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ABSTRACT

THE PURPOSE OF THIS BOOKLET IS TO HELP HIGH SCHOOL STUDENTS LEARN HOW TO STUDY MATHEMATICS MORE EFFECTIVELY. SUGGESTIONS ARE MADE ON HOW TO PLAN FOR DOING HOMEWORK, HOW TO TAKE NOTES IN CLASS, HOW TO REVIEW FOR TESTS, AND HOW TO TAKE TESTS. SPECIAL TIPS ARE GIVEN REGARDING SOME SPECIFIC TOPICS IN ALGEBRA, GEOMETRY, COORDINATE GEOMETRY, AND TRIGONOMETRY. DEVELOPING A BETTER MATHEMATICS VOCABULARY, AVOIDING CARELESS ERRORS, REVIEWING, USING A NOTEBOOK, AND LEARNING BY DISCOVERY ARE A FEW OF THE OTHER TOPICS DISCUSSED. (FL)

HOW TO STUDY MATHEMATICS:
A HANDBOOK FOR HIGH SCHOOL STUDENTS

HENRY SWAIN

N C T M

HOW TO STUDY

Mathematics

A Handbook for
High School Students

by Henry Swain

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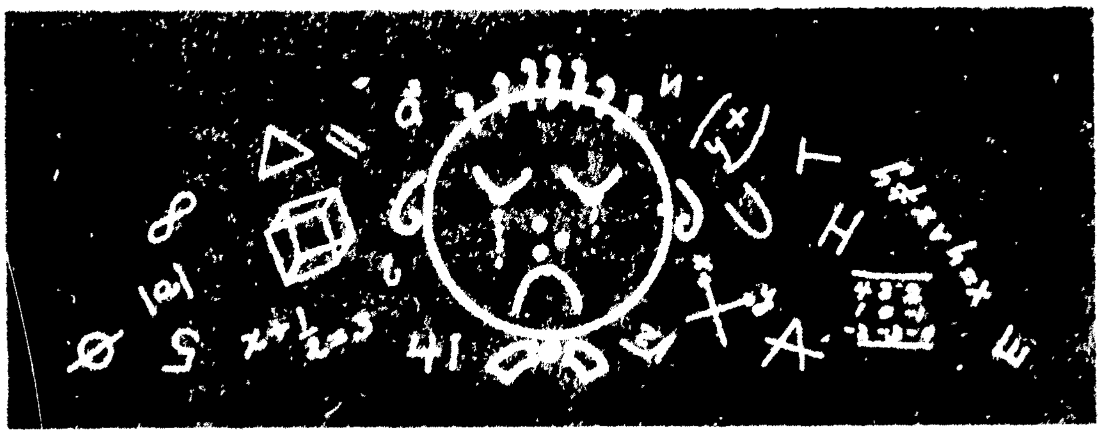


Introduction

MATHEMATICS WITHOUT TEARS

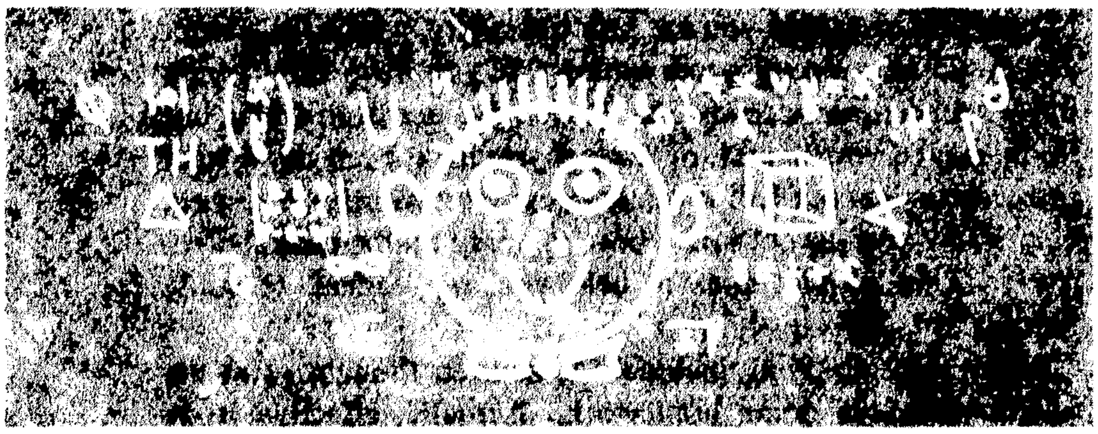
Is your mathematics bothering you? If not, maybe it should. Have you been doing your best to make sense out of it and still without success? Or are you just lazy? If you *are* lazy and not willing to try, this handbook is not for you—unless you can pick up some ideas from it which suggest that making an effort (with some helpful guidance) might be worth doing. Maybe you just *think* you are lazy but really, with a little help, you could be a pretty good worker. If you have been trying but have been unsuccessful, or if you have not been trying but think maybe you should, or even if you have been doing well in mathematics but feel it really does not mean much to you, very likely you do not know how to study mathematics *effectively*. This booklet aims to help you, its readers, to study mathematics effectively.

What does studying include? It is often thought of in the narrow sense of doing homework. Actually, effective study includes not only homework but also classwork and tests—the whole learning process.

This pamphlet will not attempt to discuss the more general principles of good studying, such as planning your study time, having a good place to study (with proper equipment, lighting, etc.), having no unnecessary distractions, getting started, and having the will to learn. These points are treated thoroughly in general booklets on how to study. We are concerned here with the specific problem of how to study *mathematics.*

Some of you, especially if you are doing well in mathematics, may feel that you have successful study methods of your own, different from the ones described here. In that case you need not feel that you must change your methods, although you might profit from comparing your methods with these, and the author would be happy to hear about any improvements you can suggest.

On the other hand, some of you may feel that the suggestions on the following pages are asking too much—that they would require more time and effort than you are prepared to give. You will probably be right. We cannot expect to do everything to perfection. We can, however, do our best. Out of the suggestions offered, you can pick the ones that may help you most, and as you find your work improving, you may be able to try further suggestions. So scoff if you wish at these ambitious suggestions, but give some of them a try—a *fair* try—and watch the results.



Homework

1 A FOUR-STEP PLAN FOR DOING DAILY HOMEWORK

There is a common misunderstanding that doing homework is just writing something to hand in to the teacher. Actually, of course, the homework is first and foremost a means of developing and strengthening your understanding of the mathematics you are studying and of increasing your mathematical skills. As an extra dividend, it is a means of developing habits of clearness, neatness, and accuracy. The paper you pass in to the teacher is only a by-product of that learning process. Home study, to be effective, must concern itself primarily with understanding and skills, not just the completion of a written assignment. The following four-step plan is a suggestion for making your home study effective:

a) *Warm up.* Take a few minutes to think back, look over your notes, and look over the book to see clearly what ideas you have been exploring recently, so you will be in tune for what is coming next.

b) *Study the new material.* Think about the ideas, principles, and methods in today's lesson. To help you see them clearly, use the "study sheet" method described in section 2 below. Give your attention to any new words in your mathematics vocabulary.

c) *Write the assignment.* As you do the exercises assigned, think about the ideas they are illustrating. Remember that your goal is not just to get answers but rather, through practice, to increase your understanding and to develop your skill. Watch for new ideas that may be developed in the exercises. (See section 6 for suggestions on the written homework.)

d) *Summarize.* Before you stop, while your mind is still "in the groove," summarize the ideas on which you have been working. Try saying them aloud as though you were explaining them to someone who has not yet learned them. There is no better way to learn a topic than by trying to teach it!

2 HOW TO USE THE "STUDY SHEET" METHOD

A good way to clarify your thinking as you study is to use a "study sheet." This is a sheet of paper on which to write down in your own words the ideas of the lesson—not problems worked out, but brief answers to questions such as "What does it mean?" "Why is it true?" "How is it used?" (We might abbreviate these as the *what, why,* and *how* questions.)

Use an outline form to organize the different parts of the topic you are studying. Make lists of important words. State warnings about errors to avoid. Recall what ideas were developed in class and organize them on your study sheet. As you read material in your textbook, write down the main points and work out the details of the examples to be sure you really understand the concepts they are illustrating. By thinking on paper you are more likely to master what you are studying than by just looking at the book. Passive study is not enough. It takes *active* study for real mastery.

The main purpose of the study sheet is accomplished as you write it, and the study-sheet method is worth using even if you discard the sheet. If, however, you file such sheets away day by day, you will probably find them very helpful later on for review.

3 HOW TO USE A TEXTBOOK

If you are using a textbook in your mathematics course, have you learned how to take advantage of all the help it can give you? Some of these suggestions may be useful:

a) Leaf through frequently to bring out a broad, overall, view of what you are studying instead of just looking at two or three pages each day.

b) Look through the table of contents now and then to see what you have been doing and what is to come.

c) Use the index, especially when you have forgotten the meaning of a word.

d) Be on the watch for italics, boxes, color, stars, and other methods the book uses to call to your attention words or ideas that are important.

e) When your book gives an illustrative example, analyze it carefully for the *ideas* behind it instead of just trying to make your homework problems look like the example.

f) If you are not able to do a problem, reread the explanatory material in the book slowly and carefully instead of giving up and waiting for someone to tell you how to do it.

g) Make the most of the study helps at the end of each chapter, such as lists of important words, outlines of the material in the chapter, review questions, and self-tests.

h) Keep an open mind about the explanatory material in your book. No textbook is perfect, and you may find your teacher showing you a better explanation in some places. If so, be sure to grasp the full meaning of the new approach. If you are permitted to do so, make a note about it in the margin of the book, or, if you need more room, write it up in your notebook. If you think you have discovered for yourself a clearer way of explaining a topic or exercise, do not hesitate to discuss it with your teacher.

i) Your book may have some "discovery type" exercises. You will soon learn to recognize them. Be alert for the ideas they are leading to, and record these ideas on your study sheet as soon as you see them. (See section 15.)

4 HOW TO DEVELOP YOUR MATHEMATICS VOCABULARY

Suppose you said to me, "*Att iakttaga vid spelets montering.*" I would not be able to understand you because I do not know the Swedish language. Suppose a radio ham tells you, "A class 'A' amplifier is biased to the midpoint of the tube's E_g, I_p curve." Unless you belong to the small minority who understand the technical language of radio, you will not know what he is talking about. Similarly, you will not be able to follow what is being said in mathematics unless you have a knowledge of its language. Furthermore, you will have difficulty in expressing yourself. If you have trouble saying clearly what you mean—on tests, for example—improving your vocabulary is the first thing you need to do. Here are some suggestions for developing your mathematics vocabulary:

a) Be on the watch for new words. They usually are in italics when they first occur and are defined.

b) Whenever you come to a new word, think all around it—think what it means, how it is spelled, what other words you can associate with it.

c) Sometimes the meaning of a word you thought you knew escapes you. Try the index, or, if you want to be fancy, consult a mathematics dictionary. Your school library should have one.

d) Make a conscious effort to *use* your mathematics vocabulary regularly and intelligently.

e) The symbols in mathematics are, in a sense, part of your mathematics vocabulary. When a new symbol appears, give it the same attention you would give to a new word. In particular, note how it is drawn—for example, $\emptyset$ and not $\bcancel{0}$, $\neq$ and not $\bcancel{=}$, $\{\ \}$ and not $\{\ \ \}$.

f) As a double check on yourself, keep a page in your notebook where you list each new word or symbol as you come to it (just the word, not the definition.) You'll be surprised how a quick look at that list once in a while will help you.

5 HOW TO MEMORIZE IN MATHEMATICS

Actually there is very little pure memory work in mathematics. Most of the work is a matter of reasoning and understanding. Some people may talk about memorizing definitions, postulates, and theorems in geometry. What really happens —or *should* happen—is that you learn to *understand* the ideas and associate them with a mental picture of the geometric situation. Then you can formulate the words yourself.

A few situations do involve pure memory work. These include (1) formulas, (2) approximate values of frequently used constants, such as π, $\sqrt{2}$, and $\sqrt{3}$, and (3) the number facts of arithmetic (addition and multiplication tables, etc.).

Some formulas follow so immediately from other formulas that they can easily be recalled by thinking of how they are derived. You can quickly recall such formulas as the one for the area of a triangle, since a triangle is half of a parallelogram; or the formula for sin $2x$, which comes from sin $(x + x)$. While you want to be able to use the derived formula directly, it is good to refresh your memory by recalling the relationship with the earlier formula.

A few formulas, such as the quadratic formula, and certain constants which are often needed require actual memorization. If you have trouble remembering them, there are ways to help yourself. For instance, write the constants or formulas on a card and clip the card to a book or notebook which you carry to and from school. Then as you ride your bus, or even as you walk, first study the card, then turn it over and see how accurately you can repeat the numbers or formulas. The next day try repeating them *before* you look at the card, and then check yourself. Keep at them until you really know them. This card method can be used also at the beginning of doing your homework or in the few minutes that are so often wasted between activities at home. Try various methods, and once you have learned your formulas or constants, keep using them so you don't forget what you have learned.

Most high school students know the number facts of arithmetic, but once in a while someone slips through without having learned them. If you are such a one, realize that there *is* something you can do about it; you do not have to continue to suffer under that handicap. Probably flash cards are one of the best devices for learning your number facts. You can easily make such cards yourself, using the number combinations on which you need practice. Ask a parent or friend to help you go through them, naming results as rapidly as you can with accuracy. Keep at them until you can go through all the necessary number facts perfectly.

6 ON DOING THE WRITTEN HOMEWORK

Remember that the main objective of the written homework is to *learn*—to learn the ideas of mathematics by putting them into practice, and also to learn habits of neatness, accuracy, and clarity of expression. The written homework sometimes has also the objective of *showing how much you have learned.* For both of these purposes the following pointers will help you do a better job:

a) *Be sure you know exactly what the assignment is.* Have a definite place in your notebook where you write down the assignment for mathematics each day. Be careful to have it accurate and complete.
b) *Follow directions.* Read the instructions at the beginning of the exercises and follow them.
c) *Work neatly and accurately.* The habits of neatness and accuracy which you develop in doing this homework will be of importance to you.
d) *Show your complete work.* Show on your paper all work that is not done mentally. This will help you and your teacher when you are checking through for errors, and it develops the good habit of expressing yourself completely enough to be clear.
e) *Check the reasonableness of your answers.* Learn to make estimates to test the correctness of your work.
f) *Do your written homework promptly.* Prompt practice fixes the principles in your mind before they become "fuzzy."

7 WHAT TO DO IF YOU "GET STUCK" ON A PROBLEM

Even if you are the best of students, you will sometimes run across a problem in the homework which you are unable to understand. If you just give up and forget about that problem, you are very likely to lose out on an idea you will need. The first thing to do is to look back at the book, at your notes, or at your study sheet for ideas related to the problem. If you have been thinking of *what, why,* and *how,* you will probably find the answer to your difficulty very soon yourself. If your work on a problem seems to be completely confused, it sometimes helps to stop thinking about it and start afresh, perhaps at a somewhat later time.

If you still are not able to clear up your thinking on the problem, make yourself a reminder to ask about it as soon as possible, preferably during the class discussion, but otherwise by arranging to ask the teacher outside of class. Try to pinpoint the difficulty so you can say, when the time comes, "Right here is where I lost the line of thought," or "I followed down to this point and then I couldn't see what to do next."

8 ON HELP FROM PARENTS OR FROM OTHER STUDENTS

Parents are frequently called on to help do mathematics homework, and telephones ring every night for Sally to ask Suzy, "How do you do the ninth one in algebra?" Is this good?

Yes, if it is done the right way. Obviously if Dad or Suzy just gives the answer, Sally doesn't learn much. If, however, the help is based on a mutual discussion of the *what* and *why* and *how* of the situation—if the conversation can be on the basis, not of "What's the answer?" but of "What's the point of this?"—then it can be very wholesome. Bull sessions about the mathematics, *after* you have done your best to do the work yourself, are good experiences.

One difficulty you may encounter, especially in these days of so-called modern mathematics, is that your dad may have been taught by a method that has now been replaced by a different one. Unless you can explain clearly to him the method your class is using, his "help" may confuse you more than aid you.

Whether or not it is right to collaborate on written work depends on the nature of the work. A project that is going to be graded for individual credit, of course, should be done independently.

9 HOW TO AVOID MAKING CARELESS ERRORS

There is no magic formula that will immediately eliminate all your careless errors. Perhaps the nearest thing to it is really to care. If you are conscious of your weakness and are sufficiently disturbed every time a foolish mistake slips in, you are less likely to make such mistakes.

Some people have found that the "check back" method helps. The idea is to check back every few seconds over what you have just done, to see if it is really what you meant, instead of rushing ahead to the end of a whole problem and then trying to go through and catch your mistakes. At first it may seem to take longer to check back, but in the long run it will save you time by helping you catch your mistakes early. When checking back becomes a habit, you will be a much more accurate worker. On the other hand, some people find that constant checking makes them lose their train of thought, so use your judgment about what is best for *you.*

Another good thing to do is to look at your answer critically and see whether it is reasonable. If you have an answer of 248 miles per hour for the rate at which a man walked from his home to his office, you can know that something is wrong. Whenever you find an unreasonable answer, go back over the problem and find the error.

10 HOW TO MAKE YOUR ERRORS HELP YOUR LEARN

What do you do when an answer is wrong in your homework or on a test? Do you throw away your paper and put the matter out of your mind—and then make the same mistake the next time? If you are wise, you will make those errors teach you something. Here is what you can do:

a) Analyze the error to see what kind it is and what caused it.

b) If it is a careless error and you really knew how to do the work correctly, make a note of it and, if you find that you keep making careless errors frequently, start putting on a campaign to overcome that habit of careless work. See section 9 above for suggestions about such a campaign.

c) If you cannot see what caused the error, search your mind, the book, and your notes for material related to that problem. See whether you applied the ideas correctly.

d) If you still cannot find the cause of your error, ask a teacher to help you see what is wrong.

e) Now that you have found what you did wrong, *do* something about it. Practice working the problem the right way; think about it; try to make an association in your mind that will protect you from making the error again.

f) Perhaps writing in your notebook a list of "errors to avoid" would be helpful.

11 HOW TO REVIEW AS YOU GO ALONG

Some students do well in their daily work but then fall down on the major tests. It is very easy, if you are not careful, to get into the habit of doing each day's assignment well enough but then never thinking of it again, with the result that the ideas you are studying just pass through your mind and very few of them stay there. What is needed is some sort of systematic, regular review instead of trying to cram in all the reviewing just before the big tests. This is easier than you may think, and it pays off in the long run.

The thing to do is to take a few minutes at least once a week, say on weekends, when you survey what you have been doing. This survey should consist of two parts: (1) a careful study of what you have been doing for the previous week, emphasizing the important ideas, and (2) a broader look at the whole preceding part of the course, seeing how the recent work is related to the rest and spotting any portions of the course about which you are not confident.

In both these parts most people can review more quickly and efficiently, not by doing many exercises, but by writing outlines of the ideas. For the more recent work, the outline should appear in some detail, giving explicitly the *what, why,* and *how.* If you do practice on exercises, select them intelligently, thinking about how the ideas you have been studying are applied. For the broader survey of the earlier work, your outline will be a cumulative condensation of your more detailed outlines, giving a bird's-eye view of the whole course. If you feel uncertain about an earlier section of the work, get out the detailed outline you made when you first reviewed that part, study it over, and try a few exercises to refresh your understanding.

If two or three students get together to discuss their review outlines, the interaction of thinking will usually be particularly beneficial.

One more thing you can do is to make up your own practice test on the chapter. Then later, compare your test with what the teacher thinks is important.

12 HOW TO USE A NOTEBOOK IN MATHEMATICS

Mention has been made in these pages of a mathematics notebook. Intelligent use of a notebook can do a great deal toward making your mathematics easier and increasing your understanding. Here are some of the things you might want to keep in your notebook, with a special place for each:

(1) Assignments (see section *6a*)

(2) Class notes (see section 14)

(3) Vocabulary list (see section 4)

(4) Study sheets (see section 2)

(5) Warnings of errors to avoid (see section 10*f*)

(6) Review outlines (see section 11)

(7) Class summaries (see section 13*j*)

Classwork

13 HOW TO MAKE THE MOST OF THE CLASS PERIOD

Many students throw away much of the help they need in mathematics by not making good use of the time when they are in class. Here are some suggestions:

a) Get ready. In the minute or two before the class gets started, think over what you have been working on recently and have your mind "warmed up," ready to go.

b) Have all necessary equipment with you: book, sharpened pencils, notebook, ruler, compasses, tables, whatever you need.

c) Take down the assignment promptly and accurately.

d) Concentrate. This takes an effort if you are the kind whose mind wanders off easily to other subjects. If you start to dream, pull your mind back sharply and remember that you need to take advantage of every minute in class or you will miss something. This is especially important if you have not been doing well.

e) Ask questions when you do not understand.

f) Listen to the questions and answers of others. When another student is answering a question, think how *you* would answer it.

g) Take part in the discussion. Not only will it help the class to have you participate, but it will help you if you join in and express yourself.

h) Try hard to grasp the ideas of the lesson. In each case look for the *whole* idea, not just part of it.

i) Do not write at the wrong time. If you take any notes or make any corrections on your homework paper, be sure that you do not miss anything that is said while you are doing so. Most people cannot write and listen at the same time. (See section 14.)

j) As soon after class as you can, summarize (preferably in writing) the main ideas brought out in the class lesson.

14 HOW TO TAKE NOTES IN CLASS

Whether or not you take notes in your mathematics class will depend on what your particular class is like. If the teacher encourages you to take notes, have a place ready in your notebook to which you can turn quickly if something should be recorded. Sometimes the teacher will give you an outline or some special pointer which is important to remember and will tell you to take it down. Sometimes an idea will be developed in class which you yourself feel is worth writing down for future reference.

In either case, there are two conflicting things you must try to do when taking notes. One is to make your notes complete and clear enough to be valuable to you later. The other is to make your notes brief enough so that you can continue to listen to what is being said in class. Doing both of these at the same time is an art—an art you will need to develop for college work.

One more thing—if you do take notes, be sure that you *use* them to good advantage later. Study them soon after class to fix the ideas in your mind. In particular, let them help you in your weekly review. (See section 11.)

15 HOW TO LEARN BY THE DISCOVERY METHOD

Sometimes your teacher may help you learn by leading you to discover an idea for yourself. This discovery method may proceed by a series of short questions with discussion as you go along, or sometimes by problems, one leading to another, or by drawing figures and making observations. If you suspect that the teacher is leading you through a discovery experience, be particularly alert. Try to see the purpose of each question; try to guess what the next question will be; and look for patterns in the numbers or geometric figures which may suggest a general principle. If you yourself can make the generalization from the gradual development to the broad mathematical principle, you will find it a stimulating and enriching experience. It is in this sort of experience that you begin to see what fun mathematics can be.

Tests

(Read this section several
days *before* the test.)

16 HOW TO REVIEW FOR TESTS

Here are some pointers that can help you do your reviewing for tests effectively:

a) Start reviewing far enough in advance so you have time to do a careful, unhurried job and still are able to go to bed early the night before the examination. If you have been reviewing as you go along (see section 11), this major review will not take long.

b) If you have some review outlines you have already made, get them out and from them make a master outline of the whole section of the course on which you are being examined.

c) Go over the parts of the master outline one at a time in considerable detail, using your detailed outlines, your class notes, your textbook, your reminders of dangers to avoid, and perhaps your study sheets to make sure that you have a clear understanding.

d) If you have none of these helps from your earlier work, you will have to make your outline right from scratch, using the textbook and your head as the only sources of help—and it will be all the more important that you do so. Make your outline concise and to the point. Such an outline conscientiously worked out with the details carefully thought through can make a tremendous difference in the results of a final examination —and, what is more important, in the lasting benefit you carry away from the course.

e) In some parts of this outline you will feel absolutely sure of yourself and it is probably unnecessary to do anything more about them. Other parts may be less clear, and for these it may be good to pick out a few exercises and try doing them with the ideas of your outline in mind. This application of the ideas to some particular examples will often clarify them in your mind.

f) If you have kept your earlier test and quiz papers, look them over for reminders of good methods to follow and poor methods to avoid.

g) Remember to review your mathematics vocabulary.

h) If there are some formulas for which you are responsible, make a list of them and then practice saying them or writing them, using the list only as a check on yourself.

i) Many books have summaries, word lists, and review material at the end of each chapter or at the end of the book. These are excellent if you use them intelligently, letting them be a guide to points you need to restudy.

j) If you were the teacher, what questions would you ask on the test? Prepare yourself for these questions.

k) Get a good night's rest the night before the examination.

l) *Don't worry.*

17 HOW TO TAKE TESTS

These guidelines will help you:

a) When you take a test, go in with an attitude that will help you. Take pride in doing the best job you can. Don't try to "get by" with doing as little as possible. Have confidence in your own ability, but don't be overconfident.

b) Be serious and concerned enough about the test to do your best, but don't worry to the point of anxiety. Fear alone can make a person do poorly on a test regardless of his ability and knowledge.

c) Have all necessary equipment, such as sharpened pencils, eraser, ruler, compasses, etc.

d) Follow directions. Read carefully and listen carefully for any special instructions, such as where answers are to be written, or how many questions must be answered. Follow directions in each individual question too.

e) Look over the whole test quickly at the start and, unless you are required to do the questions in the order given, do the ones you are sure of first.

f) If you are unable to answer a question, leave it and go on to others, coming back to the hard one later. Often with a fresh start you will suddenly see light.

g) Without fussing about the time, give an occasional glance at the clock so that you can make the most of the time you have. Many tests are planned so that a large number of the students will not finish all the questions. Don't be upset if the time is running out and you still have more to do. If you have done your best and have not wasted time, nothing more is expected of you.

h) Be careful to show clearly what you are doing. Put in enough steps so that your method is perfectly clear. The teacher is not a mind reader, and your grade may depend on whether or not the teacher can see from your work that you understand what you are doing.

i) Work neatly. To work neatly is a good habit in itself; it also fosters accuracy, and it communicates to the examiner.

j) Check back for accuracy as you go along. If you can catch those silly little mistakes at once, your job is easier than if you have to go over the whole thing later and perhaps find a mistake near the start. If you do have extra time at the end, however, make the most of it to recheck and make sure you have done your best.

Some Specific Topics

18 SETS

Much of the early study of sets is a matter of vocabulary and notation, developing a way of expressing ideas that is convenient in many parts of mathematics. Be careful to master the definitions and symbols well, and watch for opportunities later to use the set language.

19 ALGEBRA

a) *Numbers:* You will want to be clearly aware of the various kinds of numbers and how they are related. Keep a cumulative chart in your notebook on which you show, as you learn about them, each basic set of numbers (such as the set of integers or the set of rational numbers), tell its characteristics, and show the subset relations between it and other basic sets of numbers.

b) *Operations and how they behave:* Be sure that you have a clear idea of what we mean by an "operation" in mathematics. You will find that a very large part of algebra has to do with the properties of addition and multiplication. Start early to keep a chart showing these properties, and add to it or modify it as you learn more about them.

c) *Sentences:* In algebra the sentences we use to express ourselves are, much of the time, equations or inequalities. You need to be sure of what you are saying in such sentences. In particular, watch for:

(1) An equation or inequality that is true for only *some* values of the variable and false for others, such as

$$x^2 - 9 = 16 \quad \text{or} \quad |x| \geqslant 3$$

(2) An equation or inequality that is true for *all* values of the variable for which the expressions are defined, such as

$$x^2 - 9 = (x - 3)(x + 3) \quad \text{or} \quad |x| \geqslant 0$$

In (1) we are interested in the truth set or solution set (the set of numbers that make the sentence true). To find the solution set, we make heavy use of the idea of *equivalent* sentences (sentences having the same solution set) to obtain simpler sentences, in which the solution set may be easier to see. You should write up a careful analysis of what operations on a sentence are sure to give you an equivalent sentence.

In (2) the sentences are general statements (called identities). We sometimes want to convert such a sentence into another which better suits our purpose, but we want to be sure that the new sentence is still true for all values of the variable in the same domain. Again, you should write up an analysis of what changes you can make in the expressions of such an equation or inequality and still know that the new statement is true for all values of the variable in the same domain. (See trigonometric identities in section 22.)

d) *Functions:* Make a real effort to be able to use functions intelligently, because they permeate much of mathematics, not just algebra. The concept of function will grow in meaning for you as you use it more and more. It is a very helpful unifying and clarifying concept. Realize that there are various ways to express a function. Make a list of the ways (table, graph, formula, verbal description, set of ordered pairs, mapping, etc.). For each way explain how it shows (1) the domain, (2) the range, (3) the rule that defines the function; and note the advantages and disadvantages of that way.

e) *Story problems:* Practice with equations and inequalities and with functions is often given in the form of story problems. If you are one of those students who think that story problems are difficult, the chances are that you have not yet learned how to read carefully and organize the data meaningfully. In the following plan of suggested procedure, steps 1, 2, and 3 are the parts where you organize your data.

(1) *Read;* don't just skim over the words. Try to picture clearly in your mind the whole situation. Draw a sketch of it if possible. Look up any words you do not understand.

(2) *Tell what your variable represents.* Decide what quantity you are going to represent by a variable. (Usually your best choice is a quantity you are trying to find.) Give it a letter and write down clearly this important part of the data, showing the unit of measure. Suppose a problem asks: "How much water must be added to 20 ounces of a solution of alcohol and water which is 80 percent alcohol to produce a solution which is 60 percent alcohol?" Don't write "Let n = the water"! A simple, clear statement identifying your variable is suggested by just answering the question in the problem. It asks, "How much water must be added?" You should answer, *and write,* "n ounces of water must be added." This automatically guides you into mentioning the ounces and gives a clear, natural sentence. If you use more than one variable, be sure that each variable is clearly described.

(3) *Represent other quantities in terms of the variable.* This part of organizing the data is often neglected. Actually you are really defining some functions which bring in the other quantities in the problem in a form that will help set up a relation. In the example above we could write: "If n ounces of water are added to the original 20 ounces, then there will be

$20 + n$ ounces of final solution,

$.80(20)$ ounces of *alcohol* in the original solution,

$.60(20 + n)$ ounces of *alcohol* in the final solution."

(4) *Write the equation.* If the preceding steps have been carefully followed, this step will be comparatively easy. You need simply to find two different ways to express one of the quantities in the problem and equate them. Since the amount of pure alcohol has

not changed, the equation for the above problem is $.80(20) = .60(20 + n)$. This is an "alcohol" equation. The problem can also be done by using a "water" equation.

(5) *Solve the equation.* Think carefully about equivalent equations.

(6) *Give the answers.* Reread the problem to be sure that you answer the questions asked. Answer clearly, showing the units of measure. For example, say that "$6\frac{2}{3}$ ounces of water must be added."

(7) *Check.* Verify your answers by seeing whether they satisfy *all* the requirements of the words in the original story problem. Substituting in the equation you made is *not* a check of the problem because your error may have been in formulating your equation.

Notice that out of the seven steps above, six involve reading or rereading the problem.

20 GEOMETRY

Courses in geometry vary considerably, from largely intuitive geometry and appreciation of geometric forms to courses with emphasis on vectors, geometric transformations, or coordinate geometry. Most courses, however, are concerned to a large extent with the logical structure of geometry. We examine methods of reasoning and use the familiar material of geometry to illustrate and practice these methods at the same time that we explore the wonders of geometry itself. This means that you should constantly be on the watch for the way a *structure* of geometry is built up, bit by bit. The building blocks are definitions, postulates, and theorems.

a) How to study theorems: The main thing to remember about studying theorems is that it is a matter of understanding rather than memorizing. If you find yourself learning by rote and not grasping the meaning of the theorem, you should realize that you are wasting your time. Start working for understanding. For studying a theorem that is written out in your textbook, the following suggestions may be helpful:

(1) Have a "study sheet" ready.

(2) Read the statement of the theorem and study the drawing for clear understanding of the "Given" and "To prove."
(3) Make your own drawing on your study sheet.
(4) Close the book without reading further and try proving the theorem yourself as you would an original exercise. If you are successful, check your proof against the one in the book. (Of course, it is possible that two different proofs may be equally correct. If you think you have discovered a new proof, discuss it with your teacher.)
(5) If you cannot prove the theorem yourself, take a quick look at the proof (or maybe just the first step or two) in the book; then try again to write a proof with the book closed.
(6) If you still cannot prove it, carefully work through the explanation in the book, carrying out details on your study sheet as you go along. Find out how it *is* proved.
(7) Finally, before you leave your study of this theorem, whether you proved it yourself or had to have help, analyze the proof sufficiently to be able to state the main plan of the proof.

b) How to build the structure: As the definitions, postulates, and theorems of geometry accumulate, you will need to have them well organized and available in your mind. Further development of the structure depends on being able to use the earlier building blocks in proving new theorems. How can you make sure that the earlier structure is ready to serve you when you need it?

We shall speak of theorems, but the same remarks apply to definitions and postulates. For many of the theorems, the frequent use you make of them fixes them in your mind automatically, but there are other theorems which you do not use often enough to master in this way. Probably the two things that help most for mastery are (1) associating the idea of the theorem with the geometric picture in such a way that you have a quick connection of picture, idea, and statement of the theorem, in that order; and (2) being aware of the grouping of theorems according to the kind of thing they can help prove, so that when you need to prove, say, that certain lines are parallel, you can quickly think over all the methods you have had of proving lines parallel.

(1) To develop your *picture-theorem association,* a periodic systematic review, perhaps once a week, is helpful. One way to do this is to leaf through the book with a card in your hand, cover with the card the statement of each theorem as you come to it, look at the drawing, and try to state the theorem it illustrates. Then reverse the process. Cover the drawing, read the theorem, and see how quickly you can sketch the drawing and state the "Given" and "To prove." This method can be improved if, as you go through the course, you make yourself theorem cards for the new theorems you have each day. Write the statement of the theorem (not the proof) on one side of a 3″ x 5″ card, and on the other side make a drawing to go with it. These cards can be shuffled and you can do the practice described above by looking at one side, trying to reproduce the other side, and checking by turning over the card. Once in a while it helps to cover several sheets of paper with a jumble of drawings in shapes and positions different from those in the book. Then try pointing to a drawing and quickly stating the theorem, the "Given," and the "To prove." Try doing this with one of your classmates.

(2) To develop your *awareness of the groups* of theorems, you need to see the theorems in those groups. The theorem cards mentioned above can help you. In one corner of each card write what that theorem can help prove. Then, once a week or so, sort the cards according to your notes in the corner, and see how the various groups of theorems grow. Some students prefer, instead of using the cards, to list the theorems by groups in a notebook as they are developed.

c) How to think out an original proof: If you have a good mastery of the theorems in geometry and know how to classify them as described above, the chances are that original proofs come rather easily to you. When you do not see quickly how to prove something, however, the most fruitful way of finding a method of proof is usually to reason backward from the conclusion. When a teacher helps you to find a method of proof, he usually does not tell you what to do but, rather, asks questions that help you discover the method. When no teacher is present to help, you can ask yourself some of the same questions, such as:

What am I trying to prove?
What are some possible methods of proving this?
Which method is likely to work here?
If I am going to use congruent triangles, which triangles are best to use?
Do I have enough information to prove the triangles congruent?
Am I making use of all the facts in the "Given"?
Does every step have a purpose in my progress toward the final conclusion?
In each of the reasons I have given (1) is the *hypothesis* of the reason satisfied by particular statements in earlier steps, and (2) does the *conclusion* of the reason support the statement I am making at this step?

Get into the habit of asking such questions of yourself as you work, and you will learn to do some good original thinking.

21 COORDINATE GEOMETRY

Many ideas in mathematics are made clearer by looking at them graphically as well as algebraically. Get into the habit of asking yourself whether a graph will help.

Drawing good graphs takes a certain amount of care and neatness. Plan ahead by considering where the axes can best be placed and how big the unit should be to make a graph big enough to show what you want and yet small enough to stay on the page. Label your graph clearly so someone else can read it easily. In computing a table of values for a graph you can sometimes save a lot of your time by having available a slide rule and a table of squares and square roots.

TRIGONOMETRY 32

In higher mathematics it is often necessary to convert a trigonometric expression from one form to another, more useful, form. To develop skill in doing this, we practice proving trigonometric identities.

An "identity" in trigonometry is an equation that is true for all values of the variables for which the expressions are defined. (See section 19*c*.) To prove that a given sentence is an identity, we try to convert it, by permissible changes, into another sentence which we *know* is an identity, such as $\sin 2x = \sin 2x$. "Permissible changes" are almost always substitutions, either from basic trigonometric identities, such as substituting $1 - \cos^2 x$ for $\sin^2 x$, or from identities of algebra, such as substituting $(1 + \cos x)(1 - \cos x)$ for $1 - \cos^2 x$ because $(1 + a)(1 - a) = 1 - a^2$. Substitutions from identities of algebra come from your knowledge of factoring, simplifying fractions, and so on. Notice that a permissible operation which often is helpful is, to multiply a fraction by the number *one* in a form such as $\frac{1 - \cos x}{1 - \cos x}$. In this way you can frequently supply needed factors. For example, in proving that $\frac{1}{1 + \cos x} = (1 - \cos x) \csc^2 x$, since there is a factor $(1 - \cos x)$ on the right, you need a factor $(1 - \cos x)$ on the left. Try it!

As you work you will watch the two members of the equation to see what permissible changes will make them alike. There are three main approaches to the problem. Watch for signposts that point to (1) the need for making the angles alike, (2) the need for making the trigonometric functions alike, or (3) the need for making the algebraic forms alike. For instance, if one side of the equation is a single fraction and the other side is the sum of two fractions, it would be natural to make their algebraic forms alike by performing the addition of the two fractions. If you see no obvious transformation to make the functions alike, try changing all of them into sines and cosines first.

If you have trouble seeing the connection between the trigonometric expressions and the corresponding algebraic ideas, try substituting a letter for a trigonometric function temporarily. For example, if you do not see that $4 \sin^2 A - 5 \sin A + 1$ is factorable, let k represent $\sin A$ for a moment and factor $4k^2 - 5k + 1$; then replace k by $\sin A$. Try, however, to develop the ability to think of the function as though it were a single letter, and to avoid depending on the actual substitution of a letter.

Solving trigonometric equations uses a combination of the substitutions described above and the operations that give *equivalent* equations. (See section 19.)

You see now why it is so important (1) to *know your trigonometric formulas well* and (2) to *be sure of your elementary algebra.* If your algebra is rusty, find an algebra book and review what you need for trigonometry. This would include (1) factoring (only the simpler cases), (2) solving equations by factoring, (3) adding and simplifying fractions, (4) solving fractional equations, and (5) simplifying radical expressions and solving radical equations.

23 LOGIC, VECTORS, TRANSFORMATIONS, STATISTICS, LINEAR ALGEBRA, CALCULUS

These and other topics will occur at various stages in your mathematical growth. The same general habits of study you have been developing will apply here too. In particular, charts seem to be very helpful in organizing your ideas.

For example, when studying logic make a chart of the connectives "and," "or," "not," "if . . . then," showing their meanings, symbols, and truth tables; and make a chart of "statement," "converse," and "contrapositive," showing their interrelations. When studying vectors make a chart comparing the various ways of representing vectors—arrows, ordered pairs, components, complex numbers—and compare the operations you can perform with vectors. When studying geometric transformations, make a chart showing what stays the same and what is different under various kinds of transformations. More generally, try making a chart showing the interrelations of vectors, transformations, matrices, and linear algebra.

These are just a few suggestions. Once you are alert to this method of analyzing the material you are studying, you will see many opportunities to put it to good use.