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AUTHOR	Bramble, William J.; Wiley, David E.
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ABSTRACT

The research on covariance structure analysis is reviewed, and various restrictions on the parameter matrices of the general model are delineated. Models employing two particular restrictions (where the matrix of weights is completely specified and is either scaled--by some unknown but estimable matrix of scaling weights--or unscaled) are discussed in particular, and their application to test development techniques is considered. It is suggested that more precise measuring instruments can be developed through these procedures by studying the characteristics of the test and the test items, the latent variables, and the ensuing relationships. An example is provided. A computer program for solving likelihood equations and for testing fit is available. (GS)

IMPLICATIONS OF A CLASS OF COVARIANCE STRUCTURE MODELS FOR THE DEVELOPMENT OF MEASURING INSTRUMENTS

William J. Bramble
University of Kentucky

David E. Wiley
University of Chicago

Covariance Structure Models

Covariance structure analysis is a generic construct describing a variety of statistical procedures for determining sources of latent variation and covariation among a set of variables. The basic data for covariance structure analysis are p-variate response vectors for each of N subjects. The generic structural model for the population covariance matrix in this type of analysis is

$$(1) \quad \underset{p \times p}{\Sigma} = \underset{p \times m}{\Lambda} \; \underset{m \times m}{\Phi} \; \underset{m \times p}{\Lambda'} + \underset{p \times p}{\Psi}$$

In equation (1) Σ is the p x p population covariance matrix, Λ is a matrix of weights relating the observed p variables to a set of m latent variables, Φ is the m x m covariance matrix for the latent variables, and Ψ is a diagonal matrix of error variances for the p observed variables. A set of covariance structural models is generated when alternative restrictions are placed on the parameter matrices of the general model.

Wiley (1967) reviewed the work of Bock and Bargmann (1966), Scheffe (1956), Lawley (1940, 1942), and Joreskog (1967) and delineated a set of sixteen possible covariance structure models based on the following restrictions on the parameter matrices of the general model:

(a) Restrictions of the matrix of weights on the latent variables (Λ);

(1) Λ may be completely unspecified (all elements of Λ to be estimated).

(2) Λ may contain some specified elements and have other elements which must be estimated (represented by Λ^*).

(3) Λ may be completely specified, but scaled by some unknown but estimable matrix of scaling weights (represented by ΓA).

(4) Λ may be completely specified and unscaled (represented by A).

(b) Restrictions on the covariance matrix for the latent variables (Φ).

(1) The latent variables are assumed uncorrelated (orthogonal case) and thus Φ is diagonal ($\Phi = \Delta^2$).

(2) The latent variables are assumed correlated (oblique case) and thus Φ is not restricted to a diagonal matrix (thus we merely use Φ in this case).

(c) Restrictions on the matrix of error variances.

(1) The p error variances are assumed homogeneous ($\Psi = I\sigma^2$).

(2) The error variances are assumed heterogeneous (Ψ general diagonal).

In the above restrictions those on Λ depend on the theoretical preconceptions of the experimenter, the situation, and the form of the data. The general Λ matrix in which all elements are estimated is typically employed in classical factor analysis (see Harman, 1967) and in maximum likelihood factor analysis (see Lawley, 1940, 1942). Including the restricted case of Λ (i.e., Λ^*) provides a variant of these procedures. In the case of Λ completely specified ($\Lambda = A$) the usual form of A is that of a reparamaterized mixed model analysis of variance design matrix where some m-factor design is employed and the p variables are the p treatment combinations of the m-factor design. If there is a problem with varying metric across the p treatment combinations the design matrix may be rescaled ($\Lambda = \Gamma A$). The present paper involves models in which the latter two restrictions on Λ are employed ($\Lambda = A$ and $\Lambda = \Gamma A$). The restrictions employed on the other two paremeter matrices (Φ and Ψ) have to do with assumptions that are made about these parameter matrices rather than preconceptions of the experimenter. In addition we want to have the opportunity to test the validity of these assumptions in empirical situations. The procedures presented in this paper provide us this opportunity.

The class of covariance structure models described here is the 2 x 2 x 2 array of eight models presented in Table 1.

TABLE 1

THE CLASS OF COVARIANCE STRUCTURE MODELS

Restrictions on Λ

	A		ΓA	
	Restrictions on Φ			
Restrictions on Ψ	Δ^2	Φ	Δ^2	Φ
$\Psi = I\sigma^2$	$\Sigma = A\Delta^2A' + I\sigma^2$	$\Sigma = A\Phi A' + I\sigma^2$	$\Sigma = \Gamma A\Delta^2A'\Gamma + I\sigma^2$	$\Sigma = \Gamma A\Phi A'\Gamma + I\sigma^2$
Ψ	$\Sigma = A\Delta^2A' + \Psi$	$\Sigma = A\Phi A' + \Psi$	$\Sigma = \Gamma A\Delta^2A'\Gamma + \Psi$	$\Sigma = \Gamma A\Phi A'\Gamma + \Psi$

Notice in Table 1 that the models range in degree of restriction from $\Sigma = A\Delta^2A' + I\sigma^2$ (most restricted) to $\Sigma = \Gamma A\Phi A'\Gamma + \Psi$ (least restricted).

The data for the set of eight covariance structure models is like that depicted in Figure 1. It is obtained from the p repeated measures of an analysis of variance design for each of n subjects. If we assume that the observation vectors $(\underline{y_i})$ are randomly drawn from a multivariate normal population with mean vector $(\underline{\mu})$ and population covariance matrix Σ, we can apply maximum likelihood procedures in estimating the

$$
\begin{array}{c|cccccccccc|}
 & 1 & 2 & 3 & \cdot & \cdot & \cdot & j & \cdot & \cdot & \cdot & p \\
\hline
1 & y_{11} & y_{12} & y_{13} & \cdot & \cdot & \cdot & y_{1j} & \cdot & \cdot & \cdot & y_{1p} \\
2 & y_{21} & y_{22} & y_{23} & \cdot & \cdot & \cdot & y_{2j} & \cdot & \cdot & \cdot & y_{2p} \\
3 & y_{31} & y_{32} & y_{33} & \cdot & \cdot & \cdot & y_{3j} & \cdot & \cdot & \cdot & y_{3p} \\
\cdot & & & & & & & \cdot & & & & \cdot \\
\cdot & & & & & & & \cdot & & & & \cdot \\
\cdot & & & & & & & \cdot & & & & \cdot \\
i & y_{i1} & y_{i2} & y_{i3} & \cdot & \cdot & \cdot & y_{ij} & \cdot & \cdot & \cdot & y_{ip} \\
\cdot & & & & & & & \cdot & & & & \cdot \\
\cdot & & & & & & & \cdot & & & & \cdot \\
\cdot & & & & & & & \cdot & & & & \cdot \\
n & y_{n1} & y_{n2} & y_{n3} & \cdot & \cdot & \cdot & y_{nj} & & & & y_{np} \\
\hline
\end{array}
$$

Figure 1. Data for a repeated measures analysis of variance design.

elements of the parameter matrices of the structural model for Σ. The log likelihood is given as

$$(2) \quad \text{Log } L = \frac{Np}{2} \log 2\Pi - \frac{N}{2} \log|\Sigma| - \frac{N}{2} \text{tr } \Sigma^{-1} S,$$

where S is the sample covariance matrix. The general expression for the first derivatives of the likelihood functions with respect to a general element x is

$$(3) \quad \frac{\partial L}{\partial x} = \frac{N}{2} \text{ tr } \{\frac{\partial \Sigma}{\partial x} (\Sigma^{-1} S \Sigma^{-1} - \Sigma^{-1})\}.$$

Using this expression the first derivatives of the log likelihood function with respect to each of the parameter matrices of the model have been obtained by Bramble, Schmidt, and Wiley (1969, 1970).

An efficient numerical method for solving the likelihood equations was found by these authors in the quasi-Newton method originated by Davidon (1959) and modified by Fletcher and Powell (1963). A FORTRAN IV program has been developed which implements these procedures obtaining point estimates for the elements of the parameter matrices of the structural model.

An estimate of the asymptotic variance - covariance matrix of the estimates is provided from the negative of the inverse of the final estimate of the matrix of second derivatives or its expected value. The computer program is written using the matrix subroutine package of Bock and Peterson (1967) and is available from either of the authors.

Testing the Fit of Covariance Structure Models

The computer program also provides a chi square approximation of the likelihood ratio test of fit of a particular model

$$(4) \quad \chi^2 = -2 \ln \lambda = -N \ln \frac{|\hat{\Sigma}|}{|S|}$$

where the degrees of freedom are equal to the difference in the number of parameters estimated between the alternative and null models. Model testing can also be carried out sequentially, that is, the relative fit of alternative models can be tested using the likelihood ratio statistic

$$(5) \quad \lambda = \frac{|\hat{\Sigma}_{i+1}|^{1/2N}}{|\hat{\Sigma}_i|^{1/2N}}.$$

In equation (5) the estimated covariance matrix for the less restricted of the two models being considered is denoted by $\hat{\Sigma}_{i+1}$ and that for the more restricted model by $\hat{\Sigma}_i$. The chi square approximation for this statistic is

$$(6)\quad \chi^2 = N\ln \frac{|\hat{\Sigma}_i|}{|\hat{\Sigma}_{i+1}|} = \chi^2{}_i - \chi^2{}_{i+1}$$

Thus the relative fit of any two models can be tested by the difference between the χ^2 approximations for the likelihood ratio tests of fit for the two models. Sequential model testing of this type has been developed by Bramble (1970) and is useful in many applications of covariance structure analysis including applications to measurement problems.

APPLICATION OF THE COVARIANCE STRUCTURE MODELS TO MEASUREMENT PROBLEMS

Using the procedures outlined in the foregoing sections we can develop measuring instruments in which we know the relationships among traits and item characteristics more precisely than has heretofore been possible. Using these procedures the item to be included in the test must be written or selected in such a way as to very exactly represent within the test the types of variance and covariance which are of significant interest to merit study. Thus, we do not begin with a loose conglomerate of unselected items, but one in which properties of interest are built into the test systematically. This procedure in and of itself is an apparent improvement in test development techniques, but the information made available from a covariance structure analysis of an instrument designed in this way is of considerably more value.

Not only can we obtain an estimate of the relative amounts of variance in the latent traits represented in a set of items, but we get an estimate of the <u>covariance</u> between these latent traits (not merely between the observed traits) and have the opportunity to test the hypothesis that the covariances between latent traits are zero. We do this by testing the assumptions about the Φ matrix in the structural model (i.e., by looking at whether this matrix should be oblique or orthogonal). The relative sizes of the diagonal elements of Φ tell us the importance of (i.e., the variance contributed by) each of the latent traits. Information about the latent traits can guide us in the future development of the instrument. The form of Ψ yields homogenei information about the error variances.

Characteristics of a test which can be studied in this way include the number and importance of the latent variables included in the test and the relationship among these variables. We may also study characteristics of the items such as phrasing, direction of wording, etc. and we can look at the relation of these characteristics, treated as latent variables, to the substantive traits which are represented.

To give a simple example of an application of the procedures to a measurement problem consider the case of constructing a test battery to measure two types of latent ability (e.g., computational ability and logical reasoning). A test constructor may be aware of several methods which have been tried out to measure each type of ability. Let us say that two measures of computational ability (C_1 and C_2) and two of logical reasoning (R_1 and R_2) seem particularly appropriate for a given application. Thus a test might be designed which contained four subtests of homogeneous items which measure each of the four traits (C_1, C_2, R_1, and R_2).

The matrix of weights (A) for the structural model is

$$(7) \qquad A = \begin{array}{c} \begin{array}{cc} C & R \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Alternatively, the matrix A could have taken the form of a reparameterized analysis of variance design matrix. An A matrix of this type would include a column for the grand mean and a column for the contrasts of the fixed effect part of the analysis of variance design. If we administered the items to a group of subjects and computed the sample covariance matrix we could then obtain maximum likelihood estimates for the elements of particular structural models. We could test the assumption that the latent variables (C and R) are uncorrelated by testing the assumption that a model with Φ

orthogonal fits the data. We could test the assumption that the subtests have equal variances by testing the assumption that Ψ is homogeneous.

Consider, for ease of explanation, the case where only the restriction on Λ is employed (i.e., four models are run) and where the most general model fits and more restricted models do not. The parameters to be estimated are those in

$$(8)\quad \Sigma = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_c^2 & \sigma_{cr}^2 \\ \sigma_{cr}^2 & \sigma_r^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & & & 0 \\ & \sigma_{e2}^2 & & \\ & & \sigma_{e3}^2 & \\ 0 & & & \sigma_{e4}^2 \end{bmatrix}$$

The relative amount of variance for the latent variables can be assessed by looking at the diagonal elements of Φ. The degree of relationship between the two latent variables is assessed by looking at the off-diagonal element σ_{CxR} in Φ. The error components can be compared by looking at the diagonal elements of Ψ.

To expand the problem somewhat, the investigator might also be interested in some item characteristics, e.g., direction of wording D and strength of the item S, and their relationship to the latent variables. Strength refers in this case to the level at which a positive response to an item manifests a particular response (e.g., agreeing with very positively phrased items may represent a stronger response than agreeing with a more mildly stated form of the same item). A 2^2 factorial design for the characteristics of the items crossed with the original design yields

$$
(9)\quad A = \begin{array}{cc}
 & \begin{array}{cccc} H & C\ \text{vs.}\ R & D & S \end{array} \\
\begin{array}{c} C_{111} \\ C_{112} \\ C_{121} \\ C_{122} \\ C_{211} \\ C_{212} \\ C_{221} \\ C_{222} \\ R_{111} \\ R_{112} \\ R_{121} \\ R_{122} \\ R_{211} \\ R_{212} \\ R_{221} \\ R_{222} \end{array} &
\begin{bmatrix}
+1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -1 \\
+1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 \\
+1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -1 \\
+1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 \\
+1 & -1 & +1 & +1 \\
+1 & -1 & +1 & -1 \\
+1 & -1 & -1 & +1 \\
+1 & -1 & -1 & -1 \\
+1 & -1 & +1 & +1 \\
+1 & -1 & +1 & -1 \\
+1 & -1 & -1 & +1 \\
+1 & -1 & -1 & -1
\end{bmatrix}
\end{array}
$$

Analyzing the model containing the above A matrix, a 4×4 Φ matrix and a 16×16 Ψ matrix we can investigate the structure of the latent traits having to do with ability, those having to do with item characteristics, as well as the interrelationships between the two types of traits. In this way more precise measuring instruments can be developed. One question we can answer has to do with the positively and negatively phrased items. Does the reflection of an item mean that we merely reflect the scoring of an item or alternative forms of response be weighted differently? Looking at the covariance between the C vs R contrast and D sheds light on this issue. Similar questions can be asked about item strength and investigated in an analogous manner.

NUMERICAL EXAMPLE

An investigation of content acquiescence in the MMPI has been reported by Bock, Dicken, and Van Pelt (1969). These authors studied response acquiescence in two scales (Pt and Hy) most representative of the two orthogonal factors which repeatedly occur in factor analyses of the MMPI. The 108 non-overlapping items of the Pt and Hy scales were administered to a group of 81 undergraduates from San Diego State College in both their original and reversed forms. Item reversals were taken from the reversal of the complete MMPI by Dicken and Van Pelt (1967). The 216 item test was administered twice to the students, a week apart, and the results analyzed separately by MMPI scale. Thus S and Σ are 4 x 4 covariance matrices.

For the two directions of wording within an MMPI scale the response models are

$$(10) \quad x_{it} = \mu_x + \gamma_i + \alpha_{it} + \delta_{it} + \varepsilon_{it},$$

and

$$(11) \quad y_{it} = \mu_y + \gamma_i - \alpha_{it} + \delta_{it} + \zeta_{it},$$

where μ_x and μ_y are the fixed means for the scales, γ_i is the content component for subject i, α_i is the acquiescence component for subject i, δ_{it} is the component on occasion t due to trait instability for subject i, and ε_{it} and ζ_{it} are the components within - form response error for the two scales. This is a typical mixed model anova and the usual restrictions on γ, α, and δ apply. The A matrix is

$$(12) \quad A = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix},$$

and Φ is given as

$$\Phi = \begin{bmatrix} \sigma_{\gamma}^2 & & \text{symmetric} \\ \sigma_{\gamma\alpha} & \sigma_{\alpha}^2 & \\ \sigma_{\gamma\delta} & \sigma_{\alpha\delta} & \sigma_{\delta}^2 \end{bmatrix} \tag{13}$$

The sample covariance matrices reported by Bock, Dicken, and Van Pelt for the two scales are given in tables 2 and 3.

TABLE 2

Pt Sample Covariance Matrix

Form	Administration	Original 1	Original 2	Reversed 3	Reversed 4
Original	1	34.819		symmetric	
	2	30.519	34.997		
Reversed	1	28.178	24.844	29.376	
	2	25.519	27.612	22.323	26.780

TABLE 3

Hy Sample Covariance Matrix

Form	Administration	Original 1	Original 2	Reversed 3	Reversed 4
Original	1	12.3698		symmetric	
	2	7.6503	14.0059		
Reversed	1	8.3625	8.1625	12.7000	
	2	5.6961	11.5017	8.6750	14.8725

A covariance structure analysis was performed on the two covariance matrices given in tables 2 and 3 using the four covariance structure models that contain an unscaled A matrix. The chi square tests of fit for the individual models are reported in table 4.

TABLE 4

Chi Square Tests of Fit for the Pt and Hy Data

		$\Psi = \sigma^2 I$			Ψ general		
Φ Orthogonal	Pt	14.63	d.f.= 6	p<.025	12.70	d.f.=3	p<.01
	Hy	9.35	d.f.= 26	p<.20	6.04	d.f.=3	p<.15
Φ Oblique	Pt	2.64	d.f.=3	p<.55	.72	---	---
	Hy	3.77	d.f.= 3	p<.35	.07	---	---

In table 4 it is seen that the most restricted model fits for Hy, but not for Pt. The model containing Φ oblique and $\Psi = I\sigma^2$ fits for Pt and the difference chi square contrasting this model with the most restricted model is 11.99 (d.f.=3). The latter value is significant beyond the .01 level. Thus, the oblique case of Φ is necessary for the Pt data, but not for the Hy data.

The estimates for the elements of the parameter matrices are given in table 5. The estimates given for Pt and Hy are those obtained by using the appropriate model for each.

TABLE 5

Parameter Estimates for the Pt and Hy Data

Parameter	Raw Estimates* Pt	Raw Estimates* Hy	S.E. Pt	S.E. Hy	Full Parameter	Estimate Pt	Estimate Hy
t_{11}	10.42	6.02	.72	.50	ϕ_{11}	108.56	36.20
t_{21}	.72	0.00	.21	--	ϕ_{21}	7.52	0.00
t_{22}	1.40	6.02	.25	.26	ϕ_{22}	2.48	2.47
t_{31}	.08	0.00	.26	--	ϕ_{31}	.86	0.00
t_{32}	-.78	0.00	.42	--	ϕ_{32}	-1.03	0.00
t_{33}	2.19	4.90	.28	.27	ϕ_{33}	5.43	6.00
σ_e	1.54	1.52	.11	.12	σ^2_e	2.36	2.32

* The raw estimates include the elements of the Cholesky factorization of Φ (which are represented by t_{ij}) and the square root of the error variance.

From the full parameter estimates we can see that the content variance for both tests was quite large (108.56 for Pt and 36.20 for Hy) relative to the acquiescence variance (2.48 for Pt and 2.47 for Hy) and variance due to trait instability (5.43 for Pt and 6.00 for Hy). The error variances for the two tests were relatively consistent (2.36 for Pt and 2.32 for Hy). The interesting off-diagonal term in Φ is ϕ_{21} in Pt (7.52). In this case even though the variance due to acquiescence is relatively small the content by acquiescence covariance is rather large.

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