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ABSTRACT

Since mathematical English (ME) differs from ordinary English (OE) in the number of symbols used, this research investigated sequential constraint (constraints on symbol choice attributed to preceding textual material) of excerpts from 18 mathematics books, both traditional and modern, to determine its relationship to readability. Findings indicated the following: (1) the length of the total passage must be considered; (2) sequential constraint did not differ for modern and traditional mathematics books; (3) sequential constraint varied between topics, which implies that no value of sequential constraint can be assigned to ME; (4) an inverse relationship existed between sequential constraint and grade level; (5) there was more constraint in the deductive style of writing; and (6) there was an inverse relationship between sequential constraint and reading comprehension of ME. Implications for teaching would place greater emphasis on topics having high constraint since such topics were associated with lower reading scores, and topics having low constraint might be developed in greater depth since they were associated with higher reading comprehension scores. Included are a bibliography, a list of definitions of technical terms, and tables. (DH)

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# Estimates of the Relative Sequential Constraint for Selected Passages from Mathematics Books and the Relationship of These Measures to Reading Comprehension



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## INTRODUCTION [1]

Human behavioral study can be thought of as an investigation of sequences of chain reactions. Humans react to the behavior of their predecessors and in turn influence others. Language is one important aspect of human behavior which is also important in the educational process. For instance, Carroll (1968) wrote:

> By far the largest amount of teaching activity in educational settings involves telling things to students, whether orally or in print. Traditional instruction characteristically uses the lecture method, along with plentiful reading assignments. Even in more "progressive" educational settings which avoid the lecture method, much of the teacher's activity consists of asking questions and imparting information verbally. ... We expect our students to learn most things by being told about them (p. 1).

A textbook is a conventional instructional aid which utilizes language. This medium is probably the most widespread learning device in the American educational system.

In contemporary mathematics textbooks the topics discussed and the related pedagogy are somewhat different than they were a decade ago. These changes are attributable in large part to the efforts of numerous mathematics curriculum groups. Among these are the School Mathematics Study Group (SMSG), University of Illinois Committee on School Mathematics (UICSM), Madison Project, University of Maryland Mathematics

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[1] Throughout the paper the reader will encounter some unfamiliar words. Formal definitions of these technical terms appear after the references.

Project (UMMaP), Greater Cleveland Mathematics Program (GCMP), and the Commission on Mathematics of the College Entrance Examination Board. Diverse topics such as sets, numeration systems, descriptive statistics, logic, and probability are presented by at least one of the groups whereas a decade ago few of these topics were even mentioned in our schools. Different methods and emphases in teaching mathematics are also evident. For example, in recent years discovery learning has appeared in varying degrees in contemporary mathematics programs and attention is given to structure, precision of language, and the spiral approach to curriculum development.

The sucess of the materials written by the proponents of curriculum reform should be measured in large part by student performance. A contributing factor to student performance is the readability of textual materials. Mathematical English (ME) differs from ordinary English (OE) in many aspects, some of which may affect readability. For example, ME includes not only the 26 symbols of the alphabet, punctuation marks, end of sentence, and space but also numerals, operation signs, and other specialized symbols. In addition, a tremendous compacting of information occurs throughout ME by the use of special symbols such as $\sum_{i=1}^{n}$ p(i) log p, $\int_a^b$ f(x)dx, etcetera. Moreover, in OE many nouns are rich in connotation whereas in ME nouns that name mathematical objects generally have a single denotatum (at least for a given author). Any extraneous meaning such nouns may evoke is "noise."

Differences within ME itself occur because of the appearance of diverse topics and the concomitant symbolism. For example, elementary school mathematics generally does not include $\Sigma$, $\Delta x$, $\int_a^b$, etcetera, symbols which occur frequently in higher grades. Moreover, even mathematics textbooks written for the same grade level have different

symbols. Contemporary programs used set notation which rarely appeared a decade ago. Close examination of a mathematics book reveals differences in writing within the book. Sections devoted to motivational materials seem to more closely approximate OE than do those concerned with proofs. Perhaps these illustrations demonstrate that the characteristics of ME and OE are not the same and therefore may require different reading skills to attain acceptable levels of reading comprehension.

The readability of OE has been subjected to experimentation for many years. Consequently, some factors which can be utilized in predicting the readability of OE have been isolated. Among these are average sentence length; average word length, number of familiar words in a passage, and the number of syllables per 100 words. Generally a combination of some of these factors is used in a multiple regression equation for predicting the readability of OE passages. However, Chall (1958, p. 202) and Kane (1967) suggest that these equations may not be applicable to the prediction problem in mathematics.

An approach to the readability problem in ME should attempt to isolate variables which are related to reading comprehension. Such predictor variables might be the syntactical complexity, cloze scores (the number of correct insertions made into the blanks obtained by deleting every Kth word of a passage, where every K is a positive integer), proportion of non-English symbols, vocabulary familiarity (the number of words in a passage that are on a list of familiar words), and sequential constraint. The current research was directed toward investigating sequential constraint. More specifically, the purpose of this study was to analyze and compare the sequential constraint of excerpts from mathematics books and determine the relationship between sequential constraint and readability.

Samples of writings from 18 mathematics textbooks were analyzed in this study. Half of the 18 textbooks used a traditional and half used a modern approach to mathematics. A listing of the books at each of three divisions (elementary, junior high, and senior high) is given in Tables 1, 2, and 3.

In addition, five passages of diverse mathematical content were utilized. Two passages involved matrices, one logic, one the metric system, and one statistics.

The pattern of language evolvement in textual material is sequential since an order of perception is established. In reading the reader progresses from left to right and, moreover, prior context influences the future appearance of letters. Thus the occurrence of a letter may depend not only on immediately adjacent letters, but constraints may extend over much of the prior context. A tool for measuring these constraints must therefore use the probabilistic statements inherent in evolving sequential data.

An appropriate mathematical model for analyzing data of a sequential nature is a Markov chain with a discrete time parameter (Binder & Wolin, 1964). A characteristic of Markov chains, making them especially appropriate for application to the entropy concept of information theory, is that the probability of occurrence of a state is contingent upon only the immediately preceding state and none before that. For example, the assumption is made that the probability of occurrence of a specific letter (state) depends only on the immediately preceding, say 20, letters and none before those (it is irrelevant to the prediction problem whether the probability of a specific letter or the probability of the state which is induced by the letter is determined). Assuming, for this example, a 28-letter alphabet (26 letters, end of sentence, and space) there would be $(28)^{20}$

states with an accompanying probability value attached to each. The probability values when substituted in the appropriate formulas from information theory give an estimate of the 21-gram entropy. Since the determination of entropy is a limiting process, the above procedure would be repeated for dependencies extending over spans greater than the 20 preceding letters.

While it would be desirable to successively approximate the entropy by using longer and longer sequences for predicting the occurrence of a letter, the problem becomes insurmountable quickly. For example, with prediction depending on only the immediately two preceding letters there are $(28)^2$ possible states (assuming a 28 letter alphabet). Tabulation of the frequency of occurrence of each of these digrams is possible (Shannon, 1951) and the resulting estimate of entropy follows readily. However to estimate the entropy from sequences of the 10 preceding letters is quite another matter. In this case there are $(28)^{10}$ states and in order to estimate the entropy the length of the English passage from which the frequency distribution arouse would be prohibitively long.

These examples illustrate the need for an alternate approach to entropy determination. Newman and Gerstman (1952) proposed the coefficient of constraint as a device for estimating the entropy. An adaptation of the latter method was used to obtain estimates of sequential constraint in this study.

A limitation of the present study is that all art work, tables, and figures were eliminated from the textual material in which they were embedded since in most cases it was impossible to decide on symbol ordering. Certainly one would expect constraints in textual material to originate from such configurations but order of perception could not be determined. For example, if a parabolic curve showing various relationships appeared, no readily attainable agreement on what or how to

analyze sequentially the information contained therein was obvious.

## RESEARCH QUESTIONS

This research is directed toward the elaboration of the sequential constraints in ME and the relation of these constraints to reading comprehension. Smaller samples of the textual material are often used to estimate the sequential constraint for entire books. Typically a researcher will sample a continuous subset of characters, calculate the values of the sequential constraint, and then extrapolate to the materials in which the sample is embedded. The proximity of informational values determined from samples to the corresponding values found in entire textbooks or chapters is a question which is investigated in the present research.

When examining the constraint existent in ME a researcher is concerned with any variation in that quantity which may occur. It is therefore appropriate to determine what differences exist in sequential constraint for modern and traditional mathematics books, within a specified mathematics book, between mathematics books at different grade levels, and for different types of discourse.

In summary, this study was designed to answer the following questions

1. Is there an optimal sample size, in symbols, which may be used to compute a measure of sequential constraint?
2. Is there a difference in the sequential constraint of modern and traditional mathematics textbooks?
3. Is there a difference in sequential constraint for topics within a single mathematics textbook?
4. Is there a difference in sequential constraint among mathematics textbooks written for different grade levels?
5. Is there a difference in sequential constraint between deductive and non-deductive mathematics textual materials?

6. What is the relationship between sequential constraint and reading comprehension of mathematics passages?

PROCEDURE

Excerpts from 18 mathematics books and five mathematical passages were used in this study. With the exception of art work, figures, and tables, all characters which appeared in any selection were keypunched on cards for analysis on a CDC 6500 computer. The coding represented not only the usual 26 letters of the alphabet but also a symbol for the end of a sentence, space, =, +, and the other symbols that commonly occur in mathematics textual materials. Internal punctuation within ordinary English sentences was disregarded as were page numbers. Chapter titles were included. Conjunctions and hyphenated words were treated as single words, for example, 'let's' would be punched in four consecutive columns with the apostrophe omitted and 'non-zero' would be treated as one word and keypunched in seven consecutive columns. Textual material was placed on cards in the order that it would be read. To illustrate, $\frac{1}{4}$ is read one-fourth so the first symbol to be placed on the card is 1, then a symbol for the vinculum, and finally the 4. After each end-stop symbol, that is a period, exclamation mark, question mark, or colon, a space was allowed. No spaces were keypunched within any equation. Decisions leading to the above rules depended on the order in which textual material is read by a reader and previous studies (Newman & Gerstman, 1952; Newman & Waugh, 1960; Paisley, 1966) on redundancy for OE.

## The Computer Program[2]

The computer program was developed in two sections, Programs A and B. Program A encoded textual material into machine characters. In Program

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[2]The researcher is indebted to Robert Cripe for his assistance in the development of the computer program.

B contingency tables were constructed showing the frequency with which each character followed every other character immediately and at distances of 2, 3, ..., 15. All computations were done in Program B. The total program is quite flexible and can, with minor alterations, accommodate up to 126 distinct characters. In addition contingency tables can be constructed for symbols separated by a maximum of 119 intervening characters.

ANALYSIS

Shannon (1948) defined (relative) redundancy as one minus the relative entropy where the relative entropy is the ratio of the entropy to the largest value it could have while still restricted to the same symbols. Thus the relative redundancy (R) is given by the formula:

$$R = 1 - \frac{H}{H_{nom}} \qquad [1]$$

where the entropy $H = \lim_{N \to \infty} F_N$, $F_N = - \sum_{i,j} p(i)\, p_{ij} \log_2 p_{ij}$, $p(i)$ is the probability of the (N-1) -gram i, $p_{ij}$ is the probability of the symbol j given the (N-1) -gram i, and the nominal value of the entropy $H_{nom} = \log_2 n$ where n is the number of distinct symbols. As mentioned previously, if statistical effects extend over N-grams, as N becomes large the calculation of the entropy is not possible.

Garner and Carson (1960) separated redundancy into two parts, the distributional constraint and the sequential constraint. For example, the model for redundancy when only the N-1 preceding variables (each letter position is a variable) are considered is:

$$H_{nom} - F_N = (H_{nom} - H_{max}) + (H_{max} - F_N). \qquad [2]$$

In Equation 2, $H_{nom}$ is defined as before, $H_{max}$ gives the uncertainty when the symbols are independent but not necessarily equally probable and is defined as $H_{max} = - \sum_i p(i) \log_2 p(i)$, $p(i)$ is the probability of the symbol i, and $F_N$ is the N-gram entropy. The left hand side of Equation 2 approximates the numerator of the Shannon formula for redundancy and, in fact, would be identical if statistical effects did not extend over

sequences longer than N-1 symbols in length, that is, if $F_N = M$. The expression within the first parenthesis on the right hand side of Equation 2 is the distributional constraint. This expression gives the reduction in uncertainty attributable to unequal frequency of occurrence of symbols. To illustrate, if there were no syntax or spelling rules and if all letters occurred with almost the same frequency, then the reduction in uncertainty would be minuscule and prediction of any letter in a sequence would be little better than chance. The expression within the second parenthesis on the right hand side of Equation 2 is the sequential constraint which gives the reduction in uncertainty due to statistical effects extending over sequences of length N-1.

The distributional constraint is, of course, easily obtainable and is independent of problems inherent in working with sequential dependencies. However, calculation of a value for the sequential constraint is more difficult since the N-gram entropy term appears. Binder and Wolin (1964) proved that the sequential constraint is equal to the multiple contingent uncertainty. Consequently, the problem of determining the sequential constraint reduces to finding the multiple contingent uncertainty.

A technique suggested by Newman and Gerstman (1952) has been adapted in the current research to the problem of estimating the multiple contingent uncertainty. The procedure consists of calculating the simple contingent uncertainties and summing these to estimate the multiple contingent uncertainty. The specific formulas utilized in this study to estimate the multiple contingent uncertainties and subsequently the relative sequential constraints from the summation of simple contingencies follow.

An estimate $C_n$ of the relative sequential constraint for sequence

of length n is given by:

$$C_n = \frac{\sum_{k=2}^{n} H(1:K)}{H(1)} \qquad [3]$$

In formula 3, $H(1:K) = H(1) - H_K(1)$, where $H(1) = -\sum_{i} p(i) \log_2 p(i)$, the summation being over the entire alphabet, and $H_K(1)$ is the uncertainty of the letter being predicted when only the (K-1)th preceding letter is used in the prediction. Formally, $H_K(1) = -\sum_{ij} p(i)\, p_{ij} \log p_{ij}$, where $p_{ij}$ is the probability of symbol j given that i occurred k-1 letters before it. The indices on this summation sign range over the entire alphabet being considered. The expression appearing in the numerator, $\sum_{k=2}^{n} H(1:K)$, is the summation of the simple contingencies, $H(1) - H_K(1)$, culminating in an estimate of the multiple contingent uncertainty when only the N-1 preceding variables in the Markov chain are utilized.

Several researchers have used a formula commonly referred to as the Miller-Madow formula and presented by Miller (1955) for correcting sample bias of certain informational functionals computed from sequential data. Among the researchers are Newman and Waugh (1960) and Paisley (1966). The correctional formula utilized by Paisley is well disguised. As pointed out by Garner (1962) this correctional formula is suitable when the underlying populations are either univariate or bivariate but seems less appropriate with overlapping data where the source is clearly Markovian (Binder & Wolin, 1964).

## RESULTS

An adaptation of the Newman-Gerstman (1952) method was employed in the present study to estimate all values of the relative sequential constraint for ME. Successive values of the simple contingencies through 16 were summed to approximate the relative sequential constraint. In most instances simple contingencies beyond 16 characters were constant.

## Proximity of Sample Estimates of 16-Letter Relative Sequential Constraint to the Mathematical Language in Which the Samples are Embedded

Estimates of relative sequential constraint calculated from samples will often vary from those of the larger textual materials in which the samples are embedded. The nature of this variability was investigated. One aspect of the problem was to determine how representative a selected sample is of other equal length samples from the same textbook. For example, if a 5,000 symbol random sample of continuous textual material leads to a certain value of the relative sequential constraint, how representative of other 5,000 symbol random samples is that sample?

Five random samples each of 5,000, 20,000, 25,000, 30,000, 35,000, and 40,000 characters were selected from each of two mathematics textbooks to answer this question. The estimates of 16-letter relative sequential constraint for the samples from Learning Mathematics (Deans, Kane, McMeen & Oesterle, 1968) and Exploring Elementary Algebra (Keedy, Jameson, Johnson & Ciechon, 1967) are tabulated in Tables 4 and 5. The variance for each sample size from each selection is reported in Table 6. The figures imply that representativeness is closely allied to sample size. For example, in Learning Mathematics and Exploring Elementary Algebra variability among the five 5,000 symbol samples was 0.173 and 0.095 while for the 40,000 symbol samples variability was 0.003 and 0.036. The variability for intermediate sample sizes decreased with increasing sample size with a few exceptions. Thus in both books variability between samples was greater for the smaller samples and smaller with the larger sample sizes.

Another problem in extrapolating from samples to larger selections of textual material was the proximity of the sample means for estimates of the 16-letter sequential constraint to the corresponding values in the

larger selection. The means of the estimates of 16-letter relative sequential constraint for the five samples from each of the six sample sizes together with estimates of 16-letter relative sequential constraint for the total selection are presented in Table 7. With one exception an increase in sample size resulted in a better estimate of relative sequential constraint for the total selection. Thus the mean values of the 16-letter relative sequential constraint for the five 5,000 symbol samples of textual material from Learning Mathematics and Exploring Elementary Algebra were 1.714 and 2.096 while the mean values for the five 40,000 symbol samples were 0.848 and 1.399. The 16-letter relative sequential constraint for the total selection from each of the two books was 0.853 and 1.354. Therefore, the 40,000 symbol samples yielded mean values of 16-letter relative sequential constraint which were closer to the corresponding entries in the total selection than were those for the 5,000 symbol samples. For these data the proximity of the approximation seems to be contingent upon both the sample size and the length of the total selection.

## Differences in Estimates of Relative Sequential Constraint Between Modern and Traditional Mathematics Books

English language depends on only 26 discrete letters together with a space and punctuation to convey meaningful ideas. Within mathematical language a larger number of symbols is utilized. Consequently, a mathematics book usually contains many symbols not found in ordinary English books.

Alphabet size and estimates of the 16-letter relative sequential constraint for two 20,000 symbol random samples of mathematical language from different books at each of four elementary school grade levels are

reported in Table 8. At each grade level the samples were selected from passages in which a common topic was presented.

In three of the four comparisons presented in Table 8 alphabet size is greater for the modern books. However, for relative sequential constraint two of the comparisons indicate more restraint for modern language while the reverse is true for the remaining comparisons. Thus while Developing Mathematics, Understanding Mathematics, and Learning Mathematics have 55, 62, and 56 distinct symbols only the latter two books have greater 16-letter relative sequential constraint, 1.656 and 1.638, than their counterparts.

Estimates of 16-letter relative sequential constraint for excerpts from each of four junior high school mathematics books are reported in Table 9. Alphabet size and relative sequential constraint are larger for modern mathematics textbooks in one of the two comparisons presented.

The corresponding structural characteristics of mathematical language for six books at the senior high school level are enumerated in Table 10. More constraint for modern mathematics books is found in only one of the three comparisons given in Table 10 while a smaller alphabet occurs in a modern book in one of the three comparisons. Thus, Exploring Elementary Algebra is the only modern mathematics textbook in the three comparisons with more 16-letter relative sequential constraint than its counterpart (1.869 versus 1.079). The only modern mathematics book reported in Table 10 having fewer distinct symbols than its counterpart (60 versus 65) is Advanced High School Mathematics.

When comparing the modern and traditional books presented in Tables 8, 9, and 10 alphabet size is greater for modern books in six of the nine comparisons but relative sequential constraint is larger in only four of the nine comparisons. It also should be noted that in seven of the nine

comparisons the selection with the smaller alphabet has the smaller estimate of relative sequential constraint. Alphabet size seems to be directly related to relative sequential constraint for these data, at least when topic is controlled.

## Within Book Differences for Estimates of Relative Sequential Constraint

One question investigated was whether relative sequential constraint varied between topics within a mathematics book. Excerpts from four textbooks were used to study this. The books, topics selected, alphabet size, and estimate of 16-letter relative sequential constraint are tabulated in Table 11. It is apparent in all four comparisons that topic is related to estimates of relative sequential constraint. For example, the two topics, fractions and geometry, in Learning Mathematics had 1.638 and 1.052 respectively as estimates of 16-letter relative sequential constraint. For this book more constraints are imposed on textual material when fractions are discussed than when geometry is presented. In addition, for three of the four comparisons there is an inverse relationship between alphabet size and estimate of relative sequential constraint. This is in contrast to the association between alphabet size and estimates of relative sequential constraint noted earlier.

## Differences in Estimates of Relative Sequential Constraint Between Grade Levels

Some insight into the variation in relative sequential constraint between grade levels can be obtained by controlling topic and authorship. The relevant data for this aspect of the study is in Table 12. While a direct relationship between alphabet size and estimates of relative sequential constraint is apparent in these data it is also noteworthy

that sequential constraint decreases with increasing grade level. To illustrate, Row-Peterson Arithmetic 4 is less constrained than Row-Peterson Arithmetic 3 (1.410 versus 1.469) and Extending Mathematics, a book for eighth graders, has less sequential constraint (0.856 versus 1.052) than its counterpart, a book for fifth grade students. More information is contained in passages that occur at higher grade levels than their counterparts of equal length at lower grade levels.

## Differences in Estimates of Relative Sequential Constraint for Two Styles of Mathematical Language

Some of the writing found in mathematics books is deductive in nature. A question of interest was whether measures of relative sequential constraint for deductive textual materials differ from the corresponding values enumerated from less directive discourse. To answer this question two 10, 414 symbol samples of textual material were selected within topics which contain both expository styles. Alphabet size and estimate of 16-letter relative sequential constraint are given in Table 13. Relative sequential constraint and alphabet size are greater for the deductive style of ME investigated in this study.

## Relative Sequential Constraint and Reading Comprehension

To ascertain the degree of association between relative sequential constraint and reading comprehension five passages from Hater's (1969) study were used. The students in Hater's study were enrolled in grades 7 through 10 of Roman Catholic parochial schools in Cincinnati, Dayton, Springfield, and Lincoln Heights, ohio. A reading comprehension test was given to approximately 125 randomly selected students from the above schools on each of the five passages. A mean was determined for the 125 reading comprehension test scores on each passage. These five means

together with a measure of relative sequential constraint for each of the five passages are reported in Table 14. Reliability indices for the reading comprehension tests were computed using the Kuder-Richardson Formula 20. For each passage the relibility coefficient was at least 0.78. The Spearman Rank Correlation Coefficient was used to obtain a measure of relationship between mean scores on reading comprehension tests over the passages and relative sequential constraint. This coefficient was -0.30.

## CONCLUSIONS

The purpose of this study was to quantify and compare the sequential constraints (constraints on symbol choice attributable to preceding textual material) extant in ME passages and to ascertain the degree of relationship between these constraints and reading comprehension. Data consisted of textual materials from 18 mathematics books. In addition, five passages from Hater's (1969) study were utilized. The textual materials were keypunched in a uniform format for processing on a CDC 6500 computer. Analysis proceeded according to a technique initiated by Newman and Gerstman (1952). Briefly, the constraint imposed on the criterion variable (symbol being predicted) by each of the predictor variables (preceding m smybols where m = 1, 2, ...., 15) was determined. These constraints were then summed resulting in an estimate of sequential constraint.

The reader's attention was directed to the problems inherent in a relative frequency interpretation of probability associated with information theory concepts. Such difficulties confront the researcher concerned with applications of information theory if only finite samples are available. These restrictive conditions exist in language analysis. A perusal of the literature revealed corrective formulas when the underlying

models are either univariate or bivariate. However, no formula could be found to correct for sample bias with overlapping sequential data. Such data are innately characteristic of language.

The usual procedure in analyzing language rests on the assumption that a continuous sample of symbols is representative of the textual material in which it is embedded. Thus a researcher typically selects a sample, computes a measure of constraint on the sample, and extrapolates to the larger selection containing the sample. The validity of this assumption for ME was examined in the present study. Five samples of continuous textual material of 5,000, 20,000, 25,000, 30,000, 35,000 and 40,000 symbols were randomly selected from each of two passages containing 47,295 and 113,097 symbols. Measures of constraint were obtained for each sample, and the mean and variance for each sample size within each passage was determined. In addition, measures of constraint on the 47,295 and 113,097 symbol passages were found. Implications drawn from these data indicate that extrapolation is enhanced with increasing length of passages since variability within samples and distance from total selection means is reduced. Moreover, the length of the total passage must be considered. While a sample of a specified length may be entirely adequate when discussing constraint for a certain length selection, it may not be adequate for longer selections.

One question investigated was whether constraint differs for modern and traditional mathematics books. At a fixed grade level textbooks illustrative of each approach were chosen. The topic was controlled between books, and 20,000 symbol passages were randomly drawn from each of the two textbooks. For each passage alphabet size and constraint were determined. Neither modern nor traditional textbooks consistently had greater sequential constraint although modern books used more symbols in six of the nine comparisons presented. Also alphabet size was directly

related to relative sequential constraint, at least when topic was controlled, in seven of the nine comparisons.

Another aspect of the research was to determine whether relative sequential constraint fluctuates between topics within a book. Passages containing 20,000 symbols were randomly selected from each of two topics for each of four textbooks. Results indicated that sequential constraint varies between topics. The implication is that a unique value of constraint for ME, even within a given textbook, is nonexistent. That is, no value of sequential constraint can be thought to be indicative of ME.

Another question was whether constraint varies with ascending grade level. To answer this question 20,000 symbol passages were randomly selected from textbooks at different grade levels, but with topic and authorship controlled. Measures of constraint on these passages revealed an inverse relationship between relative sequential constraint and grade level. Thus textual material at the third grade level was more constrained than that at the fourth grade level. Whether the increased constraint found at lower grade levels is a help or deterrent to reading comprehension is another question which will be discussed below.

In mathematics textbooks some of the language is concerned with deductive reasoning. The comparison of constraint for this language style and less directive discourse was also investigated. A passage of each language style was selected from each of two mathematics books. Alphabet size and relative sequential constraint were determined for each of the ME passages. Results indicated that the deductive style of presentation was more constrained and had a greater number of symbols than the less directive discourse.

Five passages over which reading comprehension tests had been administered were analyzed in the final phase of the study. This phase

sought to investigate the relationship between reading comprehension of ME and relative sequential constraint. Measures of constraint on each of the five passages were determined. The correlation coefficient indicated an inverse relationship between relative sequential constraint and reading comprehension. Thus more constrained textual material seems to result in lower scores on reading comprehension tests, at least for ME. Herein may lie a distinction with OE where a direct relationship exists between reading comprehension and constraint. Possibly topics which have low constraint associated with them might be developed to a greater depth since low constraint is associated with higher scores on reading comprehension tests. That is, detailed discussion of peripheral areas related to topics having low constraint may be beneficial. Furthermore, greater emphasis in teaching should be placed on those topics having high constraint since such topics are associated with lower reading comprehension scores.

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## DEFINITIONS

1. Deductive Textual Material. Material which results when an axiomatic system is applied over a sequence of steps leading a person from initial conditions to the conclusion, examples are proofs of theorems, lemmas, etcetera.

2. Entropy (H). The minimum average number of binary digits required to encode each character of textual material, formerly $H = \lim_{N \to \infty} F_N$ where $F_N$ is the N-gram entropy: information; uncertainty.

3. Information. See entropy.

4. Letter Redundancy. Redundancy measurement in which the basic sampling units are letters.

5. Markov Chain With A Discrete Time Parameter. A stochastic process such that $P[X_{t_n} = x_{t_n} \mid X_{t_{n-1}} = x_{t_{n-1}}, \ldots, X_{t_1} = x_{t_1}] = P[X_{t_n} = x_{t_n} \mid X_{t_{n-1}} = x_{t_{n-1}}]$ for any integer $n \geq 1$ and any set of n time points $0 \leq t_1 \ldots < t_n$ in the index set T where the values are discrete and T is finite. Less formally, a stochastic process which moves through a finite number of states, and for which the probability of entering a certain state depends only on the last state occupied.

6. Mathematical English (ME). The written language found in mathematics textual materials.

7. Multiple Contingent Uncertainty. The total amount of uncertainty in the criterion variable which can be predicted from simultaneous values of the preceding variables.

8. N-gram Entropy. Information when the N-1 preceding letters are used in predicting the Nth letter of a sequence N letters long, formally $F_N = -\sum_{i,j} P(i)\, P_{ij} \log_2 P_{ij}$ where P (i) is the probability of the (N-1) -gram i and $P_{ij}$ is the probability of the single symbol j when the (N-1) -gram i is given.

9. Readability of a Passage. A group mean on a reading comprehension test over the passage.

10. Redundancy. $1 - \frac{H}{H_{max}}$ where H is the entropy and $H_{max}$ is the entropy which would result if all states were independent and equally probable. The redundancy is a measure of the constraint imposed on textual material due to its statistical structure, for example, in English the tendency of H to follow T.

11. Relative Sequential Constraint. A measure of redundancy computed from the summation of contingent uncertainties.

12. Simple Contingent Uncertainty. A measure of the amount of uncertainty reduction due to the contingencies between the initial predictor variable and the criterion variable.

13. Single Letter Uncertainty (H(1)). Uncertainty when each letter is independent of every other, formally $H(1) = -\sum_i P(i) \log_2 P(i)$ where P(i) is the probability of the letter i.

14. State. Some specific set of values of all the variables of concern.

15. Stochastic Process. An arbitrary family of realvalued random variables $\{X_t \mid t \text{ in } T\}$ where T is the set of all possible times and the possible values (states) of $X_t$ at time t are $x_t$.

16. Textbooks Which Illustrate A Modern Approach To Mathematics Education. Mathematics books whose most recent copyright date is at least 1963.

17. Textbooks Which Embody A Traditional Approach To Mathematics Education. Mathematics books whose most recent copyright date is 1962 or before.

18. Uncertainty. See entropy.

TABLE 1

ELEMENTARY SCHOOL MATHEMATICS TEXTBOOKS
USED FOR THIS STUDY

| Textbook | Publisher | Grade Level | Copyright Date |
|---|---|---|---|
| Developing Mathematics | American Book Company | 3 | 1968 |
| Row-Peterson Arithmetic | Row, Peterson and Company | 3 | 1959 |
| Understanding Mathematics | American Book Company | 4 | 1968 |
| Row-Peterson Arithmetic | Row, Peterson and Company | 4 | 1959 |
| Learning Mathematics | American Book Company | 5 | 1968 |
| Row-Peterson Arithmetic | Row, Peterson and Company | 5 | 1959 |
| Unifying Mathematics | American Book Company | 6 | 1968 |
| Row-Peterson Arithmetic | Row, Peterson and Company | 6 | 1959 |

## TABLE 2

## JUNIOR HIGH SCHOOL MATHEMATICS TEXTBOOKS USED FOR THIS STUDY

| Textbook | Publisher | Grade Level | Copyright Date |
|---|---|---|---|
| Structuring Mathematics | American Book Company | 7 | 1968 |
| Row-Peterson Arithmetic | Row, Peterson and Company | 7 | 1959 |
| Extending Mathematics | American Book Company | 8 | 1968 |
| Row-Peterson Arithmetic | Row, Peterson and Company | 8 | 1959 |

## TABLE 3

## SENIOR HIGH SCHOOL MATHEMATICS TEXTBOOK USED FOR THIS STUDY

| Textbook | Publisher | Grade Level | Copyright Date |
|---|---|---|---|
| Exploring Elementary Algebra | Holt, Rinehart and Winston, Inc. | 9 | 1967 |
| Algebra - Book One | Ginn and Company | 9 | 1960 |
| Exploring Geometry | Holt, Rinehart and Winston, Inc. | 10 | 1967 |
| A First Course in Geometry | The L. W. Singer Company | 10 | 1959 |
| Advanced High School Mathematics | Charles E. Merrill Books, Inc. | 12 | 1965 |
| Elementary Mathematical Analysis | D. C. Heath and Company | 12 | 1962 |

TABLE 4

FIVE ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR EACH OF SIX SAMPLE SIZES*

| Sample Size (In Characters) | | | | | |
|---|---|---|---|---|---|
| 5,000 | 20,000 | 25,000 | 30,000 | 35,000 | 40,000 |
| 1.780 | 1.088 | 0.932 | 0.902 | 0.842 | 0.807 |
| 1.133 | 1.064 | 0.779 | 0.900 | 0.849 | 0.802 |
| 2.052 | 0.858 | 0.952 | 0.904 | 0.848 | 0.908 |
| 1.360 | 1.179 | 0.955 | 0.886 | 0.962 | 0.910 |
| 2.046 | 1.069 | 0.980 | 0.899 | 0.959 | 0.811 |

*The samples were embedded within Chapters 1 and 7 of <u>Learning Mathematics</u>. These chapters consisted in total of 47,295 characters.

TABLE 5

FIVE ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR E[illegible]H OF SIX SAMPLE SIZES*

| Sample Size (In Characters) | | | | | |
|---|---|---|---|---|---|
| 5,000 | 20,000 | 25,000 | 30,000 | 35,000 | 40,000 |
| 2.315 | 1.880 | 1.343 | 1.816 | 1.798 | 1.649 |
| 2.078 | 2.043 | 1.101 | 1.894 | 1.747 | 1.518 |
| 2.385 | 1.822 | 1.337 | 1.849 | 1.414 | 1.200 |
| 1.597 | 1.757 | 1.740 | 1.266 | 1.339 | 1.233 |
| 2.106 | 1.844 | 1.192 | 1.232 | 1.262 | 1.397 |

*The samples were embedded within Chapters 4 and 5 of <u>Exploring Elementary Algebra</u>. These chapters consisted in total of 1[illegible]3,097 characters.

TABLE 6

VARIANCE FOR ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT OF SAMPLES FROM TWO MATHEMATICS BOOKS

| Sample | Learning Mathematics | Exploring Elementary Algebra |
|---|---|---|
| 5,000 | 0.173 | 0.095 |
| 20,000 | 0.014 | 0.012 |
| 25,000 | 0.007 | 0.060 |
| 30,000 | 0.000 | 0.111 |
| 35,000 | 0.004 | 0.060 |
| 40,000 | 0.003 | 0.036 |

TABLE 7

PROXIMITY OF SAMPLE* AND TOTAL SELECTION** ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT

| Sample Size | Learning Mathematics | Exploring Elementary Algebra |
|---|---|---|
| 5,000 | 1.714 | 2.096 |
| 20,000 | 1.052 | 1.869 |
| 25,000 | 0.920 | 1.343 |
| 30,000 | 0.898 | 1.611 |
| 35,000 | 0.892 | 1.512 |
| 40,000 | 0.848 | 1.399 |
| Total Selection | 0.853 | 1.354 |

*Sample values reported represent the means of five random samples of the indicated size. Values for the random samples are reported in Tables 4 and 5.

**In Learning Mathematics the total selection consisted of all of Chapter 1 and 7, a total of 47,295 characters while in Exploring Elementary Algebra Chapters 4 and 5 consisting of 113,097 characters were used.

TABLE 8

ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR 20,000 CHARACTER RANDOM SAMPLES OF CONTINUOUS TEXTUAL MATERIAL FROM MODERN AND TRADITIONAL ELEMENTARY SCHOOL MATHEMATICS BOOKS

| Book | Alphabet Size | Estimates of 16-Letter Relative Sequential Constraint |
|---|---|---|
| *Developing Mathematics* | 55 | 1.009 |
| *Row-Peterson Arithmetic 3* | 51 | 1.149 |
| *Understanding Mathematics* | 62 | 1.656 |
| *Row-Peterson Arithmetic 4* | 52 | 1.410 |
| *Learning Mathematics* | 56 | 1.638 |
| *Row-Peterson Arithmetic 5* | 51 | 1.189 |
| *Unifying Mathematics* | 46 | 0.721 |
| *Row-Peterson Arithmetic 6* | 48 | 0.847 |

TABLE 9

ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR 20,000 CHARACTER RANDOM SAMPLES OF CONTINUOUS TEXTUAL MATERIAL FROM MODERN AND TRADITIONAL JUNIOR HIGH SCHOOL MATHEMATICS BOOKS

| Book | Alphabet Size | Estimates of 16-Letter Relative Sequential Constraint |
|---|---|---|
| *Structuring Mathematics* | 55 | 1.491 |
| *Row-Peterson Arithmetic 7* | 53 | 1.339 |
| *Extending Mathematics* | 56 | 0.856 |
| *Row-Peterson Arithmetic 8* | 57 | 1.011 |

TABLE 10

ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR 20,000 CHARACTER RANDOM SAMPLES OF CONTINUOUS TEXTUAL MATERIAL FROM MODERN AND TRADITIONAL SENIOR HIGH SCHOOL MATHEMATICS BOOKS

| Book | Alphabet Size | Estimates of 16-Letter Relative Sequential Constraint |
|---|---|---|
| *Exploring Elementary Algebra* | 61 | 1.869* |
| *Algebra Book One* | 58 | 1.079 |
| *Exploring Geometry* | 67 | 0.977 |
| *A First Course in Geometry* | 57 | 0.995 |
| *Advanced High School Mathematics* | 60 | 1.219 |
| *Elementary Mathematical Analysis* | 65 | 1.224 |

*Represents the mean of the five 20,000 character random samples reported in Table 5

TABLE 11

WITHIN BOOK DIFFERENCES IN ALPHABET SIZE AND ESTIMATE OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR 20,000 SYMBOL RANDOM SAMPLES OF CONTINUOUS TEXTUAL MATERIAL

| Book | Topic | Alphabet Size | Estimate of 16-Letter Relative Sequential Constraint |
|---|---|---|---|
| Learning Mathematics | Fractions | 56 | 1.638 |
| | Geometry | 61 | 1.052* |
| Advanced High School Mathematics | Trigonometry and Probability** | 60 | 1.219 |
| | Matrices, Vectors and Limits** | 63 | 1.179 |
| Row-Peterson Arithmetic 3 | Measurement | 51 | 1.149 |
| | Basic Operations | 53 | 1.469 |
| Row-Peterson Arithmetic 8 | Geometry | 57 | 1.011 |
| | Fractions and Decimals** | 54 | 1.812 |

*Represents the mean of the five 20,000 character random samples reported in Table 4.

**In this study these topics composed the same sample.

TABLE 12

ALPHABET SIZE AND ESTIMATE OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT BETWEEN BOOKS AT DIFFERENT GRADE LEVELS WITH TOPIC CONTROLLED

| Book | Topic | Alphabet Size | Estimates of 16-Letter Relative Sequential Constraint |
|---|---|---|---|
| Row-Peterson Arithmetic 3 | Basic Operations | 53 | 1.469 |
| Row-Peterson Arithmetic 4 | Basic Operations | 52 | 1.410 |
| Learning Mathematics | Geometry | 61 | 1.052* |
| Extending Mathematics | Geometry | 56 | 0.856 |

*Represents the mean of the five 20,000 character random samples reported in Table 4.

TABLE 13

ALPHABET SIZE AND ESTIMATES OF RELATIVE SEQUENTIAL CONSTRAINT FOR TWO STYLES OF MATHEMATICAL LANGUAGE

| Style | Alphabet Size | Estimate of 16-Letter Relative Sequential Constraint |
|---|---|---|
| Deductive | 61 | 1.294 |
| Non-Deductive | 56 | 1.124 |

TABLE 14

MEAN* AND ESTIMATES OF 16-LETTER RELATIVE SEQUENTIAL CONSTRAINT FOR FIVE MATHEMATICS PASSAGES

| Passage | Estimate of 16-Letter Relative Sequential Constraint | Mean* |
|---|---|---|
| Matrices | 2.669 | 17.0774 |
| Metric System | 1.579 | 12.9129 |
| Studying Matrices | 2.565 | 12.5721 |
| Statistics | 3.004 | 10.7994 |
| Logic | 2.189 | 14.3951 |

*Mean of approximately 125 students on a reading comprehension test over the passage.