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Abstract

To resolve a recent controversy between Klein and Cleary and Levy, a model for dichotomous congeneric items is presented which has mean errors of zero, dichotomous true scores that are uncorrelated with errors, and errors that are mutually uncorrelated.

## A CONGENERIC MODEL FOR PLATONIC TRUE SCORES<sup>1</sup>

Charles E. Werts, Robert L. Linn, and Karl Jöreskog

In a discussion of platonic true scores, Klein and Cleary (1967) state that the use of platonic true scores makes the assumptions of classical test theory generally untenable. They illustrate their argument with dichotomous items and a dichotomous true score and show that: "The classical test theory formulation  $\sigma_X^2 = \sigma_T^2 + \sigma_E^2$ , can only be true if the mean error is not zero" (Klein & Cleary, 1967, p. 78). This statement is based on the following definitions of observed (X), true (T), and error (E) scores:

$$T = \left\{ \begin{array}{l} 1 \text{ if phenomenon is present} \\ 0 \text{ if phenomenon is absent} \end{array} \right\},$$

$$X = \left\{ \begin{array}{l} 1 \text{ if phenomenon is rated as present} \\ 0 \text{ if phenomenon is rated as absent} \end{array} \right\},$$

and  $E = X - T$ . Klein and Cleary go on to consider two parallel dichotomous items,  $X_1$  and  $X_2$ , and show that the covariance between  $E_1$  and  $E_2$  is positive when the errors,  $E_1$  and  $E_2$ , have zero means. With correlated error scores, the correlation between two parallel items overestimates the item reliabilities. In response, Levy (1969) argued that the classical assumptions can be shown to hold for a dichotomous item if

$$X = \left\{ \begin{array}{l} a \text{ if phenomenon is rated as present} \\ b \text{ if phenomenon is rated as absent} \end{array} \right\},$$

true scores (T) are defined as above and  $E = X - T$  as before. This modification will indeed make it possible for the mean error to be zero and the covariance between T and E to be zero. As Klein and Cleary (1969) note, however, Levy does not provide a means of solving for "a" and "b" without

knowledge of  $T$ . In any practical application,  $T$  would be unknown and therefore "a" and "b" would be unknown. Also, no way of obtaining item reliabilities is presented. The purpose of this paper is to provide an alternative formulation which allows for the model parameters to be determined given the structural specification of zero mean error and no correlation among errors for different items or between errors and true scores. Our approach is drawn from latent structure analysis (Anderson, 1959) for the special case of dichotomous latent variables.

#### I. A Congeneric Model for Dichotomous Items

The equation for congeneric tests is given by Jöreskog (1968, 1970, 1971) as

$$X_{ij} = B_{jT}T_i + I_j + E_{ij} \quad , \quad (1)$$

where  $T_i$  is the true score for person  $i$ ,

$X_{ij}$  is the observed score on item  $j$  for person  $i$ ,

$B_{jT}$  is the slope of the  $X_{ij}$  on  $T_i$  regression line,

$I_j$  is the intercept of this regression line, and

$E_{ij}$  is the error for person  $i$  on item  $j$ .

To illustrate the application of this definition to the case in which  $X_{ij}$  and  $T_i$  are both dichotomous (scored 1, 0), consider the case of three items, which is the minimum number of items required to identify model parameters uniquely, given experimentally independent measures. The equations are

$$X_1 = B_{1T}T + I_1 + E_1 \quad , \quad (1a)$$

$$X_2 = B_{2T}T + I_2 + E_2 \quad , \quad (1b)$$

$$X_3 = B_{3T}T + I_3 + E_3 \quad , \quad (1c)$$

where the  $E$ 's are mutually uncorrelated and are uncorrelated with  $T$ .<sup>2</sup>

In the case of dichotomous variables

$$B_{jT} = \frac{P\{X_j = 1, T = 1\} - P\{X_j = 1\}P\{T = 1\}}{P\{T = 1\}P\{T = 0\}} = P\{X_j = 1|T = 1\} - P\{X_j = 1|T = 0\}$$

and

$$I_j = P\{X_j = 1\} - B_{jT}P\{T = 1\} = P\{X_j = 1|T = 0\} .$$

This model is somewhat more complicated than the model considered by Klein and Cleary (1967) where  $X = T + E$  with  $X$ ,  $T$ , and  $E$  all taking values of 0 or 1. In essence, the congeneric model is equivalent to the model suggested by Levy (1969) if his "a" and "b" are allowed to vary from item to item. For a given item, "a<sub>j</sub>" would equal  $(1 - I_j)/B_{jT}$ , "b<sub>j</sub>" would equal  $-I_j/B_{jT}$ , and Levy's error would equal the error of equations 1, 2, or 3 divided by  $B_{jT}$ . To illustrate the point that the congeneric model does allow for the traditional psychometric assumptions in the dichotomous case, consider the following example constructed using the equations provided by Anderson (1959, sec. 2.4).

1. The  $\theta_j$  (proportion of false negatives, i.e.,  $P\{X_j = 0|T = 1\} = P\{X_j = 0, T = 1\} \div P_T$ ),  $\phi_j$  (proportion of false positives, i.e.,  $P\{X_j = 1|T = 0\} = P\{X_j = 1, T = 0\} \div (1 - P_T)$ ), and  $P_T$  (the true proportion  $P\{T = 1\}$ ) are given as:

$$\begin{aligned} \theta_1 &= .30 , \quad \theta_2 = .40 , \quad \theta_3 = .10 ; \\ \phi_1 &= .10 , \quad \phi_2 = .50 , \quad \phi_3 = .30 ; \\ P_T &= .60 , \quad \text{and} \quad Q_T = 1 - P_T = .40 . \end{aligned}$$

2. The expected marginal distributions ( $P_j = \text{Prob}\{X_j = 1\}$ ) are  $P_j = (1 - \theta_j)P_T + \phi_j Q_T$ , i.e.,  $P_1 = .46$ ,  $P_2 = .56$ , and  $P_3 = .66$ .

3. The expected joint probabilities for pairs of items,  $P_{jj'} = \text{Prob} \{X_j = 1, X_{j'} = 1\} = (1 - \theta_j)(1 - \theta_{j'})P_T + \phi_j\phi_{j'}Q_T$  ( $j \neq j'$ ) are:  
 $P_{12} = .272$  ,  $P_{13} = .390$  , and  $P_{23} = .384$ .

4. The expected joint probability for three items,  $P_{jj'j''} = \text{Prob} \{X_j = 1, X_{j'} = 1, X_{j''} = 1\} = (1 - \theta_j)(1 - \theta_{j'})(1 - \theta_{j''})P_T + \phi_j\phi_{j'}\phi_{j''}Q_T$  ( $j \neq j' \neq j''$ ) is  $P_{123} = .2328$  .

5. The regression weights ( $B_{jt} = 1 - \theta_j - \phi_j$ ) are  $B_{1T} = .60$  ,  
 $B_{2T} = .10$  , and  $B_{3T} = .60$  .

6. The intercepts ( $I_j = P_j - B_{jT}P_T = \phi_j$ ) are  $I_1 = .10$  ,  $I_2 = .50$  ,  
and  $I_3 = .30$  . The possible events for combinations of the three items and the proportion of people in each event are shown in Table 1. The means of the errors are zero, the true score is uncorrelated with the errors and the errors are uncorrelated with each other.

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Insert Table 1 about here  
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II. Identification

In an actual problem the situation would be reversed from the example shown in section I, i.e., the probabilities  $P_1, P_2, P_3, P_{12}, P_{13}, P_{23}$ , and  $P_{123}$  correspond to observed scores, and it would be desirable to identify the seven parameters,  $\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3$ , and  $P_T$  . In principle, one could solve the seven equations for this purpose:

$$P_1 = (1 - \theta_1)P_T + \phi_1Q_T , \tag{2a}$$

$$P_2 = (1 - \theta_2)P_T + \phi_2Q_T , \tag{2b}$$

$$P_3 = (1 - \theta_3)P_T + \phi_3Q_T , \tag{2c}$$

$$P_{12} = (1 - \theta_1)(1 - \theta_2)P_T + \phi_1\phi_2Q_T, \quad (2d)$$

$$P_{13} = (1 - \theta_1)(1 - \theta_3)P_T + \phi_1\phi_3Q_T, \quad (2e)$$

$$P_{23} = (1 - \theta_2)(1 - \theta_3)P_T + \phi_2\phi_3Q_T, \quad (2f)$$

$$P_{123} = (1 - \theta_1)(1 - \theta_2)(1 - \theta_3)P_T + \phi_1\phi_2\phi_3Q_T. \quad (2g)$$

The solution to these equations is facilitated by noting that in the congeneric model the expected covariance ( $C_{jj'}$ ) between two items is given by

$$C_{jj'} = B_{jT}B_{j'T}V_T,$$

where  $V_T$  is the variance of  $T$ . Translating into probabilities:

$$(P_{jj'} - P_jP_{j'}) = (1 - \theta_j - \phi_j)(1 - \theta_{j'} - \phi_{j'})P_TQ_T. \quad (3)$$

This means that

$$C_{12} = P_{12} - P_1P_2 = (1 - \theta_1 - \phi_1)(1 - \theta_2 - \phi_2)P_TQ_T, \quad (4a)$$

$$C_{13} = P_{13} - P_1P_3 = (1 - \theta_1 - \phi_1)(1 - \theta_3 - \phi_3)P_TQ_T, \quad (4b)$$

$$C_{23} = P_{23} - P_2P_3 = (1 - \theta_2 - \phi_2)(1 - \theta_3 - \phi_3)P_TQ_T. \quad (4c)$$

These equations can be solved for

$$(1 - \theta_1 - \phi_1)^2 P_TQ_T = \frac{C_{12}C_{13}}{C_{23}} = B_{1T}^2 P_TQ_T, \quad (5a)$$

$$(1 - \theta_2 - \phi_2)^2 P_TQ_T = \frac{C_{12}C_{23}}{C_{13}} = B_{2T}^2 P_TQ_T, \quad (5b)$$

$$(1 - \theta_3 - \phi_3)^2 P_TQ_T = \frac{C_{13}C_{23}}{C_{12}} = B_{3T}^2 P_TQ_T. \quad (5c)$$

The triple covariance  $C_{123}$  is defined (Boudon, 1968, p. 226) as the expectation of the products of the deviations of all three variables simultaneously, which is equal in the dichotomous case to  $C_{123} = P_{123} - P_1(P_{23} - P_2P_3) - P_2(P_{13} - P_1P_3) - P_3(P_{12} - P_1P_2) - P_1P_2P_3$  . (6)

Using equations (2a, b, c, d, e, f) equation (6) may be translated to  $C_{123} = B_{1T}B_{2T}B_{3T}P_TQ_T(Q_T^2 - P_T^2)$  and from equations (5a, b, c) we obtain

$$C_{123} = \frac{\sqrt{C_{12}C_{13}C_{23}}(Q_T^2 - P_T^2)}{\sqrt{P_TQ_T}} . \quad (7)$$

Applying these equations to our example,

1. Compute covariances by equations (4a, b, c):

$$C_{12} = .0144 , \quad C_{13} = .0864 , \quad \text{and} \quad C_{23} = .0144 .$$

2. Using equation (6) compute  $C_{123} = -.001728$  .

3. From equation (7),

$$\frac{C_{123}}{\sqrt{C_{12}C_{13}C_{23}}} = -.4082 = \frac{(Q_T^2 - P_T^2)}{\sqrt{P_TQ_T}} .$$

4. Solving for  $P_T = 1 - Q_T$  we obtain  $P_T = .60$  .

5. From equations (5a, b, c) and substituting in this value of  $P_T$  ,

$$B_{1T} = .60 ,$$

$$B_{2T} = .10 ,$$

$$B_{3T} = .60 .$$

6. It can be shown (equations 2a, b, c) that  $\phi_j = P_j - B_{jT}P_T$  permitting calculation of  $\phi_j = I_j$  :

-7-

$$I_1 = \phi_1 = .10 ,$$

$$I_2 = \phi_2 = .50 ,$$

$$I_3 = \phi_3 = .30 .$$

7. Since  $\theta_j = 1 - B_{jT} - \phi_j$  ,

$$\theta_1 = .30 ,$$

$$\theta_2 = .40 ,$$

$$\theta_3 = .10 .$$

8. Item reliabilities  $R_{jj}$  are  $R_{jj} = B_{jT}^2 P_{jT} Q_j / P_j Q_j$  , i.e.,

$$R_{11} = .3478 ,$$

$$R_{22} = .0097 ,$$

$$R_{33} = .3850 .$$

In the case of three congeneric items the model parameters are just identified, i.e., there are seven equations in seven unknowns, which is the reason that the parameters may be obtained as an exact function of the observed probabilities. In the case of overidentified models one of the estimating procedures discussed by Anderson (1959) can be used. One procedure minimizes a  $\chi^2$  function of the observed probabilities ( $P_O$ ) and the expected probabilities ( $P_E$ ) generated as a function of the parameter estimates (Cochran, 1968; Mote & Anderson, 1965). In the general case of  $J$  items there will be  $(2^J - 1)$  independent observed probabilities in the cross-tabulation table from which  $(2J + 1)$  parameters are to be estimated. In the special case of two items of equal accuracy the reliability is the correlation between these items, but the model parameters cannot be identified

(Cochran, 1968, sec. 6) since  $P_E\{X_1 = 1, X_2 = 0\} = P_E\{X_1 = 0, X_2 = 1\}$ , i.e., there are only two independent probabilities to estimate three parameters  $(\theta, \phi, P_T)$ .

### III. Variations

It is sometimes the case that three items with errors that are uncorrelated with true scores or errors of other items are available but one of these measures another variable, i.e.,

$$\begin{aligned} X_1 &= B_1 T_1 + I_1 + E_1, \\ X_2 &= B_2 T_1 + I_2 + E_2, \\ X_3 &= B_3 T_2 + I_3 + E_3. \end{aligned} \tag{8}$$

In econometrics  $X_3$  is called an "instrumental" variable (Johnston, 1963, p. 165). The equation for  $X_3$  can be transformed into

$$X_3 = B_3^* T_1 + I_3^* + E_3^*, \tag{8a}$$

where

$$B_3^* = B_{T_2 T_1} B_3.$$

$B_3^*$  is identified but  $B_{T_2 T_1}$  and  $B_3$  are not. In the case of dichotomous variables, therefore, the true proportion  $P_{T_1}$  may be estimated as shown in section II by treating  $X_3$  as a congeneric measure of  $T_1$  and  $B_3^* = (1 - \theta_3 - \phi_3)(1 - \theta_{T_1} - \phi_{T_1})$ , where  $\theta_{T_1} = P\{T_2 = 0 | T_1 = 1\}$  and  $\phi_{T_1} = P\{T_2 = 1 | T_1 = 0\}$ . The validity of such an analysis is dependent on the correctness of the independence assumption.

The above analysis can be extended to the case of four items with mutually uncorrelated errors and no correlation between error and true scores, two of each measuring different variables:

$$\begin{aligned} X_1 &= B_1 T_1 + I_1 + E_1 , \\ X_2 &= B_2 T_1 + I_2 + E_2 , \\ X_3 &= B_3 T_2 + I_3 + E_3 , \\ X_4 &= B_4 T_2 + I_3 + E_3 . \end{aligned} \tag{9}$$

Following the above line of reasoning all parameters in this model ( $P_{T_1}$ ,  $P\{T_1 = 1, T_2 = 1\}$ ,  $P_{T_2}$ ,  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ ,  $\theta_4$ ,  $\phi_1$ ,  $\phi_2$ ,  $\phi_3$ , and  $\phi_4$ ) may be identified. There are 15 independent proportions in the cross-tabulation table, so that the minimized  $\chi^2$  would have four degrees of freedom. In principle, a measure of the tenability of certain assumptions is obtained from changes in the  $\chi^2$ . For example, if it were desired to test the hypothesis that  $X_1$  and  $X_2$  were of equal accuracy, increases in the total  $\chi^2$  (with two degrees of freedom), resulting from setting  $\theta_1 = \theta_2$  and  $\phi_1 = \phi_2$ , would be an indicator of the tenability of this hypothesis.

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Footnotes

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<sup>2</sup>The true scores are not independent of the error scores or errors of each other, as is assumed in Anderson's (1959) derivations however, for our purposes the assumption that these variables are uncorrelated yields the same formulas.

Table 1  
Possible Events for Three Congeneric Dichotomous Items

Proportion of People	T	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	E <sub>1</sub>	E <sub>2</sub>	E <sub>3</sub>
.2268	1	1	1	1	.3	.4	.1
.0252	1	1	1	0	.3	.4	-.9
.1512	1	1	0	1	.3	-.6	.1
.0168	1	1	0	0	.3	-.6	-.9
.0972	1	0	1	1	-.7	.4	.1
.0108	1	0	1	0	-.7	.4	-.9
.0648	1	0	0	1	-.7	-.6	.1
.0072	1	0	0	0	-.7	-.6	-.9
.0060	0	1	1	1	.9	.5	.7
.0140	0	1	1	0	.9	.5	-.3
.0060	0	1	0	1	.9	-.5	.7
.0140	0	1	0	0	.9	-.5	-.3
.0540	0	0	1	1	-.1	.5	.7
.1260	0	0	1	0	-.1	.5	-.3
.0540	0	0	0	1	-.1	-.5	.7
.1260	0	0	0	0	-.1	-.5	-.3