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AUTHOR Pruzek, Robert M.
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ABSTRACT

Guttman's "rank-reduction" theorem is employed to demonstrate how most of the algebra and computational work for standard correlational and analysis of variance methods can be examined within a single matrix system. Analyses of both correlational and experimental data are illustrated with an easy-to-use computer program. This application and the corresponding discussion emphasize distinctions and qualifications needed for appropriate interpretations of both types of data. Since this theorem was initially presented within the framework of factor analysis, it is intended that this paper will contribute to a better understanding of relationships among factor analytic, general correlational, and experimental design methods. (Author/CK)

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A Unification of Correlational and Experimental Methodology: An Application of a Guttman Theorem

Robert M. Pruzek
State University of New York at Albany

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The principal concern of this paper is to facilitate a rapprochement of experimental and correlational methodology. While others have discussed various relationships between experimental and correlational methods (see Burt, 1947 and 1966; Creasy, 1957; Bock, 1960; McKeon, 1965), the present unification, based explicitly on the work of Louis Guttman, seems not to have been adequately discussed.

My specific aim is to show algebraically the major relationships among conventional types of product-moment correlation coefficients (including simple, multiple and canonical r's as well as partial and part coefficients) and the standard methods for analysis of variance (univariate and multivariate, with or without covariates, fixed or random effects). These relationships are shown through application of a rank reduction theorem which Guttman (1944) first presented within the context of factor analysis. In focusing on an algebraic system no explicit attention is given to statistical inferential aspects of the respective methods. Nevertheless, it is hoped that this paper will contribute to a better understanding of certain inferential procedures and their interrelationships. Restricting the discussion to algebra should facilitate computations for these methods and an easy-to-use computer program based on Guttman's theorem is briefly described for such applications.

Suppose U_o is an $N \times p$ matrix which is associated with p measures on each of N entities and that it is reasonable to examine certain linear associations among the columns. If the pairs of variables are approximately linearly related, it will often be fruitful to examine the product $U_o' U_o$ which here will be designated as V_o. Few assumptions about

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the nature of the respective variables are required in so far as the algebra of multivariate systems is concerned. Measures may be random or fixed variables; they may be associated with qualitative or quantitative responses; or they could be fallible or infallible, original or derived. Moreover, certain variables in a given system might be labeled independent, dependent or instrumental and the marginal or joint distributions may have an immense variety of forms even if scatterplots are taken to be approximately linear. Of course the interpretive uses of the derived statistics can never be guaranteed even when the variates under study do satisfy major assumptions for a formal method of analysis. No attempt is made in this paper to elaborate on appropriate uses of the statistics, or to specify when subject-matter specialists are apt to find them of interpretive value, despite the ultimate importance of such questions.

The rank reduction theorem states that for a $\underline{p} \times \underline{p}$ symmetric, Gramian matrix V_o of rank $\underline{r}$, and $\underline{p} \times \underline{s}$ weight matrix W (such that $W' V_o W$ is non-singular), the matrix

$$(1) \quad V_1 = V_o - V_o W (W' V_o W)^{-1} W' V_o$$

is a residual matrix of rank $\underline{r} - \underline{s}$. Guttman (1952) pointed out that the theorem can be repeatedly applied to successive residuals so that at the jth stage one can generate a $\underline{p} \times \underline{p}$ residual

$$(2) \quad V_j = V_{j-1} - V_{j-1} W_j (W_j' V_{j-1} W_j)^{-1} W_j' V_{j-1}$$

where V_j is of rank $\underline{r} - (\underline{s}_1 + \underline{s}_2 + \ldots + \underline{s}_j)$.
If there are k stages, then V_k will necessarily be null and $\sum_{j=1}^{k} \underline{s}_j = \underline{r}$.
All W_j's must be distinct and each product $(W_j' V_{j-1} W_j)$ should be non-singular.

My interest centers only indirectly on the rank-reduction aspects of the theorem; the main interest is to identify certain entries

of various members of (1) and (2) under differing specifications for measures in the initial matrix U_o. It will be assumed for convenience throughout this paper that columns of U_o are scaled to have means of zero, although this assumption may be relaxed at later points. Also, for convenience, it will be assumed that V_o is non-singular as well as Gramian; this assumption is usually realistic whenever $N > p$, the latter being almost universally desirable in practice. The remainder of this paper is divided into three sections, on Correlational Methods, Experimental Methods and a Computational Synthesis.

Correlational Methods

In this section I shall first examine a special case of equation (1); next a special case of equation (2) is considered; finally, brief discussion is given with regard to some general factor analysis problems.

Begin by partitioning U_o by columns: $U_o = [X \vdots Y]$. X is of order $N \times s$ and Y is $N \times t$ where $t = p - s$. This leads to $V_o = U_o' U_o = \begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix}$. While it was noted that in terms of algebra no restrictions are required for the variables in U_o as to type, it may initially be helpful to think of U_o as a set of p random variables. V_o will usually be taken as a general variance-covariance matrix although it will be useful at times to designate $V_o = R$, a $p \times p$ matrix of simple product-moment correlations.

Recalling (1), which states that $V_1 = V_o - V_o W (W' V_o W)^{-1} W' V_o$, let the $p \times s$ weight matrix be defined as follows: $W = \begin{bmatrix} I_s \\ \cdots \\ 0 \end{bmatrix}$. I_s is an identity matrix of order s, and 0 is a $t \times s$ null matrix. The following identifications may be made:

(1) V_oW is a $\underline{p}$ x $\underline{s}$ covariance matrix for the first $\underline{s}$ variables in the system with respect to the entire set of variables. If $V_o = R$, then RW is clearly a rectangular matrix of correlations. $V_o W$ may thus be written as $\begin{bmatrix} V_{xx} \\ V_{yx} \end{bmatrix}$ or $\begin{bmatrix} R_{xx} \\ R_{yx} \end{bmatrix}$.

(2) $W' V_o W$ is the covariance matrix V_{xx} for the first $\underline{s}$ variables. If W is replaced by W_n where $W_n = WD_n$ and D_n normalizes columns of V_oW, then $W_n' V_o W_n$ is a simple correlation matrix for the first $\underline{s}$ variables.

(3) $V_o W (W' V_o W)^{-1}$ is a $\underline{p}$ x $\underline{s}$ matrix of estimated least squares regression coefficients. This matrix may be written as $\begin{bmatrix} I_s \\ \hline \hat{B}_{y.x} \end{bmatrix}$ where rows of $\hat{B}_{y.x}$ are sets of $\underline{s}$ estimated least squares regression coefficients for predicting the $\underline{t}$ respective <u>criteria</u> (rows). If $V_o = R$ and $W' V_o W$ is replaced by $W_n R W_n'$, then rows of $\hat{B}_{y.x}$ are standardized regression coefficients; the latter matrix may be written as $R_{yx} R_{xx}^{-1}$.

(4) $V_o W (W' V_o W)^{-1} W' V_o$ is the covariance matrix of the entire set of predicted portions of the variables. This matrix may be considered as an approximation to the initial V_o matrix, written $\hat{V}_o = \begin{bmatrix} \hat{V}_{xx} & \hat{V}_{xy} \\ \hat{V}_{yx} & \hat{V}_{yy} \end{bmatrix}$. Note, however, that the first $\underline{s}$ variables predict themselves perfectly so all elements of $\hat{V}_o$ except $\hat{V}_{yy}$ are equivalent to the corresponding entries in V_o when using this particular W. If $V_o = R$, then each diagonal entry of $\hat{V}_{yy} = \hat{R}_{yy}$ is a squared <u>multiple</u> correlation coefficient associated with $\underline{s}$ predictors for one of the respective $\underline{t}$ criteria. $\hat{R}_{yy}$ may be written as $R_{yx} R_{xx}^{-1} R_{xy}$.

(5) $V_1 = V_o - \hat{V}_o$ is the residual matrix which in this case will be null except for $V_{yy} - \hat{V}_{yy}$. The normalized version of the residual $V_{(yy)} = (V_{yy} - \hat{V}_{yy})$ is the matrix of sth order partial correlations between all pairs of t criteria. If s = 1, all t(t - 1)/2 coefficients are 1st order partials.

(6) Given that $V_o = R$, the "half-normalized" matrix $R_{yy} - \hat{R}_{yy}$ is a matrix of part correlations. For example, if $R_{(yy)1} = R_{yy} - \hat{R}_{yy}$, which contains t complements of squared multiple conditions as its main diagonal entries, is multiplied on the right by the diagonal normalizing matrix $D^{\frac{1}{2}}_{(1 - smc_i)}$, then any column of coefficients in $(R_{yy} - \hat{R}_{yy}) D^{\frac{1}{2}}_{(1 - smc_i)}$ will contain sth order part correlations--correlations between an original criterion variable and the error portions of the t criteria with respect to the set of s predictors.

(7) Canonical correlations between the sets of s predictors and t criteria may be found as the positive square roots of the eigenvalues of either the product $\hat{V}_{yy} V_{yy}^{-1}$, or $\hat{V}_{yy}$ in the metric of V_{yy} (see Dempster, 1968). The same roots are associated with $\hat{R}_{yy} R_{yy}^{-1}$. See Meredith (1964) for a discussion of the algebra of canonical correlations under corrections for unreliability of the variables.

(8) Eigenvectors (right-hand) of $\hat{V}_{yy} V_{yy}^{-1}$, here called $\vec{q}_j$, define the canonical variates for the t criteria with respect to the set of s predictors. The t dimensional column characteristic vectors $\vec{q}_j$ may be computed as $\vec{q}_j = G_y' \Lambda_y^{-1} \vec{v}_j$ where G_y is the set of (unit-length) columns of eigenvectors of V_{yy}, Λ_y^{-1} is the diagonal matrix of reciprocal square roots of eigenvalues

of V_j and each $\bar{V}_j$ is an eigen vector of the symmetric matrix $\Lambda_y^{-1} G' \hat{V}_{yy} G_y \Lambda_y^{-1}$. (Most algorithms for generating roots and vectors require symmetric matrices.) The set of canonical variates for the $\underline{s}$ predictors can be found by using $W = \begin{bmatrix} 0 \\ \cdots \\ I_{\underline{t}} \end{bmatrix}$ and proceeding analogously--thus reversing the roles of the predictors and the criteria.

Paragraphs numbered (9) - (16) are analogous to those numbered (1) - (8) except that equation (2) of Guttman's theorem is now considered, this leading to partial statistics of various kinds most of which are direct, albeit complex, counterparts of the statistics already identified. Equation (2) states that $V_j = V_{j-1} - V_{j-1} W_j (W_j' V_{j-1} W_j)^{-1} W_j' V_{j-1}$. Each matrix W_j shall be taken to be of the simple form $\begin{bmatrix} 0 \\ I_{\underline{s}_j} \\ 0 \end{bmatrix}$ where $\underline{s}_j$ is the dimensionality of the matrix $I_{\underline{s}_j}$; thus W_j is of order $\underline{p} \times \underline{s}_j$. It will be convenient here to assume that $\sum_{j=1}^{k} \underline{s}_j = \underline{s}$, so that the set of p variables in U_o can be considered as a set of $\underline{s} = \underline{s}_1 + s_2 + \ldots + \underline{s}_k$ predictors (X) which is adjoined to $\underline{t} = \underline{p} - \underline{s}$ criteria (Y). Further, let $\underline{s}^*$ be $\sum_{j=1}^{j'} \underline{s}_j'$ for a particular $j' \leq k$.

Given the above specifications, the following identifications may be made:

(9) $V_{j-1} W_j$ is a $\underline{p} \times \underline{s}_j$ rectangular matrix of partial covariances. Analogous to $V_o W_1$ in paragraph (1), $V_{j-1} W_j$ is just a vertical slice of V_{j-1}, the latter being a matrix which is described in paragraph (13) below.

(10) $W_j' V_{j-1} W_j$ is an $\underline{s}_j \times \underline{s}_j$ matrix of $\underline{s}^*$th-order partial

covariances for the set of predictors at the j'th stage. If columns of W_j are scaled to normalize columns of $V_{j-1} W_j$, as in $V_{j-1} W_{\underline{n}j}$, then at stage j', $W'_{\underline{n}j'} V_{j'-1} W_{\underline{n}j'}$ is a matrix of $(s^* - \underline{s}_{j'})$th order <u>partial</u> correlations among the $\underline{s}_{j'}$ predictors.

(11) $V_{j-1} W_j (W_j' V_{j-1} W_j)^{-1}$ is a $\underline{p} \times \underline{s}_j$ matrix of partial regression coefficients. In general, the $\underline{s}_{j'}$ predictors at the j'th stage are <u>error</u> portions of a particular set of predictors, viz., those portions of $\underline{s}_{j'}$ variables which are not linearly predictable from the preceding $\underline{s}^* - \underline{s}_{j'}$ variables. Thus, the lower $\underline{t} \times \underline{s}_{j'}$ portion of $V_{j'-1} W_{j'} (W_{j'} V_{j'-1} W_{j'})^{-1}$ is a matrix each row of which contains semi-partial regression coefficients for predicting a single criterion from certain <u>residuals</u> associated with the $\underline{s}_{j'}$ criteria.

(12) $V_{j-1} W_j (W_j' V_{j-1} W_j)^{-1} W_j' V_{j-1}$ may be viewed as an approximation $\hat{V}_{j-1}$ to the V_{j-1} matrix with which it is associated. At stage j', $\hat{V}_{j'-1}$ is properly regarded as a set of partial covariances among the <u>portions</u> of $\underline{s}_{j'}$ predictors which are not linearly predictable from the preceding $\underline{s}^* - \underline{s}_{j'}$ predictor variables and the remaining residuals in the system. If the initial $V_o = R$, then diagonal entries in $\hat{V}_{(yy)_{j'-1}} = \hat{R}_{(yy)_{j'-1}}$ contain squared <u>multiple-partial</u> correlations--those (error) portions of the respective $\underline{t}$ criteria which can be predicted, using a multiple linear equation, from the <u>errors</u> associated with the $\underline{s}_{j'}$ variables as described above. Given that $\sum_{j=1}^{k} \underline{s}_j = \underline{s}$, the sum of all k such squared multiple partial coefficients for a particular criterion variable is necessarily

equal to the squared multiple correlation coefficient associated with the entire set of $\underline{s}$ predictors. If $s_1 = s_2 = \ldots = s_k = 1$ so that $k = \underline{s}$, then principal diagonal entries in $\hat{R}_{j'-1}$ are squares of conventional semi-partial correlations. It should be noted that values of a set of k semi-partial or multiple-partial correlations (or covariances) are <u>conditional</u> on a particular ordering of the k sets of predictors unless the k sets are mutually orthogonal. Rozeboom (1966) is an excellent source on this variety of statistics.

(13) $V_j = V_{j-1} - \hat{V}_{j-1}$ is a p x p matrix of partial covariances also. At the j'th stage of application of equation (2), $V_{j'}$ is <u>null</u> in its first $\underline{s}^*$ rows and columns. The lower $\underline{t}$ x $\underline{t}$ portion $V_{(yy)_{j'}}$ is a partial covariance matrix for the $(\underline{s}^* - \underline{s}_{j'})$th order residuals of the t criteria with respect to the portions of the $\underline{s}_{j'}$ predictors which are orthogonal to the preceding $\underline{s}^* - \underline{s}_{j'}$ predictors. If $V_0 = R$ at the onset, then the diagonal entries of $V_{(yy)_{j'}}$ contain proportions of variance of the respective criteria which remain to be predicted after employing $\underline{s}$ predictors in a linear prediction system. As one moves from the j'- 1 to the j'th stage, equation (2) states that increments of error variability for each criterion variable must be reduced by the amount of the squared multiple-partial correlation described in paragraph (12). The normalized form of either $V_{(yy)_{j'}}$ or $R_{(yy)_{j'}}$ contains semi-partial correlations among the pairs of $\underline{t}$ criteria after holding the $(\underline{s}^* - \underline{s}_{j'})$th order error portions of the set of $\underline{s}_{j'}$ predictors constant.

(14) If $V_o = R$ at the onset then the "half-normalized" version of the $\underline{t} \times \underline{t}$ matrix $R_{(yy)_{j'}}$ contains semi-part correlations. For example, if $R_{(yy)_{j'}}$ is multiplied only on the right by its normalizing diagonal, then a column in the product matrix contains correlations between an $(\underline{s}^* - s_{j'})$th order error portion of a particular criterion variable and the set of $\underline{t} - 1$ criterion residuals which cannot be linearly predicted from the $(\underline{s}^* - \underline{s}_j)$th order errors of the $\underline{s}_{j'}$ predictors at stage j'.

(15) Canonical partial correlations between a particular set of $(\underline{s}^* - \underline{s}_{j'})$th order residuals for $\underline{s}_{j'}$ predictors and the associated set of criterion residuals can be obtained as in paragraph (7): such canonicals are the positive square roots of the eigenvalues of a matrix of the form $\hat{V}_{(yy)_{j'}} V^{-1}_{(yy)_{j'}}$. Such canonicals are indeed general since they subsume practically all of the preceding correlation coefficients as special cases.

(16) Eigenvectors of $\hat{V}_{(yy)_{j'}} V^{-1}_{(yy)_{j'}}$ define the canonical variates at the j'th stage for the (error portions of) criterion variables with respect to $(\underline{s}^* - \underline{s}_{j'})$th order error portions of a particular set of $\underline{s}_{j'}$ predictors. Such canonical variates may also be generated from a symmetric matrix as described in paragraph (8).

The foregoing correlational analysis is based on equations (1) and (2) of Guttman's rank reduction theorem where the weight matrices have particularly simple form. Much of what was presented has been "available" for over fifty years. In addition to the synthetic aspects of the present elaboration, its chief virtue may be to automate the

computation of practically all of the preceding statistics through straightforward application of a simple theorem, a computer program for which may be readily developed for any digital computer.

If the W_j matrices are allowed to range more broadly, an immense variety of other methods can be identified with respect to equations (1) and (2). For instance, starting with observed variables in U_o, where $V_o = U_o' U_o$, if the set of k weight matrices W_j (each of the order $p \times s_j$) are allowed to range over all possible distinct arrays, all types of <u>component analysis</u> can be defined. If each W_j is a distinct eigenvector which corresponds to the initial V_o (of rank $k = r$), then equation (2) specifies the exhaustive set of <u>principal components</u> of V_o. If $V_o = R$, then principal component analysis is defined as it is typically employed. Of course, the usual approach to PC analysis, and common factor analysis as well, does not involve <u>a priori</u> specification of weight matrices; derived factors can, nevertheless, be <u>defined</u> in terms of the rank reduction theorem.

If V_o is replaced by a Gramian variance-covariance matrix of the "common portions" of variables, say, $V_o - D_u^2$, where D_u^2 is a $p \times p$ diagonal matrix of uniqueness variances, then any set of (Thurstonian) common factors can be generated for a given $V - D_u^2$, using a specific set of weight matrices $W_1, \ldots\ldots, W_k$. Guttman (1952) discusses certain aspects of the latter variety of applications in some detail.

Other varieties of "factor analysis" can also be associated with the rank-reduction theorem. V_o may be taken, for example, as the variance-covariance matrix of the "images" of the initial p variables (see Guttman 1953) or some "reproduced" portion of this matrix using a limited number of factors (e.g. see Harris, 1962).

For any such approach to factor analysis, based on either (1) or (2), Guttman noted that one could, by choosing weight matrices appropriately, avoid completely the task of "rotation" -- or factor transformation. Indeed, Guttman (1952) argued that interpretable factors based on weight matrices chosen a priori, presumably using a particular scientific justification, are apt to be most compelling. While space limitations preclude a discussion of this argument, it is clear that there are special virtues of a direct analysis of a complex of variables using Guttman's rank reduction theorem. Those who are interested in a modern-day version of such "Procrustes" factoring are encouraged to study the monumental work of Karl Jöreskog (1969,1970)on analysis of covariance structures.

Equations (1) and (2) can be shown to be a sufficient basis for defining other methods as well, but we shall omit discussion of such possibilities in order to proceed with applications associated with analysis of variance.

Experimental Methods

In this section Guttman's rank reduction theorem is used as a vehicle for generating sums of squares and cross-products which form the basis of computations for a wide variety of inferential tests in the analysis of variance. It will be seen that the algebra of correlational methods, as presented in the preceding section, subsumes the algebra of practically all of the standard forms of analysis of variance including those involving multiple dependent variables and covariates for fixed or random effects. Moreover, many (balanced) incomplete designs may be included within this paradigm.

It should be noted again in passing that questions of "meaningfulness" of sample statistics are not being considered here. Any

interpretative use of derived statistics involves distributional theory as well as assumptions about the nature of data for any given study. Needless to say, such factors should be given careful attention if any of the extant methods are to be employed in data analysis. Depending on what the investigator determines to be a reasonable choice of independent and dependent variables in a study, the procedures of this paper can be readily applied using a simple computer program as discussed in the final section.

Again it is convenient to consider an N x p matrix U_o which is partitioned into a set of $\underline{s}$ predictors X (usually called independent variables) and a set of $\underline{t} = \underline{p} - \underline{s}$ criteria Y (usually called dependent variables). Thus, $U_o = [\, X \,\vdots\, Y \,]$ and $V_o = U_o' U_o = \left[\begin{array}{c:c} V_{xx} & V_{xy} \\ \hdashline V_{yx} & V_{xx} \end{array}\right]$; again, also for convenience, columns of U_o will be taken to have zero means, which implies that V_o is a sum of products matrix or a covariance matrix. The rank reduction theorem of equations (1) or (2) will be used with weight matrices W or $W_1, W_2, \ldots, W_k$ of the same simple form $\begin{bmatrix} 0 \\ I_{s-j} \\ \Omega \end{bmatrix}$ as were employed in the foregoing section.

The reader may prefer to think of dependent variables Y as continuous random variables while X may best be viewed as a set of fixed <u>or</u> random variables. In particular, let independent variables be partitioned into a set of contrasts X_A (which may correspond to "planned comparisons", "main effects" or "blocking variables"), contrasts X_B (which may be associated with classical "interactions" of two or more main effects) and random variables X_C (which may be associated with "covariates" as in covariance analysis). Further, each of the subsets X_A, X_B and X_C will in turn be vertically partitioned to allow for designs of relatively more

complexity as when, say, three main effects (X_{A1}, X_{A2}, X_{A3}) are studied, using four interactions (X_{B12}, X_{B13}, X_{B23}, X_{B123}) and two sets of covariates (X_{C1}, X_{C2}).

The plan for the remainder of this section is to enumerate in paragraph form how the rank reduction theorem may be employed to generate sums of squares and cross-products for the major types of designs (one factor, factorial or nested each being with or without covariates) for either univariate analysis (ANOVA/ANCOVA) or multivariate analysis (MANOVA/MANCOVA). Attention will be initially restricted to orthogonal factorial designs. After discussing the formal algebra for major design categories several issues which pertain to applications are briefly examined. Reference is made to examples in the Appendix to facilitate the exposition.

One Factor (M)ANOVA

In the case of conventional one factor ANOVA or MANOVA, as in the fully randomized design, equation (1) of Guttman's theorem is sufficient to generate desired product matrices. The matrix X, or X_A, of U_o is taken as a set of $\underline{s} = J - 1$ contrast vectors when there are J groups Y is a set of $\underline{t}$ response variables; $\underline{t} = 1$ for univariate (ANOVA) studies or $\underline{t} > 1$ for multivariate (MANOVA) studies. Example 1 in the Appendix, which is also examined in Pruzek (1971), depicts contrast vectors for a simple four group MANOVA where $\underline{t} = 2$. The Guttman theorem may be used as

$V_1 = V_o - V_o W (W'V_o W)^{-1} W'V_o$ for $W = \begin{bmatrix} I_s \\ \underline{0} \end{bmatrix}$; the lower right $\underline{t} \times \underline{t}$ portion of the three respective matrices may be written $V_{(yy)} = V_{yy} - \hat{V}_{yy}$ or, in more conventional terminology, E = T - B, where T is the total sums of squares and products matrix, B is the between or among matrix of sums of squares and products; E is the residual or within matrix. Univariate

analyses correspond to these being 1 x 1 matrices, i.e. scalars. The particular scaling of columns of Y are irrelevant with respect to statistical tests since all standard test results are known to be invariant with respect to linear transformations of dependent variables.

It should also be noted the $\underline{s}$ contrast vectors may be associated with individual degrees of freedom; <u>if</u> the contrast vectors correspond to particularly "meaningful" comparisons of various combinations of groups, then equation (2) designates the usual <u>planned comparison</u> approach where $s_1 = \underline{s}_2 = \ldots = s_{J-1} = 1$. If the contrast vectors are not mutually orthogonal, then difficulties will typically arise in interpretation of the planned comparisons for a particular ordering of contrasts. Orthogonal planned comparisons are therefore apt to be most useful. The reader may wish to satisfy himself that the basic computations for many <u>post-hoc</u> tests may also be carried out within the present context.

<u>One Factor (M)ANCOVA</u>

The fundamental distinguishing feature of a covariance analysis is that the independent variable set includes one or more (antecedent) random variables in addition to fixed variables. For one-factor analysis of covariance, X is comprised of X_A and X_C matrices as in $X = (X_A \mid X_C)$. In this case two stages of equation (2) are required to generate matrices for the analysis of covariance. First, for a set of $\underline{s}_1$ covariates, use

$$W_{(1)} = \begin{bmatrix} \underline{0} \\ \underline{I}_{\underline{s}_2} \\ \underline{0} \end{bmatrix}$$

and note that the lower $\underline{t} \times \underline{t}$ portions of the three product product matrices in $V_{(1)} = V_o - V_o W_{(1)} (W'_{(1)} V_o W_{(1)})^{-1} W'_{(1)} V_o$ correspond to the equation $V_{(yy)(1)} = V_{yy} - \hat{V}_{yy}$ or, in standard terminology, $T^* = T - \hat{T}$ where $\hat{T}$ represents the portion of the total variation for $\underline{t}$ variables which is linearly associated with the covariates. While one might naturally

be led to find V_2 using equation (2) where the weight matrix selects contrast vectors in X, covariance analysis is not appropriately done this way since to do so would be to improperly specify regression coefficients. Rather one generates $V_1 = V_o - V_o W_1 (W_1' V_o W_1)^{-1} W_1 V_o$ where $W_1 = \begin{bmatrix} I_{s_1} \\ \underline{0} \end{bmatrix}$ for $\underline{s}_1 = J - 1$, the number of contrast vectors in X_A and follows by finding $V_2 = V_1 - V_1 W_2 (W_2' V_1 W_2)^{-1} W_2' V_1$ where $W_2 = \begin{bmatrix} \underline{0} \\ I_{s_2} \\ \underline{0} \end{bmatrix}$, i.e. $W_2 = W_{(1)}$. Since the lower $\underline{t} \times \underline{t}$ portion of V_1 is of the form $E = T - B$, it follows that the lower $\underline{t} \times \underline{t}$ portion of V_2 is $E^* = E - \hat{E}$ where $\hat{E}$ is the portion of the within groups variation linearly predictable from the set of $\underline{s}_2$ covariates. One factor MANCOVA thus requires T^* and E^* as well as $B^* = T^* - E^*$ as its sums of products matrices for any of the standard MANOVA tests with appropriate degrees of freedom. See Bock and Haggard (1968, p. 130) for a further discussion of this method. For an illustration consider Appendix example 1 where for purposes of illustration the 1st response variable is taken as a covariate with respect to the second response variable.

Factorial Design (M)ANOVA

Computations for factorial designs may be generated by using X_A and X_B matrices which are comprised of aggregates of contrast vectors. Any particular aggregate of contrasts may be viewed as specifying comparisons among defined groups of (vector) observations. Example 2 in the Appendix includes an X matrix which might be employed for a 2 x 3 factorial design. Reference will be made to this example in the following general discussion.

If all subgroups associated with a factorial design are of equal size, then contrast vectors for different main or interaction effects will be mutually orthogonal. This implies that the ordering of effects is

inconsequential with respect to the magnitudes of the derived sums of products matrices or the associated test statistics. If one desires a set of k_1 main effects, he simply specifies $X_A = \left[X_{A1}, \ldots, X_{Ak_1} \right]$ as a set of appropriate contrast vectors, as in example 2. Each aggregate of contrasts X_{Aj} corresponds to a row partitioning of (vector) observations in the dependent variable set Y. Successive applications of Guttman's theorem, following equation (2), generate V_o, V_1, ... V_{k_1} as well as the approximations $\hat{V}_o$, $\hat{V}_1$, ... $\hat{V}_{k_1}$; the lower right $\underline{t} \times \underline{t}$ portions of these products contain useful matrices for (M)ANOVA computations. The set V_{yy}, $\hat{V}_{(yy)1}$, $\hat{V}_{(yy)2}$, ..., $\hat{V}_{(yy)k_1}$ and the final residual $V_{(yy)k_1} = V_{(yy)} - (\hat{V}_{(yy)1} + \ldots + \hat{V}_{(yy)k_1})$ are simply the total sums of products for k_1 factors (main effects) and, ultimately, a residual matrix. If interaction effects are desired for any combination of effects, these are obtained using applications of equation (2) with respect to $X_B = \left[X_{B12}, X_{B13}, \ldots, X_{Bk_2} \right]$ for k_2 the number of interaction effects. Each aggregate of interaction contrast vectors may be found using elementwise products of entries in contrast vectors associated with a particular combination of main effects. The necessary sums of products matrices now will be V_{yy}, $\hat{V}_{(yy)1}$, ..., $\hat{V}_{(yy)k_1}$, $\hat{V}_{(yy)k_1+1}$, ..., $\hat{V}_{(yy)k_1+k_2}$ and the final residual $V_{(yy)k_1+k_2}$. The degrees of freedom for a particular effect are always equal to the number of vectors in a particular aggregate; this is true for main effects or interactions. See example 2 below and the analyses of this 2 x 3 design by Morrison (1967) as well as by Pruzek (1971). If, as is being assumed here, the subsets of contrasts in X are mutually orthogonal, then any $\underline{t} \times \underline{t}$ matrix of the form $\hat{V}_{(yy)j}$ will be the same, whether computed as $V_{j-1} W_j (W_j' V_{j-1} W_j)^{-1} W_j' V_{j-1}$ or as $V_o W_j (W_j' V_o W_j)^{-1} W_j' V_o$; the latter fact

may be especially useful in applications where only a computer program for product-moment coefficients is available. See Bock and Haggard (1968) for an excellent review of the statistical considerations for MANCOVA applications. Also see Mendenhall (1968) for more discussion of the linear models approach to ANOVA, and Graybill (1961) for technical details.

Factorial Design (M)ANCOVA

This class of methods simply adds a special wrinkle to the methods reviewed in the preceding paragraph. The major null hypothesis with which either ANCOVA or MANCOVA is concerned is that particular effects are null when dependent variables are linearly adjusted to the values they ostensibly would have if all relevant groups of entities were equivalent to one another on the antecedent variables. Since Bock and Haggard (1968) present the general matrix approach to MANCOVA, it is necessary here only to express their results in the terminology of this paper.

Once all of the sums of products matrices for effects of the preceding paragraph are computed, as well as a $\underline{t} \times \underline{t}$ residual matrix E, then one proceeds to generate a "corrected" residual matrix $E^* = E - \hat{E}$ where $\hat{E}$ represents the portion of the residual variation which may be attributed to the set of covariates in X_c and a special "corrected" total matrix for each effect of interest. A separate subtotal matrix must be computed for each effect in order properly to generate the minuend with respect to which E* is the subtrahend for producing the corrected effects matrix of sums of products. That is, $T_j^* = T_j - \hat{T}_j$ where $\hat{T}_j$ represents the portion of the matrix $T_j = E + \hat{V}_{(yy)j}$ which is linearly predictable from the set of covariates in X_c. Then $\hat{V}^*_{(yy)j} = T_j^* - E^*$ as required for the jth effect in MANCOVA. Bock & Haggard point out that any of the standard multivariate tests may be applied to the matrices $\hat{V}^*_{(yy)j}$ and F* using the usual degrees of freedom for effects and error degrees of freedom $d.f._e - k_3$ for k_3 the

number of covariates.

Nested Design MANOVA/MANCOVA

The concept of nesting has already been tacitly discussed in the foregoing paragraphs of this section. Whenever one replicates a particular experimental combination in order to generate a group of (vector) observations, the observations within the group are properly regarded as nested. If, as in orthogonal designs, one nests the same number of observations in every experimental group then the final residual matrix, say E, may be regarded as an aggregate of nested effects associated with entity variation within groups.

This concept may be generalized, however, as when a set of factors for a design is hierarchically arranged. The computations for nested designs are entirely straightforward with respect to the results already given for general factorial designs. The major distinguishing feature of computations for nested designs is that interaction effects have no meaning so that depending on the "tier" of a factor in a hierarchical design, computation of the effect matrix involves pooling contrast vectors which for factorial designs had been associated with interactions and with main effect contrast vectors. As an example if three factors 1, 2 and 3 are hierarchically arranged as 1, 2 within 1, 3 within 2 within 1, then contrast vectors for the three effects of this design would require three W_j matrices using equation (2) which selected $\left[X_{A1} \right]$, $\left[X_{A2} \vdots X_{B12} \right]$, and $\left[X_{A3} \vdots X_{B13} \vdots X_{B23} \vdots X_{B123} \right]$ of the contrast matrix X. See Winer (1962) for details of the discussion of computations and interpretations of effects for these designs. Multivariate extensions of his discussion are immediate.

Covariance analysis for nested designs could be done along the

same lines as discussed in the preceding paragraph were an investigator to regard this as useful.

Incomplete Design MANOVA/MANCOVA

The limited objective of this section is to present an approach to the analysis of balanced complete designs under the rubric, fractional factorials. These designs formally include latin square designs and balanced incomplete block designs so to cover fractional factorials in detail is to be reasonably comprehensive. It should be noted, however, that there are incomplete designs, particularly unbalanced incomplete set-ups, which do not lend themselves to a matrix-based analysis.

Fractional factorials may be characterized as factorials from which selected experimental combinations have been systematically deleted to allow study of a relatively large number of factors using a relatively small number of experimental units. The deletions amount to taking rows away from the design matrix X to yield X* so that contrasts remain in X* (i.e. means of columns of X* are zero). Thus, equation (2) of this paper suffices to generate sums of products matrices which are commonly used in statistical tests as in the paragraph on factorial designs above. For the most used variety of fractional factorials, the 2^{k-p} designs, the W_j matrix for the jth effect is simply a vector of the form $\begin{bmatrix} \underline{0} \\ \underline{1}_1 \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{1} \\ \underline{0} \end{bmatrix}$ where the value of unity corresponds to the jth contrast vector in X* -- the jth effect. See Mendenhall (1968) for a discussion of 2^{k-p} designs within the context of linear models and McLean (1966) for an example which, because the author gives raw observations, is readily susceptible to analysis using the approach suggested here.

As with nested designs, covariance analysis can be conducted for these incomplete designs using the methods of the paragraph on MANCOVA.

Although the foregoing discussion is abbreviated and largely restricted to computational methods, I do not wish to convey the impression that I condone a cavalier approach to choosing either independent or dependent variables. Some commentary regarding issues and general problems of experimental analysis may therefore be useful.

Regarding the approach generally, which is based on Guttman's rank reduction theorem, two references should be cited. Suits (1957) discussed the issue of dummy variables in regression pointing out that two or three different regression approaches may be taken to the specification of independent variables for the analysis of variance. It was made clear that the contrasts approach is not uniquely appropriate despite the fact that it lends itself to a particularly simple analysis for general applications and facilitates a comparison of experimental and correlational analyses of data. The second reference is Beaton (1964) which includes a discussion of a "Sweep" operator for matrix calculus which is in certain respects closely related to the present approach. Dempster (1968) discusses Beaton's operators in some detail and applies them in the analysis of multivariate data.

Another issue is that of selecting dependent variables for analysis. If an investigator is primarily interested in statistical tests, especially those which are likely to require less stringent assumptions about data than conventional parametric tests, he may study the approach of Gabriel and Sen (1968). These authors present a test statistic based on ranked scores which may be derived using the algebra of the present paper where each column of observations in Y is taken as a set of ranks. Their chi square statistic is shown formally to specialize to the Kruskal-Wallis statistic for ANOVA using ranks as well as the Wilcoxon rank scores test.

The more general problem of selecting metrics for dependent variables continues to be studied. In the case of experimental data, strong arguments have been made (see Tukey and Wilk, 1966) for transformations which reduce the portion of variance for non-additivity of a variable. That is, one should choose the metric for a variable in such a fashion that interaction effects will be relatively small in relation to main effects. Pruzek (1971) discusses the general issue of choosing dependent variables further with some attention to the issues of variable unreliability and choices among transformations. Various plots of residual values (see Wilk and Gnanadesikan, 1968) become essential when examining in detail the appropriateness of a model for analysis; in the next section a computational approach is outlined for computing residuals for various sums of products matrices of the present paper. Textbooks such as Mendenhall (1968), Winer (1962) and Hicks (1964) should be consulted for discussion of issues relating to the selection of independent variables in ANOVA, and Bock (1963) and Bock and Haggard (1968) are strongly recommended for discussion of the matrix approach to MANOVA and MANCOVA.

Computational Synthesis

The major objective of this paper has been to unify classical correlational and experimental approaches to the analysis of data with respect to the algebra of computation. It has been shown that a single theorem, originally due to Guttman (1944, 1952), can be seen to provide a vehicle for drawing these well-known classes of methods together. References have been given throughout the paper to facilitate the reader's further study of these relationships with respect to a wide variety of issues and problems which are commonly encountered in scientific research.

A computer program has been written to generate all of the matrices of equations (1) and (2) for Guttman's rank reduction theorem.

The program is presently written to read in data in the form of a cross-products (or covariance, or correlation) matrix V_o and a series of weight matrices, W_j, which are to be specified according to the user. (The program is being modified to read in "data" matrices of the form U_o and to allow "on-line" specification of a sequence of weight matrices; these improvements should substantially facilitate instructional uses of the present approach besides permitting more convenient computation of various residuals, as is noted below).

Outputs of the program are most conveniently labelled in the language of factor analysis; however, the reader may wish to make reference to either of the first two sections of this paper to help interpret entries of the respective matrices. Equation (1) corresponds simply to a special case of equation (2) so no further attention is needed for the simpler equation.

Given V_o and $W_1, W_2, \ldots, W_k$, the following matrices are included in the output:

(a) $V_{j-1} W_{\underline{n}j}$ for $j = 1, \ldots, k$

This is a pattern matrix for the primaries in general.

(b) $W'_{\underline{n}j} V_{j-1} W_{\underline{n}j}$ for $j = 1, \ldots, k$

This is the correlation matrix of the "primaries" in the language of factor analysis.

(c) $V_{j-1} W_{\underline{n}j} (W'_{\underline{n}j} V_{j-1} W_{\underline{n}j})^{-1}$ for $j = 1, \ldots, k$

This is the structure matrix for the "primaries"; entries are properly regarded as regression coefficients.

(d) $V_{j-1} W_{\underline{n}j} (W'_{\underline{n}j} V_{j-1} W_{\underline{n}j})^{-1} W_{\underline{n}j} V_{j-1}$ for $j = 1, \ldots, k$

This is an approximation, $\hat{V}_{j.1}$, to the matrix V_{j-1} which is based on variables which are selected by W_j.

(e) $V_j = V_{j-1} - V_{j-1}W_{\underline{n}j}\,(W'_{\underline{n}j}V_{j-1}\,W_{\underline{n}j})^{-1}W'_{\underline{n}j}V_{j-1}$ for $j = 1, \ldots, k$

This is a residual matrix associated with the comparison of V_{j-1} and $\hat{V}_{j-1}$. The lower $\underline{t} \times \underline{t}$ portion of this matrix may be of special interest in applications if W_j matrices are chosen in the fashion of this paper.

(f) $D_n\, V_j\, D_n$ for all $j = 1, \ldots, k$

This is a matrix of partial correlations associated with any pairs of variables for which the diagonal elements of V_j exceed zero.

The set of products (a), (b) and (c) may also be generated without scaling W_j to normalize V_{j-1}, as when raw score regression coefficients are desired.

Example 1 of the Appendix gives the initial $U_0 = \left[\, X \;\vdots\; Y \,\right]$ for a $\underline{t} = 2$ one factor MANOVA as well as the associated V_0 and the derived matrices $E = T - B$, each of order 2×2. Columns of X have been chosen to correspond with orthogonal polynomials since doing the analysis in this way allows quick hand checking of the results; $V_{xx} = X'X$ is diagonal as is V_{xx}^{-1}. (The reader may wish to satisfy himself that in this context orthogonal polynomial coefficients offer great potential for general applications of factorial design MANOVA regardless of whether factor "levels" are quantitatively related to one another.) The required (corrected) sums of squares for one-factor MANCOVA are also given for this example following the strategy of paragraph (2) in the second section. For this illustration the first dependent variable is used as a covariate with respect to the second.

The next example in the Appendix gives the raw data for Y and two alternate design matrices, labelled X_a and X_b, for a 2×3 factorial design

MANOVA. The data are taken originally from Morrison and are also included in Pruzek (1971). The V_o matrix is scaled to correlation metric and this V_o is used to derive sums of products matrices for testing the effects for Rows (SEXES), Columns (DRUGS) and Interaction (SEXES & DRUGS).

It should be clear from these examples that the algebra is equally appropriate for many more general applications of either correlation or regression analysis.

Finally, it should be helpful to indicate how one might generate matrices of residuals given matrices of the form V_j, V_{j-1} or $\hat{V}_{j-1}$, and the algebra of Guttman's theorem. Such quantities are easily derived using matrix equations.

The classical least squares estimates of factor (or component) scores in factor analysis may be expressed in present terms as

$$\hat{U}_o = U_o V_o^{-1} V_o W (W' V_o W)^{-1}$$

However, within the context of this paper, where estimates of dependent or criterion scores Y are of special interest, one can use the equation

$$\hat{Y} = Y_o V_{yy}^{-1} V_{yx} V_{xx}^{-1} V_{xy}$$

which may also be written

$$\hat{Y} = Y_o V_{yy}^{-1} \hat{V}_{(yy)j}$$

Thus, to generate residuals at any stage j', one needs only to compute $Y_o V_{(yy)j}^{-1} V_{(yy)j}$ which can be shown to be equivalent to $Y_o V_{(yy)j'}^{-1} (V_{j'-1} - V_{j'})$, for the differences between estimates at stage j'-1 and stage j'.

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APPENDIX

Example 1

Given the data in the upper portion of Table 1, the sums of products matrices, of order 2x2, were derived for error, total and between sources of variation. These matrices are designated E, T and B and are presented in the lower portion of Table 1. For covariance analysis, using the 1st dependent variable as a covariate for the second, three corrected sums of squares are $T^* = 28 - (40)^2/174$, $E^* = 3 - (4)^2/30$ and $B^* = 18.80 - 2.47 = 16.33$.

Example 2

Table 2 includes the raw scores for Y as well as the deviation scores Y (either of which will generate the desired effects matrices below) and two possible design (X) matrices for generating sums of products for a 2x3 MANOVA where $\underline{t} = 2$. The data are taken from Morrison (1968). Table 3 includes the V_o matrix scaled in correlation metric as well as the associated sums of products matrices from which statistical tests may be carried out (see Pruzek (1971) for an analysis of these data which follows these lines). The matrix X_a was used for this particular V_o.

Table 1

Independent and Dependent Variable Values and Sums of Products
For a $\underline{t}$ = 2 MANOVA with n = 2 for Each of 4 Groups

$$X \qquad\qquad Y$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \\ \hline 1 & -1 & -3 \\ 1 & -1 & -3 \\ \hline -1 & -1 & 3 \\ -1 & -1 & 3 \\ \hline -3 & 1 & -1 \\ -3 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ \hline 2 & -1 \\ 6 & -1 \\ \hline 1 & 0 \\ -5 & -1 \\ \hline -5 & -2 \\ -7 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 30 & 4 \\ 4 & 3 \end{bmatrix}, \quad T = \begin{bmatrix} 174 & 40 \\ 40 & 28 \end{bmatrix}, \quad B = \begin{bmatrix} 144 & 36 \\ 36 & 25 \end{bmatrix}$$

Table 2

An Example of Data and Two Possible Design Matrices for a 2 x 3 Orthogonal MANOVA where p = 2 and n = 4

	Y^*	X_a	X_b
I	5 6 5 4 9 9 7 6	(1 ¦ 1 0 ¦ 1 0)	(1 ¦ 1 1 ¦ 1 1)
II	7 6 7 7 9 12 6 8	(1 ¦ 0 1 ¦ 0 1)	(1 ¦ 0 -2 ¦ 0 -2)
III	21 15 14 11 17 12 12 10	(1 ¦ -1 -1 ¦ -1 -1)	(1 ¦ -1 1 ¦ -1 1)
IV	7 10 6 6 9 7 8 10	(-1 ¦ 1 0 ¦ -1 0)	(-1 ¦ 1 1 ¦ -1 -1)
V	10 13 8 7 7 6 6 9	(-1 ¦ 0 1 ¦ 0 -1)	(-1 ¦ 0 -2 ¦ 0 2)
VI	16 12 14 9 14 8 10 5	(-1 ¦ -1 -1 ¦ 1 1)	(-1 ¦ -1 1 ¦ 1 -1)

Table 3

Certain Matrix Products[1] Associated with
Morrison's 2 x 3 MANOVA with t = 2

$U_o'U_o$ (rescaled in correlation metric)

$$\left[\begin{array}{ccccc|cc} 1.00 & & & & & & \\ 1.00 & 1.00 & & & & & \\ .00 & .50 & 1.00 & & & & \\ .00 & .00 & .00 & 1.00 & & & \\ .00 & .00 & .00 & .50 & 1.00 & & \\ \hline .040 & -.765 & -.716 & -.173 & -.148 & 1.00 & \\ .060 & -.443 & -.258 & -.406 & -.295 & .714 & 1.00 \end{array}\right]$$

$$V_{yy} = \hat{V}_{(yy)1} + \hat{V}_{(yy)2} + \hat{V}_{(yy)3} + V_{(yy)3}$$

$$\begin{bmatrix} 1.00 & .714 \\ & 1.000 \end{bmatrix} = \begin{bmatrix} .002 & .002 \\ & .004 \end{bmatrix} + \begin{bmatrix} .733 & .355 \\ & .198 \end{bmatrix} + \begin{bmatrix} .035 & .078 \\ & .176 \end{bmatrix} + \begin{bmatrix} .230 & .279 \\ & .622 \end{bmatrix}$$

[1] For symmetric matrices, only upper-right portions are given

Errata for Pruzek's "Unification of Methodology"

Page	Correction	
5	Para. (8), l. 4.	Should read "computed as $\vec{q}_j = G_y \Lambda_y^{-1} \vec{v}_j$"
6	Line 1.	Should read "of V_{yy} and.."
7	Para. (11), last l.	Should read "with the $\underline{s}_j$' predictors."
13 & 16	Top p. 13, top and middle of p. 16.	Contrast matrices should be vertically partitioned, as in $(X_{A1} \vdots X_{A2} \vdots X_{A3})$.
17	l. 3.	Should read "considerations for MANOVA.."
19	l. 4.	Should read "balanced incomplete designs.."
24	Last equation & sentence which follows	$\hat{v}_j = Y_o Y_{yy}^{-1} \hat{V}_{(yy)j}$. and "$Y_o V_{(yy)j'}^{-1} \hat{V}_{(yy)j'}$ which can be shown to be equivalent to $Y_o V_{(yy)j'} (V_{j'-1} - \hat{V}_{j'-1})$"
Appendix	Ex. 2, l. 1 & 2	delete words between "raw scores for Y.." and "..and two possible design (X) matrices"
Table 3	footnote	Should read ".. only triangular portions are given." .. also upper portion of $U_o'U_o$ should read $\begin{bmatrix} 1.00 & \\ \underline{.00} & 1.00 \end{bmatrix}$ (etc.)