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ABSTRACT

The results of an investigation into some special cases of the general "orthomax" formulation are presented. In particular, the effects of manipulating a parameter in this formulation on various aspects of factor solutions are identified through the use of four sets of data, varying considerably in size, and reliability and factorial complexity of the variables. The implications for practical purposes of the results are discussed. (Author)

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ON THE GENERAL "ORTHOMAX" CRITERION FOR ORTHOGONAL FACTOR TRANSFORMATION

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ON THE GENERAL "ORTHOMAX" CRITERION FOR ORTHOGONAL FACTOR TRANSFORMATION[1]

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In this paper, the results of an investigation into some special cases of the general "orthomax" formulation are presented. In particular, the effects of manipulating a parameter in this formulation on various aspects of factor solutions are identified through the use of four sets of data, varying considerably in size, and reliability and factorial complexity of the variables. The implications for practical purposes of the results are subsequently discussed.

The appropriateness of orthogonal transformation has, in the authors' opinion, been somewhat misrepresented over the years, by the large number of studies conducted with little methodological consideration, using the standard computing center package providing either principal components or common-factors, rotated using the varimax technique. It is seldom true that anything approaching an optimal simple structure will result when orthogonality is imposed upon a solution. This is largely because it is unlikely that, if allowed unconstrained expression, the important factors underlying a set of variables will turn out to be mutually orthogonal, although if this condition does, in fact, obtain, a solution resulting from a proven oblique transformation, such as provided by the methods of Harris and Kaiser (1964), will reflect this. Orthogonality of factor axes should most properly be considered a constraint imposed upon a set of data for a particular purpose.

[1]This book was completed while both authors were at the University of Massachusetts.

The purpose may involve the development of an instrument or set of theoretical constructs in which the independence of the component parts is an important feature. Also, an orthogonal solution may be desired as an intermediate step from which to proceed to an ultimate oblique resolution; in the aforementioned Harris-Kaiser technique, for example, such a step permits oblique solutions with no risk of transforming to singularity. In the context of the general factor analytic study, however, an orthogonal solution--as the final result--will seldom permit the maximum clarity of factor interpretation for the data at hand, and it is only for the special purposes noted that this paper was written.

The history of automatic or non-subjective orthogonal transformation has followed two major paths. The first, in the direction of "blind" transformation, has had, as its guideposts, the _quartimax_ (Carroll, 1953; Ferguson, 1954; Neuhaus and Wrigley, 1954; Saunders, 1953), _varimax_ (Kaiser, 1958), and _equamax_ (Saunders, 1962) criteria. The second, directed toward hypothesis confirmatory transformation, generally referred to as the orthogonal Procrustes problem, has been most thoroughly charted in the work of Schönemann (1966a). A possible third path, having the same goal as the first but crossing the second in places, is represented by the _varisim_ technique (Schönemann, 1966b). The work reported in this paper is clearly an extension of the first alternative.

The three aforementioned analytic criteria in the "blind" approach can be seen as special cases of a more general, "orthomax" criterion[1] (Harman, 1960; Harris and Kaiser, 1964):

$$\underline{n} \sum_{j=1}^{n} \sum_{p=1}^{m} \underline{b}_{jp}^{4} - \underline{w} \sum_{p=1}^{m} \Big(\sum_{j=1}^{n} \underline{b}_{jp}^{2} \Big)^{2} = \text{maximum},$$

where $\underline{b}_{jp}$ is the loading of variable $\underline{j}$ on orthogonally transformed factor $\underline{p}$,

[1]The idea and name for this criterion appear to have originated with John B. Carroll. See Harman (1960, p. 334).

n is the number of variables, and m, of factors. The parameter w, regulating the weight given the second term, determines the special case of the formulation, with a value of 0 giving the quartimax criterion, 1 giving varimax, and $m/2$, equamax. The work reported in this paper was directed at assessing the effects of varying w widely, on three aspects of the final solution--variance dispersion, exemplification of simple structure, and interpretation of the obtained factors.

Method

Comparing the Solutions

For each set of data, the unrotated centroid matrix was transformed to several orthogonal simple structure solutions, the w parameter being varied between 0 or less than 0 and m or greater than m (the exact values of w for each data set are given with the results). The obtained factors were then matched with the factors of a graphically transformed solution, the latter determining the positions of the columns of all obtained factor matrices. The matching was accomplished by cross-correlating the factors of each analytic solution with those of the graphic, using the following rationale. Let A, of order $n \times m$, be the matrix of unrotated (centroid) factors, B, of order $n \times m$, the final transformed orthogonal solution, and T, of order $m \times m$, the orthonormal ($T'T = TT' = I$) transformation matrix, such that $B = AT$. Further, let subscripts a and g denote, respectively, analytic and graphic solutions. Then

$$\begin{aligned} B_a &= AT_a \\ B_g &= AT_g. \end{aligned} \tag{1}$$

Now, since both B_a and B_g are the results of orthonormally transforming the same initial matrix A, then there exists an orthonormal matrix K, of order $m \times m$, that maps B_a into B_g. Thus,

$$B_g = B_a K, \text{ or, from (1)} \quad (2)$$

$$AT_g = AT_a K, \text{ and} \quad (3)$$

$$K = T_a' T_g. \quad (4)$$

Since both B_a and B_g are orthogonal solutions, element p,q of K (p,q = 1, 2, ..., m) is the cosine of the angle between graphic factor q and analytic factor p. Thus, $K = T_a'T_g$ is also the matrix of correlations ($r_{pq} = \cos \theta_{pq}$) between the factors of the two solutions. The analytic factors were matched with the graphic, for each data set, by maximizing tr(K).

Once the factors of a given solution were arranged to correspond to those of the graphic solution, the common-factor variance accounted for by each factor in the solution was determined (Var[Factor p] = $\sum_{j=1}^{n} b_{jp}^2$; p = 1, 2, ..., m Solutions were compared by studying the particular allotments of variance to the factors in each and the overall equalization of the variance among the factors.

Next, an attempt was made to assess the degree of simple structure exemplified by a given solution by studying (1) the hyperplane-counts (number of loadings, by factor and for the total solution in the range 0 ± .10) and (2) the previously mentioned correlations of the obtained factors with those of the graphic solution--converted to angular separations (θ_{pq} = arccos r_{pq})-- and the mean angular separation for a solution (over the m factors in the solution). The assumption was thus made that a graphic solution was likely to be the best manifestation of simple structure for a data set.

Finally, the interpretation of each factor for a given solution was studied. A factor was interpreted in terms of the variables found to load .30 or higher, in absolute value, on the factor. Solutions for a given data set were compared in terms of how each factor was interpreted.

Data Used

Four sets of data were used in this study--varying in (1) the number of variables and factors and (2) the reliability and factorial complexity of the variables:

(A) Eight Physical Variables (8 x 2). These anthropometric variables were highly reliable, with a median communality of .81. Each variable had factorial complexity of one in all solutions. The centroid matrix and graphic orthogonal solution were obtained from Harman (1960).

(B) Twenty-Four Psychological Tests (24 x 4). These well-known variables were of typical reliability for mental tests, with a median communality--an underestimate, of course, of reliability--of .47. Their factorial complexity--as determined for each variable from the number of loadings of .30 or larger, in absolute value, in the graphic solution (an imperfect index, of course, being somewhat dependent upon the communality and largely dictated by interpretive practices)--was moderately high, with 17 variables having complexity greater than one. The centroid matrix and graphic orthogonal solution (obtained by a graphical method due to Zimmerman) were found in Harman (1960.

(C) Wittenborn Data (20 x 7). These variables, representing measures of attention, were analyzed by Wittenborn (1943), using a graphic orthogonal solution. The reliability of the variables was moderate--the median communality was .44--and they were factorially quite simple, only 5 of the 20 variables having complexity greater than one.

(D) PMA Data (57 x 13). These well-known variables were first analyzed by Thurstone (1938). The graphic orthogonal solution used in this study, however, was performed by Zimmerman (1953), and was accomplished by starting from the point at which Thurstone had stopped (not having rotated all 13 factors) and finishing the rotational procedures, obtaining a clearer resolution of the factors than in the earlier study. The 57 variables were highly reliable, with a median communality of .71. They also tended to be factorially quite complex, with 45 having complexity--as assessed by the procedure just described--greater than one. This apparent complexity was a function, in part, of the large communalities.

Results

Eight Physical Variables

Four orthomax solutions were obtained for this data set, with w set to 0, 1, 2, and 4. The summary results are presented in Table 1. It can be seen from this table that the larger the value of w, the more evenly the variance was dispersed across the two factors, the quartimax solution (w = 0) showing the most unequal variance allotment.

On the two simple structure criteria--hyperplane-counts and overall closeness to the graphic solution--discrimination among the orthomax solutions was difficult. No solution, including the graphic, had any entries on the hyperplane of either factor. The correlations between the orthomax and graphic factors were taken to five places of decimals to permit some discrimination. The quartimax solution was considerably further (4° 17') from the graphic position than were the other orthomax solutions, the latter being almost identical to the graphic. Overall, it would seem that the solutions with w set to 1, 2, and 4 exemplified simple structure equally well, with the quartimax solution perhaps slightly inferior on this criterion.

With this particularly simple data set, the four analytic solutions and the graphic admitted to the same interpretation of the factors. That is, Factor I would be interpreted in terms of variables 1, 2, 3, and 4, and Factor II, in terms of variables 5, 6, 7, and 8.

Twenty-Four Psychological Tests

Ten orthomax solutions were obtained for these data, with w set to

TABLE 1

Dispersion of Variance, Hyperplane-Counts, and Correlations and Angular Separations with the Graphic Solution for the Orthomax Solutions of the Eight Physical Variables

Variance Dispersion

Solution	Factor I	Factor II	Range
Graphic	3.352	2.612	.740
w = 0	3.556	2.411	1.145
1	3.317	2.649	.668
2	3.314	2.651	.663
4	3.311	2.654	.657

Hyperplane-Counts

Solution	Factor I	Factor II	Total
Graphic	0	0	0
w = 0	0	0	0
1	0	0	0
2	0	0	0
4	0	0	0

Correlations with Graphic Factors

Solution	Graphic Factor I	Graphic Factor II
w = 0	.99721	.99721
1	.99990	.99990
2	.99990	.99990
4	.99980	.99980

Angular Separations with Graphic Factors

Solution	Graphic Factor I	Graphic Factor II	Mean
w = 0	4° 17'	4° 17'	4° 17'
1	0° 49'	0° 49'	0° 49'
2	0° 49'	0° 49'	0° 49'
4	1° 9'	1° 9'	1° 9'

-8, -2, 0, 1, 2, 4, 8, 24, 48, and 96. Summary results appear in Table 2. The actual graphic, quartimax, and varimax solutions appear in Table 3. Solutions obtained with w set to 2 (yielding an equamax solution) and 24 are presented in Table 4. From Table 2, it can be seen that, again, as w was increased, the variance dispersion tended to become increasingly more level, although with these solutions, the relationship was not perfect. The variance equalization of the graphic solution was exceeded by only two orthomax solutions--those with the highest values of w, 48 and 96. It is interesting to note that not only was there variability among the solutions in terms of equalization of variance over the factors, but the factor receiving the largest allotment of variance varied (Factor III for w = -8, -2, and 0; Factor I for w = 1, 2, 4, 8, 24, 48, and 96) as did that accounting for the least variance (Factor IV for w = -8, -2, 0, 1, 2, 4, and 8, and the graphic solution; Factor III for w = 24, 48, and 96).

The hyperplane-counts presented would seem to have little correspondence with simple structure for the solutions with w = -8, -2, and 0, since, in these solutions, large counts, as one might expect, were recorded for factors accounting for very small amounts of variance. For the solutions with fairly equitable variance dispersion, however, the varimax solution (w = 1) had the largest hyperplane-count (20), even larger than the graphic (18). The varimax solution was also closest overall to the graphic, although the solutions with w = 2, 4, and 8 were almost as close and clearly in the same general position. The solutions with w = 24, 48, and 96 were quite different, however, although somewhat similar among themselves. These latter solutions may well exhibit as clear a simple structure as those closer to the graphic, suggesting, perhaps, that closeness to a graphic position is not the only possible ...gonal position exemplifying a simple structure.

...robably the most relevant basis of comparison lies in the comparability

TABLE 2

Dispersion of Variance, Hyperplane-Counts, and Correlations and Angular Separations with the Graphic Solution for the Orthomax Solutions of the Twenty-Four Psychological Tests

Variance Dispersion

Solution	Factor I	II	III	IV	Range
Graphic	3.240	2.570	3.272	2.374	.898
w = - 8	1.525	1.315	7.563	.980	6.583
- 2	1.560	1.343	7.490	.990	6.500
0	2.056	1.759	6.245	1.323	4.922
1	3.504	2.441	3.082	2.356	1.148
2	3.579	2.667	2.775	2.361	1.218
4	3.586	2.830	2.605	2.362	1.224
8	3.582	2.923	2.509	2.368	1.214
24	3.523	3.036	2.404	2.420	1.119
48	3.114	3.029	2.327	2.913	.787
96	3.051	3.040	2.339	2.952	.712

Hyperplane-Counts

Solution	Factor I	II	III	IV	Total
Graphic	6	7	3	2	18
w = - 8	9	8	0	8	25
- 2	13	10	0	8	31
0	12	12	0	9	33
1	3	8	4	5	20
2	3	5	3	5	16
4	3	3	3	4	13
8	3	2	3	4	12
24	5	2	4	4	15
48	7	2	4	6	19
96	7	1	4	6	18

TABLE 2—Continued

Correlations with Graphic Factors

Solution	Graphic Factor I	II	III	IV
w = -8	.8149	.7560	.6319	.7164
-2	.8236	.7663	.6660	.7137
0	.8618	.8274	.8779	.7899
1	.8968	.8358	.9990	.8374
2	.8902	.8412	.9853	.8473
4	.8853	.8429	.9809	.8542
8	.8827	.8419	.9778	.8598
24	.7637	.6166	.9288	.6865
48	.5952	.7379	.6782	.7985
96	.5305	.6919	.6360	.7744

Angular Separations with Graphic Factors

Solution	Graphic Factor I	II	III	IV	Mean
w = -8	35°25'	40°53'	50°49'	44°15'	42°51'
-2	34°33'	39°59'	48°14'	44°23'	41°49'
0	30°29'	34°10'	28°37'	37°49'	32°46'
1	26°16'	33°18'	2°34'	33° 8'	23°49'
2	27° 6'	32°44'	9°50'	32° 5'	25°26'
4	27°43'	32°33'	11°13'	31°18'	25°42'
8	28° 2'	32°40'	12° 6'	30°42'	25°53'
24	40°13'	51°56'	21°45'	46°39'	40° 8'
48	53°28'	42°27'	47°18'	37° 1'	45° 4'
96	57°58'	46°13'	50°30'	39°15'	48°29'

TABLE 3

Orthogonally Rotated Solutions, using the Graphic, Quartimax ($w = 0$), and Varimax ($w = 1$) Techniques, for the Twenty-Four Psychological Tests (Leading Decimal Points Omitted)

	Graphic Factor				Quartimax Factor				Varimax Factor			
	I	II	III	IV	I	II	III	IV	I	II	III	IV
1	14	16	68	15	-10	03	72	-07	14	19	67	17
2	09	06	44	11	-05	-03	46	-05	10	07	43	10
3	11	-02	54	14	-03	-10	55	-10	15	02	54	08
4	18	02	55	13	01	-04	58	-11	20	[illegible]	54	07
5	72	05	23	33	62	14	51	03	75	21	22	13
6	65	-01	23	44	62	02	52	11	75	10	23	21
7	77	-03	22	33	70	09	50	-01	82	16	21	08
8	55	12	40	23	38	15	59	-03	54	26	38	12
9	56	-07	21	52	67	-07	52	15	80	01	22	25
10	36	67	00	-01	11	69	21	21	15	70	-06	24
11	[illegible]	63	13	14	08	56	36	28	17	60	08	36
12	25	61	28	-15	-09	62	38	-01	02	69	23	11
13	37	45	46	-09	01	48	56	-11	18	59	41	06
14	15	32	05	44	14	13	29	45	22	16	04	50
15	02	27	14	44	02	02	33	42	12	07	14	50
16	00	24	41	38	-09	-01	54	27	08	10	41	43
17	05	43	07	52	05	15	33	57	14	18	06	64
18	-03	46	33	37	-16	18	50	40	00	26	32	54
19	08	27	24	33	01	08	40	29	13	15	24	39
20	28	10	47	33	16	-02	62	08	35	11	47	25
21	22	38	45	15	-02	27	57	09	15	38	42	26
22	25	10	41	44	18	-07	59	20	36	04	41	36
23	32	16	58	26	13	06	72	01	35	21	57	22
24	40	44	25	26	21	36	50	22	34	44	22	34
Variance	3.24	2.57	3.27	2.37	2.06	1.76	6.25	1.32	3.50	2.44	3.08	2.36

TABLE 4

Orthogonally Rotated Solutions, using Values of 2 (=m/2--Equamax) and 24 (=6m) in the Orthomax Criterion, for the *Twenty-Four Psychological Tests* (Leading Decimal Points Omitted)

	w = 2				w = 24			
	Factor				Factor			
	I	II	III	IV	I	II	III	IV
1	15	23	65	18	17	34	62	12
2	11	09	42	11	14	17	40	05
3	17	05	53	09	20	17	51	01
4	21	11	53	08	23	24	49	01
5	75	23	18	13	72	37	08	12
6	75	12	20	22	75	25	13	16
7	82	18	17	08	79	35	06	05
8	55	28	35	12	52	41	25	11
9	80	03	19	26	82	17	13	18
10	14	70	-10	22	03	62	-17	40
11	17	62	04	34	09	54	00	48
12	01	70	20	09	-09	68	10	24
13	18	61	38	06	10	68	27	14
14	22	18	01	49	24	10	06	50
15	12	09	12	51	17	02	19	47
16	09	12	39	44	15	12	44	38
17	14	20	03	64	17	08	11	64
18	00	29	29	54	03	22	34	54
19	13	17	22	39	16	15	25	38
20	36	14	45	26	38	23	43	19
21	16	40	39	26	14	43	35	28
22	37	07	39	37	42	14	40	28
23	36	24	54	23	37	36	49	17
24	34	46	18	33	30	46	14	39
Variance	3.58	2.67	2.78	2.36	3.52	3.04	2.40	2.42

of interpretation given the factors in the different solutions. The solutions presented in Tables 3 and 4 will be used for this purpose. The following tabulation gives the identifying number of the variables that would be used to provide an interpretation of the factors in each of the five solutions (that is, those variables with loadings greater than .30, in absolute value):

	Factor			
Solution	I	II	III	IV
Graphic	5, 6, 7, 8, 9, 10, 11, 13, 23, 24	10, 11, 12, 13, 14, 17, 18, 21, 24	1, 2, 3, 4, 8, 13, 16, 18, 20, 21, 22, 23	5, 6, 7, 9, 14, 15, 16, 17, 18, 19, 20, 22
Quartimax	5, 6, 7, 8, 9	10, 11, 12, 13, 24	all but 10 and 14	14, 15, 17, 18
Varimax	5, 6, 7, 8, 9, 20, 22, 23, 24	10, 11, 12, 13, 21, 24	1, 2, 3, 4, 8, 13, 16, 18, 20, 21, 22, 23	11, 14, 15, 16, 17, 18, 19, 22, 24
w = 2 (Equamax)	5, 6, 7, 8, 9, 20, 22, 23, 24	10, 11, 12, 13, 21, 24	1, 2, 3, 4, 8, 13, 16, 20, 21, 22, 23	11, 14, 15, 16, 17, 18, 19, 22, 24
w = 24	5, 6, 7, 8, 9, 20, 22, 23, 24	1, 5, 7, 8, 10, 11, 12, 13, 21, 23, 24	1, 2, 3, 4, 16, 18, 20, 21, 22, 23	10, 11, 14, 15, 16, 17, 18, 19, 24

The varimax, equamax, and w = 24 first factors would be interpreted identically, but differently from those of the graphic and quartimax solutions. The second factors of the varimax and equamax (and probably the quartimax) solutions would be interpreted identically, but again quite differently from those of the graphic solution (which has, additionally, variables 14, 17, and 18) and the w = 24 solution (with variables 1, 5, 7, 8, and 23 additionally). The same situation is true for Factors III and IV, with the quartimax factors quite different, largely because of the very unequal variance dispersion (more than 50% on Factor III, and less than 12% on Factor IV). It is probably true, of course, that the differences in interpretation noted are to a large extent a function of the differences among the solutions in both equalization and specific allotments of the total variance. Thus, variance dispersion and factor interpretation are to some degree two sides of the same coin.

Wittenborn Data

Four orthomax solutions were obtained for this data set, with w set to 0, 1, 3.5, and 7. Summary results appear in Table 5. It can be seen from this table that, again, the size of w was directly related to the degree of variance equalization. The graphic solution for these data brought about far less variance equalization relative to the orthomax solutions than have the previously presented graphic solutions. As before, there was considerable variability over the solutions in terms of which factors accounted for the most and least, etc., variance.

With the possible exception of that of the quartimax solution, hyperplane-counts were very similar for the graphic and orthomax solutions. The varimax solution again was closest--in terms of mean angular separations--to the graphic, followed by the quartimax, equamax, and the w = 7 (m) solutions. The fact that the quartimax solution was closer, overall, to the graphic than were the equamax and w = 7 solutions would appear to be further evidence that closeness to a graphic solution may be a very imperfect index of simple structure for *orthogonal solutions, since it is unlikely that the quartimax solution* with the very unequal variance dispersion represents a superior simple structure to the equamax and w = 7 solutions. As with the Twenty-Four Psychological Tests, the interpretation of a given factor, for these data, was somewhat dependent upon the particular solution in which it was found.

PMA Data

Six orthomax solutions were obtained for this well-known data set, with w set to -10, 0, 1, 6.5, 13, and 26. Summary results appear in Table 6. As might be expected with a large number of factors, the matching of the thirteen factors obtained in each solution, with those of the graphic solution was fairly difficult, and was accomplished, in several cases, only by strict

TABLE 5

Dispersion of Variance, Hyperplane-Counts, and Correlations and Angular Separations with the Graphic Solution for the Orthomax Solutions of the Wittenborn Data

Variance Dispersion

	Factor							
Solution	I	II	III	IV	V	VI	VII	Range
Graphic	1.265	.910	.954	2.665	1.240	1.510	.819	1.846
w = 0	1.303	.590	.919	3.291	1.222	1.082	.760	2.531
1	1.427	.915	.959	2.254	1.320	1.426	.866	1.388
3.5	1.351	1.102	1.031	1.655	1.401	1.558	1.068	.624
7	1.308	1.112	1.082	1.485	1.452	1.574	1.154	.492

Hyperplane-Counts

	Factor							
Solution	I	II	III	IV	V	VI	VII	Total
Graphic	10	10	12	3	11	8	7	61
w = 0	10	9	11	0	13	13	10	66
1	8	7	12	5	10	9	11	62
3.5	8	7	12	9	10	9	7	62
7	8	7	9	11	8	9	6	58

Correlations with Graphic Factors

	Graphic Factor						
Solution	I	II	III	IV	V	VI	VII
w = 0	.9966	.7848	.9566	.9628	.9738	.9470	.7628
1	.9873	.8818	.9588	.9344	.9759	.9614	.8590
3.5	.9753	.7654	.9529	.8696	.9764	.9555	.8577
7	.9715	.7340	.9471	.8452	.9766	.9491	.8488

Angular Separations with Graphic Factors

	Graphic Factor							
Solution	I	II	III	IV	V	VI	VII	Mean
w = 0	4°45'	38°17'	16°56'	15°20'	13° 8'	18°44'	40°17'	21° 4'
1	9° 9'	28° 8'	16°30'	20°52'	12°36'	15°58'	30°47'	19° 9'
3.5	12°45'	40° 3'	17°39'	29°35'	12°29'	17°10'	30°56'	22°57'
7	13°43'	42°46'	18°43'	32°18'	12°25'	18°22'	31°55'	24°19'

TABLE 6

Dispersion of Variance, Hyperplane-Counts, and Correlations and Angular Separations with the Graphic Solution for the Orthomax Solutions of the PMA Data

Variance Dispersion

Solution	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	Ra
	Factor													
Graphic	3.919	2.541	3.274	5.339	3.094	2.438	3.316	3.447	2.320	2.660	3.373	2.692	2.241	3.[illegible]
w = -10	4.784	1.892	3.053	19.354	1.095	1.632	1.416	1.587	1.101	1.111	1.343	.852	1.200	18.504
0	8.314	1.238	3.457	14.684	1.620	1.485	1.414	1.549	.904	1.117	1.508	1.218	1.907	13.780
1	7.684	2.920	4.358	6.081	2.081	4.055	1.871	3.162	2.517	1.000	1.265	1.717	1.707	5.081
6.5	3.705	2.828	3.360	3.578	2.608	3.159	3.034	3.027	3.114	2.599	3.236	3.379	2.789	1.106
13	3.389	2.821	3.336	3.385	2.711	3.104	2.924	3.146	3.107	3.160	3.097	3.242	2.996	.678
26	3.277	2.818	3.267	3.090	2.665	3.014	2.958	3.171	3.395	3.088	3.214	3.303	3.156	.730

Hyperplane-Counts

Solution	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	Total
	Factor													
Graphic	21	25	34	17	21	30	14	26	21	20	16	23	22	290
w = -10	15	31	25	0	34	27	31	34	34	37	35	36	35	374
0	14	32	29	2	33	35	31	31	34	33	34	35	33	376
1	13	23	20	14	26	13	27	24	18	34	33	28	28	306
6.5	30	25	23	23	21	22	23	21	16	18	18	23	20	283
13	28	25	27	21	23	23	14	20	15	23	23	22	17	281
26	22	21	25	23	25	24	17	22	20	26	18	18	15	276

TABLE 6--Continued

Correlations with Graphic Factors

Solution	Graphic Factor I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
w = -10	.6433	.6453	.8567	.3829	.6614	.8007	.7730	.3340	.6975	.5247	.4464	.4601	.3353
0	.7255	.4387	.9746	.6833	.8458	.7912	.7536	.8663	.7826	.7891	.5015	.5328	.6966
1	.7756	.7378	.9389	.8154	.8705	.7736	.8612	.7779	.8809	.4950	.6811	.5471	.3684
6.5	.8220	.7461	.9547	.6470	.8107	.8142	.8703	.8828	.8563	.3199	.5496	.6014	.4313
13	.8568	.7303	.9558	.5098	.9651	.8629	.8062	.8767	.8960	.9152	.6290	.9262	.3748
26	.7484	.8027	.9604	.4493	.9237	.6779	.6994	.9029	.5263	.6489	.7400	.6955	.3958

Angular Separations with Graphic Factors

Solution	Graphic Factor I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	Mean
w = -10	49°56'	49°48'	30°57'	67°29'	48°35'	36°48'	39°23'	70°29'	45°46'	58°21'	63°29'	62°37'	70°25'	53°23'
0	43°29'	63°59'	12°56'	46°54'	32°15'	37°42'	41° 6'	30° 4'	38°30'	37°50'	59°54'	57°48'	45°51'	42°11'
1	39° 8'	42°27'	20° 8'	35°22'	29°31'	39°19'	30°33'	38°56'	28°15'	60°20'	47° 4'	56°50'	68°23'	41°15'
6.5	34°43'	41°45'	17°20'	49°41'	35°50'	35°29'	29°30'	28° 1'	31° 6'	71°21'	56°39'	53° 2'	64°27'	42°13'
13	31° 2'	43° 5'	17° 6'	59°20'	15°11'	30°21'	36°16'	28°45'	26°22'	23°46'	51° 1'	22° 9'	67°59'	34°48'
26	41°32'	36°37'	16°11'	63°18'	22°32'	47°19'	45°37'	25°27'	58°10'	49°32'	42°16'	45°56'	66°41'	43°10'

adherence to the rule of maximizing tr(K).

As with previous data sets, the graphic solution of the PMA Data was obtained with less equalization of common-factor variance than were several of the orthomax solutions. Again also, almost a perfect inverse relationship can be seen between size of w and variance equalization, using the range: (largest variance allotment - smallest allotment) as the index of equalization. One could, of course, use the variance or standard deviation of the variances as an alternative index. As before, the variance was dispersed not only more or less equitably as a function of w, but also differently. One can see, for example, that Factor X in the graphic solution had 2.66 units of variance associated with it--resulting in eight loadings large enough (greater than .30) to serve in interpreting the factor. The corresponding varimax factor, however, accounted for only 1.00 unit of variance--resulting in only two loadings greater than .30. The corresponding equamax factor had roughly as much variance (2.60 units) associated with it as had the graphic, and consequently had nine marker loadings. The orthomax solutions with w set to 13 and 26 had more variance associated with this factor (3.16 and 3.09 units, respectively) than had the graphic and, consequently, would allow a broader interpretation of the factor--with 11 and 13 significant loadings, respectively. Thus, with w set to 1, Factor X accounted for little variance (the least variance of the 13 varimax factors) and would be narrowly interpreted, whereas with w set to 13, for example, Factor X accounted for more variance than eight of the remaining 12 factors and would be broadly interpreted. Conversely, varimax Factor VI can be seen to account for more variance (4.06 units) than nine of the remaining factors, whereas the corresponding factor with w set to 13 accounted for more of the variance (3.10 units) than only five of the remaining factors.

As with the previous data sets, it appears true with the PMA Data that simple structure was probably equally well exemplified in the orthomax solutions with w = 1 or greater. Again, there is evidence that hyperplane-counts signify little with orthogonal solutions. Also, with as many factors as in the PMA Data and the restriction to orthogonality, it seems true that, rather than there existing one optimal position for the axes, to which the various analytic functions transform, more or less well (as would appear to be true for oblique solutions), there exist many possible positions that exemplify simple structure equally, but not very, well. It is seen, for example, from Table 6, that virtually none of the orthomax solutions had axes in the same general position as the graphic solution, although the orthomax solution with w set to 13 was the closest.

Since simple structure would appear to be somewhat of a constant--for w = 1 or greater--and quite likely a characteristic on which little if any choice can, for practical purposes, be made among several orthomax solutions, it may well be true that such a choice most logically should be made in terms of interpretability and specific interpretation of the factors among the several possible solutions. In Table 7, the graphic Factor XII and the factors from the orthomax solutions with w set to 1, 6.5, 13, and 26 that were matched with this factor are presented. The reader will recall that this matching was accomplished by maximizing tr(K); obviously, from Table 6, the match involving this graphic factor was not very close for the solutions with w = 1, 6.5, and 26, although the factor matched from these solutions was, in each case, that which was closest to graphic Factor XII. From Table 7, it can be seen that, if the factor is interpreted in terms of the variables loading on it to the extent of .30 or larger, in absolute value, the Factor XII from the orthomax solution with w set to 13 is almost identical to the corresponding graphic factor, the interpretation in either case being a Visualization factor. The max Factor XII, however, is much more narrowly defined (1.72 units of

TABLE 7

Factor XII of the Graphic and Orthomax PMA Solutions (Decimal Points Omitted)

Variable	Graphic	w = 1	w = 6.5	w = 13	w = 26
1	01	04	01	10	37
2	16	22	05	24	49
3	07	-03	12	10	20
4	00	07	15	1[illegible]	21
5	15	02	32	23	13
6	-14	-14	09	-05	-03
7	-11	06	02	00	07
8	16	09	27	16	13
9	24	29	18	22	08
10	04	02	19	10	-01
11	-13	12	-07	-04	-02
12	-01	10	07	-03	-21
13	-06	12	04	02	06
14	30	-04	53	34	19
15	30	-03	34	26	13
16	54	30	31	61	66
17	37	06	23	32	32
18	62	34	64	65	40
19	30	14	17	30	30
20	35	01	32	33	29
21	62	29	54	68	61
22	40	12	66	47	25
23	19	-05	20	13	14
24	15	-22	29	16	10
25	26	12	48	29	10
26	26	17	33	23	19
27	13	11	28	18	14
28	-03	-05	04	-02	-01
29	01	09	-10	07	15
30	11	04	02	02	-03
31	04	-16	01	-05	04
32	11	02	21	11	12
33	-04	04	07	06	15
34	-03	03	18	09	11
35	21	05	13	15	19
36	13	-01	33	15	12
37	14	21	17	30	38
38	-02	04	24	16	28
39	-02	03	06	10	20
40	21	20	21	25	37
41	24	13	46	33	21
42	13	-13	18	15	27
43	04	11	08	02	-03
44	08	26	06	13	12
45	-01	20	09	-10	-06
46	34	50	15	37	25
47	27	59	07	27	19
48	04	16	05	01	-09
49	13	22	-05	20	35
50	11	19	02	07	-02
51	05	06	-07	08	15
52	-04	11	-12	05	19
53	-06	03	-06	00	02
54	-09	07	-07	-01	04
55	31	07	11	34	56
56	00	-08	03	01	05
57	10	21	12	10	17

variance, as opposed to ?.24 for the $\underline{w}$ = 13 factor), and has large loadings by only two of the variables that defined the Visualization factor--the non-verbal Lozenges A (variable 16) and Form Board (variable 18) tests. This varimax factor may be somewhat better characterized as a Visual Memory factor since its most significant loadings are the memory tests, Word Recognition (variable 46) and Figure Recognition (variable 47). Each of the remaining two orthomax solutions--with $\underline{w}$ set to 6.5 and 26--has a Factor XII that would be both more broadly and somewhat differently interpreted than would the graphic factor. In the $\underline{w}$ = 26 solution, for example, a verbal facet has been added with large loadings by the Reading I and Reading II tests (variables 1 and 2). In summary, it may well be true that the most important feature of an orthogonal solution is not how well the interpretation-free criteria of simple structure are fulfilled, but rather how meaningful an interpretation each obtained factor permits.

Conclusions and Implications

The following conclusions appear warranted from the results.

(1) In general, as $\underline{w}$ is increased, the variance dispersion among the factors tends to become increasingly more level (this possibility was first suggested by Saunders, 1962). Solutions with small values of $\underline{w}$ (for example, less than 1) have large first factors, precluding a clear-cut simple structure.

(2) Because of conclusion (1), hyperplane-count is a poor index of simple structure for orthogonal solutions, at least if one includes solutions with $\underline{w}$ very small, since these solutions yield large counts because of small variance allotments to factors other than the first.

(3) There is little evidence to suggest that one special case of the orthomax criterion will, in general, yield solutions more closely aligned with a graphic solution for the data than any other, or, for that matter, to

suggest that, for orthogonal solutions, this criterion corresponds closely to exemplification of simple structure.

(4) Simple structure would appear to be somewhat of a constant for orthogonal solutions with *w* set to 1 or larger.

(5) Interpretation of the factors (as with variance dispersion),however, can be expected to change substantially--partly because of the differences in (a) equalization of factor variance and (b) the variability in order of allotment of variance to the factors in a given solution--as *w* is varied.

The implications appear clear. Since any orthomax solution represents as mathematically legitimate an orthogonal transformation as another, it would seem reasonable, if an orthogonal solution is desired, to obtain several orthomax solutions with *w* varied between 1 and 2*m* or even larger (values of *w* less than 1 do not appear promising). If one can specify *a priori* (perhaps from the purposes to which the factors are to be put) an optimal variance allotment (strict equalization is seldom optimal), the choice will be clear-cut. In the construction of a multi-factorial test, for example, the user could conceivably desire either a broader or narrower interpretation for a given factor than afforded by a single given solution. Barring the possibility of a preferable variance allotment, one obtained solution will undoubtedly have factors that are, in some sense, more interpretable, interesting, or in line with theory than those in the other solutions. Choosing this solution (which stops far short of a procrustean approach), then, would appear to be a less "blind" approach to orthogonal transformation than accepting the solution obtained by only one special case, for example, varimax, of this general criterion.

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