DOCUMENT RESUME

ED 050 047 24 SP 004 974

AUTHOR Talavage, Joseph
TITLE A Model for the Social Aspects of Classroom Organization. Final Report.
INSTITUTION Georgia Inst. of Tech., Atlanta.
SPONS AGENCY Office of Education (DHEW), Washington, D.C. Bureau of Research.
BUREAU NO BR-0-D-002
PUB DATE Oct 70
GRANT OEG-4-70-0010(057)
NOTE 43p.

EDRS PRICE EDRS Price MF-$0.65 HC-$3.29
DESCRIPTORS *Classroom Communication, *Classroom Techniques, Rewards, *Social Attitudes, *Student Teacher Relationship, Systems Concepts, *Teacher Behavior

ABSTRACT

An initial effort is made to investigate social aspects of the classroom within a mathematical framework called general system theory. The objective of the study is to set the stage for a theory of social behavior in the large which, when verified, may be employed to guide computer simulations of detailed social situations. A model of a goal-seeking and learning individual (a P-model) is constructed, at which point the interconnection of several such P-models (an n-group) is formalized. The notion of an n-group may represent teacher-class interaction. Analogously to other system-theoretic developments, the dynamic behavior of n-groups is investigated. In particular, the stability and controllability of mutually rewarding behavior in such groups are the objects of discussion. Further, the notion of "status" within an n-group is formalized and change in status is related to the learning capabilities of group members. Illustrative examples are given for each of these investigations in order to provide intuitive appeal to the formalism. The results of the investigation include theorems which state necessary and/or sufficient conditions for stability and controllability of n-groups. Though the conditions are somewhat restrictive, the framework for relating them to the aspects of dynamic behavior is established. Within this framework, the investigation may be extended in several directions and these recommendations for further action are included in the report. (Author)

Final Report

Project No. 0-D-002

Grant No. OEG-4-70-0010(057)

A Model for the Social Aspects of Classroom Organization

Joseph Talavage

Georgia Institute of Technology

Atlanta, Georgia

October 1970

The research reported herein was performed pursuant to a grant with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE

Office of Education
Bureau of Research

Table of Contents

Summary

An initial effort is made to investigate social aspects of the classroom within a mathematical framework called general system theory. The objective of the study is to set the stage for a theory of social behavior in the large which, when verified, may be employed to guide computer simulations of detailed social situations.

A model of a goal-seeking and learning individual, called a P-model, is constructed at which point the interconnection of several such P-models, called an n-group, is formalized. The notion of an n-group may represent teacher-class interaction.

Analogously to other system-theoretic developments, the dynamic behavior of n-groups is investigated. In particular, the stability and controllability of mutually rewarding behavior in such groups are the objects of discussion. Further, the notion of "status" within an n-group is formalized and change in status is related to the learning capabilities of group members. Illustrative examples are given for each of these investigations in order to provide intuitive appeal to the formalism.

The results of the investigation include theorems which state necessary and/or sufficient conditions for stability and controllability of n-groups. Though the conditions are somewhat restrictive, the framework for relating them to the aforementioned aspects of dynamic behavior is established. Within this framework, the investigation may be extended in several directions and these recommendations for further action are indicated in the appropriate sections of the report.

Introduction and analysis

The study of the social aspects of classroom organization may be broadly characterized as the study of interaction among learning and goal-seeking individuals. Such a viewpoint forms the conceptual basis of this report.

Due to the difficulties presented by experimental investigations of social behavior, it may be useful to establish simulations or models of such behavior, either via mathematics or on a computer. Computer simulations of social behavior have been developed by the Gullahorns (11) who, however, state that verification of such a model is not feasible. This difficulty with verification is apparently due to the very large number of computer runs that it would entail in order to reliably generalize the results. On the other hand, mathematical models inherently deal in such generalities and so would apparently present less difficulty in this regard with respect to verification. Once verified, a mathematical theory of social behavior provides a sound and rigorous foundation on which to build computer simulations of more detailed situations. In addition, some of the results derived from such a theory may provide insight and point to new directions for experimentation.

Mathematical models of social behavior were suggested by Simon (12) more than a decade ago. Those models were "numerical" in the sense that they employed differential equations. Recently, nonnumerical formalisms such as the algebraic models of Fararo (5) have been developed and used to investigate the notion of status in groups. The theory presented in this report is essentially nonnumerical and is employed largely to study the dynamic behavior of groups.

The following section of the report displays several examples of classroom behavior which serve as interpretations of later mathematical results. In sections B and C, the person-model and model of group interaction are developed within the framework of a mathematical system theory. The remaining sections consider several pertinent aspects of group behavior including its dynamic characteristics, where the latter development parallels that of other developments of system theory such as linear system theory.

A. Social aspects of classroom organization - examples.

The study of social behavior on a mathematical basis, whether in the classroom or elsewhere, requires that rigor is pursued at several different conceptual levels. At the broadest level, the effects of interaction among individual persons with different psychological characteristics is considered (this is the level at which "sociological results" are obtained). But prior to study at that "interaction-level", one must first formalize (i.e., make rigorous) what is meant by "psychological characteristics." Such a formalization may be said to constitute the "psychological-level" of study. If we wished, we could proceed on to a next level, perhaps called the "physiological-level", and so forth. Rather, this study begins at the psychological-level with an eye toward application of that formalism at the interaction-level.

The choice of the formalism at the psychological-level, herein-after referred to as the "person-model" or simply P, was influenced by the efforts of Homans (1,2), Miller, Pribram and Galanter (3), Kelly (4) and Fararo (5).

In particular, since Homans' more recent book considered certain aspects of elementary psychological behavior in a social context, it is these aspects which are explicitly represented in the person-model. Such aspects include the notions of reward, punishment (or withdrawal of reward), expectation, and decision-making behavior. In addition, notions such as change in expectation, perception of "justice," perception of "status," and change in perception may be incorporated into a more complete person-model.

To illustrate how Homans (2) employs the above notions, consider five of his typical propositions regarding social behavior (i.e., his propositions of behavior are at the interaction level; we will develop notions of interaction level behavior later):

(1) the more valuable (i.e., rewarding) to a man a unit of activity (e.g., response) another gives him, the more often he will emit activity rewarded by the activity of the other.

(2) a man in an exchange relation (i.e., in interaction) with another will expect that the rewards of each man be proportional to his costs. If this distributive justice fails, anger and/or guilt arises.

(3) if in the past the occurrence of a particular stimulus-situation has been the occasion on which a man's activity has been rewarded, then the more similar the present stimulus situation is to the past one, the

more likely he is to emit the activity, or some similar activity now.

(4) the more often within a given period of time a man's activity rewards the activity of another, the more often the other will emit the activity.

(5) withdrawal or holding back of a reward is a punishment.

In addition, Kelly's work seems to imply another proposition:

(6) it is rewarding to receive an activity which is expected in the light of previously emitted activities.

These propositions may be used to provide insight into several forms of classroom behavior.

With regard to the notion of status, we may consider a typical classroom situation. Studies indicate that status is essentially an unknown concept to first through about fifth graders. It does, however, become evident during about the sixth grade that what people think of a child matters to the child. We shall consider the simple premise that to the average person it is rewarding to him if his status is increased, and costly if it is decreased.

With this simple premise, we can gain insight into the role status can play in the classroom if we note that an instructor can manipulate the status of students and thereby control their behavior.

For example, suppose an instructor desired to limit the interruptions during his lectures. He may believe that questions asked during class slow down the progress of the class. It often happens that a question is relevant to one or two students, while the others already know the answer. In this case lecture time is taken to teach only a few.

If a student were to ask a question, and he were made to look foolish (i.e., incurred a decrease in status) for asking this question, then the "punishment" version of Homan's proposition (4) predicts that he will be less likely to ask questions in the future. For example, student A may politely ask the instructor a question during the lecture to which the instructor replies, "Obviously Mr. A, you haven't been keeping up with the outside reading. If you will please read the assigned material, we won't have to waste the rest of the class's time, while I bring you up to date on the things you should be learning out of class." Without answering the question, the instructor then returns to his planned lecture. Student A incurred a cost to ask this question. Even though the other students may not have known the answer to the question either, student A has lost

some status in the exchange with the instructor. The instructor in this example has manipulated the status of one of his students to discourage questions during the lecture.

This example is considered again later in this report, but in a much more rigorous context.

Consider a second example of the application of propositions (1) - (6) to the classroom situation, with particular regard to the evaluation of teaching performance. That is, the opinion of the evaluator (e.g., principal or department chairman) may be shaded by any of the following:

1. The opinions of other teachers

2. The like or dislike which the students show toward the teacher

3. The parents' opinion of the teacher

4. The behavior and interest displayed in faculty meetings

5. The teacher's willingness to give time to extra-curricular activities connected with the school.

Since a teacher may be aware that he is evaluated and rewarded on the above criteria, then Homans propositions (1), (3) and (4) predict that these aspects may be of overriding significance to the teacher. In that case, his teaching efficiency may suffer. For example, it may be that, in certain teaching situations, the teacher has to require an unusually large amount of hard work from the students. This may even require the teacher placing the students in a stress situation. When these situations occur, the students' opinions of the teacher will likely worsen. This may in turn cause parent opinion and colleague opinion to suffer. If the students, parents, and other teachers complain to the principal or department chairman about this teacher, then his opinion of this teacher may well be reduced. Therefore, in cases where strict academic discipline is required, the teacher may be in a dilemma. If he does the best thing for the students, he harms himself; if he does the best thing for himself, his students may suffer.

This conflict between teacher and student behavior may result in a "non-cohesive" class-teacher relationship which is the subject of a theorem later in this report.

Finally, consider a situation where the presentation of knowledge to a class alters the social position of certain class members. That is, suppose that a boy impresses his friends with his ability to win female companionship. It is Homans' conjecture that such a

group member, if he offers his rare and valuable skill in exchange for other activities, will obtain "status" in the group, where this status is considered rewarding by the skill-holder. For example, the boy's friends may do him favors in return for introductions to his girl-friends. If now information regarding this skill is presented to all other group members in such a way as to teach them the skill (perhaps via a "dating" course), then that skill is obviously no longer rare in which case the status of the original skill-holder decreases in perhaps a dramatic fashion (depending upon what other rare and valuable skills he may possess). This decrease in status constitutes a withdrawal of reward, which Homans indicates is usually accompanied by anger. Thus, the distribution of knowledge in the group created some unrest in the group in addition to the beneficial effects of educating the group members with respect to a particular skill.

The above example will be formalized later and a theorem displaying the reduction of status will be derived.

B. Development of a person-model

Apparently, the aforementioned psychological notions such as reward, punishment, etc. are directly pertinent to Homans-type social behavior, and so they are taken as a minimal set of notions to be explicitly represented in the person-model. The structure of the person-model P is such that each of the pertinent notions is related to a "subsystem" and these subsystems are interconnected in a logical manner. In addition to the influence of other authors from a psychological standpoint, the structure of P was also affected by mathematical considerations. For example, the author's experience has been such that in order to achieve an elegant presentation (roughly, "elegance" is synonymous with "as few symbols as possible," and is desirable for clarity and efficiency), no feedback structures should be incorporated explicitly into P (6). It should be emphasized that feedback is implicit in some of the subsystems of P and so its effect is present. However, the structure itself is "hidden."

The block diagram below shows the basic structure of P.

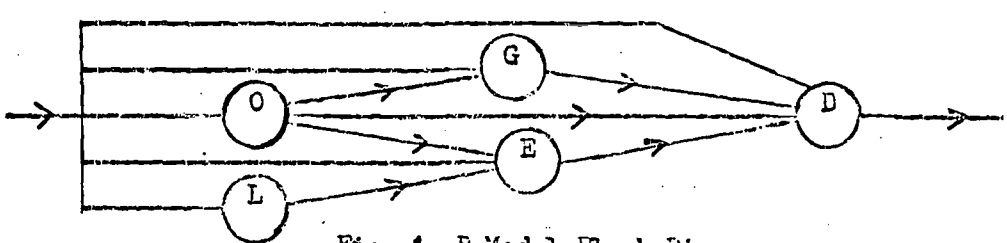

Fig. 1 P-Model Block Diagram

Consider the manner in which each of the subsystems relates to one or more pertinent psychological notions;

(a) the system O (a subsystem is also a system) represents the internal activity of generating candidates for actual response to an input "event," where such an "event" may be spoken word or sentence or perhaps a visual stimulus. The set of responses is usually limited in real life situations to responses that are appropriate to the event, either socially or otherwise. For example, a possible response to the statement "thank you" is another statement such as "you're welcome." It is quite unlikely however that one would respond to "thank you" by flapping one's arms like a bird while at the same time yodeling and stomping one's feet.

(b) the system G represents the reward-seeking (and punishment-avoiding) activity of P. This activity is assumed to place the current input event (i.e., current context) in association with each response generated by O so as, in essence, to complete the following statement: "given the current input event, if I were to exhibit this response, then I would like (not like) to receive in return an input event according to the following ordering of such events _____." Thus, G associates two orderings with the current input event and each candidate for response. One of these orderings is associated with reward-seeking and so places one input event above another if the former element is perceived as more rewarding than the latter. The other ordering places one input event above another if the former element is seen as more punishing, or more to be avoided, than the latter. It may appear that each of these orderings is the inverse of the other; however, the notions of "reward" and "punishment" are ill-defined here and so prevent a proof of such a hypothesis.

Note again that G associates two orderings to each candidate for response. Thus, for four candidates for response, the output of G displays eight orderings.

(c) the system E represents the forecasting or predictive activity of P based upon "experience." As in G, the input event is associated with each generated response, but in this case the essence of operation is to complete the statement; "given the current input event, if I were to exhibit this response, then I expect to observe in return an input event according to the following (subjective) probabilities on the input events _____." The system E associates to the current input event and each candidate for response a list containing the input events and the probability of occurrence of each element. So for the set of responses from O, E produces one such list for each response.

One of the ways that "learning" may occur is via a change in expectation, either in the event itself or in its probability number. Such a change may come about through "experience" (i.e., observing and assimilating the "world's" reaction to one's responses over a period of time). On this basis, the system L is postulated.

(d) the system L represents the activity of changing expectations based upon the history of the input events (and memory of past responses). This capability of P provides for "dynamic" behavior at the interaction-level as will be indicated later.

(e) the system D represents the decision-making activity of P. That is, D employs the current input event, the generated responses, the rewards (punishments) that are being sought (avoided), and the likelihood of receiving such rewards and punishments, so as to select <u>one</u> of the generated responses. For example, given the aforementioned information, D may select the response which provides the most desirable reward that has an associated high probability of occurrence. Such behavior on an economic basis may be characterized as maximizing expected utility.

The discussion to this point has been largely intuitive so as to give the reader a "feeling" for the structure and behavior of P. Let us now consider the formalization of P in mathematical terms, using Windeknecht's development of system theory (7). (See app. a).

A person-model P is an ordered septuple (I, R, O, G, L, E, D) where;

(i) I and R are sets representing sets of input events and responses, respectively. (Strictly speaking, we should differentiate between the events and responses themselves versus their <u>representation</u> within P. However, that will be left as a refinement for a later model.)

(ii) $O: I^N \rightarrow (\pi_R)^N$ such that for all $x \in \mathcal{D}O$ and all $n \in N$, $O(x)(n) = o(x(n))$ where $o: I \rightarrow \pi_R$.

Here, the function o represents the activity of generating candidate responses for a given input event.

(iii) $G: (I * \pi_R)^N \longrightarrow (\pi_{R * \pi_{I * I}} * \pi_{R * \pi_{I * I}})^N$

such that for all $x \in \mathscr{D}G$ and all $n \in N$,

$$G(x)(n) = (g(x(n)), \bar{g}(x(n))) \text{ where}$$
$$g: I * \pi_R \longrightarrow \pi_{R * \pi_{I * I}} \text{ and}$$
$$\bar{g}: I * \pi_R \longrightarrow \pi_{R * \pi_{I * I}}$$

The function g represents the activity of associating a "reward"-ordering of I with each candidate for response. Thus, the above expression for g is to be interpreted such that the elements of R which appear in the argument of g also appear in the result from application of g, and further that each subset of $I * I$ which results from an application of g satisfies the axioms of a partial ordering relation.

Similar remarks may be made for the function $\bar{g}$.

(iv) $L_q: I^N \longrightarrow Q^N$ such that for all $n \in N$ and all $x \in \mathscr{D}L_q$,

(i) $L_q(x)(0) = q$

(ii) $L_q(x)(n + 1) = (x(n), L_q(x)(n))$

where Q is a set and

$$\lambda: I * Q \longrightarrow Q.$$

Here, Q represents the set of "memory-states" reflecting the past experience of P, especially regarding expected rewards versus actual input events. This experience will be employed to affect the expectation of P. The function λ represents the activity of changing the memory-state on account of the current input event (i.e., the current input event may represent a reward or punishment for a previous response, and this reward or punishment may have been expected or quite unexpected). The memory-state q represents the initial memory-state of P at the beginning of our observation. Therefore

$$L = \bigcup_{q \in Q} L_q$$

(v) $E: (I * Q * \pi_R)^N \longrightarrow (\pi_{R * \pi_{I * I}})^N$ such that

for all $x \in \mathscr{D}E$ and all $n \in N$, $E(x)(n) = e(x(n))$ where

$$e: I * Q * \pi_R \longrightarrow \pi_{R * \pi_{I * \bar{1}}}$$

and $\bar{1}$ represents the closed interval on the real line, [0,1].

The function e represents the activity of associating a probability of occurrence to each element of I, given a candidate for response from R and a current input event and a specified state of knowledge. The result from e is to be interpreted as a set containing several subsets of $I * \bar{I}$, one for each candidate response, where each such subset of $I * \bar{I}$ is a <u>function</u> from I to $\bar{I}$.

(vi) $D = D' \cdot M$ where

(a) $D' : (I * \pi_R * \pi_R * \pi_{I * I} * \pi_R * \pi_{I * I} * \pi_R * \pi_{I * \bar{I}})^N \rightarrow$

$(\pi_{(I * R)} * (I * R))^N$ such that for all $x \in \mathcal{D}D'$ and all $n \in N$,

$D'(x)(n) = d'(x(n))$ where

$d' : I * \pi_R * \pi_R * \pi_{I * I} * \pi_R * \pi_{I * I} * \pi_R * \pi_{I * \bar{I}} \rightarrow$

$\pi_{(I * R)} * (I * R)$

(b) $M : (\pi_{(I * R)} * (I * R))^N \rightarrow R^N$ such that for all $x \in \mathcal{D}M$

and all $n \in N$, $M(x)(n) = m(x(n))$ where $m : \pi_{(I * R)} * (I * R) \rightarrow R$

The functions d' and m represent the decision making activity of P insofar as P employs the outputs of the subsystems O, G and E to select a response. As shown, the subset of $(I * R) * (I * R)$ which results from d' is assumed to satisfy the axioms of a simple ordering relation with respect to R. Thus, d' produces an <u>ordering</u> of responses (and the most desirable reward associated with each response by g or most undesirable punishment associated via $\bar{g}$), where this ordering is obtained via the <u>strategy</u> of P (e.g., ordering via expected utility). That is, an element in $I * R$ is above another element in $I * R$ if the response in the former pair is perceived as more goal-achieving (i.e., more reward-achieving or punishment-avoiding) than the response in the latter pair. Then, the function m represents the selection of the greatest pair in the ordering and produces the response contained within that pair.

The interconnection of the subsystems may be formalized as follows:

$$P = (\ (\ (O \# G) \ // \ (\ (\ O \ // \ L) \# E\ ) \ // \ O) \# D)$$

The expression on the right-hand side of the "equals" sign represents the block diagram interconnection for the person-model as shown previously. This expression, denoted by P, is the mathematical representation of the person-model. Note that this representation incorporates the notions of stimulus, response, reward, punishment, expectation, decision-making strategy, and "learning" (where the last notion is limited to a <u>change</u> in expectation). An example of person-model behavior will be given in the latter part of this section. P, as a <u>relation</u>, may be characterized by the following conditions on the ordered pairs of time functions;

for all $x \in I^N$ and all $y \in R^N$ and all $q_o \in Q$ and all $y_o \in R$,

$(x,y) \in (P)_{(q_o,y_o)}$ if and only if (i) $x \in I^N$

(ii) $(\exists w \in Q^N)$ $(\forall n \in N)$ $(w(0) = q_o$ and $w(n+1) = \lambda(x(n), w(n)))$ and (iii) for all $n \in N$, $y(0) = y_o$ and $y(n+1) = m(d'(x(n), o(x(n)), g(x(n), o(x(n))), g(x(n), o(x(n))), e(x(n), o(x(n)), w(n))))$.

Let $y(n+1) = f(x(n), w(n))$ where $f: I * Q \longrightarrow R$ as in condition (iii) above.

Since we wish the model to apply for <u>all</u> initial states $q_o \in Q$ and <u>all</u> initial outputs $y_o \in R$, let $P = \bigcup_{(q_o,y_o) \in Q * R} (P)_{(q_o,y_o)}$.

Technically, the systems O, G, E, D' and M are <u>static</u> systems with transfer functions as specified. Thus, the capacity for learning resides mainly within system L.

The model P represents one out of many possible structures for simulating the behavior of a person. P does not necessarily represent the manner in which "thinking" occurs. Rather, P represents a logical structure which incorporates several significant psychological notions and which, to an <u>external</u> observer recording only input events and output responses, might appear to exhibit behavior similar to a person.

Our aim is to study this <u>external</u> behavior for several person-models <u>in interaction</u> (i.e., in groups), and then to relate their external behavior to the <u>internal</u> characteristics of each person-model. It may ultimately be possible to <u>predict</u> group behavior given the characteristics of each group member.

As a means of displaying the operation of the P-model, consider a formalization of the previous example concerning manipulation of status.

With the interconnected subsystems for P as described above, we may display an example of the operation of a P-model. Consider a hypothetical course in linear programming at Georgia Institute of Technology for which the class is in the middle of the quarter and has just completed the chapter on duality in linear programming. Assume that there are n possible inputs which can be given to the professor. Therefore in terms of the model, $I = \{i_1, i_2, i_3 \ldots, i_n\}$.

Assume that there are m possible different responses that the professor can emit such that

$$R = \{r_1, r_2, \ldots, r_m\}.$$

As the professor is beginning to start his lecture (at time zero, say), some student asks the following question which is the input for the model at time zero.

i_{97} = Why would one ever apply the dual simplex algorithm when the primal simplex will solve all linear programming problems?

Note: The above input is i_{97} in the set I.

System O $\quad O: I \rightarrow \pi_R$

Suppose that of all the responses to i_{97}, the professor thought only the three r_{14}, r_{53}, and r_{82} to be appropriate where

r_{14} = The dual simplex algorithm is more efficient in some applications when solving an LP problem than the primal simplex algorithm. This often occurs when one must add artificial variables to the initial tableau. Mr. Student, if you had worked homework problem 3 in the last assignment, you would know the value of the dual simplex algorithm.

r_{53} = That is a very good question Mr. Student. It is certainly not obvious, but in certain applications, the dual simplex proves to be more efficient than the primal simplex. This is often the case when one must add artificial variables to obtain a starting point for the primal simplex. Excellent question! Any other questions?

r_{82} = Obviously, Mr. Student, you haven't been keeping up with the homework problems. If you will please work the assignments, I won't have to waste lecture time bringing you up to date.

Thus $o(i_{97}) = \{r_{14}, r_{53}, r_{82}\}$

System E $\quad e: I * Q * \pi_R \rightarrow \pi_R * \pi_{I * \bar{I}}$

In practical situations, the set I may be very large. For simplicity, this example deals with only those elements of I for which the subjective probability of occurrence is greater than 0.01. Suppose

i_{201} = The attention of all members of the class and no questions

i_{102} = Another question

i_{53} = An expression of boredom from some members of the class

i_{34} = An expression of animosity from some members of the class

and the conditional probabilities associated with the appropriate elements of R are as follows:

$Pr\{ i_{201} \mid r_{14} \} = 0.75$	$Pr\{ i_{102} \mid r_{14} \} = 0.24$
$Pr\{ i_{53} \mid r_{14} \} = 0.00$	$Pr\{ i_{34} \mid r_{14} \} = 0.00$
$Pr\{ i_{201} \mid r_{53} \} = 0.15$	$Pr\{ i_{102} \mid r_{53} \} = 0.75$
$Pr\{ i_{53} \mid r_{53} \} = 0.09$	$Pr\{ i_{34} \mid r_{53} \} = 0.00$
$Pr\{ i_{201} \mid r_{82} \} = 0.95$	$Pr\{ i_{102} \mid r_{82} \} = 0.01$
$Pr\{ i_{53} \mid r_{82} \} = 0.00$	$Pr\{ i_{34} \mid r_{82} \} = 0.03$

The probabilities for each later input event were chosen arbitrarily, but it is likely that they reflect some similarity to reality.

Notice that $\sum_k [Pr\{i_k \mid r_j\}] = 0.99 \quad j = 14, 53, 82.$

This implies that there may exist later inputs not considered here. Even though they are not included, their expectation probabilities are so small that we may assume they do not influence the decision process.

The system E produces at time zero an output of the following form:

$$e(i_{97}, q_o, \{ r_{14}, r_{53}, r_{82} \}) =$$

$$\{ \{ r_{14}, \{(i_{201}, 0.75), (i_{102}, 0.24)\} \},$$

$$\{ r_{53}, \{(i_{201}, 0.15), (i_{102}, 0.75), (i_{53}, 0.09)\} \},$$

$$\{ r_{82}, \{(i_{201}, 0.95), (i_{102}, 0.01), (i_{34}, 0.03)\} \} \}$$

where q_o represents the present set of expectations of the professor, and where for example the expectations associated with r_{14} are listed first, those associated with r_{53} are listed second, etc.

<u>System G</u> $\quad g: \; I * \pi_R \longrightarrow \pi_R * \pi_{I * I}$

It is convenient to assume that, for r_{14} and r_{53}, g orders the elements of I linearly according to increasing subscript numbers whereas for r_{82}, g produces the inverse ordering. Thus

$g(i_{97}, \{ r_{14}, r_{53}, r_{82} \}) =$

$\{ \{r_{14}, \{ (i_{201}, i_{201}), (i_{201}, i_{102}), (i_{201}, i_{53}), (i_{201}, i_{34}),$
$(i_{102}, i_{102}), (i_{102}, i_{53}), (_{102}, i_{34}), (i_{53}, i_{53}), (i_{53}, i_{34}), (i_{34}, i_{34})\} \},$
$\{ r_{82}, \{ (i_{34}, i_{34}), (i_{34}, i_{53}), (i_{34}, i_{102}), (i_{34}, i_{201}),$
$(i_{53}, i_{53}), (i_{53}, i_{102}), (i_{53}, i_{201}), (i_{102}, i_{102}), (i_{102}, i_{201}), (i_{201}, i_{201})\}\},$
$\{ r_{53}, \{ (i_{201}, i_{201}),$ --- etc.

The effect of $\bar{g}$ will not be specified in this example.

<u>System D</u> Let us assume that the decision process is based upon utility theory (8) whereby a number may be associated with each element of I in each ordering from g. Assume this number to be 4 if the element is highest in the ordering, 3 if it is second highest, etc. Then the following decision tree describes the decision process, neglecting the effect of $\bar{g}$.

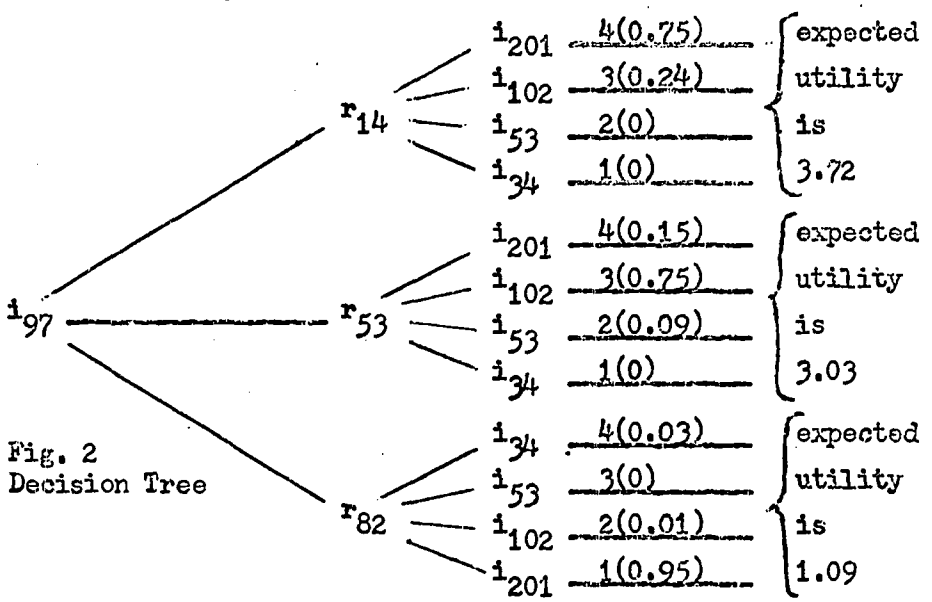

Fig. 2
Decision Tree

Apparently, the most promising candidate for response is r_{14} for which the most desirable input event in return is i_{201}. The second most promising candidate for response is r_{53}. Thus, at time zero,

$d'(i_{97}, \{ r_{14}, r_{53}, r_{82} \}, g(i_{97}, \{r_{14}, r_{53}, r_{82}\}), e(i_{97}, q_o, \{ r_{14}, r_{53}, r_{82} \}))$
$= \{ ((i_{201}, r_{14}), (i_{201}, r_{14})), ((i_{201}, r_{14}), (i_{201}, r_{53})), ((i_{201}, r_{14}), (i_{34}, r_{82})),$
$((i_{201}, r_{53}), (i_{201}, r_{53})), ((i_{201}, r_{53}), (i_{34}, r_{82})), ((i_{34}, r_{82}), (i_{34}, r_{82}))\}$

where q_o is professor's state of knowledge at time zero. Clearly then,

$m(d'(i_{97}, - - - , e(i_{97}, q_o, \{ r_{14}, r_{53}, r_{82} \}))) = r_{14}$.

Thus, the professor chooses to emit response r_{14} at time instant one. Meanwhile, i_{97} also causes the professor to change his state of knowledge from q_o. This change is not exemplified here.

C. Development of an interaction model.

In order to model interaction of person-models so as to simulate group behavior, the notion of "interaction" itself need be formalized. This notion can be represented for two person-models by the following diagram:

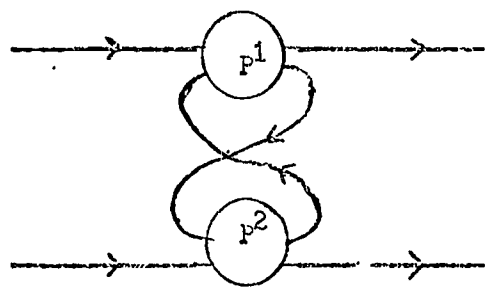

Fig. 3
2-Group Model
Block Diagram

The above diagram represents interaction between P^1 and P^2 insofar as it is characterized by the following properties;

(i) P^1 can emit events to P^2 and can receive them from P^2, and similarly for P^2 re P^1,

(ii) both P^1 and P^2 may receive events from the "environment" and both may emit events to that environment.

Thus, property (i) provides for "intra-group activity" while property (ii) allows for the presence of task inputs from the environment. The interaction diagram may be considered from several points of view by an external observer;

(a) the observer may have access only to the events received from the environment and those emitted to it,

(b) the observer may see only those events emitted from P^1 to P^2 and vice versa,

(c) the observer may note the events received from the environment as well as both the events emitted to the environment, and the events emitted by P^1 to P^2 and by P^2 to P^1,

(d) the observer may have access to the events received from the environment as well as both the events emitted to the environment and the events received by P^1 from P^2 and by P^2 from P^1.

It is assumed in this model that all events emitted by P^1 are received by P^2, and vice versa. Thus, viewpoints (c) and (d) above are equivalent.

Consider the formalization of viewpoint (c). Designate the person-models as P^1 and P^2. Therefore, the set of input events to P^1 is called I^1, and the corresponding set for P^2 is called I^2. But note that, <u>in interaction</u>, P^1 may receive events from two sources; namely, from the environment and from P^2. Denote this situation by defining $I^1 = I^1_1 * R^2_1$. That is, the input events to P^1 may now occur in pairs, where the first member of the pair is the environmental input event and the second member of the pair is an event emitted from P^2 to P^1 (thus the superscript "2" and subscript "1"). In a similar manner, we can define $I^2 = R^1_2 * I^2_2$. Repeating this procedure, we define $R^1 = R^1_1 * R^1_2$ and $R^2 = R^2_1 * R^2_2$. It is not assumed that $R^1_2 = R^2_1$ so P^1 may emit events that P^2 cannot, and vice versa. Thus, difference in skill or intellectual level is maintained in the model.

Denote the observation of interaction of P^1 and P^2 from viewpoint (c) as $P^1 \wedge P^2$. Now $P^1 \wedge P^2$ is a relation, i.e., a set of ordered pairs of time functions. The nature of this relation is as follows,

for all $x \in (I^1_1 * I^2_2)^N$ and all $y \in (R^1 * R^2)^N$, and all $q \in Q^1 * Q^2$ and all $r \in R^1 * R^2$, $(x,y) \in (P^1 \wedge P^2)_{(q,r)}$ iff

(i) $x \in (I^1_1 * I^2_2)^N$

(ii) $(\exists w \in (Q^1 * Q^2)^N)(w(0) = q$ and for all $n \in N$

$w(n+1) = (\lambda^1((x_1(n), y^2_1(n)), w_1(n)), \lambda^2((x_2(n), y^1_2(n)), w_2(n))))$

(iii) $y(0) = r$ and $y(n+1) = (f^1((x_1(n), y^2_1(n)), w_1(n)),$ $f^2((x_2(n), y^1_2(n)), w_2(n))))$, where λ^1, Q^1 and f^1 refer to P^1, and similarly for P^2, and where the notation w_1, x_1, y^2_1, w_2, x_2 and y^1_2 is as defined in the appendix b. w is called the (q,r)-state-trajectory.

Since we wish this relation to hold for all initial states $q \in Q^1 * Q^2$ and all initial outputs $r \in R^1 * R^2$, define

$$P^1 \wedge P^2 = \bigcup_{(q,r) \in (Q^1 * Q^2) * (R^1 * R^2)} (P^1 \wedge P^2)_{(q,r)}$$

Note that

(a) x_1 and x_2 are the time functions of first and second members, respectively, from x.

(b) for all $n \in N$, $w(n)$ is an ordered pair with first member being the "memory-state" of P^1 at time n, and second member being the "memory-state" of P^2 at time n.

It is of interest to glean from the above expression that the "memory-state" and the emitted response of P^1 depend upon the event emitted by P^2, and vice versa. But these are two of the relationships implied by the intuitive statement that "P^1 and P^2 are interacting". Therefore, $P^1 \wedge P^2$ has some intuitive appeal as a representation for "group interaction."

We may simplify our formal consideration of $P^1 \wedge P^2$ by making the following definitions of functions associated with $P^1 \wedge P^2$;

(a) $\lambda: (I^2 * I^1) * (Q^1 * Q^2) \longrightarrow Q^1 * Q^2$ such that

$$\lambda((i', i), (q, q')) = (\lambda^1(i, q), \lambda^2(i', q'))$$

Note that λ represents the "learning" function of $P^1 \wedge P^2$.

(b) $f: (I^2 * I^1) * (Q^1 * Q^2) \longrightarrow R^1 * R^2$ such that

$$f((i', i), (q, q')) = (f^1(i, q), f^2(i', q')).$$

Note that f represents the "output" function of $P^1 \wedge P^2$.

Henceforth, $P^1 \wedge P^2$ will be designated as a "2-group." The above definition can be extended to an "n-group", i.e., $P^1 \wedge P^2 \wedge \cdots \wedge P^n$, in a straightforward manner.

The formalization of $P^1 \wedge P^2$ (and its extension to $P^1 \wedge P^2 \wedge \cdots \wedge P^n$) provides a framework for the formal study of both the static and dynamic characteristics of group behavior. That is, the static behavior of the model is that behavior which is independent of time, and so is not associated with a learning capability for any of the individuals. This sort of behavior approximates the "steady state" or "equilibrium" situation for a group. Thus, the conditions under which a group may exhibit cohesive (as opposed to clique-ish) behavior in equilibrium may be studied in terms of the characteristics of the individuals that make up the group. In the next section of this report, the conditions on the individual goals and expectations of each member of a dyad are given such that the dyad be a cohesive

group in steady state. It should be noted that the investigation of static behavior as given here is related to the "algebraic" models used by other authors.

The dynamic characteristics of group behavior are those which involve time as a dependent variable, and so can be associated with a learning capability for some or all of the individual members. This type of behavior approximates the situations in groups where at least one member is still assimilating information and possibly modifying his behavior accordingly. Results from systems theory indicate that the most pervasive dynamic characteristic of systems is that of stability. That is, behavior is stable when it "settles down" to some steady state or equilibrium. Later in this report, necessary conditions on the relationship between the learning capabilities of each member of a dyad are given so that the dyad may settle down into some steady state situation. Another important characteristic of dynamic systems is the notion of controllability. The behavior of a group may be said to be controllable if the "state" of the group may be modified (e.g., changed from "clique-ish" group to "cohesive" group) by some sequence of "external" or "environmental" inputs. Later in this report, the controllability of n-groups is investigated.

In addition to the study of the static and dynamic characteristics of group behavior, the formalization of n-groups allows the precise definitions of such notions as status, justice, power, and other notions which are relevant to group behavior. Further, it allows the derivation of relationships among these notions in a rigorous manner. For example, it is shown later that certain "control" inputs to a group whose members have learning capabilities can affect the status of specific group members.

D. Property of interaction model behavior: cohesiveness

$P^1 \wedge P^2$ represents behavior at the interaction-level. What sort of behavior at this level is of interest?

Results obtained in general systems theory indicate that the most pervasive property of systems is that of stability. System behavior may be called "stable" whenever the behavior is maintained over a long period of time, or is repetitive over such a period of time.

Since stability is concerned with behavior that is maintained over a long period of time, then, in a social context, we may be most interested in stable behavior that is "good" behavior, i.e., rewarding behavior. Consider the conditions necessary and/or sufficient for the group behavior to be mutually rewarding, in which case the group may be said to be "cohesive."

For the interaction model designated $P^1 \Lambda P^2$, the conditions for stable cohesive behavior depend upon the environment input events and the learning processes of P^1 and P^2. Initially, let us simplify our considerations by assuming that the effects of the environment input and output events and the learning processes are negligible, in which case, static behavior is being investigated.

Mathematically, these assumptions correspond to the requirements that (a) I_1^1 and I_2^2 and R_1^1 and R_2^2 are unit sets and (b) that Q^1 and Q^2 are unit sets, respectively. Then, the subsystem transfer functions in each person model may be simplified in the following manner:

for $i \in \{1,2\}$ and $j \in \{1,2\}$ and $i \neq j$,

(a) $o^i : R_i^j \longrightarrow \pi_{R_j^i}$

(b) $g^i : R_i^j * \pi_{R_j^i} \longrightarrow \pi_{R_j^i} * \pi_{R_i^j} * R_i^j$ and similarly for $\bar{g}$.

(c) λ^i is a constant function and may be ignored,

(d) $e^i : R_i^j * \pi_{R_j^i} \longrightarrow \pi_{R_j^i} * \pi_{R_i^j} * \bar{1}$

(e) $(d')^i : R_i^j * \pi_{R_j^i} * \pi_{R_j^i} * \pi_{R_i^j} * R_i^j * \pi_{R_j^i} * \pi_{R_i^j} * R_i^j * \pi_{R_j^i} * \pi_{R_i^j} * \bar{1} \longrightarrow \pi(R_i^j * R_j^i) * (R_i^j * R_j^i)$

(f) $m^i : \pi(R_i^j * R_j^i) * (R_i^j * R_j^i) \longrightarrow R_j^i$

The form for f^i may now be written as;

for all $\rho \in R_i^j$, $f^i(\rho) =$
$m^i((d')^i(\rho, o^i(\rho), g^i(\rho, o^i(\rho)), \bar{g}^i(\rho, o^i(\rho)), e^i(\rho, o^i(\rho))))$.

Note that P^i is now a static system with transfer function f^i.

Thus, the previous definition for $P^1 \Lambda P^2$ may be simplified to;

for all $x \in I_1^1 * I_2^2$ and all $y \in (R_2^1 * R_1^2)^N$ and all $y_o \in R_2^1 * R_1^2$,
$(x,y) \in (P^1 \Lambda P^2)_{y_o}$ iff

(i) $\{x\} = (I_1^1 * I_2^2)^N$ and

(ii) for all $n \in N$, $y(0) = y_o$ and

$y(n+1) = (f^1(y_2(n)), f^2(y_1(n)))$.

It is convenient to define another function associated with P^i, namely the function which relates the input event of P^i to the distinguished "goal"-input in the ordering produced by $(d')^i$ from that input event. That is, define $\delta^i : R_i^j \longrightarrow R_i^j$ such that, for all $\rho \in R_i^j$, $\delta^i(\rho) = [\max((d')^i(\rho, o^i(\rho), g^i(\rho, o^i(\rho)), \bar{g}^i(\rho, o^i(\rho)), e^i(\rho, o^i(\rho))))]_1$.

Notice that $\delta^i(\rho)$ is the first element of an ordered pair, this first element being a later input event (from P^j) that is most desired by P^i (according to P^i's decision-making strategy), and the second element being P^i's response which is expected to produce that desirable input event from P^j.

In order to formalize the notion of a "cohesive" situation, the notion of a "reward-producing event" must be made rigorous; for any $\rho \in R_i^j$, ρ is <u>P^j-rewarding to P^i</u> iff $f^j(f^i(\rho)) = (\delta^i(\rho))$. That is, the event ρ gives rise in P^i to a most desired later input to P^i, namely $\delta^i(\rho)$. If it turns out that the response emitted by P^i to achieve that input event elicits a return response from P^j (which is itself determined by the goals and expectations of P^j) which is the same as $(\delta^i(\rho))$, then the event ρ has given rise to a reward <u>for</u> P^i <u>from</u> P^j. (Note the emphasis on g rather than $\bar{g}$).

Then define; for any $y \in \mathcal{R}(P^1 \wedge P^2)$ and any $n \in N$, $P^1 \wedge P^2$ is <u>mutual-reward cohesive on y from n</u> iff for all $n' \geq n$, $y_1(n')$ is P^1-rewarding to P^2 and $y_2(n')$ is P^2-rewarding to P^1.

Finally define; $P^1 \wedge P^2$ is <u>mutual-reward cohesive</u> iff there exists an $n \in N$ such that for all $y \in \mathcal{R}(P^1 \wedge P^2)$, $P^1 \wedge P^2$ is mutual reward stable on y from n.

The above definitions formalize the notion of an interaction between two person-models which are "perfectly matched" to each other. That is, after an "initial adjustment" period, every response from one person model gives rise to a reward for the other. Such a 2-group may be called cohesive.

The following two theorems indicate the conditions under which such cohesiveness occurs in 2-groups.

A more elegant presentation of the two theorems results from the following definitions of functions associated with $P^1 \wedge P^2$;

(a) $f: R_2^1 * R_1^2 \longrightarrow R_2^1 * R_1^2$ such that
$f(r,r') = (f^1(r'), f^2(r))$.

(b) $\delta: R_2^1 * R_1^2 \longrightarrow R_2^1 * R_1^2$ such that
$\delta(r,r') = (\delta^2(r), \delta^1(r'))$.

Theorem D-1.

If $f \cdot f = \delta$. then $P^1 \wedge P^2$ is mutual-reward cohesive (from time zero).

Proof: Let $f \cdot f = \delta$. Then for all $r \in R_2^1 * R_1^2$, $f \cdot f(r) = \delta(r)$. Since $\mathcal{O}(P^1 \wedge P^2) \subseteq R_2^1 * R_1^2$, then for every $c \in \mathcal{O}(P^1 \wedge P^2)$, $f \cdot f(c) = \delta(c)$. Pick any $y \in \mathcal{R}(P^1 \wedge P^2)$ and any $n \in N$. Clearly $y(n) \in \mathcal{O}(P^1 \wedge P^2)$, and so $f \cdot f(y(n)) = \delta(y(n))$.

$$\begin{aligned} \text{Therefore } f(y(n+1)) &= (f^1(y_2(n+1)), f^2(y_1(n+1))) \\ &= (f^1(f^2(y_1(n))), f^2(f^1(y_2(n)))) \\ &= \delta(y(n)) = (\delta^2(y_1(n)), \delta^1(y_2(n))) \end{aligned}$$

Thus $f^1(f^2(y_1(n))) = \delta^2(y_1(n))$ and

$f^2(f^1(y_2(n))) = \delta^1(y_2(n))$.

So for any $y \in \mathcal{R}(P^1 \wedge P^2)$ and any $n \in N$, $y_1(n)$ is P^1-rewarding to P^2 and $y_2(n)$ is P^2-rewarding to P^1. In that case $P^1 \wedge P^2$ is mutual-reward cohesive on y from zero, for any $y \in \mathcal{R}(P^1 \wedge P^2)$. Then, $P^1 \wedge P^2$ is mutual reward cohesive (from time zero).

QED

Theorem D-1 states a sufficient condition for obtaining mutual-reward cohesive behavior in $P^1 \wedge P^2$. This condition specifies a precise relationship among f^1, f^2, δ^1 and δ^2 which, if it holds, guarantees that mutual-rewarding behavior is achieved. In words, the theorem states that, after some time n, every input to P^i from P^j which induces, via g^i, a desired later input to P^i also causes P^i to emit a response which, in turn, causes P^j to produce a response, where the latter response is precisely the aforementioned

later input which P^i desired to receive. In other words, if the goals, expectations and decision-making processes of P^1 and P^2 are "matched" in precisely the manner indicated, mutual-reward cohesive behavior is achieved. Thus, group behavior has been related to the psychological characteristics of members of the group.

In terms of our second example in section A, theorem D-1 indicates that cohesive behavior between teacher and students may have occurred if their goals were compatible. That is, imagine a situation where the students desire to be disciplined in return for which they will be attentive and work hard. In addition, suppose that the teacher desires hard work from his students and is convinced that discipline produces this effort. In this case, the student's goals are in accord with the teacher's goals because an input event to the students which represents discipline from the teacher (who emits the event in search of harder work from the students) results in the response from the students of "working harder". Thus, the discipline-event is "student-rewarding to the teacher". On the other hand, an input event to the teacher which represents "hard work" from the students (who emit the event in search of discipline) results in the teacher continuing his discipline so as to sustain the "hard work". Thus, the "hard-work" event is "teacher-rewarding to the students". If these exchanges continue for a long period of time, this group may be called mutual-reward cohesive in the sense of our definition. Generally, however, students' goals are not likely to include receiving strict discipline in which case the goals of the above teacher and his students conflict, thus preventing cohesive behavior. The latter situation may be formalized by the contrapositive of theorem D-2.

Theorem D-2. If $P^1 \wedge P^2$ is mutual-reward cohesive, then there exists an $n \in N$ such that

$$f \cdot f \,/\, \mathcal{O}(P^1 \wedge P^2)_n = \delta \,/\, \mathcal{O}(P^1 \wedge P^2)_n$$

Proof: Assume $P^1 \wedge P^2$ is mutual-reward cohesive. Then there exists an $n \in N$ such that for all $y \in \mathcal{R}(P^1 \wedge P^2)$, $P^1 \wedge P^2$ is mutual-reward cohesive on y from n. So

$$\begin{aligned}(\exists\, n \in N)(\forall\, y \in \mathcal{R}(P^1 \wedge P^2))(\forall\, n' \in N)(n' \geq n \Rightarrow \\ y_1(n') \text{ is } P^1\text{-rewarding to } P^2 \text{ and} \\ y_2(n') \text{ is } P^2\text{-rewarding to } P^1)\end{aligned}$$

thus

$$\begin{aligned}(\exists\, n \in N)(\forall\, y \in \mathcal{R}(P^1 \wedge P^2))(\forall\, n' \in N)(n' \geq n \Rightarrow \\ f^1(f^2(y_1(n'))) = \delta^2(y_1(n')) \text{ and} \\ f^2(f^1(y_2(n'))) = \delta^1(y_2(n'))).\end{aligned}$$

Choose such an $n \in N$, and pick any element $r \in \mathcal{O}(P^1 \wedge P^2)_n$. It must be that there exists a $y \in \mathcal{R}(P^1 \wedge P^2)$ and an $n' \in N$ such that $y(n + n') = r$.

Let $n + n' = n''$ and clearly $n'' \geqq n$. Thus,

$$f \cdot f(r) = f \cdot f(y(n'')) = f(f(y(n''))) = f(f(y_1(n''), y_2(n''))) =$$
$$f(f^1(y_2(n'')), f^2(y_1(n''))) = (f^1(f^2(y_1(n''))), f^2(f^1(y_2(n''))))$$
$$= (\delta^2(y_1(n'')), \delta^1(y_2(n'')))$$
$$= \delta(y(n'')) = \delta(r).$$

So $f \cdot f / \mathcal{O}(P^1 \wedge P^2)_n = \delta / \mathcal{O}(P^1 \wedge P^2)_n$.

QED

Theorem D-2 states that it is necessary for the goals, expectations and decision-making processes of P^1 and P^2 to related in the precise manner shown in order to ever achieve mutual-reward cohesive behavior.

Corollary D-1 $\quad f \cdot f / \mathcal{O}(P^1 \wedge P^2) = \delta / \mathcal{O}(P^1 \wedge P^2)$
if and only if $P^1 \wedge P^2$ is mutual-reward cohesive (from time zero).

Proof: The proof follows from the proofs of theorems D-1 and D-2.

E. Property of interaction model behavior: stability

The previous section studied certain aspects of the behavior of $P^1 \wedge P^2$ where the learning capabilities of P^1 and P^2 were neglected. In this section, we will continue to ignore the effects of inputs from and outputs to the environment, but let us relax the restriction on Q^1 and Q^2 being unit sets. Since Q^1 and Q^2 may now be sets of large cardinality, the individuals P^1 and P^2 can exhibit modified behavior with time and so may appear to be "learning".

Since the P^i can now exhibit varied behavior with time, the dynamic characteristics of group behavior may be fruitfully studied. In particular, the property of stability may be considered. Previously, it was noted that a particular kind of stable behavior was desirable, namely mutual-rewarding cohesive behavior. In that case, no learning capability was present, and conditions were given such that a 2-group exhibited cohesive behavior for all time beyond some time n. In the present case where learning capabilities are present, cohesive behavior may be temporarily achieved, but behavior may later become non-cohesive due to the acquisition of knowledge (reference, say, the final example in section A). Let us consider the somewhat restricted case where a "learning" 2-group becomes cohesive and remains so for all time.

Since I_1^1, I_2^2, R_1^1, and R_2^2 are unit sets, the functions which characterize the subsystems of the P-models may be simplified as shown in appendix c.

The conditions necessary to achieve stability to cohesive behavior can be investigated once we have a rigorous notion of stability. Consider the following definition.

For all $n \in N$ and all $q \in Q^1 * Q^2$ and all $r \in R_2^1 * R_1^2$ and all $y \in \mathcal{R}(P^1 \wedge P^2)_{(q,r)}$,

$(P^1 \wedge P^2)_{(q,r)}$ is <u>cohesive-stable on y and the (q,r)-state-trajectory w from n</u> iff for all $n' \in N$,

$n' \geqq n \Longrightarrow f_{w(n'+1)}(f_{w(n')}(y(n'))) = \delta_{w(n')}(y(n'))$

Then $P^1 \wedge P^2$ is <u>cohesive-stable</u> iff there exists an $n \in N$ such that for all $q \in Q^1 * Q^2$ and all $r \in R_2^1 * R_1^2$ and all $y \in \mathcal{R}(P^1 \wedge P^2)_{(q,r)}$, $(P^1 \wedge P^2)_{(q,r)}$ is cohesive-stable on y and the (q,r)-state-trajectory w from n.

Note that the notion of cohesive-stability is based upon mutually rewarding exchanges. This notion differs from that of mutual-reward cohesion in the previous section because, in this case, the <u>state</u> of the P-model must be accounted for during each exchange. For example, since $w_j(n)$ may not be the same as $w_j(n+2)$, the input which was most desired by P^j at time n may no longer be so distinguished at time n + 2. The above definition assumes that the state of knowledge does not change so appreciably as to cause the latter situation to occur.

The following theorem specifies a sufficient condition on the functions which characterize each of P^1 and P^2 such that their 2-group is cohesively stable.

Theorem E-1. If for any $\rho \in R_2^1$ and $\rho' \in R_1^2$ and $q \in Q^1$ and $q' \in Q^2$,

$$\delta((\rho, \rho'), (q,q')) = [f^1(f^2(\rho, q'), \lambda^1(\rho', q)), f^2(f^1(\rho', q), \lambda^2(\rho, q'))]$$

then $P^1 \wedge P^2$ is cohesive stable.

Proof: Assume the antecedent is true. Pick any $n' \in N$, any $q \in Q^1 * Q^2$, any $r \in R_2^1 * R_1^2$ and any $y \in \mathcal{R}(P^1 \wedge P^2)_{(q,r)}$. Designate the (q,r)-state-trajectory as w.

So, since $y(n') \in R_2^1 * R_1^2$ and $w(n') \in Q^1 * Q^2$,

$$\delta_{w(n')}(y(n')) = \delta((y_2(n'),y_1(n')),(w_1(n'),w_2(n'))),$$
$$= [f^1(f^2(y_1(n'),w_2(n')), \lambda^1(y_2(n'),w_1(n'))),$$
$$f^2(f^1(y_2(n'),w_1(n')), \lambda^2(y_1(n'),w_2(n')))]$$
$$= f((f^1(y_2(n'),w_1(n')), f^2(y_1(n'),w_1(n'))),$$
$$(\lambda^1(y_2(n'),w_1(n')), \lambda^2(y_1(n'),w_2(n'))))$$
$$= f(f((y_1(n'),y_2(n')),(w_1(n'),w_2(n'))),$$
$$\lambda((y_1(n'),y_2(n')),(w_1(n'),w_2(n')))$$
$$= f(f(y(n'),w(n')), \lambda(y(n'),w(n')))$$
$$= f(f(y(n'),w(n')), w(n' + 1))$$
$$= f_{w(n' + 1)}(f_{w(n')}(y(n'))).$$

Thus $(P^1 \wedge P^2)_{(q,r)}$ is cohesive-stable on y and the (q,r)-state-trajectory w from zero. Since this is true for any $n' \in N$, $q \in Q^1 * Q^2$, $r \in R_2^1 * R_1^2$ and $y \in \mathcal{R}(P^1 \wedge P^2)$, then $P^1 \wedge P^2$ is cohesive-stable.

QED

The above theorem displays a relationship among δ^i, f^i and λ^i $(i = 1,2)$ such that, if the relationship holds, it is guaranteed that the 2-group will exhibit cohesive-stability. Notice that this condition bears some resemblance to the condition of theorem D-1. However, the above condition is complicated by the presence of changing states of knowledge.

The investigation of cohesive behavior for all time, as just considered, is a relatively simple consideration. In fact, the more general case is that in which cohesive behavior occurs over some intervals of time and does not occur at other times. The investigation of such "cycles" of behavior, which are more characteristic of real-life situations, appears to be a quite fruitful area for future study. For example, it may be possible to display those relationships among the subsystems of each of the P^i which result in relatively lengthy intervals of cohesive group behavior.

Another useful refinement of the above development would involve extending the notion of cohesive-stability beyond a "next-input" operation so as to allow a longer lapse of time between response and reward.

F. Property of interaction-model behavior: controllability

The previous section considered the conditions which were conducive to cohesive-stable behavior. It was pointed out that, in real-life situations, cohesive behavior is usually not maintained for very long periods of time. In this section, we begin to consider the possibility of exerting external influence on the group in order, for example, to achieve or maintain cohesive group behavior. Clearly, since we are concerned with external influence, we need relax the constraint of I_1^1 and I_2^2 being unit sets as made in the previous section.

External influence, via inputs from the sets I_1^1 and I_2^2, obviously affects the responses from P^1 and P^2. A more significant effect of external inputs is to modify the state of knowledge of P^i. Such a change can in turn affect the choice of response for a given input event. Thus, a change in state of knowledge may be employed for response-control. Such influence appears to characterize the efforts of parents to teach their children to exhibit selected responses and repress others so that the whole of behavior may be socially acceptable.

Consider a formalization of the notion of response-control, first for a P-model and later for a 2-group. Note that, since we are dealing with responses, the function d' which provides an ordering regarding responses is the center of interest. Now, the form for d' may be written as $d': I * Q \longrightarrow \pi_{(I * R) * (I * R)}$ since, given $a \in I$ and $q \in Q$, d' may be computed via o, e, g and $\bar{g}$.

Assume that I and R are finite sets and that $\mathcal{R}_o = \{R\}$. The latter assumption may be interpreted as stating that every response is considered "appropriate", no matter what input event occurs. Further, recall that the elements of d' are not only partially ordered with respect to the responses, but in fact simply ordered.

In order to clarify what is meant by an ordering "with respect to the responses", let C be the set of simple orderings of the set R and consider the function

$$H: \pi_{(I * R) * (I * R)} \longrightarrow C$$

such that for any $r, r' \in R$ and any $b \in \pi_{(I * R) * (I * R)}$, there exists $a, a' \in I$ such that $((a,r),(a',r')) \in b$ if and only if $(r,r') \in H(b)$.

For any $b \in \pi_{(I * R) * (I * R)}$, denote H(b) as $\hat{b}$.

The notion of $\hat{b}$ is useful since it can display the ordering produced by d' as it would look if it were "uncluttered" with elements from I. The use of $\hat{b}$ has another implication in that many elements

of $\mathcal{R}d'$ may be associated with a given $\hat{b}$. This may be interpreted as not caring why P prefers one response to another (i.e., not being concerned with the goal that P is pursuing via any particular response).

Consider the following definition of controllability with respect to a given environmental context (i.e., with respect to a given input event from the environment).

For any $b \in \mathcal{R}d'$ and $a \in I$, P is response-controllable to $\hat{b}$ on a iff for all $q \in Q$, there exists an $x \in \mathcal{D}P$ and an $n \in N$ such that

$$\text{(i) } x(n) = a \quad \text{and,}$$

$$\text{(ii) } d'(x(n), L_q(x)(n)) \in H^{-1}(\hat{b}).$$

This definition may be interpreted as stating that if P is response controllable to $\hat{b}$ on a, then P can be eventually "taught" to display at least one of the orderings in $H^{-1}(\hat{b})$ when presented with input a, regardless of P's initial state of knowledge. A sufficient condition for response-controllability may be derived following several somewhat involved considerations which are intended to display a metric space related to Q.

Since R is finite (with cardinality k, say), the responses in $\mathcal{D}\hat{b}$, where $b \in \mathcal{R}d'$ may be indexed by the subset of integers $\{1, 2, ---, k\}$. Let the indexing function be called $B_{\hat{b}}$ and be defined as $B_{\hat{b}}: \mathcal{D}\hat{b} \rightarrow \{1, 2, ---, k\}$ such that $B_{\hat{b}}(r) = k$ if r is the greatest element β in $\hat{b}$, and $B_{\hat{b}}(r) = k-1$ if r is the greatest element in the sub-ordering induced by $\hat{b}$ on $\mathcal{D}\hat{b} - \{\beta\}$, etc.

The indexing function $B_{\hat{b}}$ as defined above allows the definition of a measure of distance on the set of simple orderings over the whole of R. Thus, it becomes possible to formalize the notion of being "close" to achieving the desired response ordering for a given input. Now, if the subsystems of P were constructed such that it would be possible to bring the ordering produced by d' closer to the desired one at every time instant, then it would be guaranteed that P eventually achieves the desired ordering since it may be shown that there are only a finite number of such orderings. The following development and theorem display such a condition on the subsystem of P.

The set C represents the set of simple orderings of R. Consider the function $\mu_R: C * C \rightarrow Z$ such that for any $c, c' \in C$, $\mu_R(c,c') = \sum_{r \in R} |B_c(r) - B_{c'}(r)|$, where Z is the set of real numbers.

The function μ_R is a metric, as shown in the appendix d.

This metric can be extended to relate to the elements of $\mathcal{R}d'$. That is, the function H partitions $\pi(I * R) * (I * R)$ into the set of equivalence classes $\hat{H} = \{ H^{-1}(\hat{b}) \mid \hat{b}$ is a simple ordering of $R\}$.

Define $b \sim b'$ iff $H(b) = H(b')$.

The b-equivalence class, $[b] = \{ b' \mid b \sim b'\}$. Then $\hat{H} = \{ [b] \mid b \in \pi(I * R) * (I * R)\}$. Similarly, $H/\mathcal{R}d'$ partitions $\mathcal{R}d'$. Let the set of such equivalence classes be denoted by $\mathcal{H}$. Consider the function $\mu_H : \mathcal{H} * \mathcal{H} \rightarrow Z$ such that for all $b, b' \in \mathcal{R}d'$, $\mu_H([b], [b']) = \mu_R(H(b), H(b'))$. Clearly, μ_H is a metric.

The sufficient condition to be displayed shortly is expressed via subsystem L, so consider the extension of μ_H to a metric related to Q. That is, note that for a given $a \in I$,

$$d'_a : Q \rightarrow \pi(I * R) * (I * R)$$

Thus the set Q may be partitioned by d'_a according to the following relation:

$$q \approx_a q' \text{ iff } [d'_a(q)] = [d'_a(q')].$$

Let the set of equivalence classes be denoted by $\mathcal{Q}$ and let $[q]_a = \{q' \mid q \approx_a q'\}$. Consider the function $\mu_Q : \mathcal{Q} * \mathcal{Q} \rightarrow Z$ such that for all $q, q' \in Q$, $\mu_Q([q]_a, [q']_a) = \mu_H([d'_a(q)], [d'_a(q')])$.

Again, it should be clear the μ_Q is a metric.

The stage has been almost set for the sufficient condition on the subsystems of P which guarantees that P is response-controllable on a given input. The condition of interest is associated with the "learning" subsystem L and, in particular, with the function λ. Now, for a given $a \in I$, $\lambda_a : Q \rightarrow Q$. However, the metric μ_Q is associated with $\mathcal{Q}$, so define, for any $a \in I$, $\mathcal{L}_a : \mathcal{Q} \rightarrow \mathcal{Q}$ such that for any $q \in Q$, $\mathcal{L}_a([q]_a) = [\lambda_a(q)]_a$.

The function $\mathcal{L}_a$ may have a number of properties, but the property of interest here is associated with the fact that $\mathcal{Q}$ is in fact a metric space.

That is, any function $K : M \rightarrow M$, where M is a metric space with metric μ, is a <u>contraction map under μ</u> if and only if there exists an $\alpha < 1$ such that for all $m, m' \in M$,

$$\mu(m, m') \geq \alpha(\mu(K(m), K(m'))).$$

Further, if M is a complete metric space, there exists a unique element $m \in M$ such that $K(m) = m$, in which case m is called the fixed point of K. (9)

Now it can be shown that $\mathcal{Q}$ is a complete metric space and so a sufficient condition for response-controllability is at hand. For purposes of expediency, let b_a be a representative of the set $(d'_a)^{-1}(b)$, for any $a \in I$ and $b \in \mathcal{R}d'$.

Theorem F-1: For any $b \in \mathcal{R}d'$ and $a \in I$, if $\mathcal{L}_a$ is a contraction map under μ_Q with fixed point $[b_a]_a$, then P is response-controllable to $\hat{b}$ on a.

Proof: Pick any $b \in \mathcal{R}d'$ and $a \in I$ and assume that $\mathcal{L}_a$ is a contraction map under μ_Q with fixed point $[b_a]_a$. Select any $q \in Q$, call it $\bar{q}$. Consider the $\bar{x} \in \mathcal{X}P$ such that for all $n \in N$, $\bar{x}(n) = a$. Clearly, for any $n \in N$, $d'(\bar{x}(n), L_q(\bar{x})(n)) = d'_a(L_q(\bar{x})(n))$. It has been shown (10) that the sequence $[\bar{q}]_a, \mathcal{L}_a([\bar{q}]_a), \mathcal{L}_a(\mathcal{L}_a([\bar{q}]_a)), \ldots$ converges to the fixed point of $\mathcal{L}_a$, namely $[b_a]_a$.

But $[\bar{q}]_a = [L_{\bar{q}}(\bar{x})(0)]_a$ and

$$\mathcal{L}_a([\bar{q}]_a) = [\lambda_a(\bar{q})]_a = [L_{\bar{q}}(\bar{x})(1)]_a.$$

By induction, for any $n \in N$,

$$\overbrace{\mathcal{L}_a(\cdots \mathcal{L}_a}^{n \text{ times}}([\bar{q}]_a) \cdots) = [L_{\bar{q}}(\bar{x})(n)]_a .$$

So the sequence

$$[L_{\bar{q}}(\bar{x})(0)]_a, \quad [L_{\bar{q}}(\bar{x})(1)]_a, \ldots$$

converges to $[b_a]_a$. Further since R and I are finite, then $\mathcal{R}d'$ is finite, in which case $\mathcal{Q}$ is finite. Therefore, the above sequence converges in finite time, say $\bar{n}$, such that

$$[L_{\bar{q}}(\bar{x})(\bar{n})]_a = [b_a]_a$$

Then $L_{\bar{q}}(\bar{x})(\bar{n}) \approx_a b_a$

So $[d'_a(L_{\bar{q}}(\bar{x})(\bar{n}))] = [d'_a(b_a)] = [b]$

since $b_a \in (d'_a)^{-1}(b)$.

Thus $[d'(\bar{x}(\bar{n}), L_{\bar{q}}(\bar{x})(\bar{n}))] = [b]$

So $d'(\bar{x}(\bar{n}), L_{\bar{q}}(\bar{x})(\bar{n})) \sim b$

in which case $H(d'(\bar{x}(\bar{n}), L_{\bar{q}}(\bar{x})(\bar{n})) = H(b) = \hat{b}$

Thus $d'(\bar{x}(\bar{n}), L_{\bar{q}}(\bar{x})(\bar{n})) \in H^{-1}(\hat{b})$

So there exists an $x \in \mathcal{D}P$ and $n \in N$ such that $d'(x(n), L_{\bar{q}}(x)(n)) \in H^{-1}(\hat{b})$ and $x(n) = a$.

Since this is so for any $\bar{q} \in Q$, P is response-controllable to $\hat{b}$ on a.

QED

Theorem F-1 indicates that, in order to achieve response-controllability on a given input a, it is sufficient for $\mathcal{L}_a$ to be a contraction map.

This may be interpreted in terms of "rote learning" where it is presumed that a <u>repetition</u> of presentation of given information causes the information to be learned and the associated desired response to be displayed. For example, the repetitive presentation to a second-grader of an input event consisting of a statement that "2 + 2 = 4" followed by the question "What is the sum of 2 and 2?" may eventually be rewarded by the response "the sum of 2 and 2 is 4" if the student is controllable to this response.

The above development considers response-controllability on a single input. Future investigation would likely display conditions such that response-controllability is obtained for <u>more</u> than one input event. Such a situation is closer to reality since in general it is desired to teach students more than just one fact.

To this point, the response-controllability of a single P-model has been investigated. Consider now the response-controllability of a 2-group, $P^1 \wedge P^2$. In particular, we may define for $a \in I_1^1 * I_2^2$ and for any $b \in \mathcal{R}(d')^1 * \mathcal{R}(d')^2$, $P^1 \wedge P^2$ is <u>response-controllable to b on a</u> iff for all $q \in Q^1 * Q^2$, there exists an $x \in \mathcal{D}(P^1 \wedge P^2)$ and an $n \in N$ such that $d'(x(n), L_q(x)(n)) \in H^{-1}(b)$ and $x(n) = a$, where L and H are defined appropriately for $P^1 \wedge P^2$.

Since neither I_1^1 nor I_2^2 are restricted to be unit sets, either P-model may receive <u>two</u> inputs at one time. Intuitively, it appears that a person can understand only one message at a time. Thus, to provide some intuitive appeal to the formalization of a 2-group, consider the following functions which may be termed "attention" functions.

$$(F\text{-}1)\quad \alpha_1 : I_1^1 * R_1^2 \rightarrow I_1^1$$
$$\alpha_2 : I_2^2 * R_2^1 \rightarrow I_2^2$$

$$(F\text{-}2)\quad \alpha_1 : I_1^1 * R_1^2 \rightarrow I_1^1$$
$$\alpha_2 : I_2^2 * R_2^1 \rightarrow R_2^1$$

There are two other possible sets of such functions, but the two sets shown above are of immediate interest.

The function α_i may be interpreted as indicating at a given time whether P^i pays attention only to the environmental input or only to the input from the other P-model. If α_i holds for <u>all</u> time, then the functions denoted by (F-1) above indicate that both P^1 and P^2 attend only to the environmental inputs. The functions denoted by (F-2) above indicate that P^1 pays attention only to environmental input whereas P^2 processes only those inputs received from P^1. Therefore, set (F-1) may be interpreted as characteristic of a completely non-cohesive 2-group (i.e., no communication among group members), whereas set (F-2) can be thought of as representing a "superior-subordinate" relationship such that the environmental inputs affect P^2 only via P^1.

The attention-functions need be incorporated into the P-models in order to investigate the controllability of 2-groups. This can be accomplished in the case where α_i holds for all time by modifying P^i as follows;

(i) let λ^i be replaced by $\alpha_i \cdot \lambda^i$

and (ii) let o^i be replaced by $\alpha_i \cdot o^i$

and (iii) modifying g^i, $\bar{g}^i$, e^i and d^i appropriately.

Future work should be aimed at incorporating α_i into P^i in a more precise manner than indicated above. Such a modification of P^i appears to present no problems and, once it is completed, several theorems on controllability of 2-groups may be derived. For example, for the set of attention functions called (F-1), it is likely that the following statement may be proved;

for any $b \in \mathcal{R}(d')^1 * \mathcal{R}(d')^2$ and $a \in I_1^1 * I_2^2$,

if $\mathcal{L}_a$ is a contraction map under $\mu_{Q^1 * Q^2}$

with fixed point $[b_a]_a$, then $P^1 \wedge P^2$ is

response-controllable to $\hat{b}$ on a.

Furthermore, for the set of attention functions denoted (F-2), it appears that an interesting sufficient condition may be derived such that P^2's response-ordering is first carried to the desired type via input to P^1, then remains there as P^1's response-ordering is carried to its desired kind, thus obtaining response-controllability for $P^1 \wedge P^2$.

More complex situations arise when α_i is allowed to vary in time. For example, real-life situations indicate that the form for α_i (i.e., whether (F-1) or (F-2), etc.) depends upon recent input events. Thus an extensive area of future study is opened by these considerations.

G. Impact of environmental inputs on group structure

The previous section investigated one aspect of the effect of environmental inputs on group behavior. In particular, interest was centered on the possibility of "teaching" a certain response to be produced in a given context. If such a "lesson" could be taught, the group was said to be response-controllable.

In this section, it is assumed that the group under consideration is response-controllable. However, though it is feasible under this assumption to teach a given lesson, the effect upon the structure of the group of learning that lesson remains to be investigated. For instance, such an effect was discussed in the final example of section A. We may formalize the situation of that example by the following development. Since the example under consideration involved a change in status of a group member, the notion of status need be formalized. This can be accomplished based upon Homans' stated relationship between status and "rare and valuable activities". Consider such a formalization in terms of n-groups.

A homogeneous n-group $P^1 \wedge P^2 \wedge \text{----} \wedge P^n$ is such that $I = I^1 = I^2 = \text{----} = I^n$ and $R = R^1 = R^2 = \text{----} = R^n$. Such a group may be interpreted as one in which all members "speak the same language". For a homogeneous n-group, $(P^1 \wedge P^2 \wedge \text{----} \wedge P^n)_{(q,r)}$ describes the input-output behavior of the n-group for initial state $q \in Q^1 * Q^2 * \text{----} * Q^n$ and initial output $r \in \overbrace{R * R * \text{----} * R}^{n \text{ times}}$. Associated with this input-output behavior is the (q,r)-state-trajectory $w \in (Q^1 * Q^2 * \text{----} * Q^n)^N$, as discussed previously.

Consider $(P^1 \wedge P^2 \wedge \text{----} \wedge P^n)_{(q,r)}$ for any q and any r. Then define, for any $a \in I$, $r' \in R$ and $n \in N$, P^i has status re r for a at time n if and only if

(i) for all $n' \in N$, if $n' \leq n$, then $f^i_{w_i(n')}(a) = r'$ and for all $j \in \{1, \text{----}, n\} - \{i\}$, $f^j_{w_j(n')}(a) \neq r'$

(ii) for all $j \in \{1, \text{----}, n\} - \{i\}$, there exists an $a' \in I$ and $n'' \in N$ such that, if $n'' \leq n$, then $\delta^j_{w_j(n'')}(a') = r'$.

In the above definition, condition (i) accounts for the "rareness" of activity r' since only P^i may emit r' up to time n. Condition (ii) states that, for every group member except possibly P^i, there is at least one situation in which r would have been the desired later input and so is "valuable" to each of those members.

The previous section considered the possiblity of causing P^i to prefer responses to a given input event in a certain order. Consider now a formalization of the actuality of displaying a certain response preference.

For any $a \in I$, $r' \in R$ and $n \in N$, P^i learned r' for a at time n if and only if

(i) for all $n' \in N$, if $n' \leq n$, then $f^i_{w_i(n')}(a) \neq r'$,

(ii) $f^i_{w_i(n+1)}(a) = r'$

This definition states that, prior to time n, P^i would not emit r' in response to a, but would do so at time n.

A theorem may be framed relating the learning of a "skill" by every member of a group to the subsequent status structure of the group.

Theorem G-1. For any $a \in I$, $r' \in R$, $n \in N$ and $i \in \{1,---,n\}$, if P^i has status re r for a at time n and, for all $P^j \in \{P^1, ---, P^n\} - \{P^i\}$, P^j learned r' for a at time n, then there is no $P^k \in \{P^1, ---, P^n\}$ such that P^k has status re r for a at time n + 1.

Proof. Pick any $a \in I$, $r' \in R$, $n \in N$ and $i \in \{1,---,n\}$. Assume that the antecedent of the implication is true. Then $f^i_{w_i(n)}(a) = r'$ and for all $j \in \{1, ---, n\} - \{i\}$, $f^j_{w_j(n+1)}(a) = r'$. Thus, for all $k \in \{1, ---, n\}$, condition (i) of the definition of status of P^k is not satisfied. Therefore, there is no P^k such that P^k has status re r for a time n + 1.

QED

Theorem G-1 relates to the example under consideration in that it formalizes the loss of status of a skill-holder when that skill is taught to the remainder of the group.

The notions of status and justice play an important role in the work of Homans. We have considered the concept of status in some detail, both in this section and at the end of section A. However, there are many other aspects of "status" which remain to be investigated and other authors (especially (5)) are doing extensive investigations in this area. The notion of justice is much less studied, yet apparently is of significance in every exchange among group members. Extensions of this effort should include a formalization of the concept of justice, and relate it

to other notions discussed here. For example, one relationship which should be investigated is the conjecture that "all exchanges in cohesive-stable groups conform to the notion of distributive justice as stated in proposition (2) of section A".

Conclusions and recommendations

This study indicates that it is feasible to employ mathematical models to investigate the broader social aspects of classroom organization. Further, it displays a procedure for investigating dynamic behavior which parallels other system-theoretic developments. In particular, a mathematical framework for examining static and dynamic behavior of groups of learning, goal-seeking individuals is developed. Several significant notions such as cohesiveness, stability, controllability and status are formalized within this framework and relationships are displayed between such notions and the "internal" characteristics of the group members. Further, recommendations as to future effort at establishing "deeper" relationships are given in each of sections D through F.

Due to the incipient nature of this study, there are a number of refinements and extensions that are both possible and desirable. For example, the P-model may provide a better reflection of reality if the notion of "perception" is incorporated into it, perhaps via a partition of the set I (note that, in the example of section B, the inputs i_{34}, i_{53}, i_{102} and i_{201} represent classes of inputs and so are related to some partition of input events). Also, the investigation of dynamic behavior can be more meaningful if the learning capabilities of the P-models are extended to allow modification of goals and response-choices as well as expectations.

References

1. G. C. Homans; The Human Group; New York; Harcourt, Brace and World; 1950.

2. G. C. Homans; Social Behavior: Its Elementary Forms; New York; Harcourt, Brace and World; 1961.

3. G. Miller, E. Galanter, and K. Pribram; Plans and the Structure of Behavior; New York; Holt; 1960.

4. G. Kelly, D. Bannister; The Psychology of Personal Constructs; London; Academic Press; 1968.

5. T. J. Fararo; "Theory of Status," General Systems, Vol. 12, 1968.

6. J. Talavage; Design of Adaptive Pattern Recognition Systems on the Basis of General Systems Theory; Journal of Systems Engineering; Vol 1, pp. 397-424, 1969.

7. T. G. Windeknecht, M. Mesarovic; An Axiomatic Theory of General Systems; Proc. IEEE Syst. Sc. and Cyb. Conf.; Boston, 1967.

8. H. Raiffa; Decision Analysis; Reading, Mass: Addison-Wesley, 1968.

9. A. Kolmogorov, S. Fomin; Elements of the Theory of Functions and Functional Analysis; Vol. 1; Rochester, Graylock Press, 1957.

10. J. Baum; Elements of Point-Set Topology; Prentice-Hall; 1964.

11. J. Gullahorn; Initial Efforts in Validating a Computer Model of Social Behavior; Psychological Reports, Vol. 19, pg. 786, 1966.

12. H. Simon; Models of Man-Social and Rational; New York; Wiley, 1957.

Appendixes

(a) General systems theory

(b) Definitions of w_i, x_i and y_j^i

(c) Simplification of P-model

(d) Proof that π_R is a metric

(a) General Systems Theory

A general system is any binary relation; i.e., any set of ordered pairs. If S is a general system, then the sets

$$\mathcal{D}S = \{ x \mid (\exists y)((x,y) \in S)\}$$
$$\text{and} \quad \mathcal{R}S = \{ y \mid (\exists x)((x,y) \in S)\}$$

are called, respectively, the input set and output set of S.

A time function is any function x such that $\mathcal{D}x = N$, where N is the set of positive integers (including zero).

The n-section of a time function x, denoted x_n, is the set

$$x_n = \{(n', x(n' + n)) \mid n' \in N\}$$

The set x_n may be interpreted as what x "looks like" beyond time n.

A (time) system is any set of ordered pairs of time functions $S \subseteq A^N * B^N$ such that A and B are sets and

$$A^N = \{x \mid x: N \rightarrow A\}$$
$$\text{and} \quad B^N = \{y \mid y: N \rightarrow B\}$$

The input space of a system S, denoted $\mathcal{I}S$, is the set

$$\mathcal{I}S = \{x(n) \mid x \in \mathcal{D}S \ \& \ n \in N\}$$

The output space of a system S, denoted $\mathcal{O}S$, is the set

$$\mathcal{O}S = \{y(n) \mid y \in \mathcal{R}S \ \& \ n \in N\}$$

The n-section of a system S, denoted S_n, is the set

$$S_n = \{(x_n, y_n) \mid (x,y) \in S\}$$

S_n may be interpreted as what the system "looks like" beyond time n.

For a system S, the <u>transfer relation</u> of S, denoted kS, is the set

$$kS = \{(x(n),\ y(n)) \mid (x,y) \in S \ \&\ n \in N\}..$$

A system is <u>static</u> iff kS is a function.

The <u>series interconnection</u> of systems S and S', denoted $S \cdot S'$, is the system

$$S \cdot S' = \{(x,z) \mid (\exists y)((x,y) \in S \ \&\ (y,z) \in S')\}$$

The parallel interconnection of systems S and S', denoted $S \,//\, S'$, is the set

$$S \,//\, S' = \{(x, y//z) \mid (x,y) \in S \ \&\ (x,z) \in S'\}$$

where $$y//z = \{(n,\ (y(n),\ z(n))) \mid N \in N\}$$

The <u>feedforward interconnection</u> of systems S and S', denoted $S \# S'$, is such that

$$(x,y) \in S \# S' \text{ iff } \begin{array}{l} \text{(i) } x \in \mathcal{D}S \\ \text{(ii) } (\exists y')((x,y') \in S \ \&\ (x//y',y) \in S') \end{array}$$

Other notation:

For any set A, $\pi_A = \{B \mid B \subseteq A\}$

(b) Definitions of w_i, x_i and y^i_j

For any ordered pair, $p = (a,b)$, the <u>first</u> element of p, denoted $(p)_1$, is the element a; the second element of p, denoted $(p)_2$, is the element b.

For any time function $x \in (A * B)^N$, where A and B are sets,

$$x_1 = \{(n, (x(n))_1) \mid n \in N\}$$

and $$x_2 = \{(n, (x(n))_2) \mid n \in N\}$$

For any time function $y \in ((A * B) * (C * D))^N$, where A, B, C and D are sets,

$$y^1_1 = \{(n,\ (y_1(n))_1) \mid n \in N\}$$

and $$y^1_2 = \{(n,\ (y_1(n))_2) \mid n \in N\}$$

and similarly for y^2_1 and y^2_2.

(c) Simplification of P-model

For $i \in \{1,2\}$ and $j \in \{1,2\}$ and $i \neq j$,

(a) o^i, g^i, $(d')^i$ and m^i are as given in section D;

(b) λ^i: $R^j_i * Q^i \rightarrow Q^i$

(c) e^i: $R^j_i * Q^i * \pi_{R^i_j} \rightarrow \pi_{R^i_j} * \pi_{R^j_i} * \bar{I}$

Now, f^i may be written as; f^i : $R^j_i * Q^i \rightarrow R^i_j$ such that for all $\rho \in R^j_i$ and all $q \in Q^i$, $f(\rho,q) = m^i((d')^i(\rho, o^i(\rho), g^i(\rho, o^i(\rho)), \bar{g}^i(\rho, o^i(\rho)), e^i(\rho, o^i(\rho), q)))$.

Similarly, the form for δ^i may be displayed as;

$$\delta^i: R^j_i * Q^i \rightarrow R^j_i$$

Finally, the relation $P^1 \wedge P^2$ may be expressed as follows;

for all $x \in (I^1_1 * I^2_2)^N$ and all $y \in (R^1_2 * R^2_1)^N$ and all $q \in Q^1 * Q^2$ and all $r \in R^1_2 * R^2_1$,

$(x,y) \in (P^1 \wedge P^2)_{(q,r)}$ iff (i) $\{x\} = (I^1_1 * I^2_2)^N$

(ii) $(\exists w \in (Q^1 * Q^2)^N)(\forall n \in N)(w(0) = q$

and $w(n+1) = (\lambda^1(w_1(n), y_2(n)), \lambda^2(w_2(n), y,(n)))$

(iii) for all $n \in N$, $y(0) = r$ and

$y(n+1) = (f^1(w_1(n), y_2(n)), f^2(w_2(n), y,(n)))$.

It can be shown that w is <u>unique</u> for a given pair (q,r). For this reason, w will be referred to as the <u>(q,r)-state-trajectory</u>.

Now $P^1 \wedge P^2 = \bigcup_{(q,r) \in (Q^1 * Q^2) * (R^1_2 * R^2_1)} (P^1 \wedge P^2)_{(q,r)}$

For the present case, the functions associated with $P^1 \wedge P^2$ can be simplified as follows;

(a) λ : $(R^1_2 * R^2_1) * (Q^1 * Q^2) \rightarrow Q^1 * Q^2$ such that

$\lambda((r,r'), (q,q')) = (\lambda^1(r',q), \lambda^2(r,q'))$

(b) f: $(R^1_2 * R^2_1) * (Q^1 * Q^2) \rightarrow R^1_2 * R^2_1$ such that

$f((r,r'), (q,q')) = (f^1(r',q), f^2(r,q'))$

(c) δ: $(R^1_2 * R^2_1) * (Q^1 * Q^2) \rightarrow R^1_2 * R^2_1$ such that

$\delta((r,r'), (q,q')) = (\delta^2(r,q'), \delta^1(r',q))$

(d) Proof that π_R is a metric

If $\mu_R(c,c') = 0$, then every term in the summation must be zero. Thus, for every $r \in R$, $B_c(r) = B_{c'}(r)$, in which case $c = c'$. Conversely, if $c = c'$, then $B_c(r) = B_{c'}(r)$ for all $r \in R$, and so $\mu_R(c,c') = 0$. Further, due to the nature of the magnitude operator, $\mu_R(c,c') = \mu_R(c',c)$. Finally $\mu_R(c,c') + \mu_R(c',c'') =$

$$\sum_{r \in R} |B_c(r) - B_{c'}(r)| + \sum_{r \in R} |B_{c'}(r) - B_{c''}(r)|$$

But $\sum_{r \in R} (|B_c(r) - B_{c'}(r)| + |B_{c'}(r) - B_{c''}(r)|) \geqq$

$\sum_{r \in R} |B_c(r) - B_{c''}(r)|$ as may be shown by displaying all nine cases for the relationship among $B_c(r)$, $B_{c'}(r)$ and $B_{c''}(r)$. Thus $\mu_R(c,c') + \mu_R(c'c'') \geqq \mu_R(c,c'')$. So μ_R is a metric.