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ABSTRACT

The essays in this booklet have been written by persons who had ranked in the top one percent of the 1958-60 Upstate New York Mathematical Association of America Contests. Personal accounts are given of the role of mathematics in the authors' education and career. The careers described include applied mathematics, computer research and programing, biomedical engineering, accounting, music, the ministry, and pure mathematics. (RS)

Mathematics My Career

A COLLECTION OF ESSAYS

Edited by
Nura D. Turner

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MATHEMATICS AND MY CAREER

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A COLLECTION OF ESSAYS

By

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PREFACE

In the fall of 1957 the Mathematical Association of America (MAA) extended to national status the contest given by its Metropolitan New York Section. I was asked to chair the first contest committee of the Upper New York State Section of the MAA.

After working with the administration of the 1958 contest, it occurred to me that it would be worthwhile to make a follow-up study on what happened to students who participated in the contests. Some small grants allotted to me from the contest funds enabled me to begin a piece of work that grew into a study of the academic progress of 117 students who composed two top-ranking groups. The first group consisted of those who ranked in the top 1 percent in the Upstate New York MAA Contest Section in the 1958-60 contests; the second, of those who ranked in the top 0.03 percent nationally in the 1958 contest. Later the study was extended, for some, to the developments in their careers.

As the work progressed, the idea became more and more appealing that the experiences and opinions of these outstanding young people (many of whom continued or are still continuing academic work to the doctoral level) should be shared—that what they had to say would encourage and inspire high school and college students with interest and aptitude in mathematics to continue their mathematical studies. I therefore sought and, fortunately, obtained the cooperation of a number of former contestants in the project of writing essays on careers using mathematics. This booklet is the result of that collaboration.

It has not been possible, of course, to cover the complete range of careers using mathematics. In these personal accounts information is limited to just a few fields: applied mathematics, computer research and programming, biomedical engineering, accounting, music, the ministry, and pure mathematics. Nevertheless it is hoped that this sampling, limited though it is, will whet the appetite of high school and college students for some career that is directly or indirectly connected with mathematics.

The authors have made little attempt to outline what education one needs for a particular position. "Learn more of the mathematics that is being used today" is the sum of their advice to those who want to be prepared to work effectively tomorrow in any of the possible fields using mathematics.

Likewise, the authors have not tried to associate specific ranges of salary with levels of academic training. There are no upper limits on academic training at any level; and salaries, like taxes, are steadily increasing regardless of level, so any current information would soon be outdated.

Again, the authors have made no attempt to write in a uniform style, nor were they encouraged to do so. I have not tried to edit their writing to produce any semblance of uniformity. There is something refreshing about the different approaches as one goes from essay to essay.

In all these essays, however, whether by accident or design, the authors convey to you their liking—their *love*—for their work, and this enthusiasm may provide a new outlook for you.

Each essay is preceded by a brief introduction to give background information about the author. It is with pleasure that, as editor, I express deep gratitude to all the young men who contributed these essays. My heartfelt thanks go to those who have been so helpful in giving the advice I have sought. I thank the State University of New York at Albany for allowing me the time to devote to compiling this work. I am indebted to the Upper New York State Section of the MAA for supplying me with summer grants. And I extend my thanks to Norman G. Gunderson and Elmer E. Haskins for their part in my appointment as contest chairman; for without that appointment this project would not have been envisioned. I am particularly grateful to Frederick J. De La Fleur and Malcolm Smiley, whose critical reading of the manuscript did much to help me prepare this collection of essays.

Nura Dorothea Rains Turner

"If you think you might be interested in applied mathematics as a career, . . . why not try it as a bobby?"

APPLIED MATHEMATICS

JON CHRISTIAN LUKE

Editor's Introduction

Often in the evolution of science, so it has been said, praxis comes first and theory later. Perhaps it is not inappropriate, then, to start this series of essays with one on applied mathematics.

The author is an assistant professor of mathematics at the University of California, San Diego, under a joint appointment between the Department of Mathematics and Institute for Geophysics and Planetary Physics. In addition to teaching freshman calculus courses, he engages in research directed toward mathematical methods in mechanics, nonlinear waves (e.g., water waves), and other applications of mathematics in geophysics.

Professor Luke received two B.S. degrees in 1962 from MIT (Massachusetts Institute of Technology), one in electrical engineering and one in math natics; an M.S. in applied mathematics in 1963 from MIT; and a Ph.D. in applied mathematics from Cal Tech (California Institute of Technology). His doctoral thesis, submitted in 1966, was entitled "Nonlinear Dispersive Wave Problems." He continued at Cal Tech on a postdoctoral fellowship during the academic year 1966/67.

At the end of this essay the author speaks of his own experience, from the time of his early youth in Minneapolis. I shall mention, in anticipation, only one thing. In spite of his having disliked "many things that were actively encouraged," he has come to realize the importance of fields of study other than mathematics and science and has begun to correct for years of neglect. For example, he has become very much interested in the study of violin, which he once considered "little more than forced labor."

To avoid sounding like a college catalog or a thesis, I am writing as informally as possible; I hope that no one will be offended by this approach.

A person's life work really begins when he is born, not after he completes his formal education. Mathematical and scientific interests often develop at a fairly early age; for that reason I am beginning with a section on mathematics and science as a hobby. This section tells what I wish I had undertaken in junior or senior high school!

MATHEMATICS AND SCIENCE AS A HOBBY

If you think that you might be interested in applied mathematics as a career, the best suggestion that I can make is this: "Why not try it as a hobby?" Instead of just spending time doing work in mathematics courses, it is much more important—and enjoyable—to devote time to independent reading and research, and it is certainly not too early to start in junior or senior high school. You can wait longer, of course, but you might be missing much of the fun. Also, it is a sad fact of human nature that you might never get around to starting.

A good place to start a mathematical and scientific hobby is at a public or school library. You might look through some issues of *Scientific American* or try to read something in *What Is Mathematics?* by Richard Courant and Herbert Robbins; in *The World of Mathematics,* edited by James R. Newman; or in *The Feynman Lectures on Physics,* by Richard P. Feynman, Robert B. Leighton, and Matthew Sands. Also you might enter a science fair or a mathematics contest. All of these activities can be very useful if they stimulate your interest, but it is important not to be discouraged or overwhelmed. Mathematics and science are fundamentally cooperative efforts among human beings, and the competitive aspects are less important than the cooperation.

It is good, of course, to explore all of the resources available to you, to ask questions and to read books; but it is also useful to sit back and think for yourself about the problems. That is not to say that you should try to discover the calculus or special relativity from scratch—that would be pretty hard. (Well, it might be worth a try, at that. You probably wouldn't succeed, but who knows what else you might discover?) Of course if you set out to discover things, you are bound to have a lot of failures. If you do discover something that you think is new, it will probably turn out to be a standard, well-known result. Even the professionals have that trouble. Just the same, the concepts that you struggle with are likely to be useful later on. In fact, many people find that books and teachers can only give hints at best. To understand mathematics and science deeply, people need to start from the hints and discover the concepts for themselves.

In the next section I have tried to give a few hints about some fundamental concepts in mathematics. Although details are left out, I hope that the treatment is not superficial. If those paragraphs seem obscure, though, that is very natural; it is unreasonable to expect that anyone can make real sense of such brief explanations unless he understands the material already or else has the insight of a Newton or an Einstein.

A BRIEF TASTE OF APPLIED MATHEMATICS

Most mathematicians and scientists are intensely interested in their work. For a person who did not enjoy the subject, applied mathematics would be a difficult and unrewarding way to earn a living. I hope that the following material will give a glimpse of what applied mathematics is really about and will show how fascinating the subject can be. In any case, it should at least give some idea of the wide range of subjects to which mathematics can be applied, and of its power to unify many different fields.

First a word about calculus, since this is basic to much of applied mathematics. Suppose some numerical quantity is known at each moment of time. From this information one may wish to calculate how fast that quantity is changing. This process of finding the rate of change of a quantity is called differentiation (with respect to time). For example, if the position of an automobile along a road is known at each instant, then by differentiation it is possible to deduce its speed at each instant. A speedometer is a mechanical device that performs this differentiation and gives a reading of the velocity along the road. Just as velocity is the rate of change of position, so, too, acceleration is the rate of change of velocity.

The other fundamental concept of the calculus is integration, which is essentially the opposite of differentiation. Thus if the automobile speedometer reading is known at each instant and the starting point is known, by the process of integration it is possible to deduce the position of the automobile along the road at each instant.

Differential equations

In order to describe the orbit of a satellite, the oscillation of an electrical system, or the motion of a spinning top, the scientist or engineer uses what are called "differential equations." To understand what the applied mathematician does it is useful to understand, first, what it is like to formulate a differential equation and, second, what it means to solve the differential equation. Consider the following example of a differential equation:

$$\frac{d}{dt}\,\frac{d}{dt}\,u = -u.$$

This equation describes the motion of a pendulum, although only in an approximate way. The quantity u represents the position of the pendulum bob, measured horizontally from its rest position. The expression on the left side of the equation gives the acceleration of the pendulum bob. (The symbol

$$\frac{d}{dt}$$

placed on the left of a quantity means the rate of change of that quantity. As explained earlier, the rate of change of the rate of change of the position is the acceleration.) The equation states that the pendulum bob experiences an acceleration back toward the rest position. The farther the pedulum bob is from the rest position, the greater will be that acceleration. For example—if, at some moment, u is equal to 1 (which means that the pendulum bob is at a position one unit to the right), then it follows from the equation that the acceleration at that instant must be one unit negative; that is, there is one unit of acceleration to the left.

Now it is easier to see the distinction between what the differential equation says directly and what its solution says. If we ask "What if the pendulum bob is one unit to the right at some moment?" then the differential equation tells about the behavior of the pendulum bob (its acceleration) at that same moment. However, if we release the pendulum bob from some position and want to know what will happen in the future, we must solve the differential equation. Such information is contained within the differential equation. It is often well hidden, but the solution of this very simple differential equation is known. Suppose that the pendulum bob is released from a point on the right side. The solution says that it will swing to the rest position, overshoot, and go to the same distance on the left. Then it will swing back to the right, and so on. In this simple model there is no friction, so the pendulum keeps swinging indefinitely. A graph of u made for various values of time forms what is known as a *cosine curve*. If this graph of the solution is known, one says in mathematical jargon that "u is a known function of time." That statement means that u depends on time in a known way. A mathematical function of the time is really just a rule so that, for each instant of time, one can find the corresponding value of the quantity u.

The formulation of differential equations is often done by the scientist or engineer rather than the applied mathematician. Whoever does this needs a firm understanding of the forces at work within the system and the way the various parts of the system interact. In complicated problems it may require insight to see what is significant for an accurate mathematical model of the system and what can be neglected. For example, whether one can neglect friction in the study of a pendulum depends on what aspects of the problem are really under investigation.

To formulate differential equations is often difficult; to solve them is generally impossible. At least there is no known way to do it exactly, except in some fairly simple cases. This is where the applied mathematician comes in. Because the same differential equations often arise in physics, astronomy, electrical engineering, and many other fields, it is useful to have people who s[illegible] in dealing with the equations themselves no matter where the equations are to be used.

Here are some possible ways for the applied mathematician to proceed: (1) Find an exact way to solve the differential equation. Sometimes this can be done by cleverness or trickery. (2) Solve a simple but related differential equation. It may require insight to know if the simple equation gives a valid model of the physical problem. (3) Prove that a well-behaved solution exists, even though no way is known to find it. This may be useful if it is not known whether the original equation is well formulated. (4) Use an approximation method to try to obtain sufficiently accurate results or a useful insight into the problem. Some of these methods are numerical and are usually carried out on a digital computer, although many approximation methods are not connected with computers at all. In the case of routine numerical work on a computer, an applied mathematician is seldom involved except possibly in an advisory way. (5) Investigate the accuracy of an approximation method. (6) Develop a new approximation method. (7) Find a new way to do any of the things listed above.

Some of the major uses of differential equations are in electronics, control of automatic devices, guidance of rockets, and calculation of orbits in space.

Partial differential equations

The differential equations described in the previous section are called *ordinary differential equations* (ODEs) to distinguish them from the more complicated equations known as *partial differential equations* (PDEs). A typical problem of the kind that can be described with a PDE is that of the motion of a guitar string. By the use of ODEs the mathematician describes how a single quantity (like the position of a pendulum bob) or how several quantities change with time. The motion of a guitar string is clearly complicated, for at any moment of time it is necessary to be concerned with the behavior of each small bit or increment of the string. To do this with ODEs one might consider the position of each molecule separately, but that would certainly be a very cumbersome problem. It is better to describe the position of the string in a different way.

Suppose the string is vibrating. Consider its position at one instant of time (or imagine a photograph taken at that instant). Call that instant of time t and call the distance that the string is away from its rest position u. Of course the value of u may well be different at different places on the string. For example, the bit of string 10 centimeters from the end might be 0.2 centimeters away from the rest position. Instead of writing down numbers, let us just say that the bit of string x centimeters from the end is u centimeters away from its rest position. One could draw a graph so that for each value of x it would be possible to read off a value of the displace-

"A typical problem of the kind that can be described with a partial differential equation is that of the motion of a guitar string."

ment u. Then one would know how u depends on x at time t. Using mathematical jargon again, one says that "u is a *function of x*." A graph of this function of x looks like a photograph of the string at that instant.

A PDE can be used to tell how such a graph or function changes with time. The PDE often used to describe a guitar string says that if the string is curved at some point, the bit of string there will be accelerated. One might be led to guess this because if a guitar string is grabbed between two fingers and moved to the side so that it becomes bent, there is a force pulling on the fingers (due to the tension in the string); and, if the string is released, this force causes that bit of string to accelerate back.

To determine the bending or curvature at some point on a graph of u, one differentiates the function of x twice. A beautiful analogy now becomes apparent—for, as explained in the previous section, this is exactly the way one would determine acceleration for a function of time. Incidentally, this is the same analogy between space and time that comes into play in Einstein's special theory of relativity.

The distinction between formulation and solution of PDEs is much the same as for ODEs. Although the PDE for a guitar string states directly what happens at any given instant, it is more difficult to find out what happens if the string is released and allowed to vibrate. To find out, one must solve the PDE. For this simple problem it can be done. The solution shows that the guitar string can vibrate at its fundamental frequency and at various higher frequencies. An alternate description is that waves can travel in both directions along the string.

The same PDE that is used to describe a guitar string also occurs in many other problems. For example, this PDE has applications in the theory of sound and in the theory of electromagnetism, which includes the phenomena of light and radio waves.

Probability

Of the other fields of applied mathematics, I'll mention only one, the field of probability. A simple classical example in the theory of probability is that when two fair dice are rolled there is one chance in six that the total number of spots will be seven, but only one chance in thirty-six that the total will be two. The study of probability arose historically from its use in gambling, and even now this application seems to receive the most publicity. Some other important applications of probability and statistics have to do with random motion of gas molecules, messages in noisy communication channels, and interpretation of experimental results.

THE EDUCATION NEEDED FOR A CAREER IN APPLIED MATHEMATICS

Mathematics as a hobby can be useful no matter what field a person finally enters, but suppose now that mathematics grows into a career. Then the various formal educational requirements must be faced. The standard way to become an applied mathematician is to complete high school, to spend about four years as an undergraduate to earn a bachelor of science degree, and then to spend several years in graduate school.

An early decision regarding a major field is not really necessary. Mathematics, physics, or branches of engineering are all very acceptable undergraduate major fields for a person who goes into applied mathematics in graduate school. However, a solid foundation of calculus and physics is essential during the first years as an undergraduate. For the person who wants to be an applied mathematician within the academic community, the degree of Ph.D. is now almost a requirement. This means, incidentally, that it is wise to do some language study as early as possible, since a reading knowledge of two languages in addition to English is a common requirement for this degree.

Many graduate schools do not have a separate applied mathematics department, so a person with interests in this direction may find himself in the mathematics department (or possibly elsewhere).

WHAT APPLIED MATHEMATICIANS DO FOR A LIVING

The bread-and-butter aspects are probably the least interesting part of an applied mathematics career, but it is only fair to say a little about them. Let us suppose for the moment that the authors of papers in applied mathematics journals are representative applied mathematicians and glance through one or two journals to see what topics are of interest to applied mathematicians. Let us see, also, what information the acknowledgments give about the institutions the authors are associated with and about the financial support received for their research.

A quick look at almost any applied mathematics journal shows that the large majority of the authors are associated with some academic institution. In one journal I looked at, these academic institutions were widely scattered across the United States and Canada, with a few European institutions also represented. Within these institutions the mathematics departments were most frequently mentioned, but physics departments, electrical engineering departments, and computing centers were each referred to several times. Other departments mentioned ranged from astronomy, biology, and chemistry to economics and, unexpectedly, philosophy. Many authors acknowledged financial support from government agencies such as the National Science Foundation, the Atomic Energy Commission, the Air Force, or the National Aeronautics and Space Administration. Authors not connected with an academic institution were typically associated either with an industrial corporation or with a government or private research organization like the National Bureau of Standards or the Jet Propulsion Laboratory.

The next two paragraphs are brief sketches of jobs of applied mathematicians in the academic community and in industry, respectively. The first paragraph is specifically about the job of a professor; however, it might be remarked that a sizable number of the authors of applied mathematics papers describe research done while they were still graduate students.

As a university professor, an applied mathematician would normally teach one or two courses and spend the rest of his working time preparing lectures, doing research, and helping students with course work and research. Most academic institutions expect that a professor will publish research work from time to time. Some professors have administrative or extra counseling duties, and some take on consulting work in industry in addition to their regular duties.

An applied mathematician who works in industry typically spends much of his time consulting with and advising engineers and others who have specific problems of a mathematical nature. An ability to work in close cooperation with other people is essential in this work. Because of his firm grounding in mathematics, the applied mathematician is able to enter many fields of science or engineering rather quickly, and he may be called upon to do so if the need arises. He will probably write frequent reports and documents dealing with mathematical problems. Most of these are for circulation within the company, but some may well be published. Farsighted organizations will probably allow him to spend some time on research of his own choice. The educational requirements for work in industry are more flexible than in the academic community, but people with advanced degrees customarily receive higher salaries and, in general, more freedom to work on their own initiative.

ONE CASE HISTORY

The previous sections have dealt in generalities, so it may be useful to tell how one individual happened to become interested in applied mathematics. Unfortunately, I haven't enough information to describe any case history but my own and am forced to write on more personal terms than I would have preferred.

I had the good fortune to grow up in a stable and happy family environment. My father teaches vocational agriculture at the high school level, and my mother also has a teaching background, but my interest in mathematics and science has developed more or less independently—not under the influence of a close relative, a particular teacher, or some acquaintance. Actually, it is probably fortunate that no one in the family circle tried to influence me toward a career in science or mathematics, for I came to dislike many things that were actively encouraged. (Despite the best efforts of my parents I became unsociable and unathletic, and my study of the violin was little more than forced labor.)

In the fourth or fifth grade I developed an interest in electricity, astronomy, and related subjects. My interest in mathematics gradually evolved from this. During the eighth grade I became interested in an algebra book in the library. I was able to read only part of it and gained little factual knowledge, but I feel now that this attempt affected my later career greatly. First, if I had not had a certain amount of experience in advance, I think that I might have become confused and discouraged when algebra was first presented to me in ninth grade. More important, I realized then that for me it was enjoyable and useful to study mathematics by myself rather than to have it taught to me.

My enjoyable experiences in mathematics have been almost entirely outside of the classroom. These include both the study of standard material and a certain amount of carefree speculation and daydreaming. (It is usual to glorify this with the title of research.) When studying standard material I frequently spent many hours to realize what a teacher might have explained in a few minutes; however, the excitement of a few hard-won victories made up for what might seem an incredible waste of time. Electrical engineering remained my major field until I entered graduate school, when I finally decided that my chief interest was in applied mathematics.

SOME REMARKS ABOUT LEARNING

A friend just happened to walk by and glance at the title of this essay. His immediate reaction was "Aren't you about twenty years too young to write about a topic like that?" That is correct, of course, for as I write this essay I am just a beginner in my line of work; but even though my experience is very limited I want to make a few concluding remarks—again of a general nature.

In the past year I finally began to realize how important it is for a person to develop interests and skills in many different areas. Here are a few of the areas where I have been very negligent and in which I am now trying to develop some proficiency: (1) reading and writing, listening and speaking, (2) music, (3) being interested in people and remembering their names and faces, (4) using time efficiently and productively, (5) cultivating a cheerful and enthusiastic outlook instead of cynicism and intellectual snobbishness. A lengthy explanation would probably not be very useful here; people have been trying to explain such things to me for many years, but I didn't listen.

Maybe what I'm seeking is really the knack of learning things quickly and with pleasure. For the first time, that almost seems within my grasp. It seems to me that the key to the development of that knack is simply that one should never be too much in a hurry, or too lazy or proud, to correct simple habits if they are the real stumbling blocks to progress. For example, if my violin bow scrapes across the wrong strings, it would be wise for me to concentrate all of my efforts and correct this in a short period of time, for it is not likely to improve if I just keep on practicing in the same old way. That is obvious, of course, but somehow it is only recently that I realized it for myself.

Finally, I have come to be impressed most of all by the enormous, untapped reserve of thinking capacity a human being possesses. There are those wonderful moments when the circumstances are just right for new understanding to occur or great bursts of learning to take place. Maybe it is foolish to speak of creating such moments, but I am convinced that there is adventure in store when human beings begin to understand thought processes in greater detail. Perhaps someone who reads this essay will have a [illegible] adventure.

THE COMPUTER AND THE PROGRAMMER

"Today there are more than 200 different computers, speaking more than 1,000 dialects in over 300 languages."

FRANK RUBIN

Editor's Introduction

The present era is aptly called the computer age; for computers are now an impressive reality (though the computing machines of today are merely the realization of some portion of the visionary scheme of Leibniz). We are so computerized that there is even some thought of eliminating schools and taking education into the home through television and computer hookups; there would be no age or grade classification and no limit to the speed with which a child could proceed with his education. It is fitting that an essay on the computer and the programmer be included here.

The author's earliest awareness of possessing mathematical skill came when a junior high school mathematics teacher demonstrated the supposed advantage in speed in the use of a slide rule over hand calculation and the young student found that he could consistently beat the slide rule. Later, when the teacher made him a gift of the instrument, he was able to figure out part of the principle on which it works.

The next indication of his aptitude in mathematics came in high school when he received his geometry textbook. He read and absorbed the information on many pages in a single sitting. He then came across the intermediate and advanced algebra textbooks his mother had used in high school and found he could easily absorb that subject matter also. Attending college (Massachusetts Institute of Technology), he repeated this kind of feat by learning probability theory, vector calculus, and ordinary and partial differential equations during a single Christmas vacation. For all the subjects he studied in this way, he took the regular final examination and received credit for the course as a prerequisite for other courses. In this way he was able to take graduate-level courses in mathematics beginning in his sophomore year.

As an undergraduate he was twice

a member of the MIT team in the Putnam competitions, taking fourth place in 1961 and second in 1962. Four times he placed in the top forty individual scores in the Putnam competitions in the U.S. and Canada.

He received a B.S. in mathematics from MIT in 1962 and an M.A. in mathematics from Brandeis University in 1964. He is well on his way to a Ph.D. in systems and information science, having spent the year 1970/71 in residence at Syracuse University in the IBM Resident Study Program, on leave from his regular employment as a staff programmer with IBM at Poughkeepsie, New York.

Mr. Rubin's work is in design automation, a field in which present computers are used in almost every phase of designing, building, simulating, and evaluating future computers. When he accepted the position, he looked upon it as a rather awesome project, but he did so with the awareness of possibilities for its signaling notable advances in the technologies of many design fields.

In his work he is engaged in the same type of problem solving he began enjoying as early as the 1958 MAA contest, in which he ranked, on a national basis, eighth among eighty thousand contestants. The general problem-solving techniques he developed throughout his formal training have proved invaluable in computer problems. He has found remarkable similarities between such problems and formal mathematics. For example, he finds "stable construction," which involves the abstraction of large volumes of data in various ways to suit preassigned ends, similar to the construction problems of plane and solid geometry. Again, he finds the abstraction of information in compact form similar to the method of breaking down a long mathematical proof into a series of steps called lemmas.

I studied mathematics in high school, liked the subject, and liked it so well that I continued the studying in college as both an undergraduate and a graduate student to the master's degree. Then I began wondering what I was going to do with the mathematics I had learned. I found I had the basis to go into computing, a field which had aroused my curiosity. I was offered a good position with IBM and, with nothing to lose and perhaps a great deal to gain, accepted it under the condition that I could continue working in mathematics while working with computers. I find my work tremendously challenging because computer programming *is* mathematics. I am convinced that anyone who likes mathematics would find this field as challenging as I have found it.

Now I should like to tell you about computers, what a programmer does, the kinds of computer programmers there are, and the mathematical preparation required for each type of programming.

WHAT IS A COMPUTER?

Before we can discuss careers in the field of computers, we should try to answer one basic question: "What is a computer?"

Most people have one of two ideas about computers. On the one hand, many people have heard that computers are virtually miraculous machines that can forecast elections, track space vehicles, design bridges, play near-championship checkers, spot income tax cheaters, and even design new computers. On the other hand, many people have heard that computers are nothing more than machines that do arithmetic very quickly. Both ideas are correct!

How can a mere "arithmetic machine" do such amazing things? The answer is programming. A program is a list of detailed instructions to the computer telling it how to solve a particular problem. This list is made up of the most elementary operations: add two numbers; subtract one number from another; compare two numbers; turn a switch on or off; go to another part of the program as a result of a size comparison between two numbers.

The job of the programmer is to express a problem to the computer. However, not every problem can be expressed in the same way. For example, designing a bridge involves the solving of large numbers of mathematical equations that represent the forces in each of the parts of the bridge, but playing checkers requires methods for representing the positions of the pieces on the board and rules for moving the pieces and evaluating board positions.

This question of how to express problems to the computer was one that faced the earliest users of computers. They devised a number of different languages for computers. Some languages closely resemble mathematics, others closely resemble business terminology, a few even resemble English somewhat. Today there are more than 100 different computers, speaking more than 500 dialects in over 200 languages. Most experienced programmers understand several different computer languages in detail.

WHAT DOES A PROGRAMMER DO?

Now that we have some idea of what a computer is, let's try to find out what a programmer does by imagining that a major problem has been defined and we want to solve it on a computer.

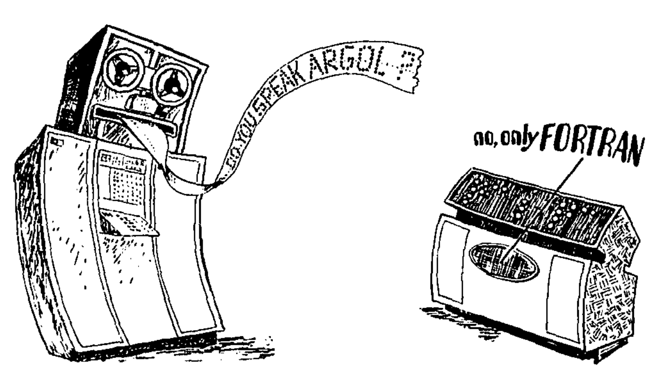
The first step is to decide on a general approach: if the problem is mathematical, we must try to estimate the number and complexity of the equations involved; if the problem is commercial, we must try to estimate how much data is involved. We will need an idea of the type of information we will be working with, how we will represent the information, and what types of results are expected.

The next step is to decide on the type of computer required for such a solution. We must consider (1) what special equipment is needed, (2) whether the problem can be solved better by some means other than computers, and (3) what language or combination of languages is suitable for expressing the solution.

Once all of these questions have been answered, a specific solution must be found. The exact equations must be formulated. The mathematical and statistical methods must be worked out. The forms in which data will be represented must be determined. Any tables or other forms for internally handling and storing data must be specified. It is also necessary to decide what kind of answer is wanted: a table of numbers, a map or drawing, a filmstrip showing stages of the computer output in successive frames, verbal answers.

Before we can convert a mathematical solution into computer instructions, we must investigate the limits of the system. If our problem involves 1,000,000 numbers, but the computer can store only 10,000 numbers, we must find a way to reduce the amount of data or to work with just a fraction of the total at one time. If our solution involves 500 hours of computation but our budget allows only 10 hours, we must develop shortcut methods. It is at this stage, usually, that we get the opportunity to be most creative. This is the point at which we first develop procedures unique to our problem.

Finally we put into practice the method for solving the problem. That is,



"Most experienced programmers understand several different computer languages in detail."

we are ready to write computer instructions. If the problem is large, we will want to divide the program into parts. Each part, called a subroutine, will do a single job such as multiply matrices, convert shillings and pence to a decimal fraction, or compute the possible moves of a bishop on a chessboard.

Generally it is best to draw a diagram of the various components of the program. This diagram is called a flow chart, and it serves to establish the major control functions in the program. This will give us an overall view of the entire system. It is best to begin our coding (the writing of computer instructions) with the simplest subroutines first. This gives us the opportunity to test the subroutines individually. Later, when we are trying to test larger problems, we can do the testing with more certainty that the individual parts work.

Once we begin to write instructions, the biggest problem is testing. Test programs must be written to test the individual subroutines. Both the programs being tested and the testing programs may have errors. Detecting, analyzing, and removing such errors constitute 90 percent of a programmer's job. Errors may continue to crop up as much as five years after the program has been in general use. The larger the program is, the more errors there will be and the longer they will persist. It is for this reason that so much emphasis must be put on advance planning. By correctly designing the whole system, errors in later stages may be restricted to small sections of the program.

When the various positions of the program are written and tested, the sections of the program are combined, and the whole system is operating, a programmer's task is still not completed. There are manuals to be written about how the program works and other manuals to be written about how to use the program. Personnel must be taught to use the program. Management must be advised of the long-range requirements for maintaining and improving the program.

A working program is not always enough. It is frequently necessary to make the program extremely efficient so that the long-term cost of using it is reduced. This is done by using the facilities of the machine that have not been used in the basic solution. For instance, suppose our computer can store 10,000 numbers, but we have used only 8,000 numbers of this capacity; by processing more numbers at a time it may be possible to speed up the total operation.

WHAT KINDS OF PROGRAMMERS ARE THERE?

We have now seen what a programmer does, from the start of a long project to its end. Let us look at some different types of programmers, their problems, and their background.

Commercial

Commercial programming generally deals with the accounting and inventory problems of business such as the following: (1) billing customers, (2) accounting for receipts, (3) calculating payrolls, (4) computing inventories, (5) keeping track of bank accounts, insurance policies, automobile registrations, and stock market prices, and (6) figuring odds at racetracks.

The basic problem in commercial programming is the amount of data involved. Storing and processing very large amounts of data in an efficient manner requires an intimate knowledge of the computer's capability to acquire, store, and retrieve data.

Some examples may show just how much data may be involved in a commercial problem. One company may have 1,000 employees on the payroll; another company may manufacture 10,000 different parts; one bank may have 100,000 savings accounts; a state may have 1,000,000 registered autos; some insurance company may issue 10,000,000 policies. As a final example, the Internal Revenue Service may process 50,000,000 income tax returns, comparing this with a list of 200,000,000 Social Security numbers and with the 20,000,000 income statements filed by employers.

Commercial programming in smaller systems generally requires the least mathematics. High school algebra plus business arithmetic may get one by. However, detailed accounting or actuarial programming will require college-level work in these fields. Work on larger systems, like those in government applications, requires more mathematics and advanced college degrees.

Scientific

Scientific programming deals generally with the design of experiments, the prediction of results, and the testing of theories against these results. Some problems may be: predicting satellite orbits, determining scattering of electrons in an electromagnetic field, estimating the size, speed, and distance of quasars or other astronomical objects, analyzing windflow patterns around a model airplane, predicting hurricanes from satellite photographs, and simulating the effects of various airplane controls in a model cockpit to train pilots.

Basic to most of these problems is the need to formulate physical data in terms of equations and then to solve these equations. Special techniques must be developed to solve large numbers of equations involving many variables.

This type of programming requires a very solid mathematics background. In high school mathematics, plane and solid geometry and trigonometry should be studied. At the college level, calculus, differential equations, partial differential equations, vector and tensor calculus, and numerical methods, particularly, are needed.

A college degree in mathematics or science is essential, and advanced degrees are preferred.

Management

The use of computers to advise management is a new and growing computer field. In order to make decisions about their businesses, management needs accurate statistics abou* current operations, forecasts of future trends, and estimations of the effects of various proposed courses of action such as mergers, investments in new plants or equipment, and relocation of distribution centers for their products. Management may also want to design new plants for the best work flow, decide when to scrap obsolete machines, or locate an executive to fill a key position.

One basic technique for management problems is to build a model of the process being studied. Such a model may give the value in dollars of operating a particular machine. It may take into account the cost of breakdowns and overtime. It may consider the amount of floor space some machines may use and how this affects people trying to work in the area. It might consider the number of defective articles manufactured by the machine, the cost of inspecting for defective articles, and even the bookkeeping involved in replacing defective merchandise for customers.

Such complex models for computer use are inevitably mathematical in nature. A programmer in this field should have a solid grounding in mathematics through calculus, probability theory, and statistics. In addition, he should study some of the special mathematical fields that have grown up in the last twenty years in this area: modeling theory, queuing theory, operations research, game theory, and decision theory. Besides mathematics, study in the area of industrial management, now available in many universities, is also valuable.

Engineering

Computer programming plays a very great role in all branches of engineering and architecture. Problems include the designing of bridges, the planning of roadways, the logical testing of computer circuitry, the determining of optimum proportions of ingredients in concrete, the simulating of effects of airplane-wing designs, and the planning of plumbing, heating, air conditioning, and electrical layouts of large buildings.

Another form of engineering programming is called *process control.* This is a "real time" problem because the computer monitors a process as it is actually happening. Processes may be the baking of bread, the smelting of iron, or the printing of newspapers. The computer tests the process and sets controls to adjust temperature, proportion of flour, or the drying time for the ink. It may also be set to warn humans when something that it cannot handle has gone wrong.

Computer jobs in the fields of engineering and architecture generally require a degree in mathematics or engineering. Mathematics, at least through differential equations, is required; and some knowledge of statistics and modeling theories may be desirable.

Systems programming

Earlier I mentioned that many different languages may be used on a single computer. Before the computer can "understand" these languages, they must be translated into the computer's simple instructions. This is done by having a special program that does the translation. This type of translation program is called a "compiler" because it compiles a list of computer instructions. Compilers constitute one type of systems program.

Other systems programs are needed to get a programmer's job into the computer, to stop it if something goes wrong, to get the job off the computer, to help print the results, to allow several programs to run at the same time without interfering with each other, to compute common mathematical functions like square roots or logarithms, to aid in the storage and retrieval of data, to do sorting, and to help diagnose errors when a mistake is made.

Since these programs are very frequently used, they must be extremely efficient. The systems programmer must be intimately familiar with the computer and all the devices that are used with it. Great abstraction must be done in many of the programs. For example, to analyze the grammar of a computer language, the langauge may be abstracted in complex tables that show the relations between language elements. "Language elements" are similar to the parts of speech: nouns, verbs, adjectives, and so forth, that are used in human languages.

Generally, a strong formal mathematical background is not needed for this type of programming. Some college-level mathematics subjects, such as symbolic logic or heuristics, are highly desirable. For a programmer who wishes to write mathematical subroutines, numerical analysis is required. To write the compiler for a computer language involving equations and scientific computations, the programmer needs a general knowledge of differential equations and linear algebra in order to understand how the language is used in these fields. Thus in systems programming there is a wide range of mathematical background that is essential, but all fields are those taught at the college level.

Consulting

Many firms and individuals today engage in the selling of programming services to business and industry. Programmers in this area work on a wide variety of problems. They may work in one, several, or all of the programming fields just described. This type of work offers great opportunities to travel and to meet people in all levels of business. Thus many programmers find this the most desirable form of work. To prepare for this type of job, a broad understanding of all the mathematical subjects mentioned above is needed.

A FINAL WORD

I hope the field of programming sounds exciting, and that many readers will enter it. If the reader does not plan to enter programming specifically, he should remember that computers are beginning to touch almost every area of human endeavor. Whether one goes into engineering, science, or business, a knowledge of computers and programming is likely to be a significant help in knowing how computers can aid in one's chosen field. Computers speed decision-making and cut overhead in so many lines of work that one who does not know how computers can help will be handicapped. His competitor will know, and will win the race.

A good foundation in mathematics is a necessary basis for understanding what computers are all about, what they can do, and what they cannot be expected to do.

"Probably the most important research tool of the biomedical engineer is mathematics."

BIOMEDICAL ENGINEERING

THOMAS G. COLEMAN

Editor's Introduction

"There is no new thing under the sun." This familiar saying from Ecclesiastes comes to mind as I think of biomedical engineering; for it is a union, or a cross-fertilization, of engineering and medicine, both of which are sciences of great age.

The author of our essay about the uses of mathematics in this field is an associate professor of physiology and biophysics in the School of Medicine at the University of Mississippi Medical Center. His work—research with some teaching—deals with the mathematical analysis of physiological systems, with application to (*a*) acquiring a better fundamental understanding of control of the human body and (*b*) designing new animal experiments aimed at obtaining new biological information. While this seems rather specialized, it is typical of the work being done by others in the field.

Professor Coleman received a B.S. in electrical engineering from the University of Rochester in 1962, an M.S. in electrical engineering from Mississippi State University in 1964, and a Ph.D. in biomedical engineering from the University of Mississippi in 1967.

He claims that he was no childhood math whiz; but early in his life he became fascinated with problems and puzzles. With great enthusiasm, though at times with little success, as early as the fourth or fifth grade he was attempting to solve some of the simpler games and puzzles found in a number of books. In this way he learned something about algebra. With one development leading to another, he entered high school with a profound interest in things mathematical. That interest resulted in a desire as a freshman college student to study electrical engineering and to plan for a career in research in a "growing field." While he has deviated from a strict application of electrical engineering, he has clung to his plans for research, which is part of his work at the medical center.

As a high school senior, Professor Coleman had only a nebulous idea of what a mathematics major could do. He feels that many high school seniors find themselves in the same position and hopes that his comments on biomedical engineering may be of help to some.

The first question that ought to be answered is "What is biomedical engineering?" Biomedical engineering is a relatively new field that is dedicated, as medicine is, to improving the care of the ill. The knowledge and techniques of engineering are being used to aid the medical profession. This uniting of engineering and medicine offers exciting careers for persons who have a suitable education in both areas.

Biomedical engineers work either as independent researchers or as integral parts of research teams. What, then, do they do? They work in such areas as development of artificial organs, patient monitoring, human control systems, and instrumentation, as well as in many other areas.

The artificial kidney is a typical example of an artificial organ. The artificial kidney control unit (see fig. 1) is a complex machine that allows people with little or no kidney function to live comfortably. Periodically these people are connected to an artificial kidney, and their blood is cleaned in a way that is similar to the way that it is cleaned by a real kidney. An artificial kidney must have temperatures, flow, and chemical concentrations controlled precisely to ensure maximum efficiency of operation and maxi-

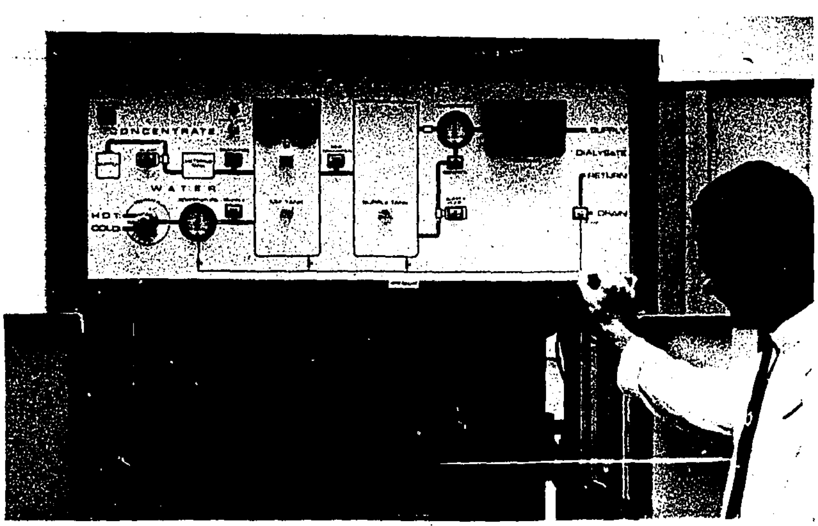


Fig. 1. Artificial kidney control unit

mum safety of the patient. The design and construction of artificial kidneys has challenged both the medical and engineering professions. The success of artificial kidneys has been immense, yet there is still much to be done.

Better care of the ill can also be achieved fundamentally and generally, although not in individual cases, by basic research. Basic research leads to a better understanding of the human body and consequently a better understanding of illness and how to cure it. An example of the role that biomedical engineering plays in basic research is found in the study of human control systems (or, simply, human systems) such as the respiratory system. The type of control that causes the respiratory system to deliver not too much and not too little oxygen to each of the body's 100 trillion cells to allow them to function properly is not altogether different from the type of control that guides a rocket into orbit. A study of this kind of system often involves the use of high-powered analog or digital computers. Figure 2 shows part of an analog computer center that is, in turn, part of a medical research complex. A mathematical description of the system under study is made, and the resulting equations can then be studied on the computer. The computer results, in turn, lead to a better understanding of the system.

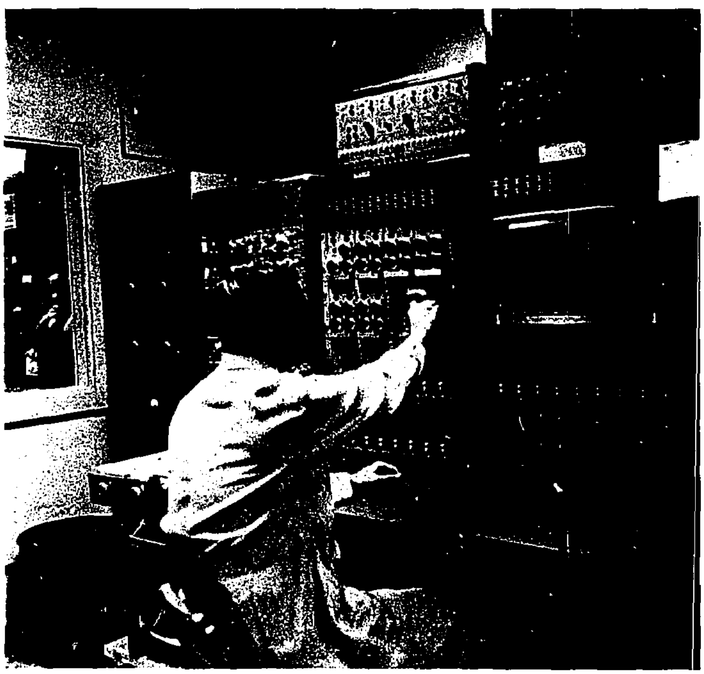


Fig. 2. Part of an analog computer center

Another example of the role of biomedical engineering in basic research is to be noted in figure 3, which shows equipment used for detecting and

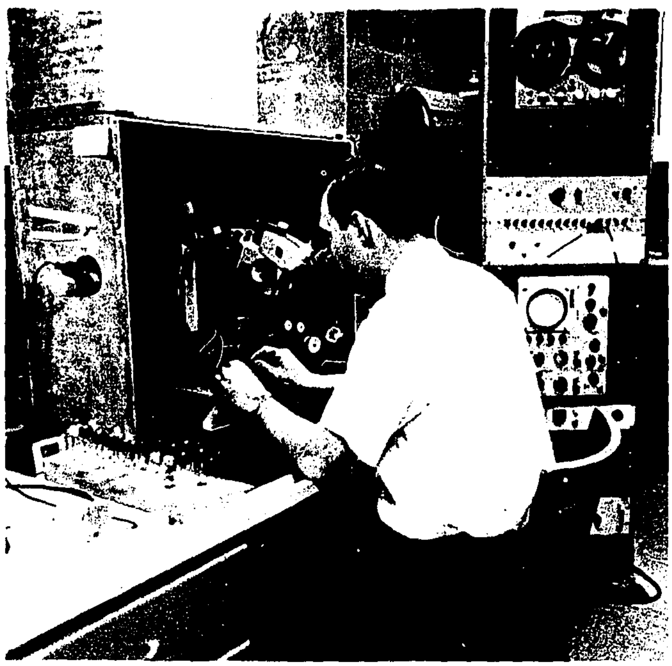


Fig. 3. Equipment used for detecting and recording the electrical activity of the eye

recording the electrical activity of the eye in an experimental laboratory that is equipped to study the eye and the electrical signals it sends to the brain. The researcher must be able to design and build the many special instruments associated with detecting and recording the small electrical signals generated by the eye in response to various images and lights. He must also be able to decode these signals to determine exactly what information is there—much as coded "enemy" messages are decoded by government agencies. In this case, the electrical signals are stored on magnetic tape so that they may be studied a number of times if initial decoding attempts fail.

One particular area of medical research involves both mathematical and experimental analysis of the circulatory system in normal and diseased states. All available data concerning the system in question are collected. These data are then converted into a great number of equations that are solved using a digital computer. These equations must be changed into a form that the computer can understand before an answer can be obtained. Figure 4 shows part of a listing of slightly less than a hundred equations being fed into a digital computer. The form is somewhat different from the basic algebraic equations, but all of the meaning is there. For instance, in this case the first column gives the number (rather than a name) of a variable or unknown, and the second column describes the type of equation that this variable is part of. The computer has been programmed in advance to correctly interpret this type of coding. It should be added that studies of this sort were impossible before the advent of modern high-speed computers, since it is virtually impossible to solve such a large mathematical problem by hand. The equations are solved by the computer, and some predictions are made.

```
26  ADDER      13   20
27  MULT       12   17
28  ADDER      27  -42
29  INT        28   39   38.3
30  GAIN       29          .2
31  ADDER     -30   78
32  MULT       12   23
33  INT        32   35   49.15
34  GAIN       33          .2
35  ADDER     -34   84
36  DIVIDE     30   78
37  OFFSET     36            -.04
38  GAIN       37            -125.
39  GAIN       31             .47
40  ADDER     -38  -39
```

Fig. 4. Mathematical equations coded for computer analysis

Equivalents to some of the algebraic equations in figure 4 can be written as follows, starting at 33:

$$x_{33} = \int_0^t (x_{32} + x_{35})\,dt + 49.15.$$
$$x_{34} = .2 \cdot x_{33}.$$
$$x_{35} = x_{84} - x_{34}.$$
$$x_{36} = x_{30} / x_{78}.$$
$$x_{37} = x_{36} - .04.$$

These particular equations describe the dynamics of the kidney, and a knowledge of the physiology and anatomy of the kidney is necessary for complete understanding. In general the variables represent fluid flows and volumes and electrolyte concentrations. The equations have indicated that certain changes will take place in some of the variables after the beginning of kidney disease. (It should be noted that the original graph seen in fig. 5 is computer-drawn but that the heavy lines were added afterward by the author for emphasis.) The predictions are then compared with what is already known, and any differences are investigated fully.

In figure 5 the lowest line refers to the urine output of the kidney. There are two pairs of numbers at the bottom. The first pair gives the value of

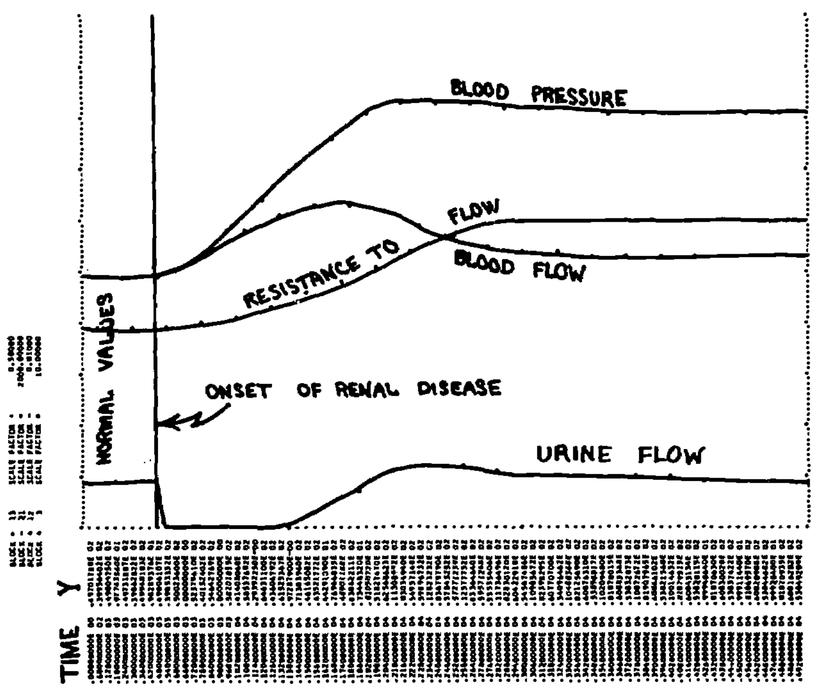


Fig. 5. Computer predictions resulting from a mathematical study

time (the abscissa) and the second pair gives the value of Y (the ordinate). Each pair is read as $x_1 \cdot 10^{x_2}$. The second number in the time column, then, is $.6 \cdot 10^2$ or 60. The predictions of this particular study were for humans. The verification is being done on dogs, and the data extrapolated to humans using standard techniques. This type of mathematical research allows large, complex systems, such as the circulatory system, to be studied in a very sophisticated manner. Such research improves our understanding of how the system works normally and, more importantly, how it works when a person is ill.

These examples have been offered to give you some idea of the many possibilities in the field of biomedical engineering; it would be impossible to make a complete listing. The next generation of biomedical engineers will have entirely different problems to solve—problems both harder to solve and more important to humanity. At the same time they will be better equipped to solve these problems, because they will have use of better research tools and a better education.

Biomedical engineers need a broad background in both medicine and engineering. A number of universities have formal graduate programs in biomedical engineering and offer both master's and doctor of philosophy degrees in this area. Most universities will accept students into their biomedical engineering programs from a number of college majors, but a major in either engineering or mathematics is generally preferred. Subjects studied in these biomedical engineering programs are varied, and the student generally has some choice in picking the curriculum that best fits his own interests. Some typical courses might include the theory of control systems, analog and digital computer programming, physiology, biochemistry, and advanced mathematics. As in most other fields, each university has a certain area or areas in which it is especially good. When selecting a university, a student should try to match one of these areas of excellence with his own interests.

Probably the most important research tool of the biomedical engineer is mathematics.

ACCOUNTING

"The information explosion in business today requires imaginative, analytical minds to systematize, report, and review the multitude of data. . . . Accounting is one profession where there is action today."

CARL R. ERNST

Editor's Introduction

At one time little more than a knowledge of arithmetic was necessary as preparation for accounting. That no longer is the case. Accounting has lost its provincialism and become sophisticated in its demand for mathematical preparation.

The author of this essay has led a varied career, with sometimes intense and sometimes indifferent interest in academics—although as long as he can recall, he has enjoyed mathematics.

It took him longer to find an occupation to his liking than was the case with our other essayists. At the seventh-grade level he wanted to be an engineer. By the tenth grade, he wanted to become a teacher of mathematics. He attended Albany Academy, the oldest boys' school in the United States, on a scholarship; and, as valedictorian of his class, he was admitted to Princeton University. He left there after two and a half years. He did not flunk out; he did rather well; but his *desire* to do well deteriorated to the point where he left the university dissatisfied alike with his own attitude and with the system of education he was experiencing. (Nevertheless, after obtaining a bank teller's position, he continued his formal education by attending night school.) Six months later he was a life insurance agent, and after still another six months he was lucky enough to be hired as a junior accountant by a firm of certified public accountants. One of the stipulations of employment with the firm was that he obtain a bachelor's degree in accounting, so he went ahead with night school sessions and obtained the degree from Russell Sage College in 1967. (Incidentally, he says that this is doing things the hard way: "Don't work full time during the day until a degree is obtained. There is nothing more tiresome than having to sit through 100 minutes of lecture after having worked all day.") Then he began studying for the C.P.A. examination in New York State. He passed all four parts of this examination on the first try, something rarely done.

From the accounting firm Mr. Ernst went to IBM as a computer systems analyst, assisting customers in designing and implementing accounting and information systems using IBM/360 computers.

After three and a half years with IBM, he combined his accounting and computer experience as controller of First Albany Corporation, a regional brokerage firm that is a member of the New York Stock Exchange. Presently he is treasurer and in charge of operations of the firm. During eight years he has come a long way from being teller in a bank!

Mr. Ernst is aware of a general awakening to the potentially practical uses of mathematics in accounting. Yet he feels that the profession, both public and private, has not yet fully understood the applicability of mathematical approaches using statistics and mathematical models. In his own experience he has noted the awe he has occasionally inspired among fellow professionals by the simple procedure of solving a practical accounting problem merely by the use of two simultaneous equations.

Who says accounting is dull? I don't.

Accounting today has evolved from the basic need to *keep track of transactions* flowing through a business. The word "business" in this paper includes profit, nonprofit, and governmental units, since each has a need for keeping track of transactions. Accountants neither make nor market products. They accumulate and report the results of making and marketing; they review these results and produce more reports; they review the way the accumulating and reporting is performed. Whatever they are doing, the essential fact is that they are keeping track of actual operations.

This brief general introduction may have impressed you adversely. You may even be considering tagging accounting as the gentle art of shuffling paper and figures. Good! Now the worst is over.

Accounting provides such a great variety of careers that it is impossible to convey the exciting possibilities to you by any general description. Would you consider a career described only as requiring two years of preparation to sit for many hours in an isolated room with no solid food? On the other hand, would you consider a career as an astronaut? They are the same career, you know.

WHAT AN ACCOUNTANT DOES

Let us look more closely at what keeping track of business transactions really means today.

Business structures have become fantastically complex. Income tax laws have, also. Shrinking profit margins require closer control of costs and better forecasting of future business requirements. Rapid business growth requires constant scrutiny of the way transactions are handled. Governmental requirements resulting from broad public ownership of corporations have created an expanding need for new ways of verifying and reporting results of business operations. The computer has exploded our ability to collect and process data.

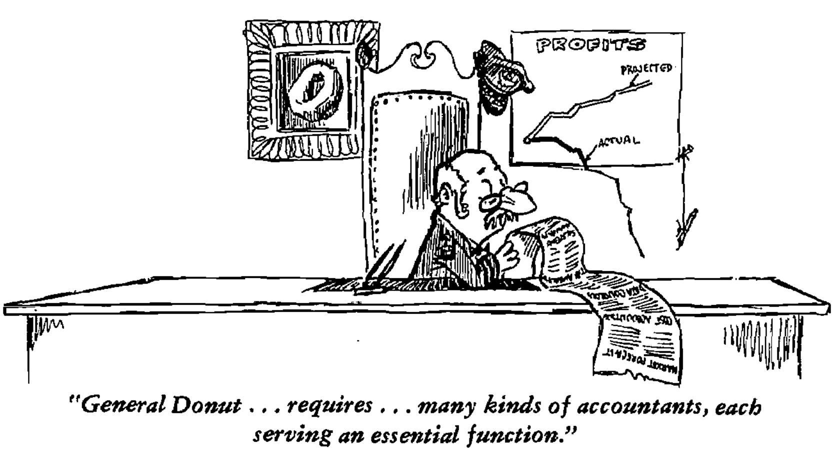
The accountant has had a central role in this business revolution.

Once, long ago and far away, there was a business that bought two cents worth of dough, hired a cook for four cents, made one doughnut, and sold it for nine cents. The bookkeeper entered the transaction in ledgers. The accountant reviewed the entries and produced this report:

Sales	$ .09
Costs	.06
Profit	$ .03

The owner was delighted, and that was the extent of the accountant's work.

At length, General Donut went into business to make one doughnut. The project manager asked the controller to have a market forecast report made, based on historical trends for like products, to determine if the doughnut could be sold. The controller's staff provided the data for a computer simulation of the potential market. By consultation that involved the cost accounting, purchasing, and engineering departments, standard costs and standard times were developed for producing the doughnut. Budgeting information was transmitted to the budget accounting department, which



"General Donut . . . requires . . . many kinds of accountants, each serving an essential function."

incorporated the forecasted sales and costs in the corporate budget. One ounce of dough was purchased for two and one-half cents. The data control department increased the inventory records and posted the cost. Since the budgeted price for dough was only two cents, cost accounting reported the variance of one-half cent to purchasing and budgeting. Subsequently a conference of high-level purchasing personnel determined from this variance report that price estimation should be improved.

A cook made the doughnut in twenty-five seconds, and this cost eight cents in wages. The payroll accounting department deducted a total of seven cents in payroll taxes and issued the cook a check for one cent. Since the cooking operation had a standard time of ten seconds, cost accounting reported a large time variance. Upon review, the cook's manager found that the cook had reported time actually spent smoking a seven-second cigarette, and records were adjusted accordingly.

The doughnut was finally sold. The very next day, the company president inquired into the computer system, from a terminal by his desk, to determine how the doughnut venture was progressing. The information was stored in the computer files according to a system developed by the accounting systems and programming departments. The president received the following report.

	Projected	*Actual*	*Variance*
Sales	$ .090	$ .090	$ 0.000
Costs			
Material	.020	.025	+.005
Labor	.040	.058	+.018
Profit	$ .030	$+.007	$—.023

(Even with the advanced analysis techniques of today, mistakes are possible.)

The president discussed the potential effect of the loss taken in the doughnut venture with the tax analyst on the controller's staff.

At the end of the year, an independent accounting firm verified the validity of the doughnut transactions as part of their tests of internal accounting control. They were satisfied with these and other tests, and issued a favorable opinion on General Donut's financial statements.

General Donut, trivial as its business might be, requires, as you have witnessed, many kinds of accountants, each serving an essential function. They not only keep track of transactions, but also influence the outcome of operations through the data and reports they generate. They provide information to every level of management from the president to the operating managers. They interpret raw data about transactions and present them in meaningful form for decision making. They not only analyze historical data, but also forecast what is to happen.

TYPES OF ACCOUNTING SPECIALTIES

Some of the various kinds of accounting specialties are briefly explained below.

Public accounting

The basic function of the certified public accountant is to certify that financial statements present fairly the financial condition and the results of the operations of a business. This function is called auditing. The C.P.A. forms his opinion by tests of accounting and source records sufficient to satisfy himself that control documents are adequate; he verifies inventory counts and valuations and scrutinizes cash, receivables, liabilities, as reported by management.

Public accountants often also provide tax consultation services and general accounting services. Lately, they have become active in management services. Management services include accounting and computer systems analysis, and costs of financing studies.

In New York State, the requirements to become a C.P.A. include a degree from an acceptable college, three years of diversified experience with a C.P.A. firm, and grades of 75 percent or more in each section of a four-part examination given twice a year. Each state sets its own standards, which may vary considerably. New York State's requirements are among the most rigid.

Internal auditing

Large corporations usually have their own staffs of auditors who use the same auditing techniques as public accountants. However, the internal auditor's tests are continuous and extensive to enforce management's transactions and to verify the correctness of recording.

Cost accounting

The cost accountant specializes in that part of accounting which records and analyzes the costs of production and other expenses of a business. This function is usually separated from other accounting functions in companies that have developed a system for allocating costs to units of production.

Budgetary accounting

Responsibilities include assisting in the preparation of income forecasts, expense budgets, and capital expenditure budgets. The budget group will also analyze variances between actual results and budgets and adjust the budgets accordingly. This function has become increasingly important. The better a business forecasts its future needs, the better chance it has of meeting its profit objectives.

Income tax specialization

Because income tax reporting requires essentially the same data used in accounting, responsibility for income tax planning and reporting is usually considered an accounting function. The income tax expert is conversant with the complexities of the corporate income tax law and may also have knowledge of other corporate taxes.

Controllership

The controller is generally responsible for the proper functioning of a company's accounting system. The budgetary, systems, and income tax accountants may report to him. He is directly responsible for the preparation of general financial statements.

Systems analyst

The systems analyst, rather than recording or interpreting accounting data, examines the way these data are handled. He studies the flow of transactions—for example, how documents like bills or purchase orders are created and processed or how accounting information is utilized by operating and planning departments. Having made such a study, the systems analyst uses his knowledge of system design, and his imagination, to suggest improvements in both work flow and reporting methods.

Beyond eliminating "red tape" procedures, the systems analyst also designs advanced systems. The computer has created almost limitless possibilities here. Information needed by all levels of business management can be stored in one central computer system and can be updated minute by minute. The design of such integrated systems requires highly knowledgeable, imaginative systems analysts.

TOOLS THE ACCOUNTANT USES

Traditionally, the basic tools of the accountant have been pencil, paper, knowledge of accounting theory and practice, and common sense. Today these tools may be supplemented by sophisticated techniques, many of which are the result of the computer's ability to process data at fantastic speeds. Some of these techniques are briefly explained below.

Flowcharting and decision tables

Flowcharting represents transaction, document, or procedural flows in symbolic form. Decision tables represent complex logic decisions in table form, sometimes supplementing flow charts. These techniques are useful to the auditor in examining internal accounting control of documents and to the systems analyst in analyzing the current system and designing a new system.

Statistical analysis

The ability to draw valid inferences from a small, selective test of voluminous data is a powerful tool in accounting. The C.P.A. when auditing, for example, cannot verify the validity of the transactions of an entire year by scrutinizing each transaction. He must often confine his tests to a small percentage of actual transactions. Properly used, statistical techniques may help give him the soundest basis for his opinion on overall financial statements.

Simulation

Simulation, a computer technique, is a recent addition to the store of accounting tools. It is the creation of a statistical model representing actuality. Once the model is validated, that is, reflects reality, various parts may be altered to determine the result of potential changes in the rules upon which the model is based. For example, in budgeting, each income and expense item may be represented in three figures: most likely, most pessimistic, and most optimistic. A simulation model may be created to determine the probable range of net income based upon these variables.

The information explosion in business today requires imaginative, analytical minds to systematize, report, and review the multitude of data for operating managers, for planning managers, for government, and for the public. The various accounting specialties mentioned are very much a part of this explosion, not only participating in it, but also guiding business in realizing its benefits.

An accountant may choose to develop his competence in one particular accounting specialty, or he may choose to gain knowledge in many areas. Accounting is one profession where there is action today.

"*Mathematics is the inspired ordering of an infinite world of numbers. . . . Music is the inspired ordering of an infinite world of sounds.*"

MUSIC AND MATHEMATICS

HOWARD P. LYON

Editor's Introduction

Music and mathematics. Do they have anything in common? To determine the answer to this question, all one needs to do is to read the biography of a musician. The chances are overwhelming that there will be at least one reference by the musician to an interest in mathematics. This essay tells you some things about the relationship.

The author came rightly by his combined interest in mathematics and music. His maternal grandfather was a coauthor of two textbooks on high school algebra which were published in the 1920s. His mother directs the choir and is the organist of the family church. His father, also, plays the organ.

Mr. Lyon entered Michigan State University intending to prepare for a career as a physicist, but he changed his plans and received a B.S. in music in 1962.

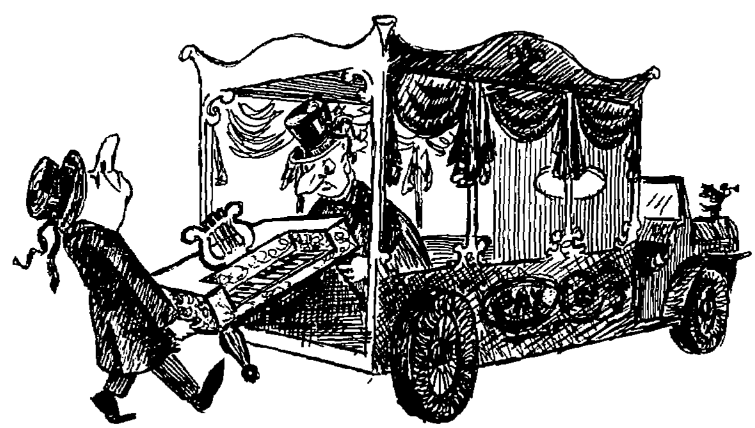
Versatility is a characteristic of this young man. With a background in mathematics, physics, and music, he earns the major part of his living employed in the console department of Organ Supply Company, Erie, Pennsylvania. He supplements his income by playing violin in the Erie Philharmonic.

Always interested in both music and mathematics, as a youngster he wondered how a certain part of a piano was held together. He could find no description in available books, and his parents would not allow him to investigate the part in the family piano. The problem rested until a neighbor's piano was ruined in a flood. The instrument had been built into the basement and could not be removed in one piece. His parents volunteered his services for dismantling the piano. He had the chance to tear the whole thing apart and to learn how a piano is put together. A few years ago he applied the knowledge gained in this way in the ambitious project of rebuilding an 1867 Steinway, putting it back into first-class condition. He uses the piano regularly.

To give you more of an insight into Mr. Lyon's personality, I shall quote from a letter received from him: "I rather like to recall that early in my schooling I thought that each new concept in mathematics—multiplication as an improvement over addition, for example—presented an alternate and often easier method of working. That's probably why I was always willing to move ahead in mathematics. Some people find that counting on their fingers works and they're reluctant to learn the reliable method. I knew I had to run out of fingers. So learning to count using pencil and paper extended my limits."

Mathematics are sometimes difficult for musicians. Musicians are inclined to do things by ear. The word "mathematics" surely *sounds* plural. Mathematicians shy away from words like "noble" and "soft." This chapter explores the middle ground between the two. I have traveled this middle ground in a hearse. The hearse, providing a noble way to travel, has carried my homemade harpsichord. I bought the hearse with an income tax refund from last year. (If you can get that large a refund, you must be a mathematician. If you spend all of your refund to carry a musical instrument in a hearse, you must be a musician.)

Now that I've presented my qualifications, let me show you some features common to mathematics and music. Both mathematicians and musicians



"The hearse, providing a noble way to travel, has carried my homemade harpsichord."

have intellectual pleasures. The mathematician plays with numbers, and he arranges four 9s to produce 100 (99 + 9/9). The composer arranges notes into a theme that can be played at half speed, in a different key, and still fit the original theme when they are played together. Proportion and logic, common to mathematics, are necessary in music as in any other discipline. The most basic chord in music, the major triad, has vibrations in 4:5:6 relationship. A major triad on middle C has a C with 256 vibrations per second, an E with 320, and a G with 384. Tempered tuning changes these a trifle to keep other proportions sounding good. A good piano tuner with a few minutes to spare can show anyone how to hear the difference between 256.0 and 256.1 vibrations per second. No oscilloscope, no computer, and no knowledge of electronics are needed.

A trumpeter can play integer products of the number of vibrations of his lowest tone. No keys are needed for these notes. The bottom 4–5 are the notes in bugle calls. The trumpeter is limited only by the condition of his lip, and if he's in good shape, he can usually get at least 7 or 8 of these notes.

Written music is dependent on mathematics and fractions. The rhythm is basic and must be counted. Several years ago one of Leonard Bernstein's second violins made an entrance a bar early in Beethoven's Fifth Symphony. The tone quality was truly excellent and equalled only by the embarrassment of the musician. A ride in a hearse would be a pleasant alternative to facing some conductors after a goof like that. . . . Counting!!!

When I entered college in the fall of 1958, I planned to major in mathematics or physics with the idea of preparing for a career as a physicist. I had always been interested in these fields, and I had had the encouragement of ranking high in the 1958 Annual High School Mathematics Competition in the Upstate New York Contest Section of the MAA. I studied mathematics and physics for two years in college. Then I returned to what really was my first love, music.

Mathematics is the inspired ordering of an infinite world of numbers. That's Oswald Spengler's definition. Music is the inspired ordering of an infinite world of sounds. That's my variation on Spengler's theme. Ordering or organization, is a necessary part of the mathematician's thinking. The organizing abilities of the mathematician and of the musician are probably similar. Because of this similarity there is probably no mathematician without some interest in music and certainly no musician without some interest in mathematics.

Editorial Comment

Some of the mathematical concepts basic to music have been known since pre-Christian times. The medieval definition of music was "numbers made audible." Music, considered as a science, was included along with arithmetic, astronomy, and geometry in the quadrivium, one of the two major divisions of the liberal arts; grammar, rhetoric, and logic formed the other, the trivium. The pre-Christian association of music and mathematics takes one as far back as at least the time of Pythagoras. Most of you are familiar with the Pythagorean theorem relating the sides of a right triangle; but do you know that Pythagoras is credited with other laws? One of these laws has to do with music: "When a string and its tension, F, remain unaltered but the length is varied, the period of vibration, T, is proportional to the length, l," that is, $T = k_1 l$. That law and two additional laws that were formulated by the early seventeenth-century French mathematician Mersenne have had a marked effect on the building of instruments with strings. Mersenne's laws state: (1) "When a string and its length remain unaltered but the tension is varied, the frequency of vibration, f, is proportional to the square root of the tension," that is, $f = k_2 \sqrt{F}$; and (2) "For different strings of the same length and tension, the period of vibration is proportional to the square root of the weight, W, of the string," that is, $T = k_3 \sqrt{W}$. These laws hold for an ideal string of negligible thickness and uniform density. For real strings the laws must be slightly modified.

The operation of these laws is to be observed in the building of a piano.

> If a piano maker relied on the law of Pythagoras alone, his longest string would have to be 150 times the length of his shortest, so that either the former would be inconveniently long or the latter inconveniently short. He accordingly avails himself of the two other laws of Mersenne. He avoids undue length of his bass strings by increasing their weight—usually by twisting thinner copper wire spirally around them. He avoids inconvenient shortness of his treble strings by increasing their tension.[1]

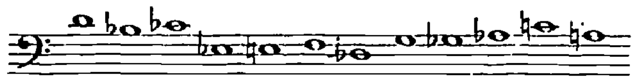
Other mathematical concepts basic to music are such familiar ones as inversion, permutations, ordered set, the identity element, and concepts that may as yet not be familiar but will become so if you continue studying mathematics: the concepts of decision theory and algebraic topology.

In the twelve-tone method of composition, the basic row must contain all twelve tones of the chromatic scale. No one tone may be repeated until all

1. The foregoing quotations are taken from "Mathematics of Music," by Sir James Jeans, in *The World of Mathematics*, ed. James Roy Newman (New York: Simon & Schuster, 1956).

twelve have been sounded. The row itself is subjected to perpetual variation. Consider the series, or ordered set, shown below.

We shall refer to this as the original form. The inversion of this series appears as follows:



The ascending interval D-G*b* of the original form is replaced by the equivalent descending interval D-B*b*; the descending interval G*b*-F is replaced by the ascending interval B*b*-C*b*; and so on. The retrograde form of the series, which follows, is obtained by reading the original form backwards, proceeding from the last tone to the first.

The retrograde inversion may be derived by reading the inversion backwards:

All these forms, found in *Studies in Counterpoint, Based on the Twelve-Tone Technique,*[2] are permutations of the original row.

Igor Stravinsky, the noted conductor, contended that music and mathematics are alike. In explaining the likeness he said:

> I have recently come across two sentences by the mathematician Marston Morse which express the "likeness" of music and mathematics far better than I could have expressed it. Mr. Morse is concerned only with mathematics, of course, but his sentences apply to the art of musical composition more precisely than any statement I have seen by a musician: "Mathematics are the result of mysterious powers which no one understands, and in which the unconscious recognition of beauty must play an important part. Out of an infinity of designs, a mathematician chooses one pattern for beauty's sake and pulls it down to earth."

Stravinsky also stated that likeness between mathematics and music can be formulated

2. Ernest Křenek (New York: G. Schirmer, 1940).

in the comparison between the mathematical and the musical conceptions of the "ordered set," or of the idea of the indispensable identity element (the element of form that will not change the other elements or change itself when combined with them).

He reported, but said he could not verify, that composers already claim to have discovered musical applications of decision theory, of group theory, and of the idea of shape in algebraic topology. To quote him further:

> Mathematicians will undoubtedly think this all very naïve, and rightly, but I consider that an inquiry, naïve or not, is of value if only because it must lead to large questions—in fact, to the eventual mathematical formulation of musical theory, and to, at long last, an empirical study of musical facts—and I mean facts of the art of combination which is composition.[3]

I wonder if the biography of any musician would not have some reference to mathematics. I shall settle for one more quotation, this one from a biography of Arthur Fiedler.[4] There Fiedler is quoted as saying, in reference to making youngsters do what they don't want to do, "It is the same with music. You have to make them practice. My father did, and I'm glad he did. . . . Musical training helps their arithmetic."

There are others besides Stravinsky, with illustrious careers, who have associated mathematics with music. One is Ernest Ansermet, the celebrated Swiss conductor of the Orchestre de la Suisse Romande in Geneva for almost fifty years, who studied mathematics with his father, a teacher of geometry. After obtaining a degree from a college in Lausanne, Ansermet taught mathematics at the high school there from 1906–1910. His mathematical background was no flighty affair. During this time, however, he pursued musical studies that had begun with training from his mother.

Maybe some of you with a strong bent for mathematics have an interest in music and will use your mathematics for a formulation of musical theory and an empirical study of musical facts. Stravinsky said that "musicians and mathematicians are both working from hunches, guesses, and examples." The mathematician George Pólya would agree with that.

3. Igor Stravinsky and Robert Craft, *Expositions and Developments* (New York: Doubleday & Co., 1956).

4. Carol Green Wilson, *Arthur Fiedler, Music for the Millions: The Story of the Conductor of the Boston Pops Orchestra* (New York: Evans Publishing Co., 1968).

MATHEMATICS AND THE MINISTRY

"To have a serious impact on the world, we must combine within ourselves the skills by which science controls the world and the insights which religion sheds on the direction this control should take. . . . Mathematics is, in a sense, the language of the modern world."

EARL B. ARNOLD

Editor's Introduction

Over the years since I began the study that resulted in this collection of essays, I have observed that a significant number of the participants in the study have chosen a religious vocation —as minister, priest, or rabbi. The author of this essay is one who made this choice.

Mr. Arnold's interest in both mathematics and the church became apparent very early. In each of the years 1959, 1960, 1961, he ranked in the top 1 percent in the high school contest in upstate New York. While still in high school he became the stated clerk of the Champlain Presbytery of the Presbyterian Church, and in 1962 he attended the Third British Conference of Christian Youth at Leicester. This simultaneous expression of the two interests has continued, as Mr. Arnold reveals in his essay; but in giving his background I shall deal first with mathematics.

Mr. Arnold's interest in mathematics became apparent when, at the end of the seventh grade, he asked the ninth-grade algebra teacher in the local high school at Saranac, New York, to explain logarithms to him. She lent him a book from which he learned the use of logarithms but not "why they work." The next year he was transferred from eighth-grade mathematics to algebra for the last twelve weeks of school. During the following summer he studied a book on calculus and achieved enough proficiency to derive formulas for the derivatives of the algebraic functions and to use them to solve maxima-minima problems. In his sophomore year he finished intermediate and advanced algebra. After his junior year he attended a National Science Foundation summer program in mathematics at Cornell. This latter experience, no doubt in conjunction with his success in the annual MAA contests, confirmed his inclination to continue pursuing the study of mathematics and his expectation of pursuing a scientific vocation.

After graduating from Cornell with a B.A. in mathematics, he spent a year in California with the Shell Development Company, where he worked in computer science in the company's mathematics department. Most of his attention was centered on information retrieval, the goal being to design a general-purpose filing system in which the format of the information could be specified by control cards rather than by programming. He had a particular interest in the linguistic side of computing; you will note a reference to language in Mr. Arnold's essay.

Mr. Arnold's interest in the church continued, however; and even while employed at Shell he did field work in Dick York's Free Church in Berkeley and attended night school at the San Francisco Theological Seminary, where he took three courses in Hebrew. In the fall of 1966 he began full-time study for the ministry and received a bachelor of divinity degree in 1969.

At present he still is combining his interests in mathematics and the ministry. Employed as a mathematician with the Shell Development Company, he also preaches in a little church on a temporary basis. He is looking for the pastorate of a Presbyterian church that desires a mathematically trained minister.

The ministry is the vocation of every Christian. Every Christian is called to be a servant to the world, and the way he exercises that servanthood is his ministry. Although my own training is aimed toward becoming a professional clergyman, my remarks about ministry apply equally well to any layman of any faith who acknowledges a basic concern for the well-being of his fellow men.

Our time is an age of technology such as has never been seen before. The imagination of modern man has been captured by the image of the scientist controlling the forces of nature to benefit mankind. Today the voice of authority is that of the technocrat, the scientifically trained person who makes decisions in the name of science that affect the lives of all of us. A person such as a clergyman who claims insight or competence in a particular field but lacks scientific training is at a considerable disadvantage if he attempts to influence these decisions. Not being qualified to meet the technocrat on his own ground, he is restricted to using nonscientific arguments, which are not generally held in high esteem by scientists. At best, he is taken for a well-intentioned but uninformed critic; at worst, he is viewed as an opponent of science and an enemy of the modern way of life. Too often the modern minister finds himself in this predicament.

It is important for the church to involve more technically competent people in its program of missions in our modern world. Ministers should be urged to add some competence in a technical field to their theological training. Laymen with technical training should be shown that their specialized knowledge makes them especially important as participants in the church's ministry. Working to help bring the kingdom of God into the world, the church tries to influence the critical decisions that affect the lives of millions. Its influence will be more effective if its representatives are well enough versed in science to deal on an equal basis with the highly

Mathematics was represented in the first worldwide Christian conference on the task of the church in a society under the impact of science and technology. The discussion attracted a hundred scientists, technologists, and theologians to the World Council of Churches in Geneva, 28 June–4 July 1970. They were asked to advise the church on the future potential impact of their discoveries on man and his life in society.

Mathematician Jeremy Bray, of the United Kingdom, who was formerly Joint Parliamentary Secretary of the Ministry of Technology, is seen above at the far left of the Panel on Technology and Society. Other members of the panel (left to right) are Professor S. L. Parmar, of India, chairman of the conference; Dr. Albert van den Heuvel, Dutch theologian; Professor L. Charles Birch, professor of biology at the University of Sydney; and Dr. Alejandro B. Rofman, director of the Centre for Urban and Regional Studies, Instituto di Tella, Buenos Aires.

trained technocrats who make these decisions. In order to have a serious impact on the world, we must combine within ourselves the skills by which science controls the world and the insights which religion sheds on the direction this control should take.

Whatever specialized training a technocrat may have, his scientific understanding rests on the foundation of mathematics. A solid grounding in mathematics is a prerequisite for further scientific study. Besides, even without specialized study in any particular science, a mathematician is, in his own right, a respected member of the scientific community. Since mathematical models are essential for conceptualizing and communicating the insights of the scientists, a mathematical background is more valuable than a nontechnical acquaintance with the particular subject matter of a scientific discipline for communicating with specialists in that field. A minister or layman with mathematical background would be well equipped to understand the arguments put forward by a technical specialist in support of or in opposition to a given program. He would be able to discuss intelligently the technical aspects of alternative program proposals. Mathematics thus becomes an important tool for the minister or layman in his relations on behalf of the church with those who make the decisions that affect our society.

On a more general level, the methodology of the sciences has influenced all disciplines, philosophy and theology as well as the social sciences. Both the process school of philosophers and the language analysts depend heavily on results achieved in physics and mathematics since the beginning of this century. Since these influential schools of philosophy contribute heavily to the patterns of thought used by modern man, a minister must take them into account in his efforts to communicate meaningfully with people. A number of theologians have attempted to incorporate some of the insights of these systems into their thought. An acquaintance with mathematics makes it easier for a minister to grasp the significance of the arguments that lie at the base of these philosophies and to use the symbols by which they express their insights. Only by thus coming to grips with the patterns of modern thought can a minister hope to be taken seriously as having something to say that is significant for our modern situation. Otherwise he will be seen as an expert in ancient texts, a promoter of esoteric rites that have nothing to do with life in the twentieth century.

There was a time when psychology was not a part of the traditional theological curriculum, just as mathematics now is not. I point out that I can see mathematics, too, as a part of that curriculum. I have used at least some of the creative mathematical ability with which I have been blessed in studying as much mathematics as I could; I have used some of the knowledge I have acquired in the work I have done in industry; and I have come to realize the relationship between mathematics and the ministry. You see, mathematics is, in a sense, the language of the modern world. The minister who has a mathematical background is in a position to associate himself with that large segment of the population which sees science as the new messiah. By employing the native tongue of these people, a mathematically sophisticated minister is better able to communicate intimately with them and to witness to them the good news of the gospel. If you, the reader, have an interest in a career in the ministry, study as much mathematics as you can consume.

MY WORLD OF MATHEMATICS

"Doing mathematics for love is the best reason of all. . . . The bait that keeps luring you on is not so much the ultimate goal . . . but the fact that the road itself is sort of fun to travel."

GEORGE T. SALLEE

Editor's Introduction

The world referred to in the title is that of "pure" mathematics, where the mathematician thinks and works as a creative artist. You will get some idea of what this means as you follow the experience of the author, who is an associate professor of mathematics at the University of California at Davis. After reading his essay, you might go on to read *A Mathematician's Apology* by G. H. Hardy, an English mathematician who could be called "the purest of the pure." (Note that the 1967 edition has a delightful preface by C. P. Snow.)

Professor Sallee was born in a farming community to parents who lacked academic training and lived in modest circumstances, but none of this was a handicap. Despite their nonprofessional status, his parents taught him arithmetic when he was four or five years old and encouraged him to study mathematics even when he had advanced enough so that they could no longer help him.

As a sophomore in high school he placed fourth in Oregon in the 1956 Annual High School Mathematics Contest. He was allowed to skip Algebra I and take Analytic Geometry. He also studied calculus, independently. Never was any pressure needed to make him study mathematics except for a single instance when his teachers insisted that he learn solid geometry because it was required for entrance to the California Institute of Technology, which he wanted to attend—and which he later did attend on a four-year scholarship.

At Cal Tech his choice of a major field of study was physics until his sophomore year, when he came face to face with the "difficulties and messiness of the subject." He then switched to mathematics essentially for aesthetic reasons; it was "just so beautiful." He received a B.A. in mathematics from Cal Tech in 1962, an M.A. in mathematics from Berkeley in 1964, and a Ph.D. in mathematics from the University of Washington in 1966. His dissertation, "Incidence Graphs of Convex Polytopes," was published in the *Journal of Combinatorial Theory* 2 (1967).

Professor Sallee poses some problems, at the end of his essay, that will challenge your creative mathematical ability.

What's your dream? Zipping down to your villa at Cannes in your Lamborghini Miura? Setting a ski-jump record? Landing on the moon?

Millionaire, athlete, astronaut—I'm none of these things, but I wouldn't trade my job to be any of them. I'm a mathematician.[1]

Most people regard mathematics as the low point of their educational career and mathematicians as harmless drudges. I've run across this attitude so often that I should be hardened into acceptance of it—but every now and then some poor soul will ask me how my work is going, and I throw caution to the winds and tell him just how great things are. Just what I'm doing and why, and how much fun I'm having doing it.

Fun. This is the thing that really surprises them. Mathematics for money is understandable, mathematics for humanity is laudable ("everyone must know some mathematics"), but mathematics for the sheer love of it is thought to be downright lunacy.

I suppose that there are a lot of good sober reasons for being a mathematician. The same can be said for being a swamp-clearer, too; but doing mathematics for love is the best reason of all. To me, mathematics is beautiful—like looking into a perfect crystal. A good proof has all of the inevitability of a falling stone, but what genius it took to see how clear it all was. And it is there for anyone to appreciate, like the superb artistry of Tennessee Williams or the Beatles. All are true. All are beautiful. Even better, there is more.

C. P. Snow put it quite well in one of his books when he said that people have always liked to solve problems and scientists are fortunate enough to get paid for doing it. So are mathematicians. There is a real thrill in trying to solve a problem that no one else in the world has ever solved before.

1. The notion of being a millionaire mathematician has a certain appeal, though.

The problems you are working on may not have a solution,[2] or (as is much more common) what you are trying to prove may not even make sense. When you are working out on the frontier of knowledge, chances are you don't know where you are going. So you keep plugging away, constructing examples and making guesses on the basis of them, and trying to see if your guesses are correct. When you finish and look back on your completed work, you say, "Well, of course, that was the way it had to be," but what it "has to be" is usually not clear when you start.

And when you have finished working on a real problem—not some made-up problem that is merely for amusement, but a real problem—you feel that you have added something to man's knowledge. Just like any scientist, you are after truth, whatever it may be. But it is a peculiar kind of truth you are seeking, for its validity rests entirely in the minds of men. While you may model what you are doing after what you think nature is, you are not bound to follow it precisely. Everyone works with his own approximation to the real world—no real plane is as flat as the mathematician's, and no real function is as smooth as mathematics often requires. But this does not really matter—we have our own set of rules which we may modify at will to make what S. K. Stein calls "the man-made universe."[3] What mathematicians are seeking to discover, I think, is the logical underpinnings of the universe—not so much what the laws might be, but how the laws might be expressed if anyone can find them. As it is, the real world is too difficult to understand, so we make approximations to simplify it into pure logic, which we hope we may someday comprehend.

Mathematics for me began the day that I found Geoffrey Mott-Smith's fascinating little book, *Mathematics Puzzles for Beginners and Enthusiasts*, in the library.[4] This was sometime during junior high school. Before that I had always liked arithmetic, but I had had no idea how interesting problems could be when they were cleverly posed. Many of the problems in the book are very simple (mathematically speaking) once you get the idea, but they are all dressed up in an enjoyable story form. Martin Gardner has the same happy knack in his *Scientific American* column. After a while, I

2. As, for example, the classical Greek problems of trisecting the angle, duplicating the cube, and squaring the circle, which have no solution with ruler and compass. This fact was not discovered until Galois came along and gave an extremely elegant proof of this result in 1830, when he was twenty. A friend of mine who is a logician tells me that there is a suspicion that Fermat's Conjecture, probably the most famous outstanding problem today ($x^n + y^n = z^n$ does not have a solution in nonnegative integers if $n > 2$) may suffer from the same defect; that is, we may be trying to solve a problem that cannot be solved as it is posed. Technically speaking, it may be formally undecidable, and this result is in a sense an axiom of the real number system.

3. Stein's book with this title is included in the list of suggested readings at the end of this essay.

4. This, also, is included in the suggested readings.

began reading algebra and geometry so that I could work on more math puzzles. In fact, a good chunk of my spare time in high school was spent doing math problems of various kinds.

The ideal solution would have been to continue learning all of my mathematics this way—finding problems of interest and then learning enough to be able to solve them. R. L. Moore at the University of Texas has had great success with this kind of problem-oriented approach to teaching. Eventually, however, there came a day when I needed to know some things that I didn't care much about. Differential equations and linear algebra I remember very well in this regard. I didn't care much about them when I first met the subjects, and I don't care much about them now (it is only fair to say that the linear algebra I have learned has been extremely valuable). But if you want to be good at something, it always takes some tears somewhere along the line. I am reasonably sure that Tommy Smith doesn't come out to the track burning with desire to practice starts, but it is necessary if he is to be a great runner. Many areas must be mastered if you want to be a great mathematician—or even just a competent one. The bait that keeps luring you on is not so much the ultimate goal (that sometimes seems so far away), but the fact that the road itself is sort of fun to travel. I am reminded of a quotation from "A Prayer," by Gelett Burgess:

> Not the quarry, but the chase,
> Not the laurel, but the race.

However, after six years of learning all kinds of advanced mathematics, I was ready for something new when I was finally in a position to begin work on my doctoral thesis. What I found was the field of combinatorial geometry, a relatively new area that has its roots in the recent interest in linear programming. The questions are all fairly easy to state once you know a few basic definitions, and then you are on your own.[5] There is not a great deal of knowledge that definitely has to be learned before you can begin work—in striking contrast to many branches of mathematics, like algebraic topology. Most of the problems do not need so much a thorough mastery of everything that has been done ("I'll bet we can use the theorem of Schmerdley that appeared in the *Rumanian Journal of Mathematical Alchemy* in 1927") as a fresh approach. In this field, a good idea is worth a lot of reading. The reason for this is simple—the field is so new that there has not yet been a chance for it to develop any structure of importance. Every new piece of knowledge looks like an island, with only an occasional bridge connecting two islands. In time, the field may develop a solid structure, so we can see what really is important and how all of these

5. Other fields that are similarly accessible to the interested person are convex sets, number theory, and graph theory. Some elementary books in these areas are cited at the end.

"Not the quarry, but the chase,
Not the laurel, but the race."

(so far) isolated results tie together. But until that time, everyone is pretty much his own guide.

Well, what is combinatorial geometry? Unfortunately, this is a tricky question to answer. The difficulty stems both from the newness of the field and the associated fact that its boundaries are so ill-defined. Some of the questions I work on could be considered fair game for graph theorists or combinatorial analysts or even topologists. So, rather than describe the field in the usual way, at the end of this essay I have listed a few "typical" questions.

As you will see by even a casual scanning of the topics, there is very little to connect them. (I am coming more and more to appreciate the definition: "Geometry is what geometers do.") About the only link is that the problems are of interest to the same group of people and that the methods used on one problem may sometimes be of use on another.

But it's an exciting field and it's an exciting life. Geometry is growing, and I want to be a part of that growth. Perhaps in a hundred years someone can look back and use a contribution I have made. There won't be many of them (von Neumann, I'm not), but if there is just one I'll be happy. And what more can a man ask out of life?

PROBLEMS

Listed below are some examples of the wide variety of problems that might be of interest to combinatorial geometers. They might very well be of interest to other mathematicians, too; this listing is not meant to exclude them. Most of these problems can also be posed in higher dimensions—things don't really start happening until you get to four dimensions[6]—but

6. I do not want a lot of gas to the effect that *the* fourth dimension is time. It is true that the world may be explained very nicely by means of the so-called space-time continuum, but this is merely one way of describing the universe. To a mathematician, to give a dimension to a space is to tell how many numbers you have to specify to uniquely identify each point. Thus each point on a line may be fixed by using only one number, while to find points in a plane requires two. Hence the line is one-dimensional, and the plane is two-dimensional. If a space is such that it requires five numbers to uniquely specify each point, then the space is five-dimensional.

In four dimensions it is possible to construct a polytope with as many vertices as you choose and such that every pair of them form an edge. Try to do this in three dimensions and then see how contrary it is to your intuition.

some of them aren't solved even in three dimensions, so there is no point in going any further yet.

Before we can pose any problems we must give a few definitions.

A *polytope* is a generalization to higher dimensions of the notion of a polyhedron (as it is usually called in three dimensions) or a polygon (in two dimensions). Polytopes have vertices, edges, and generally flat faces of all dimensions from 0 to d minus 1, where d is the dimension of the polytope.

Two polytopes are *combinatorially equivalent* (or have the same *combinatorial structure*) if they have the same number of faces of the same dimension and the faces fit together "in the same way." For example, any two triangles are combinatorially equivalent. So are any two tetrahedra. And if you take a square pyramid and cut off the apex with a plane, you get a polytope combinatorially equivalent to a cube. However, a cube is not combinatorially equivalent to a pyramid over a pentagon, although both have six faces, because that pyramid has only six vertices while the cube has eight. You might want to try the harder problem of finding an example of a polytope with eight vertices, twelve edges, and six faces (not all necessarily with four sides) which is not combinatorially equivalent to a cube.

If you are still with me, here are some known results and unsolved problems. Probably the first combinatorial theorem goes back to Euler, about 1780, who proved that for each three-dimensional polytope, if V = number of vertices, E = number of edges, and F = number of faces, $V - E + F = 2$. An analogue of this is true in higher dimensions. Using this and the fact that every vertex has at least three edges coming into it, you can show that every three-dimensional polytope has at least one face that has 3, 4, or 5 sides.

Now for some unsolved problems. Let me emphasize that these are problems that you may be able to attack successfully, as they are probably not hard if you look at them from just the right angle. (This cheery remark does not apply to problems 1 and 8, which are probably horribly difficult—problem 8 certainly is.) In problems 1 through 5, *polytope* means a three-dimensional polytope.

1. Let p_3 be the number of triangular faces of a polytope, p_4 the number of 4-sided faces, and in general p_n the number of n-sided faces. Characterize all sequences $(p_3, p_4, p_5, p_6, \ldots)$ such that a polytope exists having p_3 triangular faces, and so forth. For example, the sequence associated to the square pyramid is $(4, 1, 0, 0, \ldots)$; to the cube it is $(0, 6, 0, 0, \ldots)$.

   Remark: It is known that

$$3p_3 + 2p_4 + p_5 - p_7 - 2p_8 - 3p_9 - \ldots = 12.$$

2. Can you find any relation other than the one just mentioned which the (p_i) must satisfy to arise from a polytope?
3. Can you solve either problem 1 or problem 2 for *simple* polytopes—that is, those with *exactly* three edges coming in to each vertex?
4. The *edge distance* between two vertices of a polytope is the smallest number of edges on a path between them. A *center* of a polytope is a vertex such that the maximum edge distance to any other vertex is as small as possible, and this minimum distance is called the *radius*. (A tetrahedron has radius 1; a cube has radius 3.) Find the maximum radius of a polytope having n vertices for each $n \geqslant 4$.
5. Show that it is always possible to cut apart a polytope along edges so that the result is connected, will lie flat in the plane, and does not overlap itself.
6. Lower-bound problem: For a given number of vertices, what is the smallest number of faces of dimension d minus 1 which a polytope can have? (This problem has been solved up to dimension 9, at the last I had heard, but you might want to try it in three dimensions. It is one of the most famous problems in the field.)
7. Upper-bound problem: For a given number of vertices, what is the largest number of faces of dimension d minus 1 which a polytope might have? (I think this is also solved up to dimension 9.)
8. Four-color problem: Consider a map of countries, where each country consists of a single connected piece (like Switzerland, and not like New Zealand). Show that this may may be colored with four colors so that each pair of countries with a common border are given different colors. See *The Four-Color Problem,* listed in the suggested readings, for all that is known about this problem.

READING YOU MAY ENJOY

(This listing is given in the order in which you will probably want to read the books.)

Mathematical Puzzles for Beginners and Enthusiasts, Geoffrey Mott-Smith. 2d ed., rev. New York: Dover, 1954.

Scientific American Book of Mathematical Puzzles and Diversions, Martin Gardner, ed. New York: Simon & Schuster, 1964.

New Mathematical Diversions from Scientific American, Martin Gardner. New York: Simon & Schuster, 1966.

A Miscellany of Puzzles: Mathematical and Otherwise, Stephen Barr. New York: Thomas Y. Crowell Co., 1965.

One Hundred One Puzzles in Thought and Logic, Clarence R. Wylie, Jr. New York: Dover, 1957.

Mathematics: The Man Made Universe (An Introduction to the Spirit of Mathematics), Sherman K. Stein. 2d ed. Books in Mathematics series. San Francisco: W. H. Freeman & Co., 1969.

Mathematics and the Imagination, Edward Kasner and James R. Newman. New York: Simon & Schuster, 1940.

Combinatorial Geometry in the Plane, Hugo Hadwiger and Hans Debrunner, translated by V. Klee. New York: Holt, Rinehart & Winston, 1964.

Convex Figures, I. M. Yaglom and V. Boltyanskii. New York: Holt, Rinehart & Winston, 1961.

Graphs and Their Uses, Oystein Ore. New York: Random House, 1963.

The Four-Color Problem, Oystein Ore. New York: Academic Press, 1967.

Introduction to the Theory of Numbers, Ivan Niven and H. S. Zuckerman. 2d ed. New York: John Wiley & Sons, 1966.

EDITOR'S SUMMARY

You have read in these essays about some uses of mathematics in career activities. Many more essays could have been written. Those that have been written could have been arranged in as many orders as the number of permutations of seven things taken seven at a time. But neither the number of essays nor their arrangement really makes any particular difference. Their value, even if none has dealt with a career you are likely to choose, lies in the possibility that they have aroused in you the desire to use your mathematical ability in some way and to investigate further the uses of mathematics.

One area you may want to investigate is the field of linguistics, the study of language and languages, which has recently come into vogue in the academic world and government circles and among educated laymen. Linguistics relies strongly on mathematical theory. Mathematics itself is a language, a language common to people speaking different languages. Mathematics, however, like any field of continuing human interest, has an advantage over language in general. As G. H. Hardy said, languages die but mathematical ideas do not.

Another area, that of aeronautical engineering, is appealing because of the great interest in space exploration and the part played by mathematics in making that exploration possible.

But mathematics is common to the advanced study of practically every field. Your pursuit of its uses can go on and on.

In reading these reports of personal experience, you may have noticed common patterns: independent study that began early in youth; an early interest in puzzles and problems and the willingness to work hard at their solution; a head start on the study of both secondary and college mathematics because of an inner drive; the willingness at a more mature age to plug away even if an ultimate goal was not obvious; and, often, changes of career in midstream, not to desert mathematics but to make further use of it.

Something that you could not have noticed in your reading is a common pattern in what took place behind the scenes—a pleasant, cooperative willingness, if not eagerness, on the part of the authors to share their experiences. You must have realized, however, that these young men are ambitious and that they recognize the value and uses of their backgrounds in mathematics in fulfilling their ambitions and making their dreams come true.