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ABSTRACT

This review of recent Piagetian research in mathematics education is divided into three parts. In the first part the author states the assumptions upon which the remainder of the paper rests. These assumptions are based on Piaget's theory of cognitive development and he presents six relevant points. The second part is a review of recent research on pupil's understanding of mathematical ideas. In the research devoted to students in elementary school he examines the child's understanding of number and the growth of spacial and geometric concepts--perspective, measurement in 2 and 3 dimensions, and time concepts. In secondary school research he examines the understanding of proportion, probability, function, and proof. The third part of the paper is devoted to the implications of the first two parts for teaching. The author suggests that elementary schools should be less formal, have more manipulative materials, and provide for more peer interaction. In the secondary school the author suggests more active learning by students, utilization of concrete materials, and provision of time for student initiated questions and answers. An extensive bibliography is included. (CT)

MATHEMATICS EDUCATION REPORTS

INTELLECTUAL GROWTH AND UNDERSTANDING MATHEMATICS

by Kenneth Lovell

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This paper, which reviews twenty-five years of Piaget research in intellectual growth as it pertains to the learning and understanding of mathematics, was commissioned by the ERIC Information Analysis Center for Science and Mathematics Education. The paper was presented by Professor Lovell at a meeting of the Special Interest Group for Research in Mathematics Education at the annual convention of the American Educational Research Association on February 5, 1971. It is with great pleasure that we now make this paper available to the wider mathematics education community as a Mathematics Education Report.

F. Joe Crosswhite
and
Jon L. Higgins
editors

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INTELLECTUAL GROWTH AND UNDERSTANDING MATHEMATICS

by

Kenneth Lovell

In the view of the Bourbaki group of mathematicians, mathematics is the study of structures or of systematic patterns of relationships. This being the case, the topic discussed in this paper is, I suggest, fundamental to the mathematics educator in the school years K through Grade 12.

This paper is divided into three parts. In the first I wish to indicate very briefly, some salient features of the conceptual framework inside which I shall discuss the later sections. In the second part I review some of the evidence obtained from research carried out into pupils' understanding of mathematical ideas. Finally, I deal with some of the implications for teaching in the classroom.

Assumptions made in the paper

I turn then to part one. The Thirty-Second Yearbook of the N.C.T.M. (15) deals with the history of Mathematics Education in the U.S. and Canada. Scattered throughout the book are references to psychological and educational theories and their impact on mathematics education. The name of Piaget occurs on seven pages and some of his books are named, but the volume does not spell out the precise ways in which his work is of value to the mathematics educator. Personally I have been greatly influenced by his work, details of which have been published in a vast array of books and papers over almost 50 years, although it would be true to say that it is only in the last 10 years that there has developed a widespread interest in his work in the United States. I believe that his position regarding the acquisition of certain kinds of new knowledge is of more value to the mathematics teacher than any other position at the moment, although I equally affirm that his theory does not cover all the facts and that one day it will be replaced by or subsumed under a more all embracing one. Some of the strengths and inadequacies of this theory in relation to mathematics learning have been given by Lovell (18) and Beilin (4). While it would be wrong for me to outline Piaget's cognitive developmental system here, I must just make six points which are the assumptions, so to speak, on which the remainder of my paper rests.

1. Piaget has been concerned with the development of the general ways of knowing, or the intellect - whatever term one uses. It seems that such cannot be taken directly from the

blackboard, textbook or film by mere perception, or acquired by drill; rather these general ways of knowing have to be actively constructed by the child through interaction with the environment. When forged, these are never forgotten in mental health. For example, the child never forgets that a subclass is subsumed within a class, that if A>B and B>C then A>C. Against this Piaget distinguishes certain kinds of particular knowledge that derive primarily from the interaction between the individual and specific aspects of the environment; indeed, all kinds of teaching procedures may be employed. But the quality of the general ways of knowing or of the intellectual structures, determines the manner in which the particular knowledge is assimilated. Moreover such knowledge may be forgotten at any time and have to be revised.

2. At the core of the central mechanism of intelligence, there are in Piaget's view the basic ope. ations of uniting, seriating, equalizing, putting into one-to-one correspondence, etc. These ultimately stem from actions in the first 21 months of life, internalized with the help of language but not deriving from the latter, thus yielding implicit mental actions which eventually emerge as reversible and integrated structures around 7 to 8 years of age. The distinction he makes between physical experience and logical-mathematical experience, is also important. In the latter knowledge comes not from the objects themselves but from the actions performed on objects, as when the pupil finds that five groups of three objects yield the same total as three groups of five objects. He has to keep a constant check on the coordination of his actions to avoid contradictions. In short the pupil reflects on his activity in an auto-regulatory sense. It is this kind of experience which aids intellectual growth although it is not the only type of experience to do so as we shall note later. Moreover, intellectual structures are reorganized and linked with others even in the absence of environmental stimulants.

3. For Piaget thinking is not a representation nor a descriptive event. Rather it is an action which early on overtly, and later covertly, transforms one reality state into another thereby leading to knowledge of the state. In his view, to understand a state one must understand the transformations from which the state results. It is, of course, the implicit mental actions or the covert transformations which are important in mathematics education.

4. The concept of stage is a key one for Piaget. It indicates successive developmental periods of intelligence such as sensorimotor, pre-operational, concrete and formal operational. Each stage is characterized by a relatively stable structure that incorporates earlier structures into a higher synthesis. Pinard and Laurendeau (37) have made an explicit and detailed defense of the stage construct as used by Piaget. Experience leads me to suggest that we accept this concept of stages with the associated changes in levels of understanding in respect to new ideas to be assimilated. Of course, I well recognize transitional

stages at the borderline between one broad stage and the next, and I well recognize the unsolved problem of horizontal differentials. I also testify that the elicitation of thought characteristic of the formal operational stage is affected by familiarity with content, and credibility.

5. In the view of the Geneva school language plays a role but not a central role in the growth of thought, although I would submit that we are only at the beginning in understanding the relationship between language and thought. At the level of formal thought it is clear that propositional logic needs some inner verbal support, but the power of propositional logic is not fundamentally due to this support.

6. When the mathematical idea to be learned depends on a level of logical thought beyond that which the child possesses, the idea is either partially learned, or learned with much difficulty and his grip on the idea is tenuous (Beilin, 4).

Review of Research

Having listed six major assumptions, I turn to the second part of my paper. I will attempt to review the evidence which was accumulated over the last 25 years or so in respect to pupils' intellectual growth and their understanding of mathematics. A radical selection of the literature must be made, but I will try to give the gist of what has been determined.

Piaget and Szeminska's (35) classic book The Child's Conception of Number gave details of experiments and data obtained relating to cardinal and ordinal correspondence, to the ability of the child to differentiate and coordinate the ordinal and cardinal aspects of number, and to the beginnings of basic additive and multiplicative properties of number. In essence their view was that the concept of the natural numbers comes from the fusion of two logical entities, namely class and asymmetrical relation; with classes, relations, and numbers evolving around 7 to 8 years of age in a tightly locked mutually interdependent way. This book sparked off much research and some acute observation in the classroom. The upshot is that an understanding of classes, relations and numbers are elaborated at more or less the same time, but at the time of their emergence the child's level of performance on related tasks is uneven (Dodwell, 8). The cardinal-value property of the elements of a set is present before the ordinal value property, and it is certain that there is no sudden fusion of cardinal and ordinal properties, for numbers large and small. There is a gap in time between the child understanding the relationship N and N + 1 for small numbers, say, 8 and 9, and for larger numbers, say, 82 and 83.

Three pieces of evidence may be adduced which are relevant to the views just expressed. Apostel, Mays, Morf and Piaget (2) showed in a study involving the division of the elements of a set into subsets, that the number of elements was at first conserved only with n small, but by 8 years of age or soon after there was conservation regardless of the number of elements. Second, the recent study of American children by Almy (1) showed that during the emergence of concrete operational thought the level of the child's performance is irregular. Of 914 second grade pupils of average age 7 years 4 months all of whom were tested individually on a number of tasks, 366 were clearly operational on three tasks involving conservation of number and quantity, 181 were operational on four tasks of seriation, and 253 on two tasks involving ordination. Third, all experience at Leeds suggests that in the case of able pupils, once the logical instrument is available in one type of task it soon becomes available in related tasks, whereas in the case of less able pupils, horizontal differentials persist much longer. Thus as logical thought emerges, the child's level of performance is uneven at first, the uneveness disappearing more quickly the abler the pupil.

Evidence on another important issue has been provided by Van Engen and Steffe (46) at the University of Wisconsin. They found that of 100 first grade pupils almost all could correctly work using symbols, 2+3= and 4+5= . Yet only 54 correctly stated no preference for separate or combined piles of candies when 5 were used, and only 45 when 9 candies were employed. It would appear that many pupils in the first grade can memorize facts employing symbols, but are unable to abstract the concept of addition from physical situations. Likewise Steffe (41) showed that only 60 out of 341 first grade pupils conserved numerousness over 4 items in each of three tasks when the numbers did not exceed 8, and that 128 pupils responded incorrectly to at least one item of the 4 in each of three tasks. But on a test of addition facts with totals not greater than 8, pupils of the second group had an average score of 76%, and pupils of the first group an average score of 91%. These data again warn us that addition facts can be learned without a firm abstraction of number.

Continuing with our study of number I now turn to two British investigators. Brown (5) studied the growth of pupils' understanding of the properties of the set of natural numbers. Employing both paper and pencil, and individual tests using concrete materials, he concluded that children pass through a number of stages in grasping each property and that understanding is reached at the following ages in children of average ability using small numbers: closure at 7, identity at 7-8, commutativity at 8-9, associativity at 8-9 and distributivity at 10-11 years. But pupils' performance could be advanced or retarded by up to four years compared with the norm, and he found that the operational stage in respect to all the properties tested occurred at the earliest at 9 years in his sample.

Willington (47) gave a number of individually administered tasks to pupils of average measured intelligence of elementary school age. One of the tasks involved the understanding of the equalization of differences leading to averages, and another the understanding of the combination of odd and even numbers. Once again stages in the growth of pupils' understanding were established. A full understanding of average was available to half the 9-year-olds and all the 10-year olds; but at 10 only a little over half of the pupils could generalize that the sum of any number of even numbers is itself even, whereas the sum of a number of odd numbers varies between odd and even.

In attempting to summarize the findings in respect to number, one must be very careful, since studies are carried out with limited samples in different countries. But it may be fairly said that there is limited operational use of number to the 8th birthday. From then onwards pupils acquire a greater grasp of larger numbers with the properties of the set of natural numbers being grasped by the majority of pupils at 10-11 years of age - some years after small numbers are first conserved.

The growth of spatial concepts (Piaget and Inhelder, 32) and geometrical concepts (Piaget, Inhelder and Szeminska, 34) is, like the concept of number, a particular application of intellectual growth. But we are now dealing with infralogical operations which arise around the same time as logico-arithmetic operations but which are distinguished from the latter in that they involve proximity and continuity. That such concepts depend upon the growth of logical thinking is obvious if we consider, say, measurement - a notion which has wide applicability. A child must grasp that the whole is comprised of a number of parts added together, and understand the principle of iteration, that is, the repeated application of a unit to another entity, such as length of a line or area of a surface. Spatial and geometrical concepts then are not derived from an apprehension or 'reading' of the physical properties of objects, but are actions performed on objects in thought. Hence a conceptual space emerges from 7 years of age onwards in which the child understands spatial properties and relations, quite different from the perceptual space of the very young child who can recognize perceptual distinctions. It is true that when using concrete operational thought the image may still have a place in supporting spatial reasoning, but when formal operational thought is employed, spatial reasoning is a purely abstract affair.

In Piaget's view there arise with the beginning of the onset of concrete operational thought, simultaneously and in parallel fashion so to speak, projective structures (rectilinearity, perspectives, coordination of viewpoints), and metrical structures (measurement, systems of reference, and measurement in two and three dimensions).

A number of studies have enabled us to check on some of the Geneva findings; for example Lovell, Healey and Rowland (22), Shantz and Smock (39), Lovell (20), and above all, the outstanding study by Laurendeau and Pinard (16) at Montreal. By and large the stages in the development of the structures, proposed by Piaget, are found but there are differences. The age range for the elaboration of a particular structure is considerable even in children of comparable background and ability as judged by teachers or by test results. Again the situation, the actual apparatus employed and the mode of response, for example, drawing versus selection of prepared drawings, affects the level of thought elicited and hence the level of behavior observed. The form of analysis may also influence the interpretation given to the experimental findings. Finally we must note that a pupil's response can be irregular across related tasks for a considerable period, that is, horizontal differentials are much in evidence. No doubt differences in specific experiences of particular children, explain some of the irregularities and inconsistencies. But it is now clear that the tasks are subtle, that the relevant ideas have to be carefully devised and that analysis has to be thoughtfully considered. Elicitation of thought seems to be a more tricky problem than is the elicitation of thinking about logico-arithmetic concepts. Nevertheless, it is perfectly clear that it is the emergence and growth of logical structures which underpins pupils' ability to elaborate the spatial structures in question. The excellent study of Laurendeau and Pinard mentioned above involved a number of spatial tasks given to 50 children at each age from 2 to 12 years. From an examination of the relevant correlation coefficients, and even more from a careful examination of the scalogram analyses, the authors conclude that there is indeed a consistency or coherence in the way the particular spatial concepts in question develop. The various stages in the tasks are reached in a regular rather than in a chance order.

I would also like to mention in passing a study which Lunzer (26) carried out both in Geneva and in Manchester (England) and which yielded similar results in both cities. In essence he found that it was not until the onset of formal operational thought at around 14 years of age, that the majority of those tested were able to dissociate, completely, area and perimeter of square/rectangle, and realize that under certain changes area is conserved and not perimeter, while under other changes the reverse is true. His work also suggests that it is not until formal operational thought is emerging that a geometrical situation can be handled by going outside the given limits of a figure. This is consonant with the views of Piaget, Inhelder and Szeminska (34) in respect of the construction of lines outside the figure when copying a triangle.

The concept of volume in its three aspects, internal volume, volume as occupied space, and complementary or displacement volume, is not well understood even at the end of elementary school. The evidence first established by Piaget, Inhelder and Szeminska (34), has been broadly confirmed using different techniques by Elkind (9, 10), Lovell, and Ogilvie (23) and Uzgiris (45). It is true that individual pupil experience may well underlie situational differences and account for the observed inconsistencies across different material. But, by and large, displacement volume is not well understood until the beginning of the emergence of formal operational thought. Pupils can, of course, through the iteration of a unit volume be led to calculate the internal volume, and volume as occupied space, in 5th grade, but volume as displacement is a more difficult idea. This may not be of great consequence for pure mathematics <u>per se</u> but it is of great importance to science teachers.

The only other infralogical concept emerging during the elementary school years that I can stop to consider is that of time. The emergence of temporal operations is fundamental for mathematics as it occurs in, say, the notion of speed or that of rate of change. Unfortunately the child can use time words, and tell the time, long before he can handle temporal operations. Because of this he may appear to have elaborated temporal operations when he has not.

Research carried out by Piaget (30), and by Lovell and Slater (24), has indicated that it is not until between 8 and 9 years of age that pupils begin to carry out the following:

(i) Put events into a sequence according to their order of succession.

(ii) Mark off intervals of time between ordered points on a time scale, and place smaller ones within larger ones.

(iii) Choose some time interval as a unit and use it as a unit for measuring some other time intervals.

In (ii) and (iii) we are, of course, carrying out the operations of subdivision and displacement or iteration as in any other form of measurement. These three operations develop more or less at the same time, but the apparatus or situation used does influence the pupil's ability to evoke temporal operations at first.

Once again we see that the understanding of time at this level depends upon the growth of the logical instrument. Even so, the pupil's understanding of time at this age is confined to intuitable situations. It will be well into high school before 75 per-cent of pupils realize that time on the clock is a purely arbitrary convention and correctly answer, given good reasons, the question, "When we advance the clocks by one hour in springtime, do we grow one hour older?"

So much for the elementary school stage. Let us now consider some mathematical ideas elaborated during high school years. From around 12 years of age in the brightest pupils and from 14 to 15 years in ordinary pupils, we see the emergence of formal operational thought. The chief characteristic of such thinking is the ability to invert reality and possibility thereby leading to the ability to use a combinatorial system and hypothetico-deductive thought. It may also be characterized as second degree operations for now the pupil can structure relations between relations as in, say, metric proportion which involves the recognition of the equivalence of two ratios.

Piaget, Inhelder and Szeminska (43) argued that it is easier to study the growth of the scheme of proportion in geometric than in non-geometric forms, for before the child can think about similar figures he can perceive whether the figures having different dimensions are similar. Our work at Leeds has indicated that pupils' responses in respect to the construction of a rectangle similar but larger than a model, can be placed more or less into the categories which Piaget suggested but the ages at which the stages are reached have been somewhat higher. However, Inhelder and Piaget (14) warn us that the scheme of metrical proportion in non-geometrical form depends eventually on the emergence of the growth of formal thought and that it comes later than teachers would wish. The studies of Lovell (17), Lunzer (25), Lovell and Butterworth (21) with British pupils, also Steffe and Parr (42) and Gray (13) with American pupils, have all confirmed that apart from very able 12-year-olds, it is from the beginning of junior high school onwards - the actual age depending on the ability of the pupil - that facility is acquired in handling metric proportion. Many pupils may not be able to do this until 14 to 15 years of age and some never. This is a matter of great consequence: it has repercussions in the teaching of physics and chemistry. This inability to handle metric proportions until these ages again clearly shows the dependence of the growth of mathematical understanding on the growth of the general ways of knowing.

Allied to proportion is quantitative probability. To tackle the latter the pupil has to be able to handle, in addition to metric proportion, the permutations and combinations in which a set of elements are grouped (Piaget, 33). The recent study of Shepler (40) involving a good teaching program showed that when probability questions could be answered using multiplicative classification almost 100% of sixth grade pupils, of mean measured IQ 117, obtained correct answers. But in questions involving estimated probability using large numbers, around 1/4 to 1/2 got the answers correct although some may have done so by a rote procedure since in each problem only two numbers were given and pupils would know that the probability could not exceed 1. Such questions cannot be solved by simple multiplicative classification and require formal thought. Some bright sixth graders should have been approaching this stage, and the study does bring out what aspects of probability are assimilable by the majority of pupils in the upper classes of elementary school. I say this in spite of the recent study by the Roumanians, Fischbein, Pampa and Manzat (11) reported in an American journal. They took three groups of able pupils aged 5 to 6, 9 to 10, and 12 1/2 to 13 1/2 years of age. Subjects were asked to choose out of two sets of marbles each of 2 colors, the set which they believed offered more chances of drawing a marble of a given color. A short period of instruction was given. In the pre-school pupils judgements based on simple binary relations were prevalent, while most of the sixth grade pupils based their decisions on relations between ratios. Instruction did not produce essential changes at these levels, but instruction did bring a shift towards the type of answer given by sixth graders in the case of the 9-10 year olds. Notice, however, that the choice had to be made in the case of smallish numbers and simple ratios; e.g. 2 white 4 black and 3 white 9 black; 12 white white 4 black and 20 white 10 black; 3 white 4 black and 6 white 8 black. The instruction methods used employed grouping the marbles. The pupil with flexible concrete operational thought would be able to group the above groups as 1 white 2 black and 1 white 3 black;

3 white 1 black and 2 white 1 black; 3 white 4 black and 3 white 4 black. Even so, these able 9-10 year olds obtained only 75% of possible correct answers after instruction.

Work has also been carried out on the growth of pupils' understanding of the mathematical function. Piaget et al (31) carried out a study of functions which were linked with the scheme of proportionality, for only those functions in which laws of variation play a part were considered. A function was regarded as the relation between the magnitude of two quantities, the variation in one bringing about a variation in the other in the same proportion. The view taken of a function was, therefore, much narrower than the one currently held in mathematics. However, using ingenious experiments they have produced evidence which suggests that the pupil only slowly acquires the ability to understand a function in this limited sense. At first it is only putting into correspondence two values, e.g. the smaller the wheel the less distance travelled: or it may appear in the form of a causal dependency, e.g. the harder the surface the higher the ball bounces. But with the onset of formal operational thought the ratios between successive pairs of ordered values of a variable can be handled.

The current mathematical definition of a function is, of course, more general than that considered by Piaget et al (31) as I have already indicated. In school mathematics, function is used in the sense of single valued function so that the function $b = f(a)$, defined on A as domain and with a subset of B as range, gives a mapping of the set A into the set B such that for each $a \in A$, there is a unique image $f(a) \in B$. There are only two studies available as far as I am aware into the growth of the concept of a function. One was carried out by Thomas (44) at Columbia University and one by Orton (29) at Leeds, England. Some details of the latter's work will be published by the N.C.T.M. later this year. Both used a large number of tasks individually administered to pupils. Orton's work involved pupils of average and above average mathematics attainment of high school age. Part 1 tasks tested a wide range of situations, and presented relations in all of the major representations, by diagram, by graph, by ordered pairs, by table and by equation. The formation of the appropriate range for a given rule and domain were also considered to be important tasks, in addition to the recognition of a function. Part 2 tasks, given only to older pupils, tested their ability to handle the composition of functions, use the f-notation, and tackle harder relations throughout.

In the Part 1 tasks the stages in the growth of a concept of a function corresponded closely to those found by Thomas. Responses to Part 2 tasks also yield four stages in the growth of understanding, but these could not be aligned to the American work since the latter did not explore the understanding of the composition of functions to the same extent. Suffice it is to say that these studies have given us a far better idea of the difficulties that pupils have in the growth of their understanding of the concept of function.

An interesting study in Britain by Reynolds (38) has thrown some light on pupils' devel ping grasp of proof in mathematics. Such understanding will, of course, always be important regardless of the nature of the curriculum. He investigated among abler pupils of high school age the understanding which they have in respect to assumptions, generalizations, and proof by converse, reductio ad adsurdum, and deduction. His conclusions were that Piaget's formulations regarding stages of thinking accounts for a good deal of the variability in the nature of pupil responses for replies indicative of concrete operational thought appeared regularly, while responses indicative of formal operational thought increased with age. But his work also showed that there were discrepancies between the replies obtained, and what might be expected from Piaget's views on the nature of formal thought. For example, Reynolds' work gives evidence that the degree of structure of a problem is important in this respect. In a well structured problem such as the Geneva school used, the assumptions, variables and universes of discourse are easily identified and th pil has no need to introduce assumptions and hypotheses from outside. In the former instance solutions offered are closer to those expected from Piaget's cognitive developmental model.

Finally in this section I would like to mention the concepts of point and limit, two notions of fundamental importance in mathematics. Piaget, Inhelder and Szeminska (34) have indicated that children proceed through a number of stages in their growth of understanding of 'point', and that it is not until the onset of formal operational thought that point becomes thought of as homogeneous regardless of the original shape from which it was derived and, of course, without shape or surface area. Again Taback (43) studied aspects of the concept of a limit among American children aged 8, 10 and 12 years using a number of individually administered tasks. He tells us that his subjects were drawn from independent schools, were very sophisticated in expressing themselves and came from homes in which education was respected and books were available. It would be reasonable to assume that such pupils with a chronological age of 12 years would have a mental age of around 14 years on the average. According to Taback only about one-third of the 12-year-olds could appreciate an infinite number of points within a neighborhood; that in respect of convergence those questions which demanded a level of thought liberati g the pupil from physical materials could be answered only by the 12-year-olds; and that taking the study as a whole, with f w exceptions, only the 12-year-olds could conceptualize an infinite process.

This review has, I hope, given sufficient evidence that it is the development of the general ways of knowing which determines the manner in which taught materi l is understood.

Before I conclude this part of my paper I must say something of educational technology. I suggest we have been too much concerned with hardware and too little about teachware; i.e. too little about the preparation and presentation of material in such a way that the pupil will be motivated and helped to act on, transform and construct. In that branch of educational technology with which I have been most closely associated, computer based learning, this certainly has been the case although I believe that we are now seeing the light. Not that hardware is unimportant; indeed it is vital that the teacher be given the freedom to teach his subject matter in the way he considers best.

As I have suggested, it is the growth of the general ways of knowing that determines the manner in which new knowledge in mathematics is assimilated. I suggest that the purpose of computer based learning systems is for each individual, to provide opportunities for the evocation, organization and strengthening of the available strategies of thinking in mathematics, or other subject areas, thereby permitting an increase in level of attainment. In this I believe such systems may have great possibilities. At the same time, however, I believe we must have an open mind about the extent to which they will aid the growth of the general ways of knowing.

Again on the research side any models which are derived from hypothesizing or from pupil response, must for the forseeable future, be looked upon from the point of view of enhancing adaptive material and from the point of view of controlling practice. Such models will cover very limited areas of work and should be looked upon merely as models for optimizing instruction within that area. My colleagues at Leeds, Pat Woods and J. R. Hartley (48) are publishing this month details of such a model in respect to the addition of the natural numbers. Using criteria of probability of success and rate of working for each column of the task in vertical format, analysis of variance revealed main effects of digit size and number of rows. Following a formal development of the model, a least squares analysis derived a function which, for the experimental data, related these variables to the criteria. These are used by the computer to generate examples so that a pupil works at any specified level of success. But it must be looked upon only as a model for optimizing instruction over a very limited area, and in no sense be looked upon as a model reflecting intellectual development in the area.

A final point to note is that on such evidence as we have, teachers are slow to leave the computer system to do the things it is best suited to do and to turn themselves to more creative teaching.

Implications for teaching

I now turn to the third part of my paper. So far I have talked a great deal about research. What does it all suggest to the mathematics educator? Let me reply in this way. In 1961 an American named Mayer published a book with the title The Schools (27). It was published here by Harper and in London by Bodly Head Press. While I cannot agree with all he wrote in the book there was one sentence which I found arresting. It was this: "What future teachers need, and cannot now find, is the course which attempts to explore the profound aspects of the deceptively simple material they are going to teach, which analyzes case by case the types of difficulty that children find in approaching such material, which suggests tools and techniques and methods of presentation that may help children overcome the difficulties".

How stands the position today - 10 years later - in respect of mathematics teaching? I suggest that now we know - thanks to the Piaget-type research - much more about the profound aspects of the deceptively simple material in mathematics that children are called upon to learn. Again, if we take the trouble we can analyze in far greater detail the difficulties that children have in approaching such material. We also know that the development of the general ways of knowing will determine the manner in which the mathematical ideas are assimilated. Of course we have only just made a beginning in these matters, and far more knowledge is required. But I suggest that we can begin to write down the educational implications of all that I have said - at least the implications which relate to classroom organization and general teaching techniques - ever bearing in mind that understood knowledge results from action and the transforming of one reality state into another.

However, before I discuss such organization and techniques I would like to mention two other important findings of the last decade or so in respect to scholastic educability. First, there is now abundant evidence that the value judgments of parents, the function of language within the home and parental attitudes to education, affects scholastic educability. The evidence is so widespread that it does not need referencing. The mathematics teacher can certainly attempt to change parental attitudes but it's a hard job. Influencing what happens in the home, especially in pre-school years, is beyond the limits of the individual teacher and needs a national policy in order to attempt changes. Second, there is increasing evidence, e.g. Pidgeon (36), that teacher expectation affects pupil performance. The strong suggestion for the mathematics teacher is that he should think well of his pupils and set standards of work which are high for them - that is, high for a particular individual or small group.

Realizing then that home attitudes and teacher expectation greatly affect pupils' desire to act on and transform reality thereby yielding new knowledge, we pass to consider classroom organization and general teaching approaches that appear to aid these transformations based on a Piagetian cognitive-developmental model. In respect to the elementary school we may suggest:

1. A move from a formal classroom atmosphere with much talk by the teacher directed to the whole class, to the position where the pupils work in small groups or individually, at tasks which have been provided.

2. The opportunity for pupils to act on physical materials, and to use games in the manner suggested by Dienes. It is the abstractions from actions performed on objects and not the objects themselves that aid forward knowledge of mathematical ideas. Not until flexible formal operational thought is available in mathematics can the latter be learned using words and symbols only, and intuitive data dispensed with.

3. In the Genevan view social intercourse using verbal language is an important influence in the development of concrete operational thought. Through exchanges, discussions, agreements, oppositions, both between children, and between adults and children, the child encounters viewpoints which must be reconciled with those of his own. There is now exchange and interpersonal as well as intrapersonal coordination. These cooperative aspects of exchange are important for the pupil is forced to organize his thoughts into a coherent structure and also forced to elicit the strategies of thinking available to him. This argues a strong case for much teacher/child and child/child interaction in mathematics teaching. And since language helps the child organize his experience and carry his thinking, the case is made for dialogue and action to go alongside one another. Likewise P. I. Galperin (12), the well known Russian educator/psychologist, and a person very different from Piaget, argues for the use of numerical language and actions as opposed to language and things only.

4. Since mathematics is a structured and interlocked system of relations expressed in symbols and governed by firm rules, the initiative, and the direction of the work must be the teacher's responsibility. This was often overlooked in the progressive education movement. This does not mean that pupils should never have a choice of activities, and it does not imply that teachers should ignore naturally occurring but relevant situations. Indeed, it was 75 distinguished American mathematicians who reminded us in the American Mathematical Monthly (28) in 1962, that children wished to use mathematics as a tool with which to explore the world and not to play a game with arbitrary rules. In other words, our tasks should have the proper degree of structure and be seen by children to have relevance to real life.

5. Alongside the abstraction of the mathematical idea from the physical situation, there must be the introduction of the relevant symbolization and the working of examples, involving drill and practice and problems, on paper.

When we consider pupils over 12 years of age the position becomes more complex. With very backward children the class organization and active approach used earlier have to be continued, although new mathematical ideas have to be introduced. The nature of the activities and materials will also change to become appropriate to the pupils' emotional and physical development. But the structures that these pupils will elaborate will be those derivable from interaction with first hand reality (Lovell, 19). Alas, in the case of the weakest school educable pupils most learning in mathematics will take place using algorithms which we must give them to cope with real life situations.

In the case of ordinary pupils we can slowly move them to new topics which depend upon the emergence of formal thought. But there rests upon us the absolute necessity of introducing these structures through concrete realizations. Let me give you an example of what I mean. Take an envelope. Let its center be 0 and suppose 0X and 0Y are axes in the plane of the envelope, through 0, and parallel to two adjacent sides of the envelope. Suppose 0Z is the axis through 0 perpendicular to the plane of the envelope. The 12 or 13-year-old can build up a table showing the effects of rotating the envelope about the three axes - any one operation being followed by a second. He can understand the structure displayed by the table; hence he can understand the structure of a mathematical group in this one concrete realization. The same pupil can equally well handle the addition of the set of integers mod 4 when the operation is "add two numbers of the set", draw up an appropriate table and understand its structure, namely that of another group. But the recognition of the relationships between structures or the generalization of the structure requires flexible formal operational thought so that the group structure can now be conceived generally without a concrete realization (Dienes and Jeeves, 7). Each successive concrete realization of an abstract structure is likely to increase the pupil's awareness of the structure and when formal thought is available increase the chances of the structure being generalized. Dienes (6) also lays stress on this general point in another of his books: Experimental Study of Mathematics Learning.

Study of Mathematics Learning.

I further suggest that in the case of ordinary pupils, our knowledge of their slow move to formal thought is such that small group and individual work is still necessary, permitting opportunities for individual or small group assignments, and dialogue between teacher/pupil and pupil/pupil.

The abler the pupil, the more quickly flexible formal operational thought will be in evidence, and the more quickly can he move away from intuitable data and consider third level abstractions devoid of concrete realizations. Not only are these structures more easily attainable in these circumstances but so are the relationships between structures and hence generalization. More class teaching with verbal and symbol exposition is now possible although the need for constant discussion between teacher and pupil and between pupil and pupil remains. And pupils still need the opportunity to formulate their own questions and discuss their own answers to them. However, if there are considerable differences in attainment even in an overall high attaining class, the small group approach remains necessary. As Gagné reminded us, problem solving demands masses of structurally organized knowledge. Such knowledge is not possessed equally when pupils differ in attainment whatever their potential may be.

Having talked about the classroom and techniques in general terms I now wish to make four further points.

1. There is the question of discovery methods. I concur with Ausubel (3) that both verbal learning and problem solving through active methods can be rote or meaningful. As he points out, active methods are not meaningful unless they rest on a base of understood concepts and that the operations involved are also meaningful. Obversely, if the pupil can relate new material, given by verbal exposition, in a substantive and nonarbitrary way to what has gone before, the learning will be meaningful for the child will have transformed one reality state into another.

2. The ways in which we <u>now</u> look at mathematical ideas demand a greater degree of verbal explication than was the case in the more traditional mathematics program. Teachers and pupils now require greater powers of verbal comprehension and explication in mathematics than formerly.

3. British teachers often find it hard to change from class teaching to small group work. Materials have to be prepared, groups organized, and language and action must proceed together. It is very hard work. No doubt U.S. teachers find the same difficulties, but they must be encouraged to make the move. Again, British primary school teachers are not equally good teachers across the whole range of subjects they have to tackle: they cannot handle the content and teaching methodology with equal facility across the board. They have indeed a difficult task as have U.S. elementary school teachers. I suggest, therefore, that while we encourage American elementary school teachers as much as possible in respect to mathematics teaching, and raise the overall standard so to speak, we must not expect that all will show equal competence in this area. Some will be relatively better teachers of other subjects, and this must be accepted.

4. I would like to raise the question of whether we should wait until pupils are ready to assimilate ideas fully and formally so to speak as mathematicians would have them assimilated ideally. In my view the answer is 'No'. Indeed it is impossible to say with precision at present when pupils are ready.

> In the first place the emergence of new forms of thought are patchy and irregular at first. Some ideas of comparable structure and level of abstraction are understood better than others, and there are individual differences as well. Second, the more familiar pupils are with content, the more readily, within limits, can formal operational thought be elicited. Third, topics can be introduced in different ways and at different depths, so to speak, so that the teacher may well start with the assumption that pupils' understanding of the idea will be limited at that point in time. But the teacher can lay a framework, can make pupils feel more 'at home' with the idea, and when the teacher returns to the topic later and with a different treatment, the subject matter will be reorganized and seen in a different light for at the later date the general ways of knowing will also have advanced. It is knowledge of the subject matter and of his pupils, which allows the teacher to distinguish the level of thinking of the child in relation to this particular topic, and he will not attempt to force an understanding not yet available to the pupil.

Conclusion

In this paper I have ranged far and wide. I hope I have conveyed a note of cautious optimism in respect of mathematics education. I do not believe that by some miracle we can accelerate the growth of pupils' thinking so that what was done in college can now be done in first grade. Knowledge of Piaget-type research does not make mathematical ideas per se any easier for children to learn. But I do believe that if we will but take the trouble, and accept certain limitations for some pupils, we can bring more understanding and greater enjoyment to pupils in respect to mathematics. I look to a slow and steady improvement over the years - not to a revolution. The Coleman report in this country - said to be the second largest social science research project ever mounted - clearly showed that the differences in educational achievement between groups is there at the beginning of school. These differences will not be removed overnight.

When the Chairman of the Convention Program Committee sent the details of this conference his letter struck a somewhat pessimistic note. You will remember it ran: "These are troubled times for educational research and its practitioners. Federal support of and confidence in educational research is equivocal and diminishing. State and local educational problems are already of gigantic proportions and enlarge daily." In response to this I have tried to show that with respect to mathematics, research has given us much more knowledge of the profound aspects of the deceptively simple material pupils have to learn, more knowledge about the difficulties children have and the stages through which they pass in coming to grips with mathematical ideas, and indications concerning the form that classroom organization and teaching strategies should take. Moreover still greater knowledge could help us in curriculum development itself. If we will but take the trouble to make serving teachers rethink their position, and reshape the education of our teachers-to-be, there are grounds for some mild optimism. Not that mathematical ideas themselves can be made easier, but we may be able to produce an atmosphere in which pupils are more likely to assimilate and enjoy the ideas in question.

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