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ABSTRACT

Additive two-factor conjoint measurement is derived from axioms that do not include unrestricted solvability or a condition on interlocked standard sequences. (Author)

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RESEARCH BULLETIN

# A NOTE ON CONJOINT MEASUREMENT WITH RESTRICTED SOLVABILITY

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A NOTE ON CONJOINT MEASUREMENT WITH RESTRICTED SOLVABILITY


Abstract

Additive two-factor conjoint measurement is derived from axioms that do not include unrestricted solvability or a condition on interlocked standard sequences.

# A NOTE ON CONJOINT MEASUREMENT WITH RESTRICTED SOLVABILITY[1]

This note presents a weakened set of axioms for additive two-factor conjoint measurement. In order to discuss such measurement explicitly, the following notation will be used. The two factors will be denoted by the sets $A_1$ and $A_2$; levels of the factors will be denoted by elements a, b, c, ... in $A_1$, and p, q, r, ... in $A_2$. Joint effects of combining the two factors will be denoted by pairs of elements, such as ap, bq, ..., in $A_1 \times A_2$. These effects will be assumed to be ordered by a relation $\gtrsim$ on $A_1 \times A_2$. Unless otherwise specified, all statements about elements such as a and p will be understood to apply to all a in $A_1$ and p in $A_2$. In this situation, the representation for additive conjoint measurement states that there are functions $\phi_1$ on $A_1$ and $\phi_2$ on $A_2$, such that ap $\gtrsim$ bq if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. The usual uniqueness result states that the functions are interval scales with the same unit; that is, the functions $\phi_i$ and $\phi_i'$ both satisfy the representation if and only if there are constants $\alpha > 0$, $\beta_1$, and $\beta_2$, such that $\phi_i' = \alpha\phi_i + \beta_i$.

Adams and Fagot (1959) demonstrated that the following three axioms are necessary for the representation.

<u>Axiom 1</u>. ap $\gtrsim$ bq <u>or</u> ap $\lesssim$ bq <u>or both</u>; ap $\gtrsim$ bq and bq $\gtrsim$ cr <u>imply</u> ap $\gtrsim$ cr.

<u>Axiom 2</u>. ap $\gtrsim$ bp <u>implies</u> aq $\gtrsim$ bq; ap $\gtrsim$ aq <u>implies</u> bp $\gtrsim$ bq.

<u>Axiom 3</u>. ax $\approx$ fq <u>and</u> fp $\approx$ bx <u>imply</u> ap $\approx$ bq.

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These axioms have been called respectively weak ordering, independence, and double cancellation.

Luce and Tukey (1964) stated sufficient conditions for conjoint measurement that include two additional axioms, of which one is necessary for the representation and the other is not. The necessary condition is an Archimedean axiom, which will be stated here in the improved version given by Krantz, Luce, Suppes, and Tversky (1971).

Axiom 4. *If for some* $\bar{a}$, *a in* $A_1$ *and* p, q *in* $A_2$, *a sequence of elements* $a_i$ *in* $A_1$ *satisfies* $ap \precsim a_i q \prec a_i p \approx a_{i+1} q \precsim a\bar{q}$ *for all* $a_i$ *and* $a_{i+1}$, *then the sequence must be finite. A similar statement holds with the roles of* $A_1$ *and* $A_2$ *reversed.*

The nonnecessary condition of Luce and Tukey is a solvability axiom, which requires the sets $A_1$ and $A_2$ to be unbounded and is consequently too strong for many empirical applications. Luce (1966) therefore substituted the following restricted solvability axiom.

Axiom 5. *For any* a, $\bar{b}$, b *in* $A_1$ *and* p, q *in* $A_2$ *such that* $\bar{b}q \succsim ap \succsim bq$, *there is a* b *in* $A_1$ *such that* $ap \approx bq$; *for any* a, b *in* $A_1$ *and* p, $\bar{q}$, $q$ *in* $A_2$ *such that* $b\bar{q} \succsim ap \succsim bq$, *there is a* q *in* $A_2$ *such that* $ap \approx bq$.

In order to prove conjoint measurement under this weaker solvability condition, Luce added an axiom that assumes the existence of certain interlocked standard sequences. Fortunately, this last rather complicated axiom need not be repeated here, because the present note shows that it can be replaced by the following very weak condition, which Krantz et al (1971) formulated and called essentialness.

Axiom 6. *There are* a, b, c *in* $A_1$, *and* p, q, r *in* $A_2$, *such that* $ar \succ br$ *and* $pc \succ qc$.

In other words, an additive conjoint representation, unique up to linear transformations, will be derived from the necessary conditions of weak ordering, independence, double cancellation, and the Archimedean property, and the nonnecessary conditions of restricted solvability and essentialness.

Theorem. *If the system* $\langle A_1, A_2, \succsim \rangle$ *satisfies Axioms* 1, 2, 3, 4, 5, *and* 6, *then there are real-valued functions* $\phi_i$ *on* $A_i$ $(i = 1, 2)$, *such that* $ap \succsim bq$ *if and only if* $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$; *moreover, the functions* $\phi_i'$ *also satisfy the representation if and only if* $\phi_i' = \alpha\phi_i + \beta_i$ *for some constants* $\alpha > 0$, $\beta_1$, *and* $\beta_2$.

Proof. By the ordering and independence axioms, the sets $A_i$ can be weakly ordered as follows. For a, b in $A_1$, let $a \succsim b$ if and only if $ap \succsim bp$ for all p in $A_2$; for p, q in $A_2$, let $p \succsim q$ if and only if $ap \succsim aq$ for all a in $A_1$.

[Th]e theorem will first be proved for a square subset $A_1' \times A_2'$, bounded by points $\underline{a} < \bar{a}$ and $\underline{p} < \bar{p}$, such that $\underline{a}\bar{p} \approx \bar{a}\underline{p}$, and $\underline{a}\underline{p} \precsim ap \precsim \bar{a}\bar{p}$ for all ap in $A_1' \times A_2'$. Such a subset must exist by Axioms 5 and 6. For any a in $A_1'$, $\underline{a}\bar{p} \approx \bar{a}\underline{p} \succsim a\underline{p} \succsim \underline{a}\underline{p}$; thus, using Axiom 5, let $\pi(a)$ be such that $\underline{a}\pi(a) \approx a\underline{p}$. Let B be the set of all pairs (a,b) in $A_1' \times A_2'$ such that $a > \underline{a}$, $b > \underline{b}$, and $a\pi(b) \precsim \bar{a}\underline{p}$. For any (a,b) in B, $\bar{a}\underline{p} \succsim a\pi(b) \succsim a\underline{p}$; thus, using Axiom 5 again, let aob be such that $a\pi(b) \approx (aob)\underline{p}$. It will now be shown that if B is nonempty, then the system $\langle A_1', \succsim, B, o \rangle$ satisfies the axioms for bounded extensive measurement as stated by Krantz (1967) or Krantz et al. (1971).

To prove that o is commutative, suppose that (a, b) is in B. By definition, $a\underline{p} \approx \underline{a}\pi(a)$ and $\underline{a}\pi(b) \approx b\underline{p}$; hence, by double cancellation,

$a\pi(b) \approx b\pi(a)$. It follows that $(b, a)$ is in B, and $(aob)\underline{p} \approx (boa)\underline{p}$; thus, by independence, $aob \approx boa$.

To prove that o is monotonic, suppose that $(a, c)$ is in B, and $a \succsim b$. By independence, $a\pi(c) \succsim b\pi(c)$. It follows that $(b, c)$ is in B, and $aoc \succsim boc$. By commutativity, therefore, $(c, a)$ and $(c, b)$ are also in B, and $coa \approx aoc \succsim boc \approx cob$.

To prove that o is associative, suppose that $(a, b)$ and $(aob, c)$ are in B. Since $aob > b$, it follows that $(b, c)$ is in B. By definition and commutativity, $b\pi(a) \approx (aob)\underline{p}$ and $(boc)\underline{p} \approx b\pi(c)$; hence, by double cancellation, $(boc)\pi(a) \approx (aob)\pi(c)$. By definition and commutativity, $a\pi(boc) \approx [ao(boc)]\ \underline{p} \approx [(boc)\ oa]\underline{p} \approx (boc)\pi(a)$; consequently $a\pi(boc) \approx (aob)\pi(c)$. It follows that $(a, boc)$ is in B, and $ao(boc) \approx (aob)oc$.

The remaining conditions for extensive measurement can be verified immediately. Therefore, if B is nonempty, there is a function $\phi_1$ on $A_1'$, such that $\phi_1(aob) = \phi_1(a) + \phi_1(b)$, and $\phi_1(a) \geq \phi_1(b)$ if and only if $a \succsim b$. For any p in $A_2$, $\bar{a}\underline{p} \approx \underline{a}\bar{p} \succsim ap \succsim \underline{a}\underline{p}$; thus, let $\alpha(p)$ be such that $\alpha(p)\underline{p} \approx \underline{a}p$, and let $\phi_2(p) = \phi_1[\alpha(p)]$. By the definitions of $\alpha(p)$ and $\pi(a)$, $\underline{a}p \approx \alpha(p)\underline{p} \approx a\pi[\alpha(p)]$; hence, $p \approx \pi[\alpha(p)]$. Thus, for $ap \precsim \bar{a}\underline{p}$, it follows that $[a, \alpha(p)]$ is in B, and $ap \approx a\pi[\alpha(p)] \approx [ao\alpha(p)]\underline{p}$. Now, suppose that $ap \precsim \bar{a}\underline{p}$ and $bq \precsim \bar{a}\underline{p}$. In this case, $ap \succsim bq$ if and only if $ao\alpha(p) \succsim bo\alpha(q)$, which by extensive measurement occurs if and only if $\phi_1(a) + \phi_1[a(p)] \geq \phi_1(b) + \phi_1[a(q)]$, which by definition holds if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. In other words, $\phi_1$ and $\phi_2$ provide a conjoint representation for all $ap \lesssim \bar{a}\underline{p}$. The uniqueness of these scales follows from the uniqueness of extensive measurement.

To show that the representation also holds for the rest of $A_1' \times A_2'$ when B is nonempty, suppose first that $ap \succsim \bar{a}\underline{p}$ and $bq \succsim \bar{a}\underline{p}$. If $a \succsim b$ and $p \succsim q$, or if $a \precsim b$ and $p \precsim q$, the proof is obvious. Suppose, therefore, that $a \precsim b$ and $p \succsim q$; the proof is similar if $a \succsim b$ and $p \precsim q$. Since $b\bar{p} \succsim \underline{a}\bar{p} \approx \bar{a}\underline{p} \succsim b\underline{p}$ and $\bar{a}p \succsim \bar{a}\underline{p} \approx \underline{a}\bar{p} \succsim \underline{a}p$, let r and c be defined such that $br \approx cp \approx \underline{a}\bar{p}$. It follows from double cancellation that $ap \approx bq$ if and only if $ar \approx cq$. It also follows from independence that $ar \precsim br \approx \underline{a}\bar{p}$ and $cq \precsim cp \approx \underline{a}\bar{p}$; hence, by the representation already established, $\phi_1(b) + \phi_2(r) = \phi_1(c) + \phi_2(p)$, and $ar \approx cq$ if and only if $\phi_1(a) + \phi_2(r) = \phi_1(c) + \phi_2(q)$. These facts can be combined to show that $ap \approx bq$ if and only if $\phi_1(a) + \phi_2(p) = \phi_1(b) + \phi_2(q)$. If $ap \succ bq$, then because $bq \succsim \bar{a}\underline{p} \approx \underline{a}\bar{p} \succsim \underline{a}p$ by hypothesis, let d be defined such that $dp \approx bq$, and thus $a \succ d$ by independence; it follows from the definition of $\phi_1$ and the last sentence that $\phi_1(a) + \phi_2(p) > \phi_1(d) + \phi_2(p) = \phi_1(b) + \phi_2(q)$. If $ap \prec bq$, a similar argument gives the reversed inequality. These results mean that $ap \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$, as claimed. In the remaining case, $ap \succsim \underline{a}\bar{p} \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(\underline{a}) + \phi_2(\bar{p}) \geq \phi_1(b) + \phi_2(q)$, by the results already proved.

If B is empty, then $\underline{a}$ and $\bar{a}$ must be the only elements in $A_1'$, and it follows from Axiom 5 that $\underline{p}$ and $\bar{p}$ are the only elements in $A_2'$. By definition, $\underline{a}\underline{p} \prec \underline{a}\bar{p} \approx \bar{a}\underline{p} \prec \bar{a}\bar{p}$. Therefore, for any constants $\alpha > 0$, $\beta_1$, and $\beta_2$, let $\phi_1(\underline{a}) = \beta_1$, $\phi_1(\bar{a}) = \beta_1 + \alpha$, $\phi_2(\underline{p}) = \beta_2$, and $\phi_2(\bar{p}) = \beta_2 + \alpha$. These functions clearly satisfy the representation and uniqueness for conjoint measurement.

Next, the representation will be generalized to the hypothesis that $A_1' \times A_2'$ is rectangular. It will be assumed that $\bar{a}\underline{p} \succ \underline{a}\bar{p}$; the proof is



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similar in the other case. Let $a_1$ be such that $a_1\underline{p} \approx a\bar{p}$; if $a_n$ is defined and $a_n\bar{p} \precsim \bar{a}\underline{p}$, let $a_{n+1}$ be defined such that $a_{n+1}\underline{p} \approx a_n\bar{p}$. It has already been shown possible to define conjoint scales $\phi_1$ and $\phi_2$ on $\underline{a} \precsim a \precsim a_1$ and $\underline{p} \precsim p \precsim \bar{p}$ respectively; for convenience, let $\phi_1(\underline{a}) = \phi_2(\underline{p}) = 0$ and $\phi_1(a_1) = \phi_2(\bar{p}) = 1$. The scale $\phi_1$ can now be extended to the rest of $A_1'$ by induction on n, since by the Archimedean axiom, there must be some m such that $a_m\bar{p} \succsim \bar{a}\underline{p}$. Therefore, to perform the induction, it is assumed that $\phi_1(a)$ has been defined with the appropriate properties for all $a \precsim a_n$.

Suppose that $a_{n+1}$ is defined; the proof is similar if $a_n\bar{p} \succ \bar{a}\underline{p}$. For any a such that $a_n \precsim a \precsim a_{n+1}$, it follows that $a_{n-1}\bar{p} \approx a_n\underline{p} \precsim a\underline{p} \precsim a_{n+1}\underline{p} \approx a_n\bar{p}$; thus, let a' be such that $a'\bar{p} \approx a\underline{p}$. Since $a' \precsim a_n$ by independence, $\phi_1(a')$ is defined by the induction hypothesis; therefore, let $\phi_1(a) = \phi_1(a') + 1$. For any $a_n \precsim a \precsim a_{n+1}$ and p in $A_2'$, $a'\underline{p} \precsim a'p \precsim a'\bar{p} \approx a\underline{p}$; thus, let $\delta(ap)$ be such that $\delta(ap)\underline{p} \approx a'p$. The definitions of a' and $\delta(ap)$ imply by double cancellation that $\delta(ap)\bar{p} \approx ap$. To compare ap and bq, suppose first that $a_n \precsim a \precsim a_{n+1}$ and $a_n \precsim b \precsim a_{n+1}$. Consequently, $ap \succsim bq$ if and only if $\delta(ap)\bar{p} \succsim \delta(bq)\bar{p}$, which by independence is equivalent to $\delta(ap)\underline{p} \succsim \delta(bq)\underline{p}$, which by definition holds if and only if $a'p \succsim b'q$, which by the induction hypothesis occurs if and only if $\phi_1(a') + \phi_2(p) \geq \phi_1(b') + \phi_2(q)$, which by definition is equivalent to $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. Next, suppose that $a_n \precsim a \precsim a_{n+1}$ and $b \precsim a_n$. If $ap \succsim a_n\bar{p}$, then $ap \succsim bq$, and also $\phi_1(a) + \phi_2(p) \geq \phi_1(a_n) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. If $ap \precsim a_n\bar{p}$, then because $a_n\underline{p} \precsim ap$, let p' be such that $ap \approx a_np'$; hence, $ap \approx a_np' \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) = \phi_1(a_n) + \phi_2(p') \geq \phi_1(b) + \phi_2(q)$. Since a' is defined uniquely (up to $\approx$) for each a, the uniqueness of $\phi_1(a)$ is the same as the uniqueness of $\phi_1(a')$, which is specified by the induction hypothesis.

Finally, the representation can be extended to all $A_1 \times A_2$ by defining a series of rectangular subsets. To fix the upper boundaries of these rectangles, an infinite sequence of points $\bar{a}_n$ will be defined, such that $\bar{a}_{n+1} \gtrsim \bar{a}_n$ for all n, and given any a in $A_1$, there is an m such that $a_m \gtrsim a$. If $A_1$ is bounded by an $\bar{a}$ such that $\bar{a} \gtrsim a$ for all a, then let $\bar{a}_n = \bar{a}$ for all n; clearly this sequence has the required properties. Next, suppose that there is no such upper bound. Let three points $\bar{a}_o$, $q_o$, and q be given, with $q > q_o$. The points $\bar{a}_{n+1}$ and $q_{n+1}$ will be defined recursively, given $\bar{a}_n$ and $q_n$, with $q > q_n$. If there is an a such that $aq_n \gtrsim \bar{a}_n q$, then let $\bar{a}_{n+1}$ be such that $\bar{a}_{n+1}q_n \approx \bar{a}_n q$; also, let $q_{n+1} = q_n$. Otherwise, let $\bar{a}_{n+1}$ be any point such that $\bar{a}_{n+1} > \bar{a}_n$; since in this case $\bar{a}_{n+1}q_n < \bar{a}_n q < \bar{a}_{n+1}q$, let $q_{n+1}$ be such that $\bar{a}_{n+1}q_{n+1} \approx \bar{a}_n q$. These definitions immediately imply that for all n, $\bar{a}_{n+1} \gtrsim \bar{a}_n$ and $q > q_{n+1} \gtrsim q_n$. Now, let any a in $A_1$ be given; by the hypothesis of unboundedness, there must also be a b such that $b > a$. By the Archimedean axiom, the portion of the $\bar{a}_n$ such that $\bar{a}_n < a$ and $bq_n < aq$ must be finite; thus, there is an m such that either $\bar{a}_m \gtrsim a$, or else $a_m < a$ and $bq_m \gtrsim aq$. In the second of these cases, the portion of the $a_n$ such that $n \geq m$, $a_n < a$, and $q_n = q_m$ must again be finite; thus, there must be an $\ell$ such that $a_\ell < a$ and $aq_\ell < a_\ell q$. In this case, however, $bq_\ell = bq_m \gtrsim aq > a_\ell q$; hence, $a_{\ell+1}q_\ell \approx a_\ell q > aq_\ell$, which means that $a_{\ell+1} > a$, as claimed.

In a similar manner, it is possible to define sequences $\underline{a}_n$ , $\bar{p}_n$ , and $\underline{p}_n$ , such that if the set $A_1^{(n)}$ is defined to contain all a in $A_1$ with $\underline{a}_n \precsim a \precsim \bar{a}_n$ and the set $A_2^{(n)}$ is defined to contain all p in $A_2$ with $\underline{p}_n \precsim p \precsim \bar{p}_n$, then $A_1^{(n)} \times A_2^{(n)} \subseteq A_1^{(n+1)} \times A_2^{(n+1)}$ for all n, and for any ap in $A_1 \times A_2$, there is an m such that ap is in $A_1^{(m)} \times A_2^{(m)}$. As has already been shown, it is possible for any n to construct functions $\phi_1^{(n)}$ on $A_1^{(n)}$ and $\phi_2^{(n)}$ on $A_2^{(n)}$ with the properties required for conjoint measurement. Moreover, the arbitrary constants in these functions can be set so that for any n, $\phi_1^{(n+1)}(a) = \phi_1^{(n)}(a)$ for all a in $A_1^{(n)}$, and $\phi_2^{(n+1)}(p) = \phi_2^{(n)}(p)$ for all p in $A_2^{(n)}$. Thus, for any a in A, $\phi_1(a)$ can be defined as equal to $\phi_1^{(n)}(a)$, where n is such that a is in $A_1^{(n)}$; similarly, for any p in $A_2$, $\phi_2(p)$ can be defined as equal to $\phi_2^{(n)}(p)$, where n is such that p is in $A_2^{(n)}$. QED.

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