures on line with functions of local bounded variation. Absolutely continuous and mutually singular measures and consequences of the Radon-Nikodým theorem. Riesz representation for C(X), X compact.

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Q. Functional Analysis

The purpose of this course is to develcp some of the basic ideas of functional analysis in a form suitable to applications and to deepen the student's understanding of linear methods in undergraduate mathematics. Whenever possible, topics should be treated and applied in a setting with which the student has some familiarity; main theorems should be supported with concrete and meaningful examples.

1. Metric spaces. Review of topology and metric spaces if necessary. Completion of metric spaces. Method of successive approximations. Proof that a contraction operator on a complete metric space has a unique fixed point. Application to existence of the solution of a system of linear equations, polynomial equations, initial value problems for ordinary differential equations and integral equations.
2. Normed linear spaces. Examples (not all complete) from sequence spaces, function spaces and finite dimensional spaces. Completion of $C[a,b]$ under the L^1 norm. Proof that the unit ball in a normed linear space is compact if and only if the space is finite dimensional. Equivalence of norms in finite dimensional spaces.
3. Linear functionals. The dual space of a normed linear space. Computation of the dual for spaces R^n, c_o, ℓ_p $(1 \leq p < \infty)$ and $C[a,b]$. Contrast with algebraic dual. Convex sets and separation of convex sets by linear functionals. Support functions. Analytic, geometric and complex forms of Hahn-Banach theorem. Applications of the Hahn-Banach theorem, such as (a) computation of the distance from a point to a subspace in terms of the linear functionals which vanish

on the subspace; (b) the existence of a function in $L^{\infty}(0,1)$ of minimal L^{∞} norm which satisfies the $N+1$ relations $\int_0^1 t^n f(t)dt = a_n, n=0,1,\ldots,N$; (c) solution of the Hausdorff moment problem for $C[0,1]$; (d) the existence of Green's function for Laplace's equation in the plane for a domain with sufficiently smooth boundary (see reference [9]). Principle of uniform boundedness and applications, such as (a) existence of a continuous periodic function on $[-\pi,\pi]$ whose Fourier series fails to converge; (b) the Silverman-Toeplitz conditions for a regular matrix summability method; (c) existence of the Riemann-Stieltjes integral $\int_0^1 f d\alpha$ for every continuous f implies that α is of bounded variation on $[0,1]$. Weak (not weak*) convergence of sequences in normed linear spaces. Proof that weakly convergent sequences are bounded but not necessarily norm convergent. Characterization of weakly convergent sequences in spaces such as ℓ_p $(1 \le p < \infty)$ and $C[a,b]$. Elementary introduction to distribution theory.

4. Linear operators. Examples from matrix theory, differential and integral equations. The closed graph theorem and the interior mapping principle. Notion of an adjoint operator. Inversion of linear operators near the identity. The spectrum and resolvent of an operator.

5. Hilbert spaces. Inner products, orthogonality and orthogonal systems. Fourier expansions, Bessel's inequality and completeness. Representation of linear functionals. Self-adjoint operators on a real Hilbert space as a generalization of symmetric linear transformations on R^n. Eigenvalues, eigenvectors, invariant subspaces and projection operators. The spectral theorem for completely continuous self-adjoint operators. (Here the goal is the formula

$$Ax = \Sigma \lambda_k (x, x_k) x_k,$$

where A is a compact and self-adjoint operator, x is any point, $\lambda_1, \lambda_2, \lambda_3, \ldots$ is the sequence of non-zero eigenvalues and $x_1, x_2, x_3, \ldots$ is the corresponding set of eigenvectors.) Construction of a one-parameter family of projections E_λ which allows representation of the action of A in terms of a vector-valued Riemann-Stieltjes integral

$$Ax = \int \lambda \, dE_\lambda x.$$

Description (without proof) of the corresponding theorem for the unbounded case. Application of the theory of compact, self-adjoint operators to Sturm-Liouville systems or integral equations with symmetric kernels.

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R. Complex Analysis

The amount of material that can be covered depends very much on the amount of knowledge that can be assumed from GCMC Mathematics 13. The outline assumes that the student knows this material quite well; but some of the more advanced topics may have to be omitted or treated in less depth. Such topics are enclosed in brackets.

1. Holomorphic functions. (Much of this should be review.) Cauchy's integral theorem in a more general setting than was used in GCMC 13. According to circumstances this may be for unions of star-shaped regions, for C^1 Jordan curves, for singular cells, etc., but not for general rectifiable Jordan curves. Cauchy's integral formula. Taylor and Laurent series. Residue theorem. [Evaluation of some more sophisticated definite integrals than those in GCMC 13.] Classification of isolated singularities, Casorati-Weierstrass theorem. Liouville's theorem. Fundamental theorem of algebra. $\int f'/f = 2\pi i$ (number of zeros minus number of poles), with applications to some special cases (number of zeros of a polynomial in a quadrant, for example). Maximum modulus theorem. Schwarz's lemma. Rouché's theorem with some concrete applications (fundamental theorem of algebra again; zeros of $e^z + z$ and other special functions). [Montel's theorem, Phragmèn-Lindelöf theorems.]

2. Harmonic functions. Cauchy-Riemann equations. Mean-value property for harmonic functions. Poisson formula and Dirichlet

problem for the circle and annulus. Connection with Fourier series and Poisson summability of Fourier series at points of continuity. [Other problems on functions holomorphic in a disk: Abel's theorem, elementary Tauberian theorems.] [Fatou's theorem on radial limits.] [Positive harmonic functions, Herglotz's theorem on the integral representation of holomorphic functions with positive real part in a disk. Harnack's theorem.]

3. Holomorphic functions as mappings. Mapping properties of the elementary functions. Nonconstant holomorphic functions are open. Conformality at points where the derivative is not zero. Only holomorphic functions produce conformal maps. Conformal automorphisms of the disk and the half-plane. Normal families. Proof of the Riemann mapping theorem. [Schwarz-Christoffel formula. Conformal representation of a rectangle on a half-plane. Elliptic functions. Proof of the small Picard theorem.]

4. Analytic continuation. Schwarz reflection principle. Analytic continuation. Permanence of functional equations. Monodromy theorem. [Multi-valued functions. Elementary Riemann surfaces.]

5. Zeros of holomorphic functions. Infinite products. Entire functions. Meromorphic functions. The Weierstrass factorization theorem. Mittag-Leffler theorem. Gamma function. [Jensen's formula and Blaschke products.]

[6. Approximation. Runge's theorem and approximation by polynomials. Mergelyan's theorem.]

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TOPOLOGY

The following section includes suggested outlines for a sequence of three one-semester graduate courses in topology:

S. Topology

T. Homology and Multivariable Integration (two outlines are offered)

U. Topology and Geometry of Manifolds

Each student should include courses S and T in his program of study. Students who plan to elect course U must study the first (preferred) outline of T.

S. Topology

We assume that the students have made a brief study of metric spaces, Euclidean spaces and the notion of continuity of functions in metric spaces. (This material is covered in sections d, e, and f of GCMC 11-12.)

1. Basic topology. Topological spaces, subspace topology, quotient topology. Connectedness and compactness. Product spaces and the Tychonoff Theorem. Separation axioms, separation by continuous functions. Local connectedness and local compactness. Metric spaces, completion of metric spaces, uniform continuity. Paracompactness, continuous partitions of unity.
2. Applications to calculus. Use the results above to re-prove the basic topological results needed for calculus and the Heine-Borel and Bolzano-Weierstrass Theorems.
3. Fundamental group. Homotopies of maps, homotopy equivalence. The fundamental group π_1, functorial properties, dependence on base point. Show that $\pi_1(S^1) = Z$.
4. Applications of the fundamental group. Brouwer fixed point theorem for the disk D^2. R^2 is not homeomorphic with R^3. Relevance of the fundamental group to Cauchy's residue theorem. Fundamental theorem of algebra.
5. Covering spaces. Covering spaces, homotopy lifting and homotopy covering properties. Regular coverings, existence of coverings, universal covering. Factoring of maps through coverings. Relation with Riemann surfaces.

T. Homology and Multivariable Integration
Preferred Outline

1. Manifolds. Topological manifolds. C^k and C^∞ functions on R^n. Differentiable structure on a topological manifold. Diffeomorphisms. C^∞ partitions of unity for paracompact manifolds.
2. Functions on manifolds. The ring $\mathfrak{a}(U)$ of C^∞ real-valued functions on an open set U, the ring $\mathfrak{a}(x)$ of germs of C^∞ functions at a point x. Pull backs of these rings via a C^∞ function. Tangent bundle and cotangent bundle. Bases for tangent and cotangent spaces in a coordinate system. Vector fields, Poisson bracket, flows. Inverse and implicit function theorems. Frobenius theorem.
3. Applications to differential equations. Relation of vector fields to ordinary differential equations, and the Frobenius theorem to partial differential equations.
4. Differential forms. Differential forms, elementary forms. Exterior multiplication of forms, the differential operator d on forms; dd = 0 and d of a product.
5. Applications to classical vector analysis. The algebra of forms on R^3 contains vector algebra and with d contains vector analysis.
6. deRham cohomology. Pull back of forms via a C^∞ map commutes with d. Closed and exact forms, deRham groups as a cohomology theory.

7. Simplicial homology. Simplicial complexes, simplicial homology. Barycentric subdivision, simplicial approximation theorem. Calculation of π_1 for a simplicial complex. Singular homology and cohomology of a space.

8. Applications of simplicial homology. The Brouwer fixed point theorem for D^n, invariance of domain and the Jordan curve theorem. (Recall use of the Jordan curve theorem in complex analysis.)

9. Stokes' theorem. Integral of a p-form over a singular p-chain. Proof of Stokes' theorem. This implies that integration induces a bilinear map from singular homology and deRham groups to R. Green's theorem as a special case of Stokes' theorem.

Second Outline

Note: If a student does not plan to take course U, then the following easier version of T may be desirable. This is carried out by working in R^n instead of in general differentiable manifolds, and the result is still a fairly general version of Stokes' theorem.

1. Simplicial homology. Simplicial complexes, barycentric subdivision, simplicial maps and the simplicial approximation theorem. Simplicial homology theory, functorial properties of homology groups. Calculation of homology groups for simple complexes.

2. Differential forms. Differential forms on open sets of R^n. Properties of differential forms, the operator d on forms. Pull back of forms via a C^∞ function. Application to vector algebra and vector calculus. Closed and exact forms, the deRham groups.

3. Singular homology and applications. Singular homology theory. Applications: the Brouwer fixed point theorem, R^n and R^m are homeomorphic if and only if $n = m$, invariance of domain, Jordan curve theorem.

4. Stokes' theorem. Integration of p-forms over differentiable singular p-chains. Proof of Stokes' theorem.

U. Topology and Geometry of Manifolds

1. Chain complexes. Chain and cochain complexes (examples from T), derived groups. Exact sequences, ladders, the 5-lemma. Exact sequences of chain complexes, Bockstein exact sequence. Chain homotopies. Poincaré lemma and cone construction; derived groups of a contractible open set are zero.

2. Riemannian metrics for manifolds. Riemannian metrics for paracompact manifolds. Geodesics: existence and uniqueness. A paracompact C^∞ manifold may be covered with a star finite covering by geodesically convex sets (so that all sets in the covering and all intersections are contractible).

3. Comparison of homology theories. A lattice L of subsets of X containing $\emptyset$ and X gives a category S with elements of L as objects and inclusions as morphisms. A cohomology theory h on S is a sequence of cofunctors h^q from S to abelian groups along with natural transformations

$$\delta : h^q(A \cap B) \to h^{q+1}(A \cup B)$$

such that the Mayer-Vietoris sequence is exact. Proof that if L is a star finite covering of X by open sets, then cohomology theories h, $\hat{h}$ which agree on finite intersections agree on X. Use of these results to deduce deRham's theorem and to prove that simplicial and singular theories agree on a simplicial complex.

4. Global differential geometry. The remainder of the course is devoted to surfaces. Gaussian curvature, spaces of constant curvature. Gauss-Bonnet theorem for surfaces, non-Euclidean geometries.

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4. Spanier, E. H. Algebraic Topology. New York: McGraw-Hill, 1966.

Notes

Course S is almost exactly Chapters I, II, III of [3]. The material in T is also in [3] in a somewhat different order. Another reference for the differential forms part of the second outline of T is [2]. The treatment of deRham's theorem in U is given in [1]. The material on Riemann surfaces in U is in [3]. A good reference for the singular homology theory is [4].

ALGEBRA

The following section includes suggested outlines for two one-semester graduate courses in algebra:

V. Galois and Field Theory

W. Ring Theory and Multilinear Algebra

These courses are independent of one another and should be offered as electives. Each course outline includes a basic minimal list of topics as well as a list of optional topics from which the instructor is invited to choose.

V. Galois and Field Theory

Note: 1 and 2 are reviews of topics that should have been covered in the previous one-year algebra course D-E outlined in the appendix, pp. 8-11.

1. Review of group theory. The third isomorphism theorem. Definition of simple groups and composition series for finite groups. The Jordan-Hölder theorem. Definition of solvable groups. Simplicity of the alternating group for $n > 4$. Elements of theory of p-groups. Theorems: A p-group has non-trivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

2. Review of elementary field theory. Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a field, a an algebraic element of some extension field. Direct proof that if a has degree n, the set of polynomials of degree n-1 in a is a field, demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting an angle, duplicating the cube, squaring the circle (assuming π transcendental).

3. Galois theory. The group $G(M/K)$ of K-automorphisms of a field $M \supset K$. Fixed field H' of a subgroup H of $G(M/K)$. Subgroup F' of $G(M/K)$ leaving an intermediate field F fixed. Examples like $Q(\sqrt[3]{2})$ to show that $G(M/K)'$ may be bigger than K. An object is closed if it equals its double prime. If $M \supset F \supset F_1 \supset K$ and $G(M/K) \supset H \supset H_1$, then $[F_1' : F'] \leq (F:F_1)$ and $(H_1' : H') \leq [H:H_1]$. All finite subgroups of $G(M/K)$ are closed. M/K is galois if $G(M/K)' = K$. Fundamental theorem. Artin's theorem: if M is a field and G a finite group of automorphisms of M, then M is galois over G'. Extension of isomorphisms theorem. Applications to elementary symmetric functions. Galois subfields and normal subgroups.

4. Construction of galois extension fields. Splitting fields, several characterizations. Uniqueness. Separability. Galois if and only if separable and splitting. Galois closure of intermediate field. Galois group as a group of permutations of the roots. Examples of splitting fields. Explicit calculations of galois groups of equations. Roots of unity. Cyclotomic polynomials. Irreducibility over the rationals. Construction of regular polygons by ruler and compass.

5. Solution of equations by radicals. Definition of radical extension fields. In characteristic 0, if M/K is radical, then $G(M/K)$ is solvable. Tie-up between radical extensions and solving

equations by radicals. Insolvability of general equations of degree ≥ 5. If f is irreducible over Q, of prime degree p, and has exactly 2 real roots, then its galois group is S_p. Explicit examples. Hilbert's Theorem 90. Form of cyclic extension if ground field contains roots of unity. If G(M/K) is solvable, then M/K is radical.

6. Finite fields. Recall GF(p). A field has p^n elements if and only if it is the splitting field of $x^{p^n} - x$. $M \supset K$, finite fields, implies M is galois and cyclic. Examples from elementary number theory. The normal basis theorem.

Optional Topics

The following topics are listed with no preferential order. They are to be used at the instructor's discretion.

7. Simple extensions and separability. A finite dimensional extension field is simple if there are only finitely many intermediate fields. M separable and finite dimensional over K implies M is simple. Purely inseparable extensions and elements. Maximal separable and purely inseparable subfields. Splitting fields are generated by these. Transitivity of separability.

8. Algebraic closure and infinite galois theory. Definition of algebraically closed field. Existence and uniqueness of algebraic closure. (Use Zorn's lemma.) Point out that one cannot get the Fundamental Theorem of Algebra this way. Infinite algebraic

extensions, Krull topology on galois group. Galois group is compact and totally disconnected. Inverse limit of finite groups. Fundamental theorem of galois theory in this case.

9. Transcendental extensions. Algebraically independent subsets of field extensions. Purely transcendental extensions. Transcendental extensions. Transcendence bases done so that the proof could be used for bases of vector spaces. Usual properties of transcendence bases and transcendence degree. Transcendence degree of composite. Separable generation. MacLane's criterion.

W. Ring Theory and Multilinear Algebra

1. Categories and functors. Introduce the category of sets. Definition of a category. Examples of categories: the category of groups, the category of rings, the category of fields, the category of vector spaces, the category of modules; epimorphisms, monomorphisms, isomorphisms, surjections, injections. Examples to show that an epimorphism is not necessarily surjective and a monomorphism is not necessarily injective. A group as a one object category whose morphisms are all isomorphisms; similar ways of looking at groupoids and other algebraic systems. Dual of a category, duality, examples. Additive and abelian categories with examples. Functors and natural transformations with lots of examples, for instance viewing modules as functors. The Yoneda lemma: Nat (Hom(A,-), T) = T(A). Illustrations and

examples of universal objects. Definition and elementary properties of adjoint functors. (The language of categories will be useful throughout the course and elementary categorical notions can simplify many proofs.)

2. Introduction to algebraic number theory. Noetherian rings and their modules. The Hilbert basis theorem. Definition of integral elements. Integral closure. Integers in a number field. Examples of quadratic fields. Units. $Q(i)$, $Q(\omega)$ have integers which are UFD but integers in $Q(\sqrt{-5})$, $Q(\sqrt{10})$ are not. Fermat's last theorem for $p = 3$ using $Q(\omega)$.

3. Valuation and Dedekind rings. Definition of a discrete valuation ring as a PID with exactly one non-zero prime ideal. Valuation of quotient field associated with discrete valuation ring and converse. Examples of rank one discrete valuations. Various characterizations of discrete valuation rings including: R is such if it is a Noetherian domain which is integrally closed and has exactly one non-zero prime ideal. The ring of fractions of a domain with respect to a multiplicative semigroup. R_p for p a prime ideal. Dedekind ring is a ring R such that R_p is a discrete valuation ring for all prime ideals p of R. Unique factorization of ideals in Dedekind rings. Other characterizations of Dedekind rings. Approximation lemma. If $M \supset K$ are fields with M finite-dimensional and separable over K, K is the field of quotients of a Dedekind ring A, then the integral closure of A in M is again Dedekind. Integers of a number field are Dedekind.

4. Tensor products. Defintion of one sided module over a ring R. Examples. Free modules. Submodules, quotient-modules, exact sequences. Tensor products defined via universal properties. Uniqueness. Existence. Examples, $\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/3\mathbb{Z} = 0$. If R is commutative, tensor product is again an R-module. Tensor product of maps. Behavior of tensor products with regard to exact sequences and direct sums. Examples. Tensor products of free modules and matrix rings. Associativity of tensor product. Tensor product of n modules over a commutative ring, multilinear maps. Tensors. Tensor product of p copies of a free module and q copies of its dual, components in notation of physics. Tensors as defined in physics: R is the ring of C^{∞} functions on R^n and M is the R-module of derivations of R. M is free, generated by the usual partials. Transformation of coordinates. Express elements of $M^p \otimes_R M^{*q}$ in terms of two coordinate systems to get usual transformation rules. The tensor algebra and its universal property.

5. Exterior algebra. Multilinear alternating maps. Antisymmetric maps. Definition of the exterior algebra as a homomorphic image of the tensor algebra. Universal property of the exterior algebra. p-vectors. Exterior algebra of a free module over a commutative ring, explicit calculation of a basis and dimension of the module of p-vectors. Prove invariance of vector-space dimension once more. Determinants via exterior algebra. Usual formula for

determinant, determinant of transpose = determinant.

6. Structure theory of noncommutative rings. Ring means ring with unit. Simple left module is ring modulo a maximal left ideal. Primitive ideals. Division rings and vector spaces over them. The ring of all linear transformations, both finite and infinite dimensional case. Schur's lemma. Density theorem. Wedderburn-Artin theorem. Uniqueness of simple modules. Structure of semisimple Artinian rings. Structure of semisimple modules.

7. Finite group representations. The group algebra. Maschke's theorem: The group algebra of a group of order n over a field of characteristic prime to n, is semisimple. Representations and characters. Connection between the decomposition of the group algebra over the complex field and the simple representations. The characters determine the representation.

Optional Topics

The following topics are listed with no preferential order. They are to be used at the teacher's discretion.

8. Radicals of non-commutative rings. Radical = intersection of all primitive ideals = intersection of all left maximal ideals. Equivalent definition of radical. Examples. Behavior of radical under homomorphisms and subring formation. Nakayama Lemma. Artinian rings. Radical is nilpotent in Artinian ring. A ring

modulo its radical is a subdirect sum of primitive rings. Connection with semi-simple rings.

9. *Further group theory.* Permutation groups. Linear groups. Structure theory of linear groups. Examples of finite simple groups. Groups defined by generators and relations. Further work on representations of finite groups: one dimensional representations, the number of simple characters, orthogonality relations, applications and examples.

10. *Further algebraic number theory.* Infinite primes. The product formula. The Dirichlet unit theorem and finiteness of the class number.

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5. Freyd, P. Abelian Categories. New York: Harper and Row, 1964.

6. Jacobson, N. Lectures in Abstract Algebra. Vol. III. Theory of Fields and Galois Theory. Princeton, N.J.: Van Nostrand, 1964.

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14. Zariski, O. and Samuel, P. Commutative Algebra. Vol. I. Princeton, N. J.: Van Nostrand, 1958.

APPLICATIONS

It is essential that the prospective teacher of college mathematics know and appreciate some of the honest applications of the calculus, linear algebra, and probability. Merely as a matter of expediency a teacher of these subjects will have need of convincing examples and illustrations; but, more important, a knowledge of some applications will enable him to know best how to present mathematics and will add an extra dimension to his exposition.

There are many different ways in which the prospective teacher can acquire a background in applications of mathematics. We list here several possibilities which seem highly appropriate; each requires at least one year of course work.

1. Probability and Statistics. Some students will wish to pursue the study of probability and statistics at the advanced undergraduate or graduate level. The year long course obtained by combining the two outlines* for Mathematics 7 of GCMC will serve our purpose well, provided that due emphasis is placed upon applications of these subjects.

2. Differential Equations - Applications. In the pages that follow, three one-semester courses at the advanced undergraduate or beginning graduate level are described:

   X. Advanced Ordinary Differential Equations with Applications

   Y. Problem-Oriented Numerical Analysis

   Z. Seminar in Applications

   As a source of material in applied mathematics no subject is richer than differential equations. Hence, our alternative recom-

* See footnote, p. 7.

mendations for a year's study in applications begin with course X.

A second semester can be chosen from several possibilities: perhaps the best is the course Mathematics 10, as described in GCMC, or one of the courses outlined in the CUPM publication, A CURRICULUM IN APPLIED MATHEMATICS. Course Y, if taught in the proper manner, will also be suitable for this purpose. Another alternative for this second semester would be for the mathematics department to offer a seminar (course Z) presenting applications of the calculus, linear algebra and probability to the physical, biological and social sciences.

In summary, the suggested requirement for a year of study in applications of mathematics is one of the four sequences: 7B-7A; X-10; X-Y; X-Z. Because of the demand on students' time, we have been compelled to limit this requirement to one year. Nevertheless, we hope that many students will have the opportunity and interest to elect a third semester from among 7B, 7A, 10, X, Y, Z.

X. Advanced Ordinary Differential Equations with Applications

This course is designed to provide background for teaching the topics in differential equations that occur in the lower division GCMC courses; to give further exposure to applications via one of the most intensively used classical routes; and to provide a first course for students who may be interested in specializing in this area. Because of the nature of the subject, many different good course outlines are possible, but in any case, emphasis should be put on efficient ways of obtaining from differential equations useful information about their solutions, as distinguished say from methods for finding baroque solution formulas of little practical value.

1. Fundamentals. The vector differential equation $\dot{x} = f(t,x)$; prototypes in physics, biology, control theory, etc. Local existence (without uniqueness), by the Cauchy construction, when f is continuous. Prolongation of solutions and finite escape times. Properties of integral funnels (e.g., Kneser's theorem); extreme solutions when n = 2. Jacobian matrix of f locally bounded → Lipschitz condition → uniqueness → continuous dependence on initial values and parameters. Effects of stationarity.
2. Numerical integration. Euler, Runge-Kutta, and other methods; elements of error analysis for these methods. Practical machine computation.
3. Linear equations. Discussion of physical and other real-world models leading to linear equations. Linearization. Structure of the solution set of the vector equation (1) $\dot{x} = A(t)x + b(t)$; variation of parameters formula; the fundamental matrix. Matrix exponentials; thorough treatment of (1), using Jordan canonical

form, when A is constant. Applications in engineering system theory. Floquet's theorem.

4. Sturm-Liouville theory. The two-point boundary value problem for second order self-adjoint equations and how it arises. Existence of eigenvalues. Comparison, oscillation, and completeness theorems. Orthogonal expansions. Green's function. Applications to diffusion and wave equations. Some special functions.

5. Stability. Liapunov, asymptotic, and orbital stability; uniform properties. Basic theorems of Liapunov's direct method. Extensive treatment of the linear case. Applications in control theory.

6. Phase-plane analysis. Geometric treatment of second-order stationary systems. Classification of simple equilibrium points. Closed orbits and Poincaré-Bendixson theory.

Optional Topics

7. Power series solutions. Classification of isolated singularities of linear equations; formal solutions; Frobenius method. Asymptotic series.

8. Carathéodory theory. (Prerequisite: Lebesgue Integration).

References

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4. Hahn, W. Stability of Motion. New York: Springer, 1967.

5. Hartman, P. Ordinary Differential Equations. New York: Wiley, 1964.

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7. Lefschetz, S. Differential Equations: Geometric Theory, 2nd ed. New York: Interscience, 1963.

8. Zadeh, L. A. and Desoer, C.A. Linear System Theory. New York: McGraw-Hill, 1963.

Y. Problem-Oriented Numerical Analysis

Although the course we have in mind overlaps with standard courses in numerical analysis in some of its material, it differs fundamentally in spirit from such courses. The traditional course in numerical computation is intended to train the student to be able to compute certain specific quantities, such as the approximate value of definite integrals, roots of polynomial and transcendental equations, solutions of ordinary differential equations, by applying known algorithms to well formulated specific numerical problems. Courses in contemporary theoretical numerical analysis have tended to emphasize the technical aspects of specialized topics, such as the theory of approximation, spline interpolation, numerical linear algebra, discrete variable techniques for differential equations; here the stress is on widely applicable computational techniques, their underlying theory, and the errors arising in their application.

A problem-oriented course in numerical analysis starts with real life problems (from physics, economics, genetics, etc.), develops mathematical models (often in the form of differential or other types of functional equations), analyzes the models, and develops and applies numerical methods to the models in order to get some answers. The student's knowledge of analysis, linear algebra or differential equations is called upon in the analysis of the model; techniques of numerical analysis are studied and sifted through in the search for applicable methods; specific numerical computations are performed by the student, using a computer; and, finally, the numerical answers are examined in two ways: by means of a theoretical analysis of the errors inherent in the algorithm and in the machine computation, and by a comparison with the original problem to see whether the "answer" (often a table of values of some unknown functions) is a reasonably good approximation to reality.

It seems clear that text materials for Mathematics Y should include applied books or journal articles (as a source of real problems) and numerical analysis texts (as a source of numerical methods). Sample topics and associated texts might be:

(a) Problems in the theory of flight. Here one can find mathematical models and their analyses in works such as A. Miele, Theory of Flight Paths, Reading, Mass.: Addison-Wesley, 1962, and apply to the ensuing systems of differential equations techniques found in P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, New York: Wiley, 1962. (One sample problem on these lines can be found in

Section 10.9 of D. McCracken and W.S. Dorn, Numerical Methods and Fortran Programming, New York: Wiley, 1964. Although these authors pull a refined model of a simplified flight problem out of a hat - which the instructor in Course Y must not do - they examine at length the implications of the properties of the numerical solutions for the behavior of the physical system and use the flexibility of their computer program to vary parameters and do some interesting mathematical experimentation.)

(b) Control theory. Selected models and analyses from an applied text such as A. Sage, Optimum Systems Control, Englewood Cliffs, New Jersey: Prentice-Hall, 1968, can lead to problems of numerical solution of partial differential equations, two-point boundary value problems, and problems of numerical linear algebra. Suitable sources for numerical methods are:

G.E. Forsythe and W. Wasow, Finite-Difference Methods for Partial Differential Equations, New York: Wiley, 1960.

R.D. Richtmyer, Difference Methods for Initial-Value Problems, New York: Wiley, 1967.

H.B. Keller, Numerical Methods for Two-Point Boundary Value Problems, Waltham, Mass.: Blaisdell, 1968.

R.S. Varga, Matrix Iterative Analysis, Englewood Cliffs, New Jersey: Prentice-Hall, 1962.

A.S. Householder, The Theory of Matrices in Numerical Analysis, New York: Blaisdell, 1964.

Z. Seminar in Applications

As another approach to applications, we suggest a seminar devoted to applications of the calculus, linear algebra and probability to the physical, biological and social sciences. Fortunately, there are now several books which contain a wealth of readily accessible examples:

(1) Noble, Ben. Applications of Undergraduate Mathematics in Engineering. New York: Macmillan, 1967.

(2) Kemeny, J.G. and Snell, J.L. Mathematical Models in the Social Sciences. Boston: Ginn and Co., 1962.

(3) Thrall, Robert, et al. Some Mathematical Models in Biology. Ann Arbor, Mich.: The University of Michigan, 1967.

Further examples can be found in works on linear programming and game theory, and other references cited in the 1966 CUPM document, A CURRICULUM IN APPLIED MATHEMATICS.

The students would participate in the formulation of scientific problems in mathematical terms and in the interpretation and evaluation of the mathematical analysis of the resulting models. Due emphasis should be given to problems whose analysis rests on the use of the computer. It might be appropriate for the instructor to invite guests who could expose the student to the attitudes of users of mathematics. While such an arrangement would, perhaps, not be a traditional course in applied mathematics, it would allow the students to come into contact with a variety of serious applications of the usual mathematics of the first two undergraduate years. The following illustrate the type of examples we have in mind:

a. The formulation and analysis of a system of differential equations which serves as a model for (i) the interdependence of two species one of which serves as food for the other ((2), Chapter III), or (ii) a time-optimal navigation problem which requires that a boat be transferred from a given initial position to a given terminal position in minimal time (see page 34 of G. Leitman, An Introduction to Optimal Control. McGraw-Hill Book Company, 1966).

b. The formulation and analysis of waiting line and traffic problems involving simple calculus and probability ((1), Chapter 15).

c. Elementary matrix analysis associated with chemical mixture problems and mechanical equilibrium problems((1), Chapter 10) and matrix eigenvalue problems arising from electrical circuit analysis ((1), Chapter 5).

d. The "transportation problem" of making optimal use of a given shipping network to obtain a specified redistribution of commodities (see Chapter 14 of G. B. Dantzig, Linear Programming and Extensions. Princeton University Press, 1963). This is, of course, a special case of linear programming.

e. Game theory as applied to games of timing ("duels") in which rewards to competing strategists depend on when certain acts are performed (see Chapter 9 of M. Dresher, Games of Strategy. Prentice-Hall, 1961).

APPRENTICESHIP IN TEACHING

Every mathematician is a teacher in the sense that he must explain mathematical ideas to other people--to students, to colleagues, or to the mathematical community at large. For this reason, the graduate education of every mathematics student should include a program designed to develop skill in oral and written communication of mathematics. This program should begin as soon as the student enters graduate school and continue at increasing levels of responsibility.

Ultimately, the attitude of the graduate faculty will determine the success of any such program. If effective teaching is regarded as an important and non-trivial function of the department, and if senior mathematicians encourage excellent exposition by precept and personal interest, graduate students and younger faculty will respond accordingly. Every instructor of a graduate class should realize that his course can have a profound effect upon his students in the way it serves to strengthen the attributes of a good teacher.

Because the conditions of undergraduate and graduate instruction vary widely from one university to another, the suggestions given below offer a variety of ways in which the mathematics departments might stimulate more interest in good teaching. Each university is encouraged to create its program individually, seeking to establish an intellectual environment in which teaching and learning flourish together.

Some universities have experimented recently with special programs which bring new teaching assistants to the campus before the start of classes in the fall. Sessions are devoted to a general orientation to graduate and undergraduate study at that university and to the role of the graduate assistant. At least one program runs for the entire summer term and includes an initial involvement with graduate mathematics besides activities in preparation for teaching.

During the first stage of his training, the teaching assistant should be given limited duties, but he should be made to feel that he is a junior colleague in a profession rather than a hired hand in a work crew. At a pace which is adjusted individually to his rate of development, he should progress through a sequence of teaching assignments, acquiring more responsibility and independence as he gains in experience and confidence. He can mark homework papers, conduct office hours for undergraduates, prepare questions for tests, and assist in marking tests.

A prospective teacher can learn much by observing a skillful teacher in an undergraduate class in a subject familiar to the apprentice. This is of particular value in a class of selected students, such as freshman or sophomore honors sections where the interchange between students and the instructor is lively and challenging.

Regular consultation between an apprentice and his supervisor is essential. Each course supervisor should arrange meetings of all assistants for that course; at these meetings there should be free exchange of ideas concerning problems of instruction, alternate suggestions for presenting specific concepts, proposals for future test questions, and planning the development of the course. In addition to formal consultation, however, supervisors should maintain a running dialogue with apprentices, work with them in the marking of tests, and cooperate in performing with them the day-to-day duties which are an integral part of teaching.

After a graduate student has developed competence in these duties and has acquired a basic feeling for classroom instruction, he should be drawn more actively into teaching by conducting discussion sections, by giving occasional class lectures, or by accepting major responsibility for teaching an appropriate course at an appropriate level. His supervisor should maintain good contact through continued consultation, classroom visitation, and informal discussions. As the assistant matures in his teaching role, direct supervision should be relaxed gradually to encourage him to develop his individual classroom style and techniques; the opportunity for consultation should remain open, but the initiative should pass from the supervisor to the assistant.

Special seminars can also be used to assist students to improve their exposition. Many departments require a proseminar in which graduate students present advanced mathematical topics to fellow students and several members of the faculty. It would be equally appropriate to require each first year graduate student to present a short series of talks on some phase of undergraduate mathematics which is outside of his previous course of study. The objective should be to present the topic at a level suitable for undergraduates, emphasizing clarity in organization and expression rather than making the occasion a mathematical "tour de force."

Another possibility is to assign a few graduate students to experimental projects in undergraduate mathematical instruction instead of assigning them regular classroom duties. For example, they could help to prepare a collection of classroom examples for a calculus course, develop problems to be solved on the computer, or plan and evaluate alternative approaches to specific topics in lower division undergraduate mathematics.

As indicated in the Program Description, the apprenticeship in teaching should constitute approximately one-fourth of the total work load of a student during his first graduate component. It is conceivable that some of these activities, such as seminars, can qualify for academic credit. But whether or not academic credit is granted for this phase of graduate work, the student's performance as a teacher should be evaluated, and an informal departmental record should be kept in sufficient detail to show the work done and the level of competence attained.

Finally, any program of increased attention to the teaching role of prospective mathematicians has budgetary implications which cannot be ignored. One additional cost is for increased faculty time devoted to supervising teaching assistants. Another is for stipends for graduate students if the number of apprentice teachers is expanded. But if the quality of mathematics instruction improves in the future as a result of such efforts, the money will have been well spent. Fortunately, there is reason to believe that imaginative proposals to improve the quality of teaching by graduate students can attract the additional financial support needed to make them effective.

Although many graduate students welcome an opportunity to teach and thereby to become self-supporting, the stipend itself is not an adequate incentive for good teaching. This incentive can best be provided by the persistent concern of established mathematicians that teaching be excellent throughout the department.

APPENDIX: Outlines of Some Undergraduate Courses

Mathematics 11-12. Real Analysis. First semester – 39 lessons.

a. Real numbers. (6 lessons) The integers; induction. The rational numbers; order structure, Dedekind cuts. The reals defined as a Dedekind-complete field. Outline of the Dedekind construction. Least upper bound property. Nested interval property. Denseness of the rationals. Archimedean property. Inequalities ([7] is a good source of problems). The extended real number system.

b. Complex numbers. (3 lessons) The complex numbers introduced as ordered pairs of reals; their arithmetic and geometry. Statement of algebraic completeness. Schwarz inequality.

c. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc.; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

d. Metric spaces. (6 lessons) Basic definitions: metric, ball, boundedness, neighborhood, open set, closed set, interior, boundary, accumulation point, etc. Unions and intersections of open or closed sets. Subspaces. Compactness. Connectedness. Convergent sequence, subsequence, uniqueness of limit. A point of accumulation of a set is a limit of a sequence of points of the set. Cauchy sequence. Completeness.

e. Euclidean spaces.(6 lessons) R^n as a normed vector space over R. Completeness. Countable base for the topology. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. The open sets; the connected sets. The Cantor set. Outline of the Cauchy construction of R. Infinite decimals.

f. Continuity. (8 lessons) (Functions into a metric space) Limit at a point, continuity at a point. Continuity; inverses of open sets, inverses of closed sets. Continuous images of compact sets are compact. Continuous images of connected sets are connected. Uniform continuity; a continuous function on a compact set is uniformly continuous. (Functions into R) Algebra of continuous functions. A continuous function on a compact set attains its maximum. Intermediate value theorem. Kinds of discontinuities.

g. Differentiation.(6 lessons) (Functions into R) The derivative. Algebra of differentiable functions. Chain rule. Sign of the derivative. Mean value theorems. The intermediate value theorem for derivatives. L'Hospital's rule. Taylor's theorem with remainder. One-sided derivatives; infinite derivatives. (This material will be relatively familiar to the student from his calculus course, so it can be covered rather quickly.)

Second semester – 39 lessons.

h. <u>The Riemann-Stieltjes integral</u>.(11 lessons) [Alternative: the Riemann integral.] Upper and lower Riemann integrals. [Existence of the Riemann integral: for f continuous; for f monotonic.] Monotonic functions and functions of bounded variation. Riemann-Stieltjes integrals. Existence of $\int_a^b f d\alpha$ for f continuous and α of bounded variation. Reduction to the Riemann integral in case α has a continuous derivative. Linearity of the integral. The integral as a limit of sums. Integration by parts. Change of variable. Mean value theorems. The integral as a function of its upper limit. The fundamental theorem of calculus. Improper integrals. The gamma function ([11], 367-378; [10], 285-297).

i. <u>Series of numbers</u>. (11 lessons) (Complex) Convergent series. Tests for convergence (root, ratio, integral, Dirichlet, Abel). Absolute and conditional convergence. Multiplication of series. (Real) Monotone sequences; lim sup and lim inf of a sequence. Series of positive terms; the number e. Stirling's formula, Euler's constant ([11], 383-388). Again, see [7] for problems.

j. <u>Series of functions</u>.(7 lessons) (Complex) Uniform convergence; continuity of uniform limit of continuous functions. Equicontinuity; equicontinuity on compact sets. (Real) Integration term by term. Differentiation term by term. Weierstrass approximation theorem. Nowhere-differentiable continuous functions.

k. Series expansions. (10 lessons) Power series, interval of convergence, real analytic functions, Taylor's theorem. Taylor expansions for exponential, logarithmic, and trigonométric functions. Fourier series: orthonormal systems, mean square approximation, Bessel's inequality, Dirichlet kernel, Fejér kernel, localization theorem, Fejér's theorem. Parseval's theorem.

References.

1. Apostol, T. Mathematical Analysis. Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1957.
2. Bartle, R. The Elements of Real Analysis. New York, J. Wiley and Sons, Inc., 1964.
3. Buck, R. C. Advanced Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1964.
4. Eggleston, H. Introduction to Elementary Real Analysis. New York, Cambridge University Press, 1962.
5. Gelbaum, B. and Olmsted, J. Counterexamples in Analysis. San Francisco, California, Holden-Day, Inc., 1964.
6. Goldberg, R. Methods of Real Analysis. New York, Blaisdell Publishing Company, 1964.
7. Kazarinoff, N. Analytic Inequalities. New York, Holt, Rinehart and Winston, 1961.
8. Rankin , R. Introduction to Mathematical Analysis. New York, Pergamon Press, 1962.
9. Rudin, W. Principles of Mathematical Analysis. New York, McGraw-Hill Book Company, 1964.
10. Spiegel, M. Theory and Problems of Advanced Calculus. Schaum's Outline Series, New York, Schaum Publishing Co., 1963.
11. Widder, D. Advanced Calculus. Englewood Cliffs, New Jersey, Prentice Hall, Inc., 1961.

Mathematics 13. Complex Analysis (One semester)

a. Introduction. (4 lessons) The algebra and geometry of complex numbers. Definitions and properties of elementary functions, e.g., e^z, sin z, log z.

b. Analytic functions. (2 lessons) Limits, derivatives, Cauchy-Riemann equations.

c. Integration. (6 lessons) Integrals, functions defined by integrals. Cauchy's theorem and formula, integral representation of derivatives of all orders. Maximum modulus, Liouville's theorem, fundamental theorem of algebra.

d. Series. (5 lessons) Taylor and Laurent series. Uniform convergence, term by term differentiation, uniform convergence in general. Domain of convergence and classification of singularities.

e. Contour integration. (3 lessons) The residue theorem. Evaluation of integrals involving single-valued functions.

f. Analytic continuation and multivalued functions. (6 lessons) Analytic continuation, "multivalued functions," and branch points. Technique for contour integrals involving multivalued functions.

g. Conformal mapping. (6 lessons) Conformal mapping. Bilinear and Schwarz-Christoffel transformations, use of mapping in contour integral evaluation. Some mention should be made of the general Riemann mapping theorem.

h. Boundary value problems. (3 lessons) Laplace's equation in two dimensions and the solution of some of its boundary value problems, using conformal mapping.

i. Integral transforms. (4 lessons) The Fourier and Laplace transforms, their inversion identities, and their use in boundary value problems.

References:

Function Theory

Copson, E. T. Theory of Functions of a Complex Variable. New York, Oxford University Press, 1960.

Hille, E. Analytic Function Theory, 2 volumes. New York, Blaisdell Publishing Company, Vol. I, 1959; Vol. II, 1962.

Knopp, K. Theory of Functions. New York, Dover Publications, Inc., 1945.

Nehari, Z. Introduction to Complex Analysis. Boston, Allyn and Bacon, Inc., 1961.

Conformal Mapping

Nehari, Z. Conformal Mapping, New York, McGraw-Hill Book Company, 1952.

Contour Integration

MacRobert, T. Functions of a Complex Variable. London, The Macmillan Company, 1933.

Whittaker, E. T. and Watson, G. N. A Course in Modern Analysis. New York, Cambridge University Press, 1958.

Integral Transforms

Sneddon, I. N. Fourier Transforms. New York, McGraw-Hill Book Company, 1951.

Steepest Descent

DeBruijn, N. Asymptotic Methods in Analysis. Amsterdam, North-Holland Publishing Company, 1958.

Jeffreys, H. and Jeffreys, B. Methods in Mathematical Physics. New York, Cambridge University Press, 1956.

Morse, P. and Feshbach, H. Methods of Theoretical Physics. New York, McGraw-Hill Book Company, 1953.

Mathematics D-E. Abstract Algebra

The purpose of this year course is to introduce the student to the basic structures of abstract algebra and also to deepen and strengthen his knowledge of linear algebra. It provides an introduction to the applications of these concepts to various branches of mathematics.

1. Groups.(10 lessons) Definition. Examples: Vector spaces, linear groups, additive group of reals, symmetric groups, cyclic groups, etc. Subgroups. Order of an element. Theorem: Every subgroup of a cyclic group is cyclic. Coset decomposition. Lagrange theorem on the order of a subgroup. Normal subgroups. Homomorphism and isomorphism. Linear transformations as examples. Determinant as homomorphism of GL(n) to the non-zero reals. Quotient groups. The first two isomorphism theorems. Linear algebra provides examples throughout this unit.

2. Further group theory.(10 lessons) The third isomorphism theorem. Definition of simple groups and composition series for finite groups. The Jordan-Hölder theorem. Definition of solvable groups. Simplicity of the alternating group for $n>4$. Elements of

theory of p-groups. Theorems: A p-group has nontrivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

3. Rings. (10 lessons) Definition. Examples: Integers, polynomials over the reals, the rationals, the Gaussian integers, all linear transformations of a vector space, continuous functions on spaces. Zero divisors and inverses. Division rings and fields. Domains and their quotient fields. Examples: Construction of field of four elements, embedding of complex numbers in 2 x 2 real matrices, quaternions. Homomorphism and isomorphism of rings. Ideals. Congruences in the ring of integers. Tests for divisibility by 3, 11, etc.. leading up to Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$, and such problems as showing that $2^{32} + 1 \equiv 0 \pmod{641}$. Residue class rings. The homomorphism theorems for rings.

4. Further linear algebra. (continuing Mathematics 3) (12 lessons) Definition of vector space over an arbitrary field. (Point out that the first part of Mathematics 3 carries over verbatim and use the opportunity for some review of Mathematics 3.) Review of spectral theorem from Mathematics 3 stated in a more sophisticated form (e.g., as in reference [4]). Dual-space adjoint of a linear transformation, dual bases, transpose of a matrix. Theorem: Finite-dimensional vector spaces are

reflexive. Equivalence of bilinear forms and homomorphism of a space into its dual. General theory of quadratic and skew-symmetric forms over fields of characteristic not two. The canonical forms. (Emphasize the connections with corresponding material in Mathematics 3.) The exterior algebra defined in terms of a basis--two-and three-dimensional cases first. The transformation of the p-vectors induced by a linear transformation of the vector space. Determinants redone this way.

5. Unique factorization domains. (12 lessons) Primes in a commutative ring. Examples where unique factorization fails, say, in $Z[\sqrt{-5}]$. Definition of Euclidean ring, regarded as a device to unify the discussion for Z and F[x], F a field. Division algorithm and Euclidean algorithm in a Euclidean ring; greatest common divisor; Theorem: If a prime divides a product it divides at least one factor; unique factorization in a Euclidean ring. Theorem: A Euclidean ring is a principal ideal domain. Theorem: A principal ideal domain is a unique factorization domain. Gauss's lemma on the product of two primitive polynomials over a unique factorization domain. Theorem: If R is a unique factorization domain so is R [x].

6. Modules over Euclidean rings. (14 lessons) Definition of module over an arbitrary ring viewed as a generalization of vector space. Example: Vector space as a module over F[x] with x acting like a linear transformation. Module homomorphism. Cyclic and free modules. Theorem: Any module is a homomorphic image of a free

module. Theorem: If R is Euclidean, A an n x n matrix over R, then by elementary row and column transformations A can be diagonalized so that diagonal elements divide properly. Theorem: Every finitely generated module over a Euclidean ring is the direct sum of cyclic modules. Uniqueness of this decomposition, decomposition into primary components, invariant factors and elementary divisors. Application to the module of a linear transformation, leading to the rational and Jordan canonical forms of the matrix. Several examples worked in detail. Similarity invariants of matrices. Characteristic and minimal polynomials. Hamilton-Cayley Theorem: A square matrix satisfies its characteristic equation. Application of module theorem to the integers to obtain the fundamental theorem of finitely generated abelian groups.

7. Fields. (10 lessons) Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a field, a an algebraic element of some extension field. Direct proof that if a has degree n, the set of polynomials of degree $n-1$ in a is a field, demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting the angle, duplicating the cube, squaring the circle (assuming π transcendental). Existence and uniqueness of splitting fields for equations. Theory of finite fields.

References.

1. Barnes, W. Introduction to Abstract Algebra. Boston, Massachusetts: D.C. Heath and Company, 1963.

2. Birkhoff, G. and MacLane, S. A Survey of Modern Algebra. New York: The Macmillan Company, 1965.

3. Herstein, I.N. Topics in Algebra. New York: Blaisdell Publishing Company, 1964.

4. Hoffman, K. and Kunze, R. Linear Algebra. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961.

5. Lewis, D. Introduction to Algebra. New York: Harper and Row, 1965.

6. Mostow, G., Sampson, J.H. and Meyer, J.P. Fundamental Structures of Algebra. New York: McGraw-Hill Book Company, Inc., 1963.

7. van der Waerden, B.L. Modern Algebra, Vol. I, rev. ed. New York: Ungar Publishing Company, 1953.

Course B1. Introduction to Computing

(Reprinted from Communications of the ACM, Volume 11/ Number 3 / March, 1968.)

APPROACH

This first course in computing concentrates on the solution of computational problems through the introduction and use of an algorithmic language. A single such language should be used for most of the course so that the students may master it well enough to attack substantial problems. It may be desirable, however, to use a simple second language of quite different character for a problem or two in order to demonstrate the wide diversity of the computer languages available. Because of its elegance and novelty, SNOBOL can be used quite effectively for this purpose. In any case, it is essential that the student be aware that the computers and languages he is learning about are only particular instances of a widespread species.

The notion of an algorithm should be stressed throughout the course and clearly distinguished from that of a program. The language structures should be carefully motivated and precisely defined using one or more of the formal techniques available. Every effort should be made to develop the student's ability to analyze complex problems and formulate algorithms for their solution. Numerous problems should be assigned for computer solution, beginning early in the course with several small projects to aid the student in learning to program, and should include at least one major project, possibly of the student's own choosing. Careful verification of program operation and clear program documentation should be emphasized.

CONTENT

This outline reflects an order in which the material might be presented; however, the order of presentation will be governed by the choice of languages and texts as well as individual preferences. In particular, the treatment of some of the topics listed below might be distributed throughout the course. Although not specifically listed in the following outline, programming and computer projects should constitute an important part of the content of this course.

1. Algorithms, Programs, and Computers. The concept and properties of algorithms. Flowcharts of algorithms and the need for precise

languages to express algorithms. The concept of a program, examples of simple programs, and description of how computers execute programs. Programming languages including the description of their syntax and semantics. (10%)

2. Basic Programming. Constants, identifiers, variables, subscripts, operations, functions, and expressions. Declarations, substitution statements, input-output statements, conditional statements, iteration statements, and complete programs. (10%)

3. Program Structure. Procedures, functions, subroutine calling, and formal-actual parameter association. Statement grouping, nested structure of expressions and statements, local versus global variables, run-time representation, and storage allocation. Common data, segmenting, and other structural features. (10%)

4. Programming and Computing Systems. Compilers, libraries, loaders, system programs, operating systems, and other information necessary for the student to interact with the computer being used. (5%)

5. Debugging and Verification of Programs. Error conditions and messages, techniques of debugging, selection of test data, checking of computer output, and programming to guard against errors in data. (5%)

6. Data Representation. Systems of enumeration and binary codes. Representation of characters, fixed and floating-point numbers, vectors, strings, tables, matrices, arrays, and other data structures. (10%)

7. Other Programming Topics. Formatted input and output. Accuracy, truncation, and round-off errors. Considerations of efficiency. Other features of language(s) being considered. (10%)

8. Organization and Characteristics of Computers. Internal organization including input-output, memory-storage, processing and control. Registers, arithmetic, instruction codes, execution of instruction, addressing, and flow of control. Speed, cost and characteristics of various operations and components. (10%)

9. Analysis of Numerical and Nonnumerical Problems. Applications of algorithm development and programming to the solution of a variety of problems (distributed throughout the course). (15%)

10. Survey of Computers, Languages, Systems, and Applications. The historical development of computers, languages, and systems including recent novel applications of computers, and new developments in the computing field. (10%)

11. Examinations. (5%)

ANNOTATED BIBLIOGRAPHY

In addition to the materials listed here, there are numerous books and manuals on specific computer languages which would be appropriate as part of the textual material for this course. Very few books, however, place sufficient emphasis on algorithms and provide the general introductory material proposed for this course.

1. ARDEN, B.W. An Introduction to Digital Computing. Addison-Wesley, Reading, Mass., 1963, 389 pp.

   This text uses MAD and emphasizes the solution of numerical problems, although other types of problems are discussed. Numerous examples and exercises.

2. FORTE, A. SNOBOL3 Primer. M.I.T. Press, Cambridge, Mass., 1967, 107 pp.

   An elementary exposition of SNOBOL3 which might well be used to introduce a "second" language. Many exercises and examples. (SNOBOL4 is now becoming available.)

3. GALLER, B.A. The Language of Computers. McGraw-Hill, New York, 1962, 244 pp.

   Emphasizes "discovering" the structure of algorithms needed for the solution of a varied set of problems. The computer language features necessary to express these algorithms are carefully motivated. The language introduced is primarily based on MAD, but FORTRAN and ALGOL are also discussed.

4. GRUENBERGER, F. The teaching of computing (Guest editorial). Comm. ACM 8, 6 (June 1965) 348 and 410.

   Conveys eloquently the philosophy which should be used in developing and teaching an introductory course.

5. GRUENBERGER, F. and JAFFRAY, G. Problems for Computer Solution. Wiley, New York, 1965, 401 pp.

Contains a collection of problems appropriate for computer solution by students. Student is guided into the analysis of the problems and the development of good computational solutions, but actual computer programs for the solutions are not given.

6. HULL, T.E. Introduction to Computing. Prentice Hall, Englewood Cliffs, N.J., 1966, 212 pp.

Text on fundamentals of algorithms, basic features of stored-program computers, and techniques involved in implementing algorithms on computers. Presents a complete description of FORTRAN IV with examples of numerical methods, nonnumerical applications, and simulations. Numerous exercises.

7. MARCOVITZ, A.B. and SCHWEPPE, E.J. An Introduction to Algorithmic Methods Using the MAD Language. Macmillan, New York, 1966, 433 pp.

Emphasizes algorithms and their expression as programs, characteristics of computers and computer systems, formal definition of computer languages, and accuracy and efficiency of programs. Numerous examples and exercises.

8. PERLIS, A.J. Programming for digital computers. Comm. ACM 7, 4 (Apr. 1964) 210-211.

Description of a course developed by Perlis at Carnegie Institute of Technology which has strongly influenced the course proposed here.

9. RICE, J.K. and RICE, J.R. Introduction to Computer Science: Problems, Algorithms, Languages, and Information, Preliminary edition, Holt, Rinehart, and Winston, New York, 1967, 452 pp.

Presentation revolves around the theme of "problem solving," emphasizing algorithms, languages, information representations, and machines necessary to solve problems. Problem solution methods classified, and many sample problems included. The nature of errors and uncertainty is considered. Detailed appendix on FORTRAN IV by E. Desautels.

10. School Mathematics Study Group. Algorithms, Computation and Mathematics, rev. ed. Stanford University, Stanford, Calif., 1966. Student Text, 453 pp., Teacher's Commentary, 301 pp.; Algol Supplement: Student Text, 133 pp., Teacher's Commentary, 109 pp.; Fortran Supplement: Student Text, 132 pp., Teacher's Commentary, 102 pp. Available from A.C. Vroman, Inc. 367 South Pasadena, Pasadena, Calif. A MAD Language Supplement by E.I. Organick is available from Ulrich's Book Store, 549 E. University Avenue, Ann Arbor, Mich.

Although developed for high school students and teachers, this work contains much material appropriate for this course. Develops an understanding of the relationship between mathematics, computing, and problem solving. Basic text uses English and flow charts to describe algorithms; supplements introduce the computer language and give these algorithms in ALGOL, FORTRAN, and MAD.