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The epistemological status of factors--that is, components, common factors, and image factors--is briefly discussed. Implications for the number of factors problem of varying views of factor analysis and the particular factor analytic model employed are noted, and the rationales underlying the best known decision rules regarding the correct number of factors are discussed. The results of a study are presented, in which reanalyses were performed on 17 correlation matrices found in the literature, and eight well-known rules and one new rule for determining the correct number of factors were applied. The rules are compared in light of the factor analytic model implied in each case, and some implications for practice are noted.

# SOME EMPIRICAL FINDINGS CONCERNING THE NUMBER OF FACTORS PROBLEM

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There is probably no facet of the factor analytic process that appears more arbitrary or intractable than that concerned with determining the "correct" number of factors to represent the variables at hand. A unique solution is, indeed, not theoretically possible, in that the decision reached regarding the "true" dimensionality of a domain of variables depends entirely upon which one of a number of seemingly reasonable operationalizations of the "correct" number of factors is employed. One may be tempted to infer, from the last sentence, that he merely needs to agree upon an underlying rationale from among many, marshal several arguments to support it, and apply the associated rule of thumb, regarding the number of factors, to his data--all with unequivocable results. In fact, the problem is not so easily handled. For one thing, even though rationales may exist to support the hypotheses of $r_1$, $r_2$, and $r_3$ factors from $n$ variables, the ultimate interpretation of each factor determined will depend upon the number of additional factors extracted and transformed. That is, the final complexion of Factor I in a solution of $r_1$ transformed factors may well be quite distinct from that of Factor I in a solution of $r_2$ transformed factors. It is, in short, common knowledge among factor analysts that the transformation of axes is very sensitive to the number of factors transformed.

If we leave the procedural level for a moment and consider what is implied in our notion of a factor, the number of factors problem displays still another facet of complexity. It appears that factors may be conceived of two ways. The first conceptualization--and the more formally stated one--places factors in Feigl's nomological net (see Feigl and Scriven, 1956; Royce, 1963), giving them the status of determiners of covariation among the more phenotypic constructs in the domain of interest. Consistent with this position--which we may call an _explanatory_ view--is the notion, expressed by Cattell (1958, 1962) and Humphreys (1964), among others, that an indefinitely large number of such epistemologically defined constructs may be conceived of as influencing any variable under investigation. In addition, for the sample of variables at hand, the number of variables influenced by each factor will likely be small. The number of factors problem in this case, then, reduces to the task of identifying those factors whose influence is great upon the variables _sampled from the domain of interest_, and those whose influence, while real, is slight. A second conceptualization of factors--and in some cases a view held largely by default, that is because little theoretical charting of the domain of interest has been done--gives these constructs a somewhat lesser role than the first view, namely, as convenient groupings of variables. This position may be identified as a _taxonomic_ view of factors and factor analysis. The implication of this view for the number of factors problem is that the number of factors seen as _necessary_ to embrace the _n_ variables in an analysis is considerably less than _n_. Additionally, the notion of a universe of psychological interest, from which variables are sampled, is not generally appropriate.

Another complicating issue involving the number of factors problem has to do with the factor analytic model employed. Up to this point, we have refrained from distinguishing between "factor analysis" in a generic sense and the more specific notion of "common-factor analysis." Three linear models are available for conducting a "factor analysis" in the broadest sense of the expression. The simplest of these is the component model, and although the original development and much subsequent application of principal component analysis were concerned with conceptual matters quite distinct from those generally associated with common-factor analysis in its psychometric tradition, the taking of the first $r$ principal components, usually with subsequent transformation--a process known as incomplete component analysis--has been often used analogously to common-factor analysis. Indeed, by far the most widely used computing center program for "factor analysis" yields varimax-rotated principal components. The component model comes closest of the three models to exemplifying the taxonomic view of factor analysis.

The oldest of the three models is the common-factor model. Largely as a result of the mathematically intractable problem of communality, many users, over the years, have substituted component analyses for common-factor analyses. It is the common-factor model that is most directly implied by Guttman's fertile notion of sampling variables from a universe of interest. Although reduction of each variable (to its common part) occurs in common-factor analysis, few users would expect anything approaching exact rank reduction of the variable space, so that the implication of this model for the number of factors problem is that a small number ($n/2$ or less) of interpretable and useful factors must be identified, the remaining factors being considered insignificant. Most analyses based upon the explanatory view of factor analysis employ this model. Throughout the remainder of this paper, the expressions "factors" and "factor analysis" will be used with reference to the broad class of analyses--utilizing any of the three models; "common-factors" and "common-factor analysis" will indicate this particular model.

A third model, which combines features of both the component and common-factor models is Guttman's (1953) image model. As with the component model, the (image) factors obtained are linear combinations of the observed variables; the problem of the unknown communalities is, thus, circumvented. Conceptually, however, the image model is closer to the common-factor than to the component model. Guttman has shown, for example, that as we approach the inferential limit (as the universe of variables of interest is sampled more and more extensively), the image and the common-factor models become one. The image model, then, has much to recommend it as a substitute for the common-factor model, in the conduct of an analysis based on an explanatory view of factor analysis; a corollary is that the appropriate number of image factors corresponds to that of common-factors.

Within the traditions and models outlined above, four classes of criteria have been invoked to answer the number of factors question. These criteria, listed earlier by Kaiser (1960) are (1) algebraic criteria, establishing lower and upper bounds on the rank of the matrices involved, (2) psychometric criteria, dealing with the internal consistency of the constructs obtained, (3) statistical criteria, dealing with reproducing the observed dispersion or correlation matrix to a tolerance level ascribable to sampling error, and (4) psychological importance criteria, concerned with the meaningfulness or interpretability of the factors obtained.

Strictly speaking, the rules of thumb for the number of factors that seem to have sprung from the work on algebraic bounds on the rank of the reduced correlation matrix have not been fully justified. In such papers as those by Ledermann (1937), Albert (1944), and Guttman (1954, 1958), proofs were given regarding limits on the rank of a reduced correlation matrix, $\underline{R} - \underline{U}^2$ (where $\underline{R}$, of order $\underline{n} \times \underline{n}$, is, of course, the observed correlation matrix, and $\underline{U}^2$, of order $\underline{n} \times \underline{n}$, is the diagonal matrix of unique variances), but the bounds that have been established may be a long way from the optimal number of factors to retain. An example of the faulty interpretation that has been widespread is that concerning Guttman's (1954) weaker lower bound, in which an elegant proof on a universally weakest lower bound to the rank of a gramian $\underline{R} - \underline{U}^2$ was translated into a rule of thumb concerned with the number of principal components to retain! Kaiser's (1960) later observations regarding the "roots of $\underline{R}$ greater than unity" rule gave some psychometric justification to the aforementioned misinterpretation, but it seems safe to say that the work on algebraic considerations has been interesting as lending theoretical insight, but has not provided a practical answer to the number of factors problem.

Kaiser (1960, 1964) has employed psychometric logic in answering the number of factors question. He has pointed out that for a principal component to have positive alpha internal consistency, its associated latent root must exceed unity, thereby adding some procedural justification to Guttman's (1954) work on lower bounds; the "Kaiser-Guttman rule" is in widespread use today. The application of the preceding logic to the common-factor model led to alpha factor analysis (Kaiser and Caffrey, 1965).

To this point, problems of statistical inference to populations of persons have been avoided. Instead, the problems of inference to populations of variables--in the common-factor and image models--have been implied, with the classical problems of statistical inference largely ignored. There is, however, a long tradition of factor analysis as an inferential statistical procedure, in which the correct number of factors for a set of data is that number capable of yielding a reproduced correlation matrix in which the entries are simultaneously within normal sampling error of the observed coefficients. Inferential procedures--largely involving the distinguishability of the latent roots--have been applied to the component model by Hoel (1937), Bartlett (1950, 1951), Lawley (1956), and Anderson (1963). These procedures have generally been considerably more complicated and less useful when applied to correlation than to covariance matrices. Tests of significance for the common-factor model have been discussed by Thurstone (1938), Lawley (1940, 1953), Coombs (1941), Young (1941), McNemar (1942), Saunders (1948), Bartlett (1950, 1951), Burt (1952), Whittle (1952), Rippe (1953), Rao (1955), Anderson and Rubin (1956), Jöreskog (1962, 1963, 1967), Lawley and Maxwell (1963), and Harman (1967). One disadvantage seen for these procedures is that they are most appropriate when the factors are obtained by the method of maximum-likelihood, and this method has, in the past, often presented computing problems. With Jöreskog's (1967) recent work, however, these problems have been minimized. Other objections to the usual significance testing procedures employed focus on the non-independent nature of the sequential tests performed, and the consequent lack of rigorous control over Type I error. A perhaps more compelling objection centers around the dependence of this approach upon the total sample size, with the resulting problem of "statistical but not practical significance" with particularly large $N$.

The notion of psychological (or sociological, educational, etc.) *importance* has led to several attempts at operationalization. Based on the rationale of an indefinitely large set of influences on each variable, Cattell (1958) established rules for how much common-factor variance should be accounted for before factoring should be terminated, and arrived at the widely used "scree" test or, alternatively, test for the "break in the curve" of plotted latent roots somewhat later (see Cattell, 1966). Humphreys (1964) found that the Kaiser-Guttman rule of retaining the number of factors (actually components) corresponding to latent roots of $R$ greater than one tended to suggest fewer factors, with a large sample, than could be meaningfully interpreted. Various operationalizations of the importance rationale were given by Pennell (1968), Linn (1968), and Cliff (1970), the latter two applications being based upon monte carlo procedures. Horn (1965) presented a technique for determining the number of non-error latent roots of $\underline{R}$ greater than unity, again using random data generation. Humphreys and Ilgen (1969) discussed the generalization of Horn's (1965) procedure to the common-factor model. Informal operationalizations of importance in terms of the component model have been widely used over the years, with the usual rule involving extracting components until the percentage of variance accounted for by those extracted exceeds some arbitrary value, for example, 80%.

Interesting papers containing discussions of the number of factors problem are those by Wrigley (1958), Kaiser (1960,1964), Harris (1962, 1964), and Browne (1968), in addition to many of the papers already noted. Little work on the problem has been done from the image theoretical viewpoint. Harris (1962) showed the interrelations existing between the component, common-factor, and image models, and related each model to the matrix $\underline{S}^{-1}\underline{RS}^{-1}$, where the diagonal matrix, $\underline{S}^2 = [\mathrm{diag}(\underline{R}^{-1})]^{-1}$. A class of scale-free solutions was implied, and the number of roots of $\underline{S}^{-1}\underline{RS}^{-1}$ greater than one was shown to have applicability to both the common-factor and image models and to correspond to Guttman's (1954) stronger lower bound. Kaiser (1963) further developed this bound as a useful cutoff for image analysis and suggested further that an analytic orthogonal rotation of these image factors could be expected to result in "residualization" of the less important factors, thereby providing an importance-based rule for the number of image factors to retain. Finally, Kaiser (1970) utilized Harris's (1962) class of solutions in arriving at his "Second Generation Little Jiffy" procedure. This program includes a number of factors rule, which will be developed further in the next section.

The preceding notions, characterization, and summary of the traditions and procedures in factor analysis are presented schematically in Figure 1.

Insert Figure 1 about here

Most of what appears in this figure should be clear from the preceding text. The broken line connecting the taxonomic view of factor analysis with the common-factor and image models indicates that although these models could be employed in the light of the taxonomic view, the component model would appear to be more appropriate. On the other hand, the component model is seen as unsatisfactory in connection with the explanatory view of factor analysis, except, perhaps, when employed to get a preliminary solution for use as an indicator of what to expect in a final common-factor or image solution. The remainder of the paper is devoted to comparing the decisions reached, regarding the appropriate number of factors to be retained, by the more prominent rules already noted.

## Method

Reanalyses were performed on 17 correlation matrices found in the literature. For each data set eight well-known rules for deciding on the number of factors to retain were applied, and the decisions reached in each case were compared. The three models were used. The data reanalyzed ranged in size from a six variable set to one having 34 variables, each set having been reanalyzed by Jöreskog (1963) in connection with his early non-iterative statistical procedure. The 17 data sets--in ascending order of size, and using the identifying numbers used later in this paper--were found in (1) McLeish (1950), (2) Chapman (1949), (3) Harman (1967)--the Eight Physical Variables--(4)Karlin (1941), (5) Davis (1944), (6) Emmett (1949), (7) Denton and Taylor (1955), (8) Harman (1967)--the Thirteen Psychological Tests--(9) Bechtoldt (1961)--first sample--(10) Bechtoldt (1961)--second sample--(11) Rimoldi (1948), (12) Harman (1967)--the Twenty-Four Psychological Tests--(13) Pemberton (1952), (14) Ahmavaara and Markkanen (1958), (15) Fleishman and Hempel (1954), (16) Karlin (1942), and (17) Green, Guilford, Christensen, and Comrey (1953).

First, principal component solutions were obtained and two number of factors criteria were applied: (1) the Kaiser-Guttman rule of latent roots of $\underline{R}$ greater than one, and (2) a scree test on the plot of latent roots. Common-factor solutions were obtained for each data set by four different methods: (1) the widely used iterated principal factor analysis technique, that is, successive factoring of estimates of $\underline{R} - \underline{U}^2$, iterating on the communality estimates, with the application of a scree test after convergence, (2) Jöreskog's (1963) non-iterative statistical procedure, with its accompanying likelihood-ratio test (in fact, these solutions--with the number of factors determined--were already available in the source just noted), (3) Jöreskog's (1967) exact, iterative maximum-likelihood procedure, utilizing the program UMLFA noted in Jöreskog (1967), with its associated likelihood-ratio test for the number of factors, and (4) Kaiser's (1970) "Second Generation Little Jiffy" (referred to in the sequel simply as "Little Jiffy"). In the latter non-iterative procedure, the $\underline{S}^{-1}\underline{RS}^{-1}$ matrix, given special prominence by Harris (1962), is utilized, with the number of latent roots greater than the mean latent root value taken as the correct number of factors. This rule is based on no compelling theoretical rationale, but rather was merely seen to yield a reasonable value. Finally, "Harris factors" of the image covariance matrix (Kaiser, 1963) were obtained. It may be recalled that if we express the canonical decomposition of $\underline{S}^{-1}\underline{RS}^{-1}$ as $\underline{W\Gamma W}'$, then factors, $\underline{F}_I$, of order $\underline{n} \times \underline{r}$, of the image covariance matrix may be expressed as

$$F_I = SW_r[(\Gamma_I - I)^2\Gamma_I^{-1}]^{1/2}, \tag{1}$$

where the subscript $\underline{r}$ indicates that only the first $\underline{r}$ columns of $\underline{W}$ and elements of $\underline{\Gamma}$ have been retained. It is reasonable, as pointed out by Kaiser (1963), to let $\underline{r}$ be the number of elements of $\underline{\Gamma}$ greater than one, since for values of $\underline{\Gamma}$ less than one, the columns of $\underline{F}_I$ would be scaled inversely as to the corresponding value of $\underline{\Gamma}$, a condition obviously not desirable. Interestingly, the number of elements of $\underline{\Gamma}$ greater than one coincides with Guttman's (1954) stronger lower bound. In the subsequent reporting of findings, we include this rule, therefore, under the image model, but it should be clear that it could (probably more appropriately) be characterized as associated with the common-factor model. Kaiser's (1963) technique of orthogonally rotating this rather large number of image factors, with the goal of "residualization" of the less important factors constituted our last generally known procedure for determining the appropriate number of image factors to retain.

Three of the procedures noted above, for determining the number of factors, require a considerable amount of subjective judgment. The two scree tests were first performed by each of the authors independently. The convention was observed, in the instance of more than one "break" in the curve, of using the first such break as indicating the number of factors. After these independent tests had been performed, the results were compared. With approximately half of the data sets, the decisions reached by the authors coincided; with the other half, the task was subsequently undertaken again by both authors working together. In the end, the authors felt fair confidence in the decision reached for all but one or two data sets. A similar procedure was employed in judging whether or not, in the image analyses performed, columns of the image factor pattern matrix were, in fact, "residual." Better agreement was obtained in the independent judgments with the image analyses than with the scree tests, and, overall, greater confidence was felt in the final set of decisions.

An additional theoretical line of reasoning was initiated and pursued. It will be recalled that Kaiser's (1960) psychometric rationale for retaining the number of principal components corresponding to latent roots of $\underline{R}$ greater than one revolved around the fact that this number of components has positive alpha reliability or internal consistency. It can be easily shown that, in fact, the principal components are those components with maximum internal consistency (where the alpha coefficient for component $k$ is given by $[n/(n - 1)](1 - 1/m_k)$, $m_k$ being the $k$th latent root of $R$), subject, of course, to the orthogonality constraints within any set. It is possible, however, that maximum internal consistency may not constitute as desirable a criterion for the components to satisfy as one involving maximum reliability in the stability or equivalence sense. In general, such components will not be principal components. Maximizing the reliability of a composite--provided that the reliabilities of the parts are known--has been dealt with by Mosier (1943), Peel (1948), Green (1950), and Gulliksen (1950), and has been noted in the factor analytic context by McDonald (1970). The problem simplifies to maximization of the ratio of quadratic forms

$$\phi = w'R^*w/w'Rw, \tag{2}$$

where $R^*$, $n \times n$, is $R$ reduced by the diagonal, $n \times n$ matrix of necessarily known error variances, $E^2$, and unit-length constraints are imposed upon the unknown vector of weights, $w$. Algebraic manipulation reveals that the maximization of $\phi$ in (2) is equivalent to maximization of $\psi$ in

$$\psi = w'Rw/w'E^2w. \tag{3}$$

The formulation (3) implies that the solution involves the canonical decomposition of $E^{-1}RE^{-1}$, as

$$E^{-1}RE^{-1} = V\Lambda V', \tag{4}$$

and the matrix of component pattern coefficients is given by

$$F_c = EV_r\Lambda_r^{1/2}, \tag{5}$$

where $V_r$, $n \times r$, contains the first $r$ columns of $V$, and diagonal $\Lambda_r$, $r \times r$, the first $r$ diagonal elements of $\Lambda$. The reliability of component $k$, $r_{kk}$, is given by

$$r_{kk} = 1 - 1/\lambda_k, \tag{6}$$

where $\underline{\lambda}_k$ is the kth diagonal element of $\underline{\Lambda}$. Equation (6), then, represents a basis for deciding on r, the number of components to retain. One simply sets r at the point at which component reliabilities drop below an acceptable level.

A criticism leveled at the Kaiser-Guttman rule of latent roots of R greater than one has been that the condition of negative alpha reliability for components beyond r obtains only before transformation, and that after transformation--a procedure almost always applied--more than r components may have positive, and even acceptable, alpha internal consistency. The argument in favor of the Kaiser-Guttman rule that components associated with latent roots less than one contribute less variance to the whole than does any one (standardized) variable is subject, of course, to the same criticism. The maximally reliable components formulated in (2) through (6) also share this criticism, as well as that stemming from the fact that, in general, reliabilities of the variables are not known. We still regard this alternative to principal components, however, as potentially useful, given that the variable reliabilities are known.

If the variable reliabilities are not known, several possibilities exist, utilizing communalities as lower bounds on reliabilities. Fully iterated communalities would be ideal in that they would be, generally, the strongest lower bounds; squared multiple correlation coefficients could, instead, be used to obviate the somewhat unattractive prospect of iteration. Whereas in (2) we maximized the reliability--or that portion of total component variance that is true variance (in the generic sense)--we would, in the case of unknown reliabilities, maximize the ratio of common to total component variance, or, in a sense, establish components with maximum overlap with the common factors underlying the variables. Thus, instead of (4), we write

$$U^{-1}RU^{-1} = XBX', \tag{7}$$

with $\underline{U}$, $\underline{n} \times \underline{n}$, equal to $\underline{S}$ if squared multiple correlations are used. The component pattern matrix, then, is given by

$$\begin{aligned} F_c &= UXB^{1/2}, \text{ or} \\ F_c &= SXB^{1/2}, \end{aligned} \tag{8}$$

and the portion of component $\underline{k}$'s variance that is common-factor variance is given, analogously to (6), by $1 - 1/\underline{b}_k$.

At this point, it may be worthwhile to digress enough to anchor the material formulated in (2) through (8) to a general component formulation. In place of (3) and the analogous formulation with $\underline{U}^2$ replacing $\underline{E}^2$, we may, extending McDonald's (1970) formulation of the common-factor model, write a general ratio of quadratic forms

$$\xi = w'Rw/w'D^2w = \text{maximum}, \tag{9}$$

which implies the canonical decomposition of the matrix $\underline{D}^{-1}\underline{RD}^{-1}$, and subsequent row rescaling of the principal axes of this matrix. Equation (9) leads to a general weighted least-squares component solution, of which _principal_ component analysis is that special case with $\underline{D}^2 = \underline{I}$ (or an _un_weighted least-squares component solution). In general, (9) implies the weighting of the residuals by $\underline{D}^{-1}$, which in the most useful cases dealt with in (2) through (8) means weighting the residuals by the inverse error or unique standard deviations. Clearly, weighting the residuals inversely as to error or what is used to represent error may often be more desirable than weighting all residuals equally.

It should be clear to the reader that we had arrived, in (7) and (8), right into Harris's (1962) "Rao-Guttman" system. In Harris's paper, the expressions $1 - 1/\underline{b}_k$ were given as the squared canonical correlations between the variables and factors (in the "Rao" part of the "Rao-Guttman" relationships), and which, in our development, are interpreted identically as maximal coefficients of determination reflecting the (maximized) portions of total component variance that are common-factor variance. We, thus, see these components, which form part of Harris's (1962) scale-free system, as, in some respects, superior to principal components. Analogous to the procedure involving component reliabilities (when the variable reliabilities are known), we may establish minimal values of the common variance/total variance ratio, and retain only as many such maximum communality components as exceed this minimal value.

In the present study, such components were obtained, using non-iterated communality estimates (squared multiple correlations), for each of the 17 data sets. It was decided to set the minimal value for the communality/total variance ratio--a lower bound, of course, on the component reliability--to the mean common variance for the variables, under the (somewhat arbitrary) assumption that only those components with a portion of common (true) variance larger than the average variable communality (reliability) were interesting. It is worth noting that with many data sets, the first few components absorbed almost all the common variance, with the ratio of common variance/total variance very large. After the $r$th such component, a large drop occurred, in which the ratio dropped considerably below the mean communality used as the stopping point. With these data sets, the decision regarding the number of components was easily made. With a few data sets, however, a fairly gradual drop in the ratios occurred around the mean communality value, rendering any decision regarding the number of components considerably more tenuous.
Other criteria regarding an acceptable value for the communality/total variance ratios could be employed with this procedure, but in any case, a rationale with more conceptually meaningful anchorings than, say, the scree test or Kaiser's (1970) "Little Jiffy" procedure (for common factors, of course, as well as

components) is available. It is interesting to note that if we use positive reliability in this sense (or communality/total variance ratio) as our criterion, our rule corresponds to Guttman's (1954) stronger lower bound, whereas the requirement of positive alpha reliability has been seen to correspond to Guttman's weaker lower bound. Thus, such maximum communality components bear the same relationship to canonical factor analysis that principal components bear to alpha factor analysis.

## Results

The results of the reanalyses of the aformentioned 17 correlation matrices appear in Table 1. In this table, N refers to the number of persons, n, to the number of variables.

Insert Table 1 about here

It is apparent from Table 1 that there was little agreement among the procedures for any given data set and also, in most cases, little agreement between the decisions reached through the procedures and that reached in the source study. The most variable set was No. 16, with estimates ranging from four to 19, whereas with No. 12 (the familiar Holzinger-Harman Twenty-Four Psychological Tests), fair consensus was reached (if one ignores the Guttman stronger lower bound, which is not a widely used "working" rule) of either four or five factors.

Disregarding, for a moment, the model involved with each procedure, of all procedures, the Little Jiffy number-of-factors rule tended to suggest--over all data sets--the smallest number. Next in order was the scree test of the roots of the iterated $R - U^2$ matrix, and this test suggested only slightly fewer factors, overall, than the test associated with the maximum communality procedure described in the last section. The results of application of the Kaiser-Guttman rule and of a scree test on the roots of R suggested a larger number of factors (components). Overall, the number of non-residual columns of the image factor pattern matrix was even larger, and the two likelihood-ratio tests indicated the largest number of factors, again excluding the Guttman stronger lower bound, which, as would be expected, was, overall, largest. It is interesting to note that the results of the likelihood-ratio tests were, in general, considerably closer to the number of factors decided upon in the source studies than were the other procedures studied, the other procedures tending, consistently, to indicate fewer factors than in the source studies.

Comparing the procedures in terms of the models employed, one discovers that those procedures used with the component model tend to suggest the fewest factors, followed by those procedures used with the common-factor, and lastly, image models. Over all data sets, the number of non-residual columns of the orthogonally rotated image factor pattern matrix, the results of the likelihood-ratio tests, and the number of factors in the source study tended to be approximately one third the number of variables. From these procedures, the number of factors/number of variables ratio dropped to about .27 or so for the widely-used Kaiser-Guttman rule, and to a low--for those procedures studied--of about .22 for the Little Jiffy rule.

## Discussion

It is the authors' belief that the distinctions drawn earlier in this paper between the explanatory and taxonomic views of factor analysis and between the various models employed, when combined with the results of the reanalyses just presented, can be synthesized into the following observation regarding the number of factors problem: the appropriate number of factors for a given data set depends, in part, upon the view held regarding factors and factor analysis and the consequent linear model employed in the analysis. Put another way, the "correct" number of components may not coincide with the "correct" number of common or image factors. In addition, the "correct" number of factors may be largely a function of the interpretability of the factors and quality of simple structure obtained.

In the case of the component model, there is no "correct" number of components, but only a convenient or useful number. This is true because components are constructs representing little more than convenient groupings or categorizations of variables, and just as such constructs have little theoretical status, the number of such constructs has little or none. The application of such rules as the Kaiser-Guttman then, with implications strictly speaking more for the component than common-factor model (the weaker lower bound rationale is regarded as procedurally uninformative), is seen as inappropriate when a common-factor or image analysis is being performed. In fact, this psychometric rule breaks down even in the case component case with transformation. The same defect exists, of course, with the maximum reliability or maximum communality components discussed earlier.

It is probably true that the psychometric rationale should be applied after, rather than before transformation, if it is to be applied at all. If the investigator wished to assess either the reliability or communality/total variance ratios of the transformed components, these quantities could be obtained by

$$D_{tr} = \text{diag}[T'D_rT], \qquad (10)$$

where the $k$th diagonal element of diagonal $D_{tr}$, $r \times r$, is the reliability or communality/total variance ratio of component $k$ after transformation, diagonal $D_r$, $r \times r$, contains such values before transformation, and $T$, $r \times r$, is the transformation matrix, consisting of unit-length columns of direction cosines, i.e., the standard orthogonal transformation matrix, in the orthogonal case, or the transformation from $F_c$ in (8) to the primary-factor structure matrix, in the oblique case. It should be clear from the preceding development that if the variable reliabilities are known $\underline{D}_r$ is given, using (4), by $(\underline{I} - \underline{\Lambda}^{-1})$, and if we are dealing with communality/total variance ratios, $\underline{D}_r$, using (7), is given by $(\underline{I} - \underline{B}^{-1})$. Analogously, we may obtain the alpha reliability of the components after transformation, in the diagonal elements of diagonal $\underline{D}_\alpha$, $\underline{r}$ x $\underline{r}$, given by

$$D_\alpha = [n/(n-1)]\mathrm{diag}[T'(I - M^{-1})T], \tag{11}$$

where $\underline{M}$, $\underline{r}$ x $\underline{r}$, is the diagonal matrix of latent roots of $\underline{R}$ associated with the $\underline{r}$ components retained, and $\underline{T}$ is as defined above. The investigator, thus, may well find, if he retains more components than only those associated with positive alpha reliability before transformation, that all may have positive--and perhaps adequate--alpha reliability after transformation. Formulations (10) and (11), then, may be useful in deciding on the number of components, if the investigator wishes to transform his components as well as employ a reliability- or communality/total variance ratio-based decision regarding an acceptable value for such a coefficient after transformation.

If the component model is employed in connection with the importance or interpretability rationale, the implication appears to be that the investigator should probably transform more components than he will ultimately interpret and defer the decision regarding those that he will interpret until after transformation. Ideally, at this point, a priori considerations will dictate the number of components that will admit to informative interpretation. Because of the effects of the number of axes transformed upon the ultimate interpretation of each, the investigator would be wise to transform this a priori number as well as, perhaps, one or two more, to determine that number that is associated with optimally interpretable factors. Not all of these final factors, of course, need be given equal interpretive status.

With the common-factor and image models--often employed in connection with the more explanatory view of factors and factor analysis--the notion of a "correct" number of factors has greater validity, and may, in fact, be of considerable theoretical importance. Additionally, this number will almost invariably be greater--and perhaps considerably so--than the number of components generally deemed worth retaining. At the procedural level, the three rationales--statistical, psychometric, and importance--exist upon which to base this decision (the algebraic rational not being tightly fixed to any widely used rules). The psychometric rationale, manifested in alpha factor analysis (Kaiser and Caffrey, 1965), in the case of common factors, is generally ill-founded for the same reasons as with components (although again it could be applied after transformation) and will generally suggest fewer common factors (Guttman's weaker lower bound) than is likely to be appropriate. Additionally, as has been pointed out by McDonald (1970), the weighting of the residuals in alpha factor analysis is at variance with the most logical weighting. The shortcomings of the statistical inference rationale have been pointed out earlier. Perhaps the most serious drawback to this approach is the dependence on $\underline{N}$. The question remains, then, of how we might best employ the importance rationale. Clearly, the Little Jiffy rule is indicative of far too few common factors. The scree test would appear to suggest too few factors in many cases, and to suffer overall from an inability always to yield an unequivocable result. We are left with the image analysis-based "rotation for residualization" procedure as perhaps closest to the most fruitful operationalization of this rationale.

Again, somewhat in line with what was said regarding the component model, the most reasonable approach--in the spirit of the importance or interpreability rationale--appears to lie in the extraction of considerably more common or image factors than will be ultimately interpreted. If some elusive "correct" number exists, presumably when <u>that</u> many factors are transformed, the resulting constructs will be untimately clearest and most interpretable. Indeed, if some smaller number than this are to be eventually interpreted, these chosen factors will likely be of clearer resolution when formed as a linear transformation of these many initial coordinates than of only a few, unless, of course, the greater dimensionality results in the fission of otherwise relatively unitary constructs. The implication here is that the decision regarding the number of factors perhaps most profitably can be made after, rather than before, transformation. The use of Guttman's stronger lower bound may be most appropriate, with successively fewer factors transformed until a most interpretable solution results.

The authors are aware of the difficulties at present of operationalizing the notion of "most interpretable," as well as the inherent dependence of such a procedure upon the transformation method employed. It may be that the use of a class of solutions, such as yielded by the general "orthomax" criterion (see Harris and Kaiser, 1964, or Hakstian, 1972, for greater detail) is called for. Obviously, if several such orthogonal solutions contained $m$ columns with only one two large factor pattern coefficients, then fewer factors should probably be transformed. In summary, the number of factors problem--in the case of common and image factors--should perhaps be somewhat recast into that of finding the number of factors to transform so that an optimally clear solution results. This number, then, would represent the theoretically most justifiable dimensionality of the variables, although fewer than this many factors may ultimately be interpreted.

Kaiser (1963, 1964) suggested that the orthogonal rotation of an image or common-factor pattern matrix, with the number of columns dictated by Guttman's stronger lower bound, could be expected to reveal the important as well as insignificant factors. The authors believe that future development of the importance or interpretability rationale may profitably be directed at a more thorough investigation of this strategy, as well as at ways of detecting optimally interpretable solutions or, alternatively, those exhibiting the best simple structure, since the latter condition has been advanced, above, as a criterion of the correct number of factors.

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TABLE 1

Correct Number of Factors According to Various Procedures, Using 17 Data Sets

| | | | | Model | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Component | | | Common-Factor | | | | Image | |
| Data Set | N | n | Number of Factors in Source Study | Roots of R>1 | Scree (Rts of R) | Max. Comm. | Scree (Rts of R-U$^2$) | Little Jiffy | Likelihood-Ratio 1963[a] | Likelihood-Ratio 1967[b] | Rts of $S^{-1}RS^{-1}$ >1[c] | After Rotation |
| 1 | 100 | 6 | 5 | 2 | 2 | 2 | 2 | 2 | 1 | 2 | 3 | 3 |
| 2 | 329 | 7 | 3 | 2 | 2 | 2 | 2 | 1 | 2 | 3 | 3 | 2 |
| 3 | 305 | 8 | 2 | 2 | 2 | 2 | 2 | 2 | 5 | 4 | 4 | 2 |
| 4 | 163 | 8 | 3 | 2 | 3 | 2 | 3 | 2 | 4 | 4 | 5 | 3 |
| 5 | 421 | 9 | 2 | 1 | 3 | 1 | 3 | 1 | 4 | 2 | 4 | 3 |
| 6 | 211 | 9 | 3 | 2 | 4 | 2 | 4 | 2 | 2 | 2 | 3 | 2 |
| 7 | 170 | 13 | 6 | 3 | 3 | 3 | 3 | 3 | 8 | 6 | 8 | 3 |
| 8 | 145 | 13 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 7 | 3 |
| 9 | 212 | 17 | 6 | 5 | 6 | 4 | 7 | 3 | 6 | 6 | 7 | 6 |
| 10 | 213 | 17 | 6 | 5 | 7 | 4 | 6 | 4 | 6 | 6 | 8 | 6 |
| 11 | 138 | 19 | 7 | 7 | 5 | 6 | 5 | 6 | 3 | 3 | 11 | 5 |
| 12 | 145 | 24 | 4 | 5 | 4 | 4 | 4 | 4 | 4 | 5 | 13 | 5 |
| 13 | 154 | 25 | 8 | 6 | 6 | 6 | 3 | 5 | 8 | 9 | 14 | 8 |
| 14 | 293 | 25 | 9 | 7 | 9 | 7 | 6 | 6 | 9 | 14 | 16 | 11 |
| 15 | 197 | 26 | 9 | 4 | 4 | --[d] | 4 | --[d] | 8 | 8 | --[d] | --[d] |
| 16 | 200 | 33 | 9 | 11 | 8 | 9 | 4 | 8 | 8 | 9 | 19 | 13 |
| 17 | 283 | 34 | 12 | 8 | 6 | 7 | 6 | 7 | 14 | 11 | 20 | 9 |

[a] Jöreskog's (1963) non-iterative estimation procedure.

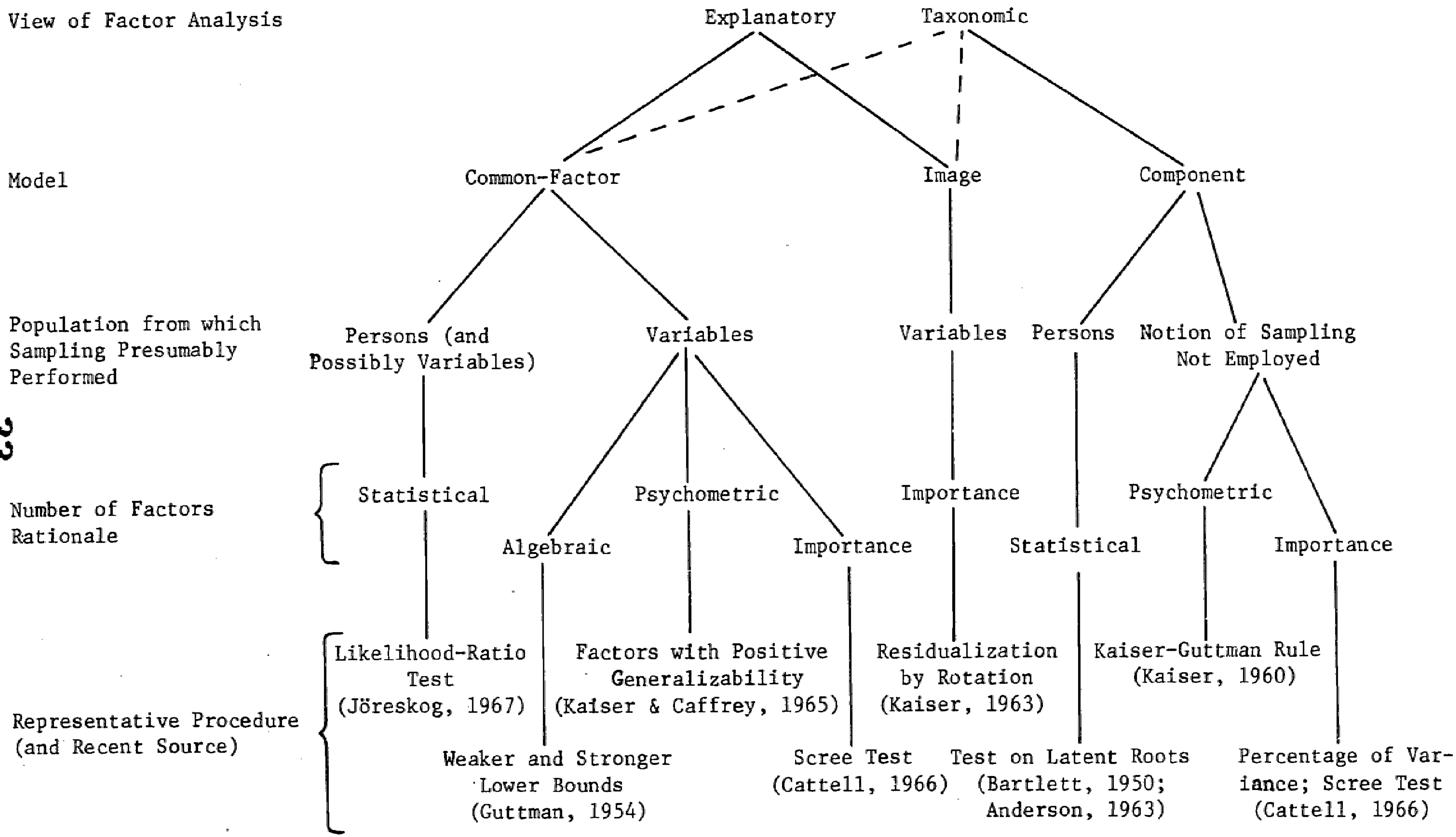
[b] Jöreskog's (1967) maximum-likelihood procedure.

[c] Guttman's stronger lower bound.

[d] Procedure not applied because of a near-zero determinant of R.

FIGURE CAPTION

Figure 1. Schematic representation of the underlying views, models, and bases for inference in factor analysis, and the rationales and procedures for deciding on the correct number of factors.


View of Factor Analysis
Model
Population from which Sampling Presumably Performed
Number of Factors Rationale
Representative Procedure (and Recent Source)
Explanatory
Taxonomic
Common-Factor
Image
Component
Persons (and Possibly Variables)
Variables
Variables
Persons
Notion of Sampling Not Employed
Statistical
Algebraic
Psychometric
Importance
Importance
Statistical
Psychometric
Importance
Likelihood-Ratio Test (Jöreskog, 1967)
Weaker and Stronger Lower Bounds (Guttman, 1954)
Factors with Positive Generalizability (Kaiser & Caffrey, 1965)
Scree Test (Cattell, 1966)
Residualization by Rotation (Kaiser, 1963)
Test on Latent Roots (Bartlett, 1950; Anderson, 1963)
Kaiser-Guttman Rule (Kaiser, 1960)
Percentage of Variance; Scree Test (Cattell, 1966)