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ABSTRACT

THIS IS ONE OF A SERIES OF UNITS INTENDED FOR BOTH PRESERVICE AND INSERVICE ELEMENTARY SCHOOL TEACHERS TO SATISFY A NEED FOR MATERIALS ON "NEW MATHEMATICS" PROGRAMS WHICH (1) ARE READABLE ON A SELF BASIS OR WITH MINIMAL INSTRUCTION, (2) SHOW THE PEDAGOGICAL OBJECTIVES AND USES OF SUCH MATHEMATICAL STRUCTURAL IDEAS AS THE FIELD AXIOMS, SETS, AND LOGIC, AND (3) RELATE MATHEMATICS TO THE "REAL WORLD," ITS APPLICATIONS, AND OTHER AREAS OF THE CURRICULUM. THE PRESENT UNIT HAS EMPHASIS IN THE AREA WHERE THE FIELD PROPERTIES FOR RATIONAL NUMBERS ARE USED IN DEVELOPING ARITHMETIC AS IT IS CURRENTLY TAUGHT IN ELEMENTARY SCHOOL. FOCUS IS ON THE ARITHMETIC OF POSITIVE AND NEGATIVE RATIONAL NUMBERS, AND ON THE FIELD PROPERTIES IN TEACHING ARITHMETIC. (RP)

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COVER PAGE--USES OF THE FIELD PROPERTIES IN ELEMENTARY SCHOOL

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I. Background Assumed

A. Previous exposure to the field properties. This does not imply thorough understanding, but does assume some familiarity with the terminolcgy and concepts.

B. Basic knowledge of the operations and algorithms used with the whole numbers, the fractional numbers (rational numbers) and the fractional numbers expressed in decimal form.

II. Placement of the Unit

Because of the above assumptions, the unit should be placed after an introduction to the field properties. It could follow such an introduction immediately, as an extension of such a unit, or it could be used somewhat later, to provide review, extension, and a spiral approach to the topic of the field properties.

III. Objectives for the Unit

A. To show teachers places where the field properties are used in developing the arithmetic of the elementary school.

B. To give teachers a greater command of the field properties.

C. To provide teachers with more practice in the use of the field properties.

D. To provide teachers with information that will be useful when pupils, parents, or other teachers question the worth of teaching one or another of the field properties.

E. To help teachers to gain deeper insight into the structure of the number systems studied in elementary school arithmetic.

IV. Books Surveyed for Writing the Unit

SMSG, Grades K-6

American Book Company (Deans, et al), Grades 1, 3-6

Modern Arithmetic Through Discovery (Silver Burdett), 1-6

GCMP, Grade 1, and Intermediate Series (1964-65), Booklets 1-12, Grades 4-6

Seeing Through Arithmetic (Scott-Foresman), Grades 1-6

Moving Ahead in Arithmetic (Holt, Rinehart, Winston), Grades 1-6

Elementary School Mathematics (Addison-Wesley), Grades 1-5

Elementary Mathematics (Harcourt, Brace, & World), Grades 1-3

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c. Chart showing the grade range where topics are studied.

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USES OF THE FIELD PROPERTIES IN ELEMENTARY SCHOOL

1. Introduction

Do such words and phrases as "reciprocal", "additive inverse", "commutative property of multiplication", "distributive property" sound familiar to you? If you have been involved with the teaching of arithmetic recently, or if you are preparing to teach, you probably answered "yes". You have probably learned these and other related terms, and the concepts which they describe. You may even have learned that a mathematical system with certain properties is called a group, while if it has these and other similar properties, we call it a field.

Suppose we press on a little bit farther. Do you know, for example, what good a reciprocal is? Do you know where this idea is used in elementary school teaching and learning? Do you know how the associative property of addition is used in adding columns of figures? Do you know what arrays, such as 3 rows of 21 dots, have to do with the distributive property?

The purpose of this unit is to give you answers to questions such as those we have just asked. You have undoubtedly been told that the field properties play a key role in developing elementary arithmetic--but it is entirely possible that no one has actually shown you what this role is.

In this unit, we will look at places where the field properties are used in developing arithmetic as it is currently taught in the elementary school. We hope that you will gain a better understanding of the field properties, of the arithmetic of positive, negative, whole, and rational numbers, and of the importance of the field properties in teaching arithmetic. These properties are basic to the study of the structure of number systems. This study is a unifying thread in mathematics, which continues in high school and college, building on the foundation laid here.

For your convenience the field properties are listed in Table I to serve as a review and a convenient reference page.

TABLE I.

THE FIELD PROPERTIES

Closure Property of Addition: For every two numbers, a and b, $a + b$ is a unique number. For example, $3 + 7$ is a unique number.

Commutative Property for Addition: For every two numbers, a and b, $a + b = b + a$. For example, $4 + 6 = 6 + 4$.

Associative Property of Addition: For every three numbers, a, b, and c, $a + (b + c) = (a + b) + c = a + b + c$. For example, $3 + (4 + 5) = (3 + 4) + 5 = 3 + 4 + 5$.

Additive Identity: There is a number 0, such that for every number a, $a + 0 = 0 + a = a$. For example, $17 + 0 = 0 + 17 = 17$.

Additive Inverse: For every number a, there exists a number, $-a$, read "negative a", such that $a + (-a) = (-a) + a = 0$. For example, $6 + (-6) = (-6) + 6 = 0$.

Closure Property of Multiplication: For every two numbers, a and b, $a \times b$ is a unique number. For example, 2×9 is a unique number.

Commutative Property of Multiplication: For every two numbers, a and b, $a \times b = b \times a$. For example, $5 \times 8 = 8 \times 5$.

Associative Property of Multiplication: For every three numbers, a, b, and c, $a \times (b \times c) = (a \times b) \times c = a \times b \times c$. For example, $2 \times (6 \times 11) = (2 \times 6) \times 11 = 2 \times 6 \times 11$.

Multiplicative Identity: There is a number 1, such that for every number a, $a \times 1 = 1 \times a = a$. For example, $32 \times 1 = 1 \times 32 = 32$.

Multiplicative Inverse (Reciprocal): For every number a (except 0), there exists a number, $\frac{1}{a}$, such that $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$. For example, $3 \times \frac{1}{3} = \frac{1}{3} \times 3 = 1$.

Distributive Property of multiplication over addition: For every three numbers a, b, and c, $a \times (b + c) = (a \times b) + (a \times c)$. For example,

$$5 \times (4 + 3) = (5 \times 4) + (5 \times 3)$$
$$5 \times 7 = 20 + 15$$
$$35 = 35$$

Before we examine the uses of the field properties systematically, we have provided a few practice exercises to help you review these properties.

Practice Set 1

1. Name the property or properties which show each of the following statements to be true:

(a) $3 \times 8 = 8 \times 3$

(b) $7 \times (10 \times 3) = (7 \times 10) \times 3$

(c) $4 + [2 + (-2)] = 4 + 0$

(d) $4 + 0 = 4$

(e) $12 \times (4 + 1) = (12 \times 4) + (12 \times 1)$

(f) $\frac{3}{2} \times \frac{2}{3} = 1$

(g) $3 + (8 + 17) = (3 + 8) + 17$

(h) $\frac{3}{7} \times 2\frac{1}{3} = 1$

(i) $3 \times 1 = 3$

(j) $(\frac{7}{9} \times \frac{4}{5}) \times \frac{3}{4} = \frac{7}{9} \times (\frac{4}{5} \times \frac{3}{4})$

2. Look at each expression below. How was it changed from the preceding form? Write the name of the property that permits the fact. The answer "numeration system" would be acceptable in cases such as $24 = 20 + 4$. Your answer will be either "numeration system" or the appropriate field property.

$16 \times 3 + 16 \times 4 = 16 \times (3 + 4)$ ____________________

$= 16 \times 7$ ____________________

$= 7 \times 16$ ____________________

$= 7 \times (10 + 6)$ ____________________

$=(7 \times 10)+(7 \times 6)$ ____________________

$= 70 + 42$ ____________________

$= 70 + (40 + 2)$ ____________________

$= (70 + 40) + 2$ ____________________

3. Classify the following sentences as true or false:

(a) $8 + (5 \times 4) = (8 + 5) \times (8 + 4)$

(b) $(6 + 7) + 4 = 4 + (6 + 7)$

(c) $3 \times 0 \times 5 = 15$

(d) $\frac{7}{9} \times \frac{9}{11} \times \frac{11}{7} \times \frac{4}{3} = \frac{3}{4}$

(e) $(\frac{13}{14} \div \frac{1}{7}) \times \frac{1}{7} = \frac{13}{14}$

(f) $5 \times (4 + 6 - 3) = (5 \times 4) + (5 \times 6) - (5 \times 3)$

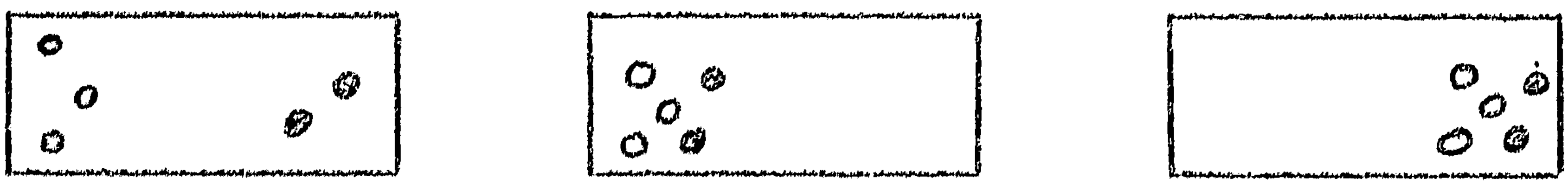
(g) $5 \times 6 + (7 + 4) = (5 \times 6 + 7) + 4$

(h) $17 \div (4 + 2) = (17 \div 4) + (17 \div 2)$

2. ADDITION AND SUBTRACTION OF WHOLE NUMBERS

2.1 MANIPULATION OF CONCRETE OBJECTS.

Some of the earliest mathematical experiences the child has in school involve joining two sets of concrete objects to form a single set. For example, a child may have 3 red disks on his desk in one bunch, and 2 black disks in another bunch. [See Figure 1 (a)].



(a) (b) (c)

Figure 1

He has several ways he can put the two sets together to get a single set. For example, he may push the black disks over to the red ones [Figure 1 (b)]. Then again, he may push the red ones over to the black ones [Figure 1 (c)]. Each of these procedures leads to one set of 5 elements. This is the sort of motivational material from which one develops an understanding of the commutative property of addition.

Of course, these are not the only choices open to the child. He could pick the red disks up in his left hand, pick up the black ones in his right at the same time, and dump the contents of his hands simultaneously into a bag. He could ignore the disks and begin to read a book. He could line the disks up, red, black, red, black, red. None of these responses changes the fact that this is an illustration of the addition fact $3 + 2 = 5$, but none of these responses have anything particular to do with the commutative property of addition.

If a person has three sets of disks (for example, red, blue, and yellow), and he wishes to push the sets together to form one large set, he has some choices to make. He may decide to leave one set fixed, and then push the others to it. Probably he also will decide which of the other sets he will move first. He realizes (or can be led to realize) that it doesn't make any difference which choices he makes--he ends up with the same number of elements in the large set. (Of course, he ends up with the large set in different places, but that has no effect on the number in the set). This manipulation is motivation for the associative property of addition.

A variation of this sort of concrete object manipulation is to use large cubes with dots painted on the faces, like dice. By throwing a pair of dice and counting the total number up, the pupils will see that the sum, for that throw, is the same regardless of which die we start counting on--another demonstration of the commutative law of addition. Three dice could be used similarly to motivate the associative law of addition.

2.2 "RINGING" SETS

In the kindergarten and early primary grades, sets and set pictures are often used to convey basic ideas. Often children are asked to "ring" sets to indicate the set under consideration.

Suppose 3 sets are given, as follows:

△ ☆	XBF	QPY
+	C	AZT

Suppose we wish to combine these three sets to form one set. We may wish to first combine the right-hand two sets, into one set, and indicate this by "ringing" them; then to combine the remaining set with this set, as follows:

△ ☆	XBF	QPY		△ ☆	XBF	QPY
+	C	AZT		+	C	AZT

Another possibility would be to combine first the two left-most sets and then combine this set with the remaining set, as shown below:

△ ☆	XBF	QPY		△ ☆	XBF	QPY
+	C	AZT		+	C	AZT

Children will soon see that the number of objects included in the final set is the same in either case. This builds readiness for the associative property of addition.

2.3 REGROUPING

The process of regrouping is strongly connected with the fact that our system of writing numbers is a place-value system and has a base of 10. Place value and the base create the need for regrouping; the base determines the size of the groups we are obtaining.

Regrouping always uses the associative property of addition; often it also involves the commutative property of addition as well. Following are a few examples of regrouping. Can you give a reason for each step below?

Example 1.
$$\begin{aligned} 3 + 8 &= (1 + 2) + 8 && \text{(since } 3 = 1 + 2\text{)} \\ &= 1 + (2 + 8) \\ &= 1 + 10 && \text{(since } 2 + 8 = 10\text{)} \\ &= 11 \end{aligned}$$

Example 2.
$$\begin{aligned} 3 + 8 &= 3 + (7 + 1) \\ &= (3 + 7) + 1 \\ &= 10 + 1 \\ &= 11 \end{aligned}$$

Example 3.
$$\begin{aligned} 23 + 9 &= (20 + 3) + 9 \\ &= 20 + (3 + 9) \\ &= 20 + ([2 + 1] + 9) \\ &= 20 + (2 + [1 + 9]) \\ &= 20 + (2 + 10) \\ &= 20 + (10 + 2) \\ &= (20 + 10) + 2 \\ &= 30 + 2 \\ &= 32 \end{aligned}$$

Example 4.

$$\begin{array}{rl} 197 = & (100 + 90 + 7) \\ \underline{+348} = & \underline{(300 + 40 + 8)} \\ & 400 + 130 + 15 \\ = & (400 + 100) + (30 + 10) + 5 \\ = & 500 + 40 + 5 \\ = & 545 \end{array}$$

Example 5.

$$\begin{array}{rl} 197 = & 1 \cdot 100 + 9 \cdot 10 + 7 \\ \underline{+348} = & \underline{3 \cdot 100 + 4 \cdot 10 + 8} \\ & (1 + 3) \cdot 100 + (9 + 4) \cdot 10 + (7 + 8) \\ = & 4 \cdot 100 + (10 + 3) \cdot 10 + (15) \\ = & 4 \cdot 100 + [1 \cdot 100 + 3 \cdot 10] + [1 \cdot 10 + 5] \\ = & (4 + 1) \cdot 100 + 3 \cdot 10 + [1 \cdot 10 + 5] \\ = & 5 \cdot 100 + [3 + 1] \cdot 10 + 5 \\ = & 5 \cdot 100 + 4 \cdot 10 + 5 \\ = & 545 \end{array}$$

Example 6.

$$\begin{array}{rl} 197 = & 1 \cdot 10^2 + 9 \cdot 10 + 7 \\ \underline{+348} = & \underline{3 \cdot 10^2 + 4 \cdot 10 + 8} \\ & (1 + 3) \cdot 10^2 + (9 + 4) \cdot 10 + (7 + 8) \\ = & 4 \cdot 10^2 + (10 + 3) \cdot 10 + (10 + 5) \\ = & (4 \cdot 10^2 + 1 \cdot 10^2) + (3 \cdot 10 + 1 \cdot 10) + 5 \\ = & (4 + 1) \cdot 10^2 + (3 + 1) \cdot 10 + 5 \\ = & 5 \cdot 10^2 + 4 \cdot 10 + 5 \\ = & 545 \end{array}$$

Example 7.

$$\begin{array}{rlrl} 276 = & (200 + 70 + 6) = & & 200 + 60 + 16 \\ \underline{-138} = & \underline{-(100 + 30 + 8)} = & & \underline{-(100 + 30 + 8)} \\ & & & 100 + 30 + 8 \\ & & = & 138 \end{array}$$

2.4 ADDING IN DIFFERENT ORDERS

A common way to have pupils check their addition is to tell them that, if they added from the top down, they should check by adding from the bottom up. Here one is relying on the associative and commutative properties; it is because these properties hold that this check works.

Example:

$$\begin{array}{r} 3 \\ 2 \\ 4 \\ \underline{+7} \end{array}$$

In adding from the top down, we follow the procedure below:

$$\begin{aligned} 3 + 2 + 4 + 7 &= [(3 + 2) + 4] + 7 \\ &= [5 + 4] + 7 \\ &= 9 + 7 \\ &= 16 \end{aligned}$$

We can think of the procedure in adding from the bottom up in either of the two ways:

$$\begin{aligned} 3 + 2 + 4 + 7 &= 3 + [2 + (4 + 7)] \\ &= 3 + [2 + 11] \\ &= 3 + 13 \\ &= 16 \end{aligned}$$

or

$$\begin{aligned} 7 + 4 + 2 + 3 &= 7 + [4 + (2 + 3)] \\ &= 7 + [4 + 5] \\ &= 7 + 9 \\ &= 16. \end{aligned}$$

In all of these procedures, the use of the associative property is evident. In comparing the second of the "bottom up" procedures with the "top down" method, we see that we have used the commutative property by assuming that $3 + 2 + 4 + 7 = 7 + 4 + 2 + 3$.

2.5 ADDITION TABLES

If we study the addition table (see Figure 2), we can find all sorts of interesting patterns. If we pursue the reasons for the patterns, we can turn up many interrelationships among the numbers in the table. Some of these patterns are directly related to the properties we are examining. Suppose, for example, we fold the table on a line extending from the upper left corner to the lower right corner of the table. This line is called the "main", or "principal" diagonal, of the table. When you fold the table in this manner, each number above the diagonal is folded against a number below the diagonal. What do you notice about the two numbers? Why do you suppose this happens?

ADDITION TABLE

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 2

Let's look at one of these pairs of coinciding numbers. For example, 2 + 4 is 6. Does the 6 in the 2 + 4 position coincide with another 6? Does that 6 represent the sum 4 + 2? What property is illustrated here?

Do you see a row in the table that is identical with the row outside the table? Which row is it? Is there a column in the table that is identical to a column outside the table? Which column is it? What is the identity element for addition? How do this row and this column illustrate that there is an identity element for addition?

In teaching students the addition combinations, how could we use the commutative property of addition and the additive identity to cut down our work as we fill in the table?

When you studied the field properties, you may have looked at abstract operations expressed in tables. An example of this sort of thing is given in Figure 8.

$\circledcirc$	A	B	C
A	B	A	C
B	C	B	A
C	A	B	C

Figure 3

To read a table of this type, let us consider B $\circledcirc$ C. Locate B on the left; go across that row until we find the column headed C. The table entry is A. Thus B $\circledcirc$ C = A.

If you did study this sort of topic, you may have been told, "You can tell from looking at the operation table whether a set is closed or not. If there is an element in the interior of the table that is not in the outside row or outside column, the set is not closed with respect to that operation. If no such element exists, the set <u>is closed</u> with respect to that operation."

According to this statement, you might think, "Addition of whole numbers is obviously not closed. For example 7 + 5 = 12, and 12 is not in the outside row or column." This would be a perfectly natural reaction, but it is a false conclusion. The above statement is true whenever you have the <u>entire set</u> you are working with represented in the outside row or column. In our case, however, the set of whole numbers is an infinite set--it is simply not possible (even theoretically) to list <u>all</u> the whole numbers. Thus, our addition table is only a part of the complete addition table for whole numbers (and, in fact, we could not write a complete table). What we call the addition table for whole numbers is a table of the basic combinations; with these and the use of place value we can get any addition result we desire.

Whenever we cannot write a complete addition table for a set of numbers, the question of whether or not that set is closed under addition must be decided by referring to the definition of closure. This definition states that a set of numbers is closed under addition if (and only if), for any two numbers we select from the set, the sum is also in that set. Let us return to the case we have been considering, which is whether the set of whole numbers is closed under addition. Our experience tells us it is. We have not proved that it is, but we will accept, as a postulate, that the set of whole numbers is closed under addition.

Exercises:

Tell, in each of the following cases, whether or not the set is closed under the operation given.

(1) $\{0, 1, 2\}$, addition

(2) $\{0, 1\}$, multiplication

(3)

+	A	B	C	D
A	A	B	C	D
B	D	C	A	E
C	C	D	B	A
D	B	A	C	D

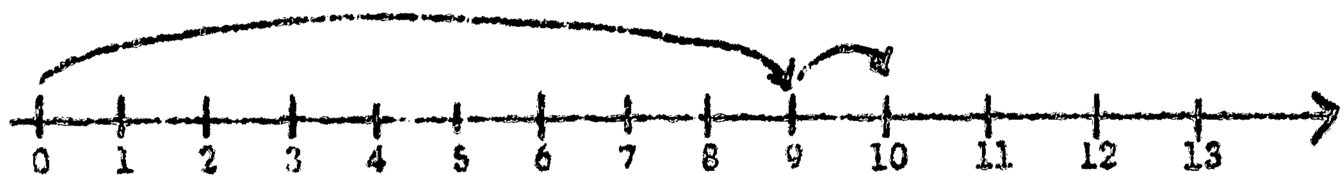
(4)

0	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5

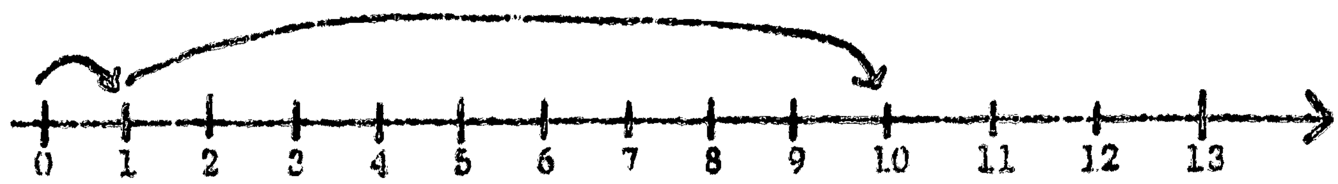
2.6 THE NUMBER LINE

The number line is an extremely useful graphic device at virtually all levels of elementary mathematics. It provides a clear picture of the order of the numbers with which one is working.

In the early grades, we can use the number line to motivate the commutative property of addition. For example, suppose we wish to get the sum $9 + 1$. We start at 0, jump 9 units to the right, then jump one more unit to the right, thus landing finally at 10. This is illustrated below:



In the same way, $1 + 9$ can be illustrated as follows:



We see that, in each case, we end up at the same spot, 10. Doing this sort of thing with various combinations will serve to make clear the concept that $a + b = b + a$.

2.7 A GENERAL REARRANGEMENT PROPERTY

It is extremely tedious and somewhat pointless for students to time and again go through all possible steps in an addition process (for example), justifying each step by the associative or commutative property. Because of this, it is common in elementary texts to develop the associative and commutative properties, and then to show that these allow us to use a general rearrangement principle, allowing us to shift the order and arrangement in a sum or product at will.

Example (Using associative and commutative properties):

$$
\begin{aligned}
35 + 27 &= (30 + 5) + (20 + 7) = [(30 + 5) + 20] + 7 \\
&= [30 + (5 + 20)] + 7 = [30 + (20 + 5)] + 7 \\
&= [(30 + 20) + 5] + 7 = (30 + 20) + (5 + 7) \\
&= 50 + 12 = 50 + (10 + 2) \\
&= (50 + 10) + 2 = 60 + 2 = 62
\end{aligned}
$$

Example (Using general rearrangement principle):

$$
\begin{aligned}
35 + 27 &= (30 + 5) + (20 + 7) = (30 + 20) + (5 + 7) \\
&= 50 + 12 = 50 + 10 + 2 = 60 + 2 = 62
\end{aligned}
$$

3. MULTIPLICATION OF WHOLE NUMBERS

3.1 THE MULTIPLICATION ALGORITHM

The algorithm which we use to find the answer to multiplication problems leans heavily on the distributive property. As we develop this algorithm in the elementary school, the distributive property is continually stressed.

To see that what we have been saying is actually the case, let us consider several examples. First, suppose we are multiplying a 3-digit number (683) by a 1-digit number (7).

(a) $683 \times 7 = (600 + 80 + 3) \times 7$ (because of the place-value numeration system we use)

$= (600 \times 7) + (80 \times 7) + (3 \times 7)$ (by an extended form of the distributive property)

$= 4200 + 560 + 21$ (using place value and multiplication facts)

$= 4781$ (by using addition principles)

(b) The person performing multiplication in the manner above is working a correct, but somewhat cumbersome, process. A step toward a more efficient notation is the following:

$$\left.\begin{array}{rcl} 683 & & \\ \times\ \ 7 & & \\ \hline 21 & \leftarrow & (7 \times 3) \\ 560 & \leftarrow & (7 \times 80) \\ 4200 & \leftarrow & (7 \times 600) \\ \hline 4781 & & \end{array}\right\} \text{ using the distributive property}$$

As another example of the use of the distributive property in multiplication, consider the following:

$$\begin{aligned} 375 \times 100 &= (300 + 70 + 5) \times 100 \\ &= (300 \times 100) + (70 \times 100) + (5 \times 100) \\ &= 30{,}000 + 7{,}000 + 500 \\ &= 37{,}500 \end{aligned}$$

A final example again illustrates the dependence of our multiplication algorithm upon the distributive property.

```
   248
 x 316
    48 ← (6 x 8)
   240 ← (6 x 40)
  1200 ← (6 x 200)
    80 ← (10 x 8)
   400 ← (10 x 40)
  2000 ← (10 x 200)
   240 ← (300 x 8)
 12000 ← (300 x 40)
 60000 ← (300 x 200)
 76208
```

3.2 MULTIPLYING BY MULTIPLES OF 10.

If a child has learned that multiplying by powers of 10 results in "adding zeros", he can develop some short-cuts to multiplication by multiples of 10. (Of course, when we use the term "adding zeros", we are indulging in arithmetical slang. Imprecise terms of this type are useful if introduced after the students understand the ideas involved. In multiplying by 10 or a multiple of 10, the actual result is that the digits of the multiplier are shifted into places designating higher powers of 10.) Suppose he faces the problem 42×20; this could conveniently be handled by either of the following procedures:

$$42 \times 20 = 42 \times (2 \times 10) = (42 \times 2) \times 10 = 84 \times 10 = 840$$

$$42 \times 20 = 42 \times (2 \times 10) = 42 \times (10 \times 2) = (42 \times 10) \times 2 = 420 \times 2 = 840.$$

In either case, we see that the associative property of multiplication is used. In the latter situation, we have also used the commutative property of multiplication.

The use of the associative property of multiplication is shown again in the following example:

$$\begin{aligned} 64 \times 600 &= 64 \times (6 \times 100) = (64 \times 6) \times 100 \\ &= [(60 + 4) \times 6] \times 100 = [(60 \times 6) + (4 \times 6)] \times 100 \\ &= (360 + 24) \times 100 = 384 \times 100 = 38,400. \end{aligned}$$

You will undoubtedly have noticed that we also used the distributive property when we chose to think of 64×6 as $(60 + 4) \times 6$ and then found 60×6 and 4×6. These are the sorts of procedures one begins to perform extremely rapidly if one practices mental calculation.

3.3 SPECIAL MULTIPLICATION ALGORITHMS

"Criss-cross", or "lightning", multiplication, is a mental short-cut to the multiplication process. It is illustrated below, and the arrows drawn clearly indicate the origin of the name "criss-cross".

$$\begin{array}{r} 3\ 2 \\ \underline{\times\ 2\ 1} \\ \underline{\underline{2}} \end{array} \qquad \begin{array}{r} 3\ 2 \\ \underline{\times\ 2\ 1} \\ \underline{\underline{7}}\ 2 \end{array} \qquad \begin{array}{r} 3\ 2 \\ \underline{\times\ 2\ 1} \\ \underline{\underline{6}}\ 7\ 2 \end{array}$$

Can you see how the distributive property is being used in this procedure?

Extending "criss-cross" multiplication to 3-digit situations makes the mental calculation a bit more complex. However, the technique works for 3 (or any number of) digits. A 3-digit example is given below.

$$\begin{array}{r} 2\ 4\ 8 \\ \underline{\times\ 3\ 2\ 7} \\ 5\underline{\underline{6}} \end{array} \qquad \begin{array}{r} 2\ 4\ 8 \\ \underline{\times\ 3\ 2\ 7} \\ 4\underline{\underline{9}}6 \end{array} \qquad \begin{array}{r} 2\ 4\ 8 \\ \underline{\times\ 3\ 2\ 7} \\ 5\underline{\underline{0}}96 \end{array} \qquad \begin{array}{r} 2\ 4\ 8 \\ \underline{\times\ 3\ 2\ 7} \\ 2\underline{\underline{1}}096 \end{array} \qquad \begin{array}{r} 2\ 4\ 8 \\ \underline{\times\ 3\ 2\ 7} \\ \underline{\underline{8}}1096 \end{array}$$

Notice that "carrying" into the place beyond the one in which we are working often occurs. We must make a mental note of the amount carried, and use that amount in the next step.

Naturally, the "criss-cross" method is not meant to be a staple in the mathematical diet of your students. It is properly an enrichment topic, which will probably appeal to some of your students, who may continue to use it. It is not an item to be taught for mastery.

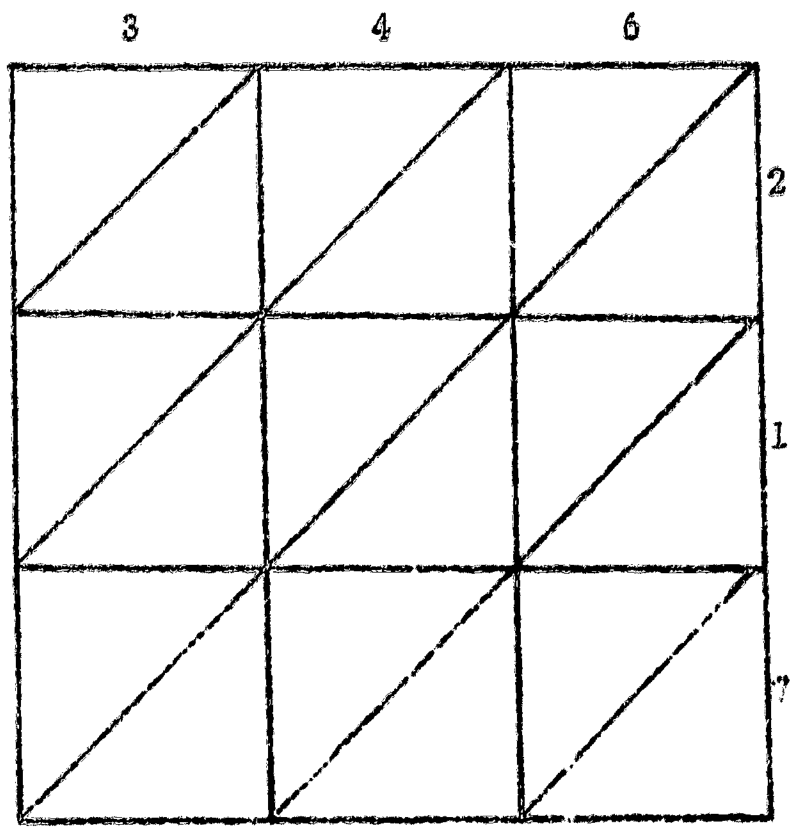
<u>Exercises</u>:

Work the following exercises, using "criss-cross" multiplication.

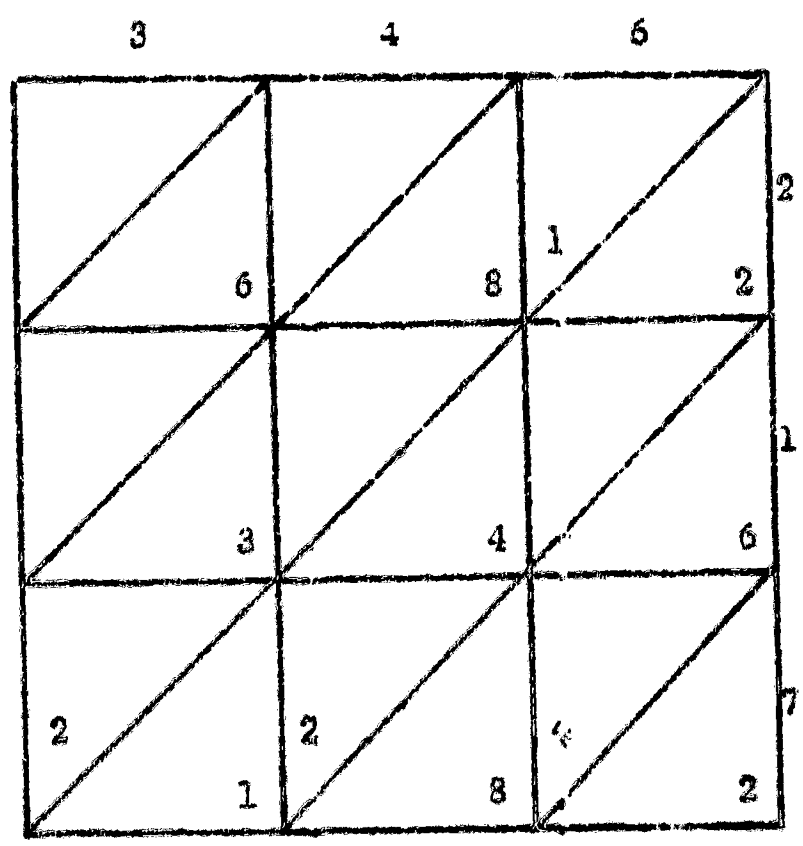
(1) 23 x 45 (2) 308 x 89 (3) 235 x 687

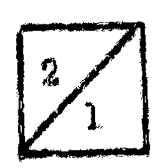
In the Middle Ages, a multiplication algorithm commonly used by the Arabs made use of a lattice diagram. This lattice consisted of a rectangle divided into squares; each square was further divided into 2 triangles by drawing the diagonal between the upper right and lower left corners of the square.

Suppose we wish to multiply 346 by 217, using the lattice method. We write the 346 above the lattice and the 217 to the right, as shown below.



The distributive property is used implicitly in obtaining the partial products, which are recorded in the interior of the lattice, as is shown below.

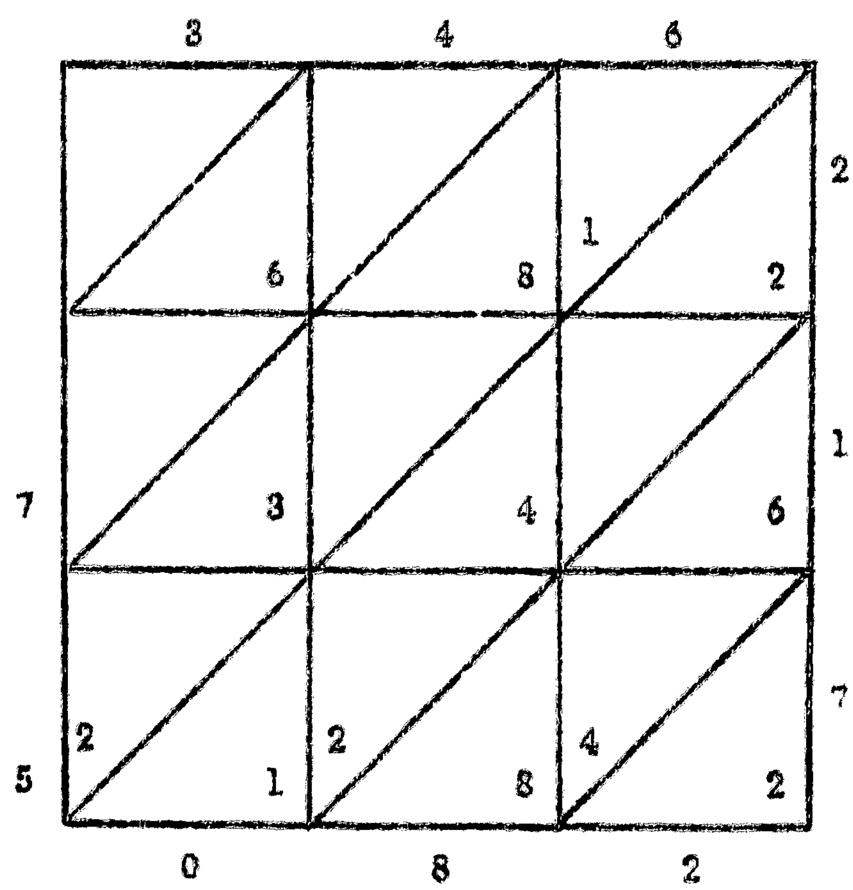


For example, the entry in the 3 column and the 7 row is 

, and 21 is the product of 3 and 7.

The answer is obtained by adding down the diagonals, with carrying, where necessary, into the next column. The role of the diagonals is to line up the units digits, tens digits, etc. For example, the diagonal starting beside the 1 in 217 has the entries 8, 4, and 6. This diagonal is the tens diagonal, and so we really have (8 + 4 + 6) x 10. The answer is written around the lattice at the base of the diagonals.

The final form of the problem is as follows:



Therefore, 346 x 217 = 75,082.

Just as with "criss-cross" multiplication, lattice multiplication is an enrichment topic--not to be taught for mastery. Lattice multiplication appears in many current elementary text series.

<u>Exercises</u>:

Work the following exercises, using lattice multiplication.

(1) 346 x 18 (2) 192 x 307 (3) 4153 x 6872

3.4 DOUBLING A PRODUCT

A common mistake made by students of all ages is their assumption that, to double a product, one must double each factor. A knowledge of the associative property of multiplication is sufficient to demonstrate that doubling one factor is all that is required. This is demonstrated in the following example:

$$2 \times (5 \times 9) = 2 \times 45 = 90$$

$$\begin{aligned} 2 \times (5 \times 9) &= (2 \times 5) \times 9 \quad \text{(b the associative property of multiplication)} \\ &= 10 \times 9 \\ &= 90 \end{aligned}$$

This could also be calculated as follows:

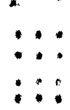
$$\begin{aligned} 2 \times (5 \times 9) &= 2 \times (9 \times 5) \quad \text{(by the commutative property of multiplication)} \\ &= (2 \times 9) \times 5 \quad \text{(by the associative property of multiplication)} \\ &= 18 \times 5 \\ &= 90 \end{aligned}$$

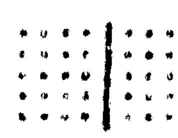
However, doubling both factors makes the product 4 times as great, as shown below:

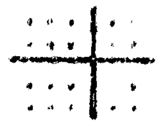
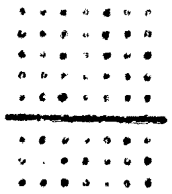
$$\begin{aligned} (2 \times 5) \times (2 \times 9) &= 10 \times 18 \\ &= 180. \end{aligned}$$

3.5 ARRAYS AND RECTANGULAR REGIONS

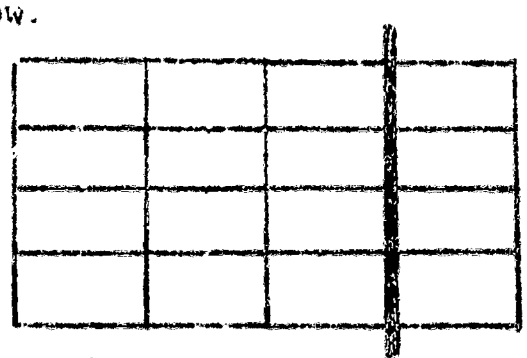
Arrays of objects and rectangular regions play important parts in the development of the concepts of multiplication and the distributive property.

At a very early stage in the child's school experience, arrays are used to motivate the multiplication of whole numbers. For example, the array  may be used to present the idea, 3×4. Merely rotating the array 90^0, so that it looks like , shows that we can also think of this as 4×3. This forms a good intuitive basis for accepting the commutative property of multiplication. Even without reorienting the array, we can see that it can be thought of either as 3 rows of 4 objects or as 4 columns of 3 objects each.

The partitioning of arrays is excellent moivation for the distributive property of multiplication over addition. For example,  shows that

5 x 7 = (5 x 4) + (5 x 3). Similarly,  demonstrates that 4 x 5 = (2 x 3) + (2 x 3) + (2 x 2) + (2 x 2) or 4 x 5 = (2 + 2)(3 + 2). In the same way,  demonstrates that 8 x 7 = (5 + 3) x 7 = (5 x 7) + (3 x 7).

In the same way, one can use rectangular regions which are partitioned. This is illustrated below.



4 x 4 = 4 x (3 + 1)

Of course, the distributive property is a two-way street. One may start with an expression like 3 x 6 and express it as (3 x 4) + (3 x 2), or one may express (3 x 4) + (3 x 2) in the form 3 x 6. This indicates that we might provide more understanding of the distributive property if we sometimes join two arrays rather than always partitioning them. This process is shown below.

 joined with  = 

(3 x 6) + (3 x 2) = (3 x 8)

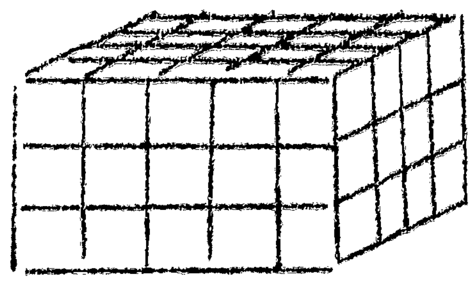
Arrays can also serve to motivate the idea that 1 serves as an identity element for multiplication. For example, clearly shows that 1 x 6 = 6; shows equally clearly that 4 x 1 = 4.

3.6 CONCRETE OBJECTS

Just as with addition and subtraction, multiplication is often motivated by using concrete objects. Actually, the array of dots and the rectangular regions used in the last section are semi-concrete objects--in using them, we have just stylized concrete objects for convenient representation on the printed page. Everything we did there could be done with concrete objects, and, if so done, would be more meaningful to some students.

One concrete device that we could use to motivate the associative property of multiplication is stacking rows of blocks in layers to form a rectangular solid. The associative principle comes into play when we wish to determine the total number of blocks in the solid.

Suppose we have a base consisting of 4 rows of 5 blocks each. Suppose also that we have built the solid 3 layers high. By examining the



box in different ways, we can show that we can find the number in one layer (4 x 5) and then multiply by the number of layers (3), or we can find the number of blocks on a "side" (3 x 5) and multiply by the number of rows (4), or we can find the number of blocks on an "end" (3 x 4) and multiply by the numbers of columns (5). That is,

$$3 \times (4 \times 5) = (3 \times 5) \times 4 = (3 \times 4) \times 5 = 3 \times 4 \times 5.$$

Notice that to get the expression (3 x 5) x 4, we have used both the commutative and the associative properties of multiplication.

3.7 MULTIPLICATION TABLES

If we examine the multiplication table, (see Figure 4 below) we can observe many of the same things that we saw when we examined the addition table.

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Figure 4

First, what happens when we fold the table along the main diagonal? Is the part of the table on one side of the diagonal a mirror image of the part on the other side? If so, what does this say about the multiplication of whole numbers? That is, which of the field properties does this illustrate?

Is there an identity element for multiplication? How did we tell for addition? Then, is there a row or column (or both) in the table identical to the row (or column) outside? What is the multiplicative identity?

Can we tell from the table whether or not the whole numbers are closed under multiplication? Support your answer with a reason.

4. DIVISION OF WHOLE NUMBERS

4.1 THE "STACKING" ALGORITHM

One feature common to most contemporary elementary mathematics programs is that they introduce students to the division process with the use of an algorithm which is appreciably different from the one we used when we learned division. This algorithm involves the stacking of partial quotients, either down the side of the problem or on the top. For this reason, this form of the division algorithm is often called the "stacking" algorithm. What gives us the right to write partial quotients and then add them up to get the final quotient? Basically, it is the same process which allows us to write partial products in multiplication and then add them up to get the final product. That is, the "stacking" algorithm makes use of the distributive property of multiplication over addition.

How can we use the distributive property of multiplication over addition when we are involved in division? First, we must remember that multiplication and division are inverse operations; thus, every division problem can be recast in multiplication form. For example, when we ask for the number x that results when we divide 1554 by 37 (in equation form, $1554 \div 37 = x$), we are looking for the number x which, when multiplied by 37, equals 1554 (in equation form, $37 \cdot x = 1554$).

If we solve for x, we find out the value of x in this case is 42. By using the distributive property, we can write the equation $37 \times 42 = 1554$ as $37 \times (40 + 2) = 1554$, or even $37 \times (10 + 10 + 10 + 10 + 2) = 1554$.

Now let us apply this to the "stacking" algorithm. Our first step to finding the quotient is to select a multiple of 37 that is less than (or possibly equal to) 1554. Often we use multiples and powers of 10 to help us. A good estimator might see that forty 37's is less than 1554 but fifty 37's is more than 1554. He would then see that the difference between 1554 and forty 37's is 74, or two 37's. Hence for him the "stacking" would look like this:

37 / 1554	
1480	40 ← (40 x 37)
74	
74	2 ← (2 x 37)
0	42

Therefore he has actually followed the procedure $37 \times 42 = 37 \times (40 + 2) = (37 \times 40) + (37 \times 2) = 1480 + 74$.

A less able estimator might perform the same division as follows:

37 / 1554	
370	10
1184	
370	10
814	
370	10
444	
370	10
74	
37	1
37	
37	1
0	42

This person has implicitly used the distributive property also. His process has been:

$$
\begin{aligned}
37 \times 42 &= 37 \times (10 + 10 + 10 + 10 + 1 + 1) \\
&= (37 \times 10) + (37 \times 10) + (37 \times 10) + (37 \times 10) + (37 \times 1) + (37 \times 1) \\
&= 370 + 370 + 370 + 370 + 37 + 37 \\
&= 1554
\end{aligned}
$$

Another example of the "stacking" algorithm is the following:

$$
\begin{array}{r|r|r}
38 \,/ & 6914 & \\
 & \underline{3800} & 100 \\
 & 1114 & \\
 & \underline{760} & 20 \\
 & 354 & \\
 & \underline{190} & 5 \\
 & 144 & \\
 & \underline{114} & \underline{3} \\
 & 30 & 128
\end{array}
$$

That is,
$$
\begin{aligned}
6914 &= (100 \times 38) + (20 \times 38) + (5 \times 38) \\
&\quad + (3 \times 38) + 30 \\
&= (128 \times 38) + 30
\end{aligned}
$$

Here, we see that our quotient is not an exact multiple of the divisor; there is a remainder. But, if we express the situation in the form $6914 - 30 = (100 \times 38) + (20 \times 38) + (5 \times 38) + (3 \times 38)$, we see that we are still, in reality, using the distributive law to help us.

In some elementary materials, the "stacking" algorithm is written in a different way--the "stacking" is done on the top. This has the advantage that the conversion to the usual method of writing quotients is easier. Its primary disadvantage is that it is difficult for the student to judge the amount of space he will need to allow above the problem.

The following is an example of "stacking" on the top.

```
        186
        ---
          1
         10
         25
         50
        100
26 / 4837          Thus, 4837 ÷ 26 = 186 R 1
     2600
     ----
     2237
     1300
     ----
      937
      650
      ---
      287
      260
      ---
       27
       26
       --
        1
```

4.2 DISTRIBUTIVITY OF DIVISION OVER ADDITION

We have often referred to "the distributive property"; that is, the distributive property of multiplication over addition. We call this property <u>the</u> distributive property not because it is the only one that exists, but because in our usual work it is the only one that we use. However, there is also a distributive property of division over addition, which is demonstrated in the following example:

$$56 \div 8 = (48 + 8) \div 8 = (48 \div 8) + (8 \div 8) = 6 + 1 = 7.$$

Symbolically, we can state the distributive property of division over addition as follows:

$$(a + b) \div c = (a \div c) + (b \div c).$$

In dealing with the distributive property of multiplication over addition, we know and use the fact that

$$c \times (a + b) = (c \times a) + (c \times b)$$

and also the fact that $(a + b) \times c = (a \times c) + (b \times c)$. Thus, in effect we are actually using <u>two</u> distributive laws; they are connected by the fact that multiplication is commutative.

Is division commutative? This is easily checked by trying an example. $2 \div 3$ is not the same as $3 \div 2$. Therefore, division is not commutative.

We have just seen that $(a + b) \div c = (a \div c) + (b \div c)$. Is there a second distributive law of division over addition? That is, is it true that $c \div (a + b) = (c \div a) + (c \div b)$? Let's try an example. If we let $c = 18$, $a = 6$, and $b = 3$, $c \div (a + b) = 18 \div (6 + 3) = 18 \div 9 = 2$, and $(c \div a) + (c \div b) = (18 \div 6) + (18 \div 3) = 3 + 6 = 9$. In this case $c \div (a + b) \neq (c \div a) + (c \div b)$.

Should we try other cases? What would be gained? We have seen already that it is <u>not</u> true for all values of a, b, c, that $c \div (a + b) = (c \div a) + (c \div b)$. Therefore, examining more cases would add nothing. We might, indeed, find special values of a, b, c, where the relation would be true. However, the one example we have used shows that it is sometimes false, and thus cannot be a general property of whole numbers.

Finding an example to show that something is <u>not</u> true is known as finding a counter-example. This is an important technique in mathematics. It is extremely easy to prove something false if one can find a counter-example. It is not possible to prove something true by citing examples, except in cases where we can examine <u>every possible example</u>. Most of the sets we work with are either infinite, in which case it is not possible to examine all cases, or they are large finite sets, where it would be highly impractical to look at every possible case; thus, we are not normally able to prove anything by citing 1, 10, 100, or 1000 cases.

Consider the addition of two even numbers. By examining several cases, such as $4 + 8 = 12$, $2 + 126 = 128$, etc., we will quickly be led to the conviction that the sum of two even numbers is always even. However, no number of cases would prove this conjecture. A proof by direct means is quite easy here. Let 2n be one even number, and let 2m be another. Then $2n + 2m = 2(n + m)$ by the distributive property. Since m and n were whole numbers, by the closure property of addition their sum is a whole number, and then by definition of even number $2(n + m)$ is even.

The alert reader may have wondered about subtraction. Do there exist distributive laws of multiplication over subtraction or of division over subtraction? This is a worthwhile idea to pursue--and we will leave it for the interested reader.

4.3 THE DISTRIBUTIVE PROPERTY AND THE EUCLIDEAN ALGORITHM

As teachers, you may have seen the technique for finding the greatest common divisor of two numbers by a sequence of divisions. This procedure is known as the Euclidean Algorithm. The basis for the procedure is an application of the distributive law of division over addition.

The Euclidean Algorithm is illustrated by the following example.

To find the greatest common divisor of 345 and 2348, we start by writing

$$2348 = 6 \times 345 + 278.$$

That is, we have divided 345 into 2348, getting a quotient of 6 and a remainder of 278. We are looking for the greatest common divisor of 345 and 2348, and now conclude that the number for which we are searching also divides 278. Thus, we look for the greatest common divisor of 278 and 345.

$$345 = 1 \times 278 + 67.$$

Continuing in the same way,

$$\begin{aligned} 278 &= 4 \times 67 + 10 \\ 67 &= 6 \times 10 + 7 \\ 10 &= 1 \times 7 + 3 \\ 7 &= 2 \times 3 + 1 \\ 3 &= 3 \times 1 + 0 \end{aligned}$$

Thus, 1 is the greatest common divisor of 345 and 2348.

How is the distributive property involved? It was involved when we said that if the divisor divided 345 and 2348, it also divided 278. That is, if $a = b \times q + r$, and a number d divides a and also divides q, it must divide r. This is true since $r = a - (b \times q)$. Symbolically, we can write a divides b as $a \mid b$. Thus, what we have said is: If $a \mid b$ and $a \mid c$, and $b \geq c$, then $a \mid (b - c)$, for whole numbers a, b, and c.

4.4 DIVIDING A PRODUCT BY TWO

It is extremely common to find that students, dividing a product by two, attempt to divide both of its factors by two. A little use of the associative property of multiplication can show them that their process is wrong. For example,

$$\begin{aligned}(7 \times 8) \div 2 &= 7 \times (4 \times 2) \div 2 \\ &= (7 \times 4) \times 2 \div 2 && \text{(by the associative property of multiplication)} \\ &= 7 \times 4 && (2 \div 2 \text{ is } 1) \\ &= 28\end{aligned}$$

and, since $7 \times 8 = 56$ and 28 is $56 \div 2$, only one factor need be divided by two to cut a product in half.

5. COMPARING AND CONTRASTING ADDITION WITH SUBTRACTION AND MULTIPLICATION WITH DIVISION

You are aware that, with our four major operations, we have two pairs of inverse operations. Addition and subtraction are inverse operations, as are multiplication and division. That is, subtracting 5 is the inverse of adding 5 (subtraction of a number "undoes" the adding of that number), and dividing by 6 is the inverse of multiplying by 6.

Since we do have these pairs of inverse operations, it is only natural to compare and contrast the operations--to find similarities and differences in the ways that they work.

In the first place, we see that closure of addition and multiplication does not guarantee closure of their respective inverse operations. Examples of this are easy to illustrate; $2 - 3 = \square$ and $4 \div 5 = \square$ have no solutions in the set of whole numbers.

In the second place, we find that subtraction and division are not commutative, as are addition and multiplication. For example, $6 - 2 \neq 2 - 6$, and $6 \div 2 \neq 2 \div 6$.

In the same manner, we see that subtraction and division do not satisfy the associative property. For an example of this, let us consider the following:

$(6 - 2) - 1 = 4 - 1 = 3$ but $6 - (2 - 1) = 6 - 1 = 5$

$(8 \div 4) \div 2 = 2 \div 2 = 1$ but $8 \div (4 \div 2) = 8 \div 2 = 4$

We have a bit better luck with the identity elements. If we subtract 0 from a whole number, we get as a result the whole number with which we started. That is, $a - 0 = a$ for all whole numbers a. However, subtracting a number from 0 does <u>not</u> give us the number; in fact, we have no solution for this problem in the set of whole numbers. This is a bit different from the way 0 works in addition; $0 + a = a + 0 = a$. Therefore, we usually say that 0 is a <u>right</u> identity for subtraction; it acts as an identity when written on the right, but not when written on the left.

Is there an identity element for division? That is, is there a number x such that $a \div x = a$ and $x \div a = a$? We see that the situation is similar to subtraction; $a \div 1 = a$ for all whole numbers a, but there is no unique whole number x such that $x \div a = a$ for all a. Hence, we say that 1 is a <u>right</u> identity for division.

You may have noticed that we have not discussed additive or multiplicative inverses. This is because we do not have inverses in the set of whole numbers. In order to get additive inverses, we would have to extend our set to include negative integers; in order to get multiplicative inverses, we would have to include the fractional numbers. In order to get a field, we would need to add the additive and multiplicative inverses to the properties we have already; thus, we would need to extend our set to include the rational numbers.

6. RENAMING FRACTIONAL AND MIXED NUMBERS

Moving from the set of whole numbers to the set of fractional numbers, we see an immediate difference. Whereas, in our Hindu-Arabic base 10 system, we have one standard form for representing a whole number, we have a variety of standard names for a fractional number. For example, 342 + 178 would usually be expressed in the standard form 520, but the fractional number named by $\frac{2}{3}$ could equally well be named by $\frac{4}{6}$, $\frac{22}{33}$, or $\frac{844}{1266}$. We call the symbols by which we represent fractional numbers <u>fractions</u>. Fractions that name the same fractional number are spoken of as <u>equivalent fractions</u>. Of course, we do in some sense have a single standard form for fractions; the "lowest-term" form (when the numerator and denominator of the fractions have no common factor except 1.) Consider $\frac{14}{18}$, which equals $\frac{2 \times 7}{2 \times 9}$. The factors of 14 are 2 and 7; the factors of 18 are 2 and 9. The common factor is 2.

Any time we wish to select a fraction which is equivalent to a particular fraction we are working with, we are using the property of 1 as multiplicative identity, where 1 is written in the form $\frac{a}{a}$ for some particular counting number a. To illustrate this, let's consider a couple of examples. Suppose we have $\frac{3}{4}$ but need an equivalent fraction with denominator 16. $\frac{3}{4} \times \frac{4}{4} = \frac{12}{16}$ is the process we use. Actually we are thinking "$\frac{3}{4} \times 1 = \frac{3}{4}$; I need to multiply 4 by 4 to get 16 as my denominator; 1 may be written as $\frac{4}{4}$; $\frac{3}{4} \times \frac{4}{4} = \frac{12}{16}$." Another example: Suppose we wish to reduce $\frac{54}{60}$ to lowest terms. We can follow the procedure $\frac{54}{60} = \frac{9 \times 6}{10 \times 6} = \frac{9}{10} \times \frac{6}{6} = \frac{9}{10} \times 1 = \frac{9}{10}$.

A special case of renaming a fractional number is found in the upper grades, when percent is introduced. Of course, when a percent is written in fractional form, it is simply a fraction with denominator 100. Thus the problem $\frac{a}{b} = \frac{x}{100}$ involves multiplying $\frac{a}{b}$ by 1 in the form $\frac{100 \div b}{100 \div b}$. That is, $\frac{3}{5} = \frac{x}{100}$ is solved by multiplying $\frac{3}{5}$ by $\frac{20}{20}$ (20 is $100 \div 5$), and x turns out to be 60.

Elementary school children are often asked to write "equivalence rows" for fractional numbers. This is a row of equivalent fractions, such as $\frac{2}{7} = \frac{4}{14} = \frac{6}{21} = \frac{8}{28} = \frac{10}{35} = \frac{12}{42} = \frac{14}{49} = \frac{16}{56}$. The students are using 1 repeatedly as the multiplicative identity, first as $\frac{1}{1}$, then as $\frac{2}{2}$, then $\frac{3}{3}$, then $\frac{4}{4}$, $\frac{5}{5}$, $\frac{6}{6}$, etc.

For example $\frac{2}{7} = \frac{2}{7} \times 1 = \frac{2}{7} \times \frac{2}{2} = \frac{4}{14}$; $\frac{2}{7} = \frac{2}{7} \times 1 = \frac{2}{7} \times \frac{3}{3} = \frac{6}{21}$; etc. Of course, when students begin to write equivalence rows, at the start of their work with fractional numbers, they are not thinking in terms of the multiplicative identity. At that point multiplication of fractional numbers has not yet been defined. Instead [illegible]

[illegible]

The [illegible] mixed [illegible] improper fractions, and the reverse process, also involve 1 as multiplicative identity.

Examples: $5\frac{1}{3} = 5 + \frac{1}{3} = (5 \times \frac{3}{3}) + \frac{1}{3} = \frac{15}{3} + \frac{1}{3} = \frac{16}{3}$

$$\frac{12}{9} = \frac{9+3}{9} = \frac{9}{9} + \frac{3}{9} = 1 + \frac{1 \times 3}{3 \times 3} = 1 + (\frac{1}{3} \times \frac{3}{3}) = 1 + (\frac{1}{3} \times 1) = 1 + \frac{1}{3}$$

$$\frac{32}{7} = \frac{28+5}{7} = \frac{28}{7} + \frac{5}{7} = \frac{4 \times 7}{7} + \frac{5}{7} = (4 \times \frac{7}{7}) + \frac{5}{7} = (4 \times 1) + \frac{5}{7} = 4 + \frac{5}{7}$$

7. ADDITION AND SUBTRACTION OF FRACTIONAL NUMBERS

As fractional numbers are developed in some elementary programs, $\frac{1}{b}$ is defined to be the number such that $b \cdot \frac{1}{b} = 1$. Then $\frac{a}{b}$ is defined to be $a \cdot \frac{1}{b}$. The algorithm for adding fractional numbers proceeds as follows:

$\frac{a}{b} + \frac{c}{b} = a \times \frac{1}{b} + c \times \frac{1}{b}$ (definition)

$= (a + c) \times \frac{1}{b}$ (distributive property--which we have to assume for fractional numbers if we follow this approach)

$= \frac{a+c}{b}$ (definition)

If we wish to add or subtract fractions with unlike denominators, the development would take the following tack:

$\frac{a}{b} - \frac{c}{d} = (\frac{a}{b} \times \frac{d}{d}) - (\frac{c}{d} \times \frac{b}{b})$ (1 as multiplicative identity, used in the form $\frac{b}{b}$ and $\frac{d}{d}$, which were chosen in order to make denominator the same.)

$= \frac{ad}{bd} - \frac{cb}{db}$

$= \frac{ad}{bd} - \frac{bc}{bd}$ (commutative property of multiplication)

$= (ad \times \frac{1}{bd}) - (bc \times \frac{1}{bd})$

$= (ad - bc) \times \frac{1}{bd}$ (distributive property of multiplication over subtract

$= \frac{ad - bc}{bd}$

Let's consider some numerical examples.

Example 1. $\frac{3}{7} + \frac{2}{5} = (\frac{3}{7} \times \frac{5}{5}) + (\frac{2}{5} \times \frac{7}{7})$

$$= \frac{15}{35} + \frac{14}{35}$$

$$= \frac{29}{35}$$

Example 2. $\frac{3}{8} - \frac{1}{16} = (\frac{3}{8} \times \frac{16}{16}) - (\frac{1}{16} \times \frac{8}{8})$

$$= \frac{48}{128} - \frac{8}{128}$$

$$= \frac{40}{128}$$

$$= \frac{5}{16}$$

This latter example was worked correctly, but the process could have been shortened by noticing that the least common denominator of 8 and 16 is 16. Thus, the example could have been worked as follows:

$$\frac{3}{8} - \frac{1}{16} = (\frac{3}{8} \times \frac{2}{2}) - \frac{1}{16}$$

$$= \frac{6}{16} - \frac{1}{16}$$

$$= \frac{5}{16}$$

Another example which shows the value (in terms of cutting down the number of steps and the complexity of computation) of using the least common denominator is the following:

$\frac{4}{12} + \frac{3}{16} = (\frac{4}{12} \times \frac{4}{4}) + (\frac{3}{16} \times \frac{3}{3})$ (48 is the least common denominator)

$$= \frac{16}{48} + \frac{9}{48}$$

$$= \frac{25}{48}$$

Once the addition algorithm is established for fractional numbers, it is possible to find many occasions where one would use the commutative and associative properties of addition. One such example is:

$$\frac{3}{4} + \frac{1}{4} + \frac{2}{4} = (\frac{3}{4} + \frac{1}{4}) + \frac{2}{4} = \frac{4}{4} + \frac{2}{4} = \frac{6}{4}.$$

Adding or subtracting mixed numbers can also involve either the associative property of addition or the commutative property of addition or both. For example:

$$3\frac{1}{4} + 5\frac{2}{3} = (3 + \frac{1}{4}) + (5 + \frac{2}{3})$$

$$= (3 + 5) + (\frac{1}{4} + \frac{2}{3})$$ (using both the associative and the commutative properties of addition)

$$= 8 + (\frac{3}{12} + \frac{8}{12})$$ (using 1 as the multiplicative identity)

$$= 8 + \frac{11}{12}$$

$$= 8\frac{11}{12}$$

8. MULTIPLICATION OF FRACTIONAL NUMBERS

As with addition, multiplication of fractional numbers provides many opportunities to use the associative and commutative properties. Most of these are quite similar to the ways already discussed for whole numbers, so there would seem to be no need for further elaboration. There are, however, a few uses of field properties in multiplication of fractional numbers which are different from those in whole-number arithmetic. Consider $\frac{3}{4} \times \frac{28}{45}$. When we teach the "cancelling" procedure, as in

$$\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{7}{\cancel{28}}}{\underset{15}{\cancel{45}}} = \frac{1 \times 7}{1 \times 15} = \frac{7}{15},$$

we are really developing a shortcut which uses the multiplicative identity, the associative property of multiplication, and the commutative property of multiplication all in one bundle.

Let us go through the above problem as we might if we wished to emphasize the properties that we have used.

$$\frac{3}{4} \times \frac{28}{45} = \frac{3 \times 28}{4 \times 45} = \frac{3 \times (4 \times 7)}{4 \times (3 \times 15)} = \frac{(3 \times 4) \times 7}{(4 \times 3) \times 15} = \frac{(3 \times 4) \times 7}{(3 \times 4) \times 15} = (\frac{3}{3} \times \frac{4}{4}) \times \frac{7}{15}$$

$$= (1 \times 1) \times \frac{7}{15} = 1 \times \frac{7}{15} = \frac{7}{15}$$

Exercise: In each of the steps in the process above, identify the property or properties used.

Suppose we wish to multiply a whole number times a "mixed" number, such as $6 \times 2\frac{1}{3}$. Of course, we could convert the $2\frac{1}{3}$ to $\frac{7}{3}$, in which case we have involved the multiplicative identity $(2 = 2 \times 1 = \frac{2}{1} \times \frac{3}{3} = \frac{6}{3})$. Then our multiplication can also use the "cancelling" procedure: $\overset{2}{\cancel{6}} \times \frac{7}{\underset{1}{\cancel{3}}} = 14$. However, we could have written $2\frac{1}{3}$ as $2 + \frac{1}{3}$ (or mentally thought of it that way). Then, we could use the distributive property in the following manner:

$$6 \times 2\frac{1}{3} = 6 \times (2 + \frac{1}{3}) = (6 \times 2) + (6 \times \frac{1}{3}) = 12 + 2 = 14.$$

Multiplication of two mixed numbers is virtually the same as the above situation. Many people would handle the problem $3\frac{3}{8} \times 2\frac{4}{7}$ by converting each mixed number to a fraction, $\frac{27}{8} \times \frac{18}{7}$, and then multiplying in the usual way: $\frac{27}{8} \times \frac{18}{7} = \frac{27}{4} \times \frac{9}{7} = \frac{243}{28} = 8\frac{19}{28}$. However, the same problem could be approached by a double use of the distributive property, and is so presented in some materials. This approach is illustrated below:

$$
\begin{aligned}
3\frac{3}{8} \times 2\frac{4}{7} &= (3 + \frac{3}{8}) \times (2 + \frac{4}{7}) = (3 + \frac{3}{8}) \times 2 + (3 + \frac{3}{8}) \times \frac{4}{7} \\
&= (3 \times 2) + (\frac{3}{8} \times 2) + (3 \times \frac{4}{7}) + (\frac{3}{8} \times \frac{4}{7}) \\
&= 6 + \frac{3}{4} + \frac{12}{7} + \frac{3}{14} \\
&= 6 + (\frac{3}{4} \times \frac{7}{7}) + 1 + (\frac{5}{7} \times \frac{4}{4}) + (\frac{3}{14} \times \frac{2}{2}) \\
&= 6 + \frac{21}{28} + 1 + \frac{20}{28} + \frac{6}{28} \\
&= 7 + \frac{47}{28} \\
&= 7 + (1 + \frac{19}{28}) \\
&= 8 + \frac{19}{28} \\
&= 8\frac{19}{28}
\end{aligned}
$$

This use of the distributive property twice is exactly the same as the use of the distributive property in algebra, as in the example below.

$$\begin{aligned}(a + b) \times (c + d) &= (a + b) \times c + (a + b) \times d \\ &= (a \times c) + (b \times c) + (a \times d) + (b \times d) \\ &= ac + bc + ad + bd\end{aligned}$$

9. DIVISION OF FRACTIONAL NUMBERS

9.1 THE RECIPROCAL OF A NUMBER

One of the most convenient features of the set of fractional numbers is that for each non-zero number $\frac{a}{b}$, there is a number, $\frac{b}{a}$, such that the product of the two numbers is 1. For a binary (2-element) operation, when there are 2 elements which, when operated upon, produce the identity, these two elements are called inverses, and each is said to be the inverse of the other. In our case, the operation is multiplication, and $\frac{b}{a}$ is said to be the multiplicative inverse of $\frac{a}{b}$. However, we often avoid the longer name "multiplicative inverse" by using the equivalent term "reciprocal".

Exercises: Find the reciprocals of the following numbers:

$3,\ 7,\ \frac{1}{7},\ 5,\ \frac{3}{5},\ \frac{8}{3},\ 1\frac{1}{16},\ \frac{14}{32},\ 1 + 1$

The reciprocal is extremely useful in most methods of developing the division algorithm for fractional numbers, as we shall see in the next few sections.

9.2 REINTERPRETING DIVISION AS MULTIPLICATION

A common device for developing the usual rule for division of fractions is to reinterpret the division problem as a multiplication problem. Thus the division problem $\frac{3}{7} \div \frac{4}{5} = \square$ means $\frac{4}{5} \times \square = \frac{3}{7}$. Assuming the students have had previous experience with number sentences, they will wish to perform operations which leave us with $\square$ on the left, and an expression on which they know how to operate on the right. Multiplying both sides of the equation by the reciprocal of $\frac{4}{5}$ (i.e., by $\frac{5}{4}$) results in the desired solution. This is demonstrated below:

$\frac{4}{5} \times \square = \frac{3}{7}$

$\frac{5}{4} \times (\frac{4}{5} \times \square) = \frac{5}{4} \times \frac{3}{7}$ (multiplying both sides by the reciprocal of $\frac{4}{5}$)

$(\frac{5}{4} \times \frac{4}{5}) \times \square = \frac{5}{4} \times \frac{3}{7}$ (the associative property of multiplication)

$1 \times \square = \frac{5}{4} \times \frac{3}{7}$ (the property of reciprocals; i.e., the product of reciprocals is 1)

$\square = \frac{5}{4} \times \frac{3}{7}$ (1 is the multiplicative identity)

$\square = \frac{15}{28}$ (algorithm for multiplying fractional numbers)

The only problem with the procedure as written, which rapidly leads to the "multiply by the reciprocal of the divisor" technique, is that the reciprocal of the divisor has been written to the left of the dividend. Since the multiplication of fractional numbers is commutative, we could just as well have written the right-hand side as $\frac{3}{7} \times \frac{5}{4}$, without any particular comment.

Another related technique is actually to fill in the box in the equation. We think in the following manner: "I want the left-hand side to be $\frac{3}{7}$, since it is to be equal to the right-hand side. Therefore, the number I write in the box must be the number which will result in $\frac{3}{7}$ when multiplied by $\frac{4}{5}$. I know $\frac{4}{5} \times \frac{5}{4} = 1$, and I know that $1 \times \frac{3}{7} = \frac{3}{7}$. Therefore I will fill the box with $\frac{5}{4} \times \frac{3}{7}$." Symbolically, this is:

$$\frac{4}{5} \times \square = \frac{3}{7}$$

$$\frac{4}{5} \times \boxed{\frac{5}{4} \times \frac{3}{7}} = \frac{3}{7}$$

$$\frac{4}{5} \times \boxed{\frac{15}{28}} = \frac{3}{7}$$

In interpreting division in terms of multiplication, we have acknowledged that in a sense division is not a basic operation—that all division problems involve situations which may be interpreted as multiplication situations where a factor is missing. Addition and subtraction are connected to each other in exactly the same way. Every subtraction problem can be interpreted as an addition [illegible] a missing addend. In Section [illegible] we discussed the relationship between addition and subtraction, and between multiplication and division, referring to these pairs as *inverse operations*.

The parallelism between these two pairs of inverse operations is shown below:

DEFINITION OF SUBTRACTION

$a - b = \square$ means $b + \square = a$

DEFINITION OF DIVISION

$a \div b = \square$ means $b \times \square = a$

When we look at the subtraction of integers (Section II 2), we will reinterpret subtraction in terms of addition, just as we have reinterpreted division in terms of multiplication in this section.

[illegible] THE COMPLEX FRACTION TECHNIQUE

Another technique for approaching the division of fractional numbers is to extend the definition of fractions to include expressions of the form $\frac{\frac{a}{b}}{\frac{c}{d}}$, where a, b, c, and d are whole numbers and b, c, d are not equal to 0. Such an expression is known as a complex fraction.

When we are faced with a division problem of fractional numbers, we can write the problem in complex fraction form and then use the property of 1 as multiplicative identity, as follows:

$$\frac{3}{5} \div \frac{1}{4} = \frac{\frac{3}{5}}{\frac{1}{4}} = \frac{\frac{3}{5}}{\frac{1}{4}} \times \frac{\frac{4}{1}}{\frac{4}{1}} = \frac{\frac{3}{5} \times 4}{\frac{1}{4} \times 4} = \frac{\frac{[illegible]}{20}}{1} = \frac{[illegible]}{20}$$

Notice that in this case we chose the form of 1, $\left(\frac{\frac{4}{1}}{\frac{4}{1}}\right)$, so that we obtained a denominator 1; that is, we used the reciprocal of the denominator (divisor). The same problem could have been solved without relying on the reciprocal, as is shown below.

$$\frac{3}{5} \div \frac{4}{7} = \frac{\frac{3}{5}}{\frac{4}{7}} \times \frac{35}{35} = \frac{\frac{3}{5} \times 35}{\frac{4}{7} \times 35} = \frac{21}{20}$$

Here we chose to write 1 in the form $\frac{35}{35}$, because 35 is a multiple of both 7 and 5, the denominators of the fractions involved. As a matter of fact 35 is the least common multiple of 7 and 5, but it is not necessary to select the least common multiple. <u>Any</u> common multiple will work. For example, we could use 105 in the above problem, as follows:

$$\frac{\frac{3}{5}}{\frac{4}{7}} \times \frac{105}{105} = \frac{\frac{3}{5} \times 105}{\frac{4}{7} \times 105} = \frac{3 \times 21}{4 \times 15} = \frac{3 \times (3 \times 7)}{4 \times (3 \times 5)} = \frac{3 \times 7}{4 \times 5} = \frac{21}{20}$$

9.4 THE COMMON DENOMINATOR TECHNIQUE

A less widely used, but convenient, technique for dividing fractional numbers, is one in which one: (a) gets common denominators for each fraction; (b) divides the numerator of the dividend by the numerator of the divisor; (c) divides the denominator of the dividend by the denominator of the divisor. This procedure is illustrated below.

$$\frac{4}{7} \div \frac{2}{3} = (\frac{4}{7} \times \frac{3}{3}) \div (\frac{2}{3} \times \frac{7}{7})$$ (use of the multiplicative identity)

$$= \frac{12}{21} \div \frac{14}{21}$$

$$= \frac{12 \div 14}{21 \div 21}$$

$$= \frac{12 \div 14}{1}$$ (1 as the multiplicative identity)

$$= 12 \div 14$$

$$= 6 \div 7$$ (1 is used in the form $2 \div 2$)

$$= \frac{6}{7}$$

In the example above, you may have been bothered by the step $\frac{12}{21} \div \frac{14}{21} = \frac{12 \div 14}{21 \div 21}$. What justification is there for this step? We justify this below.

Let $\frac{12}{21} \div \frac{14}{21} = \frac{\square}{\triangle}$.

Then, by the definition of division in terms of multiplication,

$$\frac{12}{21} = \frac{14}{21} \times \frac{\square}{\triangle}$$

By our algorithm for multiplication of fractions, $\frac{12}{21} = \frac{14 \times \square}{21 \times \triangle}$. Therefore, $12 = 14 \times \square$ and $21 = 21 \times \triangle$. Again, by the definition of division, $12 \div 14 = \square$ and $21 \div 21 = \triangle$. Therefore, substituting in the original equation for $\square$ and $\triangle$ gives

$$\frac{12}{21} \div \frac{14}{21} = \frac{12 \div 14}{21 \div 21}$$

This procedure does not explicitly involve the reciprocal, but leads toward it. Actually, there is a sense in which the reciprocal has been involved. The numerators before getting common denominators were 4 and 2. After we obtained common denominators, they were 12 and 14. If we examine $12 \div 14$ (or $\frac{12}{14}$), we see that $\frac{12}{14} = \frac{4}{7} \times \frac{3}{2}$. That is, the <u>dividend</u> has actually been multiplied by the reciprocal of the divisor, but in rather disguised form.

10. FRACTIONAL NUMBERS REPRESENTED IN DECIMAL FORM.

10.1 ADDITION AND SUBTRACTION OF DECIMALS

Fundamentally, of course, decimal fractions are means of representing the set of fractional numbers. Thus, answers to computational problems involving fractional numbers expressed in common fraction form and in decimal fraction form differ only in the notation. However, this notational difference forces us to develop special algorithms for dealing with fractional numbers when they are expressed in decimal form. The principles which allow us to develop these particular algorithms, of course, are our particular system of numeration (which has a base ten and is positional) and certain of the field properties.

Since decimal fractions are merely an extension, to the right of the decimal point, of the numeration system we use for whole numbers, many of the uses of the field properties which we cited for whole numbers also are used with decimals. However, the introduction of the decimal point produces new problems. Foremost among these is the proper placement of the decimal point in the answer.

In so-called traditional classes, the placement of the decimal point in a particular type of computational problem was handled by teaching rules; these rules were usually not explained--they were presented to students as true because they worked. The modern trend, as in all other phases of elementary mathematics, is to explain why these processes work.

Suppose we wish to add 6.7 and 4.2. If the student has been taught that the base ten place value system has been extended to the right of the decimal point, he will agree that we can write 6.7 as $67 \times \frac{1}{10}$ and 4.2 as $42 \times \frac{1}{10}$. Hence the process for adding these two numbers will follow the procedure below:

$$\begin{aligned} 6.7 + 4.2 &= (67 \times \tfrac{1}{10}) + (42 \times \tfrac{1}{10}) && \text{(numeration fact---expanded notation)} \\ &= (67 + 42) \times \tfrac{1}{10} && \text{(use of the distributive property)} \\ &= (109) \times \tfrac{1}{10} && \text{(whole number computation)} \\ &= 10.9 && \text{(numeration fact).} \end{aligned}$$

Similarly,
$$\begin{aligned} 83.25 + 79.48 &= (8325 \times \tfrac{1}{100}) + (7948 \times \tfrac{1}{100}) \\ &= (8325 + 7948) \times \tfrac{1}{100} \\ &= 16273 \times \tfrac{1}{100} \\ &= 162.73 \end{aligned}$$

where we used the distributive property at one point.

What if the numbers to be added have different numbers of decimal places? Consider the following case:

$$\begin{aligned} 2.43 + 3.7 &= (243 \times \tfrac{1}{100}) + (37 \times \tfrac{1}{10}) = (243 \times \tfrac{1}{100}) + (37 \times \tfrac{10}{100}) \\ &= (243 \times \tfrac{1}{100}) + (370 \times \tfrac{1}{100}) = (613 \times \tfrac{1}{100}) = 6.13 \end{aligned}$$

Examples of the type above lead quickly to the "rule" that we line up the decimal points and put the decimal point in the answer in line with the other decimal points.

Subtraction is identical in all respects, except that here we use the distributive property of multiplication over subtraction. An example is given below:

$3.76 - 1.49 = (376 \times \frac{1}{100}) - (149 \times \frac{1}{100})$

$= (376 - 149) \times \frac{1}{100}$ (the distributive property of multiplication over subtraction)

$= 227 \times \frac{1}{100}$ (whole number computation)

$= 2.27$

10.2 MULTIPLICATION OF DECIMALS

Developing the rule for locating the decimal point in multiplication is similar to what we have done above, but involves the associative and commutative properties of multiplication, rather than the distributive property. For example,

$6.78 \times 2.4 = (678 \times \frac{1}{100}) \times (24 \times \frac{1}{10})$

$= (678 \times 24) \times (\frac{1}{100} \times \frac{1}{10})$ (by using both the associative and commutative properties of multiplication)

$= 16{,}272 \times \frac{1}{1000}$ (we have multiplied the whole numbers together and also multiplied the fractional numbers together)

$= 16.272$

Another example:

$7.9 \times 3.6 = (79 \times \frac{1}{10}) \times (36 \times \frac{1}{10})$

$= (79 \times 36) \times (\frac{1}{10} \times \frac{1}{10})$

$= 2844 \times \frac{1}{100}$

$= 28.44$

And yet another:

$$0.21 \times 0.435 = (21 \times \frac{1}{100}) \times (435 \times \frac{1}{1000})$$

$$= (21 \times 435) \times (\frac{1}{100} \times \frac{1}{1000})$$

$$= 9135 \times \frac{1}{100,000}$$

$$= .09135$$

From these examples, it is clear to see where we get the rule that the number of decimal places in the answer to a multiplication problem is the sum of the number of decimal places in the factors.

Of course, where we have written fractions with denominator 100, we could equally well have written powers of ten, using exponents. For example, 300 could be written 3×10^2, and .547 could be written 547×10^{-3}.

If we used exponents and powers of ten, the first example we looked at would be handled as follows:

$$6.78 \times 2.4 = (678 \times 10^{-2}) \times (24 \times 10^{-1})$$

$$= (678 \times 24) \times (10^{-2} \times 10^{-1})$$

$$= 16272 \times 10^{-3}$$

$$= 16.272$$

10.3 DIVISION OF DECIMALS

The division of decimals is customarily sidestepped by converting to an equivalent fraction whose denominator is a whole number. This is the basis of the relocation of the decimal point in the division process.

Suppose we need to divide 345.23 by 7.18.

$$345.23 \div 7.18 = \frac{345.23}{7.18} = \frac{345.23 \times 100}{7.18 \times 100} = \frac{34523}{718} = 34523 \div 718$$

Notice that we have made use of 1 as $\frac{100}{100}$, and the fact that 1 is the multiplicative identity, in transforming this problem from division by a decimal fraction to division by a whole number. The division as usually written is given below:

$7.18_{\wedge}\overline{)345.28_{\wedge}}$

and here again we have implicitly multiplied by $\frac{100}{100}$.

Another illustration of this process:

$$73.866 \div 33.2 = \frac{73.866}{33.2} = \frac{73.866 \times 10}{33.2 \times 10} \qquad (\frac{10}{10} = 1)$$

$$= \frac{738.66}{332} = 738.66 \div 332$$

In the division algorithm, this would be written $33.2_{\wedge}\overline{)73.8_{\wedge}66}$ which has the same result and relies on the same property.

Another example is $2.400 \div 300.0 = \frac{2.400}{300.0}$, which could be handled in either of the following ways:

(1) $\frac{2.400}{300.0} = \frac{2.400}{300} = 2.400 \div 300$

(2) $\frac{2.400}{300.0} = \frac{2.400 \times 10}{300.0 \times 10} = \frac{24.00}{3000} = 24 \div 3000$

11. INTEGERS AND RATIONAL NUMBERS

11.1 ADDITIVE INVERSES

When we extend the set of whole numbers to the set of integers, or extend the set of fractional numbers to the set of rational numbers, we introduce another of the field properties. Now, for each number a in our set, there is another number, $-a$, read "negative a," such that $a + (-a) = 0$. For example, the additive inverse of 6 is -6, because $6 + (-6) = 0$; the additive inverse of $\frac{-3}{4}$ is $\frac{3}{4}$, because $(-\frac{3}{4}) + \frac{3}{4} = 0$.

<u>Exercises:</u> (1) Find the additive inverses of the following numbers:

$5, -3, 12, 0, -\frac{8}{7}, -\frac{4}{15}, \frac{5}{14}, 0.38217$

(2) If $a + \frac{3}{4} = 0$, what is a?

11.2 ADDITION AND SUBTRACTION OF POSITIVE AND NEGATIVE NUMBERS.

Suppose we wish to add two negative numbers, say -7 and -4. We can use the property of additive inverses to help us. That is, we know that $(-7) + 7 = 0$ and that $(-4) + 4 = 0$. If we add these two equations, we have $[(-7) + 7] + [(-4) + 4] = 0 + 0$.

By using the commutative and associative properties of addition, we can eventually arrive at the form $[(-7) + (-4)] + [7 + 4] = 0 + 0$. Since we know $7 + 4 = 11$, and since $0 + 0 = 0$ (0 is the additive identity), we have $[(-7) + (-4)] + 11 = 0$. Thus, whatever $(-7) + (-4)$ is, it must result in 0 when added to 11. However, this is merely another way of saying that $[(-7) + (-4)]$ is the additive inverse of 11. We already know that the additive inverse of 11 is -11. Therefore, $(-7) + (-4) = -11$.

The general case follows exactly the same steps. Let $-a$ and $-b$ be two negative numbers. Then we have additive inverses a and b respectively, so $(-a) + a = 0$ and $(-b) + b = 0$.

$$[(-a) + a] + [(-b) + b] = 0 + 0$$

Rearranging by means of the associative and commutative properties of addition, we get $[(-a) + (-b)] + [a + b] = 0 + 0$, and since $0 + 0 = 0$, $[(-a) + (-b)] + [a + b] = 0$. Therefore, $(-a) + (-b)$ is the additive inverse of $a + b$. Thus, $(-a) + (-b) = -(a + b)$.

Suppose we wish to add a positive and a negative number. If they are additive inverses, there is no trouble--the sum is 0. If they are not additive inverses of each other, the sum may be either positive or negative, depending on the two numbers involved. Let us examine two examples.

(a) $5 + -8 = 5 + (-5 + -3)$

$= (5 + -5) + (-3)$ (the associative property of addition)

$= 0 + (-3)$ ($5 + -5$ are additive inverses)

$= -3$ (0 is the additive identity)

(b) $17 + -4 = (13 + 4) + (-4)$

$= 13 + (4 + -4)$ (the associative property of addition)

$= 13 + 0$ (4 and -4 are additive inverses)

$= 13$ (0 is the additive identity)

Now let us consider the process of subtraction when dealing with positive and negative numbers. Again, some examples would be helpful.

$4 - 2$ is a problem we have already seen. This is the same as whole number subtraction; hence the answer is 2. We may also note that $4 + (-2) = 2$.

What about $7 - 10 = \square$? We can use our knowledge of the relationship between addition and subtraction to write this in additive form, as $10 + \square = 7$. One way of handling this is the following: we know that $10 + (-10) = 0$; we also know that $0 + 7 = 7$; therefore the answer to the problem must be $-10 + 7$, which we can calculate as -3. By the commutative property of addition, we can write $-10 + 7$ as $7 + (-10)$. This and the preceding example seem to suggest that we can get the answer to a subtraction problem by adding the additive inverse of the number we are subtracting to the number we are subtracting from. That is, $a - b = a + (-b)$. Let's see if this continues to hold in the following cases.

(a) $-5 - (-3) = \square$ can be rewritten in additive form as $-3 + \square = -5$. Since $(-3) + 3 = 0$, and $0 + -5 = -5$, $\square = 3 + (-5)$, or -2. Again, we see that $3 + (-5)$ may be written $-5 + 3$, so we can add the additive inverse of -3 (i.e., 3) instead of subtracting -3.

(b) $-18 - (-31) = \square$, rearranged, is $-31 + \square = -18$. $(-31 + 31) = 0$, $0 + -18 = -18$, so $\square = 31 + (-18) = -18 + 31 = 13$.

(c) $12 - -7 = \square$

$-7 + \square = 12$

$-7 + 7 = 0$, $0 + 12 = 12$, so $\square = 7 + 12 = 12 + 7 = 19$

(d) $15 - (-18) = \square$

$-18 + \square = 15$

$-18 + 18 = 0$, $0 + 15 = 15$, so $\square = 18 + 15 = 15 + 18 = 33$

(e) $-16 - 18 = \square$

$18 + \square = -16$

$18 + -18 = 0$, $0 + -16 = -16$, so $\square = -18 + (-16) = -16 + (-18) = -34$

(f) $-21 - 14 = \square$

$14 + \square = -21$

$14 + -14 = 0$, $0 + -21 = -21$, so $\square = -14 + (-21) = -21 + (-14) = -35$

In each case, we have seen that $a - b = a + (-b)$. This can be proved generally, so we see that we can always transform a subtraction problem into an addition problem when dealing with positive and negative numbers.

Exercises:

Find the answers to the following problems:

(a) $6 + (-7)$ (b) $(-8) + (-5)$ (c) $(-9) - 6$

(d) $(-2) - (-3)$ (e) $(-8) + 10$ (f) $2 - (-7)$

(g) $-23 - 12$ (h) $(-27) + 27$ (i) $(-11) + 10$

11.3 MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS

Some elementary programs introduce the multiplication and division of positive and negative numbers in the upper grades of the elementary school. These programs rely heavily upon the concepts of the additive inverse and the distributive property to develop the rationale for the "rules of signs" in multiplication.

Suppose we want to find the product of -3 and 7. We know that $-3 + 3 = 0$ (that is, that -3 and 3 are additive inverses). Then $7[-3 + 3] = 7 \times 0$, and $7 \times 0 = 0$. Therefore $7[-3 + 3] = 0$. Using the distributive property, we can write the left-hand side of the equation as $7(-3) + 7(3)$, so $7(-3) + 7(3) = 0$. We know that $7(3) = 21$. Thus $7(-3)$ must be the additive inverse of 21; we also know that -21 is the additive inverse of 21. So we have established that $7(-3) = -21$.

By repeating the above process with $-a$ replacing -3, b replacing 7, where a and b stand for positive numbers, we can prove that $b(-a) = -ba$.

Now consider $(-3) \times 7$. Since the commutative property of multiplication holds, this must also be -21. In the general setting, $(-a)(b) = -ba$.

Finally, let us examine $(-3)(-7)$. The same process we used above holds some promise. $-7 + 7 = 0$, so $-3[-7 + 7] = -3(0) = 0$. By the distributive property, we can write the left-hand side as $(-3)(-7) + (-3)(7)$, so we now have $(-3)(-7) + (-3)(7) = 0$. We have already established that $(-3)(7) = -21$, so $(-3)(-7) + (-21) = 0$. Thus $(-3)(-7)$ must be the additive inverse of -21; that is, $(-3)(-7) = 21$.

Again, by following the identical steps, using $-a$ for -3 and $-b$ for -7, with a and b positive numbers, we establish in general that $(-a)(-b) = ab = ba = (-b)(-a)$.

11.4 DIVISION OF POSITIVE AND NEGATIVE NUMBERS

The division of a negative number by another negative number can easily be established as giving a positive quotient. For example, $(-7) \div (-18) = \frac{-7}{-18} = \frac{7 \times (-1)}{18 \times (-1)} = \frac{7}{18} \times \frac{(-1)}{(-1)} = \frac{7}{18} \times 1 = \frac{7}{18}$. Here we have relied upon the use of 1 as $\frac{(-1)}{(-1)}$ and the fact that 1 is the multiplicative identity. In general, $(-a) \div (-b) = \frac{-a}{-b} = \frac{a \times (-1)}{b \times (-1)} = \frac{a}{b} \times \frac{(-1)}{(-1)} = \frac{a}{b} \times 1 = \frac{a}{b}$ (a and b are here considered to be positive numbers).

To get the rule of signs for a positive number divided by a negative number or for a negative number divided by a positive number, we use our old technique of changing the division problem to a related multiplication problem. We will go through this development in general; it might be useful for you to try particular examples to understand the process better. In all of the following, a and b are positive numbers. $a \div (-b) = \square$ may be rewritten $(-b) \times \square = a$. We know that $\frac{1}{-b}$ is the reciprocal of $-b$ —that is, $(-b) \times (\frac{1}{-b}) = 1$. Also, since 1 is the multiplicative identity, $1 \times a = a$. Thus, $\square = \frac{1}{-b} \times a = \frac{a}{-b}$. But $-b$ is also $(-1) \times b$. So the multiplication problem could be written $(-1) \times b \times \square = a$. We know that the reciprocal of b is $\frac{1}{b}$, $(b \times \frac{1}{b} = 1)$ and $1 \times a = a$. So we have $(-1) \times \square = (\frac{1}{b}) \times a = \frac{a}{b}$. Multiplying by -1, we have $\square = -(\frac{a}{b})$. Therefore,

$\frac{a}{-b} = \square = -(\frac{a}{b})$, or $\frac{a}{-b} = -(\frac{a}{b})$, and the quotient of a positive number by a negative number is negative.

Now consider $-a \div b = \square$. The corresponding multiplication equation is $b \times \square = -a$. Here we can use the reciprocal of b, since $b \times \frac{1}{b} = 1$, and the fact that 1 is the multiplicative identity, since $1 \times -a = -a$. So $\square = \frac{1}{b} \times (-a) = \frac{-a}{b}$. But $\frac{1}{b} \times (-a)$ is also $\frac{1}{b} \times [a \times (-1)]$. By the associative property of multiplication, $\frac{1}{b} \times [a \times (-1)] = (\frac{1}{b} \times a) \times (-1) = \frac{a}{b} \times (-1) = -(\frac{a}{b})$. Thus the quotient resulting when a negative number is divided by a positive number is negative.

12. MISCELLANEOUS USES OF THE FIELD PROPERTIES

There are a variety of other situations in which the field properties play a role in the elementary curriculum. Some of these are indicated below. You will undoubtedly encounter others as you teach elementary school arithmetic.

12.1 MEASURES

In applications involving measurement, money, quantity, time, etc., a unit is attached to the number to tell what the number means, as 4 feet 6 inches, $6\frac{1}{4}$ square miles, 75 miles per hour, 3 quarts 1 pint, 3 hours 45 minutes 12 seconds. The symbol consisting of the numeral and the unit is sometimes referred to as a denominate number.

The measure, or denominate number, has, in all cases, grown out of some physical situation, where we have been measuring in terms of some unit. In fact, where we use units and subunits, the purpose has been to avoid the use of fractions. 4 yards 2 feet 10 inches could easily be written as $4\frac{34}{36}$ yards, and we could then operate using our procedures for handling fractions. However, if we were to work with the units and subunits themselves, then we would need only our knowledge of whole numbers to handle the situation.

In actual situations, of course, our measurements are approximations. We do not have perfect measuring equipment, nor are we able to read any measuring device precisely. The mathematical quarts we deal with are exactly two pints; the physical quarts are merely approximately two pints.

In adding and subtracting using denominate numbers, regrouping is used in converting from one unit to another. This regrouping is similar to the regrouping used for whole numbers and uses the associative property of addition. Examples of this are shown below.

(a)
$$\begin{array}{r} 2 \text{ yds. } 6 \text{ ft. } 4 \text{ in.} \\ + \ 3 \text{ yds. } 7 \text{ ft. } 9 \text{ in.} \\ \hline 5 \text{ yds. } 13 \text{ ft. } 13 \text{ in.} \end{array}$$

$$\left.\begin{array}{l} = 5 \text{ yds. } 13 \text{ ft. } (12 + 1) \text{ in.} \\ = 5 \text{ yds. } (13 + 1) \text{ ft. } 1 \text{ in.} \end{array}\right\} \text{here we have converted the 12 inches to 1 foot and associated it with the 13 feet}$$

$$= 5 \text{ yds. } 14 \text{ ft. } 1 \text{ in.}$$

$$\left.\begin{array}{l} = 5 \text{ yds. } (12 + 2) \text{ ft. } 1 \text{ in.} \\ = (5 + 4) \text{ yds. } 2 \text{ ft. } 1 \text{ in.} \end{array}\right\} \text{here we have converted the 12 feet to 4 yards and associated the 4 yards with the 5 yards}$$

$$= 9 \text{ yds. } 2 \text{ ft. } 1 \text{ in.}$$

(b)
$$\begin{array}{l} \ \ 8 \text{ qt.} \\ - \ 2 \text{ qt. } 1 \text{ pt.} \end{array}$$

$$\left.\begin{array}{rl} = & \begin{array}{l} \ \ 8 \text{ qt. } 0 \text{ pt.} \\ - \ 2 \text{ qt. } 1 \text{ pt.} \end{array} \\ = & \begin{array}{l} \ \ 7 \text{ qt. } 2 \text{ pt.} \\ - \ 2 \text{ qt. } 1 \text{ pt.} \\ \hline \ \ 5 \text{ qt. } 1 \text{ pt.} \end{array} \end{array}\right\} \text{here 1 quart has been converted to 2 pints, and the 2 pints has been associated with the 0 pints}$$

Of course, in dealing with fractional parts of a measure, performing operations on denominate numbers, etc., other of the field properties may be called into play, but these would be uses previously described in one or another of the sections of this unit.

12.2 ABSTRACT OPERATIONS AND OPERATION TABLES

One way to tell if a person really understands certain of the field properties is to introduce an abstract operation, and ask questions about it. This is often done by using finite sets and allowing the operation to be characterized by exhibiting the operation table. Sometimes, however, the operation is described in words or symbols, and is an operation on certain of the sets of numbers the students are familiar with. In any case, all of the properties can be investigated in this way (of course, for a distributive property to function, we need two operations for the distributive property to connect).

Below are two examples: The first is an operation table; the second is an operation defined on whole numbers.

(a)

*	A	B	C	D
A	B	C	A	D
B	A	B	C	D
C	D	A	B	C
D	C	D	A	B

We can see the operation is closed; all elements of the set can be listed, and there is no element in the table not belonging to the set. The operation is clearly not commutative; for example, $B * A$ is A but $A * B$ is C; also this can be verified by noticing that the table is not symmetric around the main diagonal. B is a left identity ($B * x = x$ where x is any element in the set), but there is no right identity. Every element has an inverse, since B appears in every row. In fact, each element is its own inverse. The operation is <u>not</u> associative, since, for example, $A * (B * C) = A * C = A$ but $(A * B) * C = C * C = B$. There is no point in investigating for distributivity, since only one operation is given.

(b) Let the operation $\int$ mean $\frac{a + b}{2}$, where a and b are whole numbers. (In other words, we are taking the average of the two numbers). The operation is <u>not</u> closed (for example, $\frac{1 + 2}{2} = \frac{3}{2}$, which is not a whole number. It is commutative ($a \int b = \frac{a + b}{2}$, $b \int a = \frac{b + a}{2}$, and $\frac{a + b}{2} = \frac{b + a}{2}$, since addition of whole numbers is commutative). Is the operation $\int$ associative?

Let's try $4 \int (5 \int 7)$ and $(4 \int 5) \int 7$.

$4 \int (5 \int 7) = 4 \int 6 = 5.$

$(4 \int 5) \int 7 = \frac{9}{2} \int 7 = \frac{\frac{9}{2} + 7}{2} = \frac{\frac{9}{2} + \frac{14}{2}}{2} = \frac{\frac{23}{2}}{2} = \frac{23}{4}$ or $5\frac{3}{4}$.

Therefore, we have a counter-example and the operation is <u>not</u> associative. There is no identity element, since there is no number x such that $x \int y = y \int x = y$ for every whole number y. Since there is no identity element, the question of inverses is meaningless.

Exercises:

Investigate which of the field properties hold in the following situations.

(1)

#	A	B	C	D	E
A	A	B	C	D	E
B	B	C	D	E	A
C	C	D	E	A	B
D	D	E	A	B	C
E	E	A	B	C	D

(2)

$\boxtimes$	A	B	C	D
A	A	A	A	A
B	A	B	A	A
C	A	A	C	A
D	A	A	A	D

(3) $\vdash\!\dashv$ means the smaller of a and b where a and b are whole numbers. That is, $10 \vdash\!\dashv 3 = 3$, $11 \vdash\!\dashv 11 = 11$, $4 \vdash\!\dashv 5 = 4$.

(4) $\Rightarrow$ means $\frac{2a + 2b}{2}$, where a and b are fractional numbers. For example, if $a = 7$ and $b = 12$, $a \Rightarrow b = \frac{2a + 2b}{2} = \frac{(2 \times 7) + (2 \times 12)}{2} = \frac{14 + 24}{2} = \frac{38}{2} = 19$. Similarly, if $a = \frac{1}{3}$ and $b = \frac{5}{8}$, $a \Rightarrow b = \frac{2a + 2b}{2} = \frac{(2 \times \frac{1}{3}) + (2 \times \frac{5}{8})}{2} =$

$$\frac{\frac{2}{3} + \frac{10}{8}}{2} = \frac{\frac{16 + 30}{24}}{2} = \frac{\frac{46}{24}}{2} = \frac{23}{24}.$$

12.3 APPLICATIONS

Various types of applied, or practical, problems, may involve one or more of the field properties. It would be impossible to anticipate all possible uses of the field properties in applied problems. We will merely show one example (taken from page 256 of the Unifying Mathematics text put out as a sixth grade text by the American Book Company). A batch of papers were being sold by a boy, who received a certain commission on each paper. The boy chose to figure the commission on each paper, and then multiply by the number of papers sold; his father, on the other hand, figured the total amount of money received for the papers, and figured the total commission by multiplying this amount by the appropriate percentage. The fact that the result is the same in both cases is, of course, due to the fact that multiplication is associative.

You will undoubtedly find many similar situations where the field properties figure prominently in applications.

12.4 MENTAL ARITHMETIC

The shortcuts used in mental arithmetic are usually based on one or another of the field properties. For example, to multiply 247 x 13, one might think:

"100 x 13 = 1300, 200 x 13 = 2600, 50 x 13 = 650, 3 x 13 = 39, so 247 x 13 = 2600 + 650 - 39 = 3250 - 39 = 3211." In doing this, we have used the distributive property in a general form (distributivity of multiplication over both subtraction and addition) to think of 247 x 13 as (200 + 50 - 3) x 13 = (200 x 13) + (50 x 13) - (3 x 13). We also used the associative property of multiplication, as we thought of 200 x 13 as 2 x (100 x 13).

Another example is the mental calculation of 4 x 199. We can think of 199 as 200 - 1, and use the distributive property of multiplication over subtraction, as follows:

$$4 \times 199 = 4 \times (200 - 1) = (4 \times 200) - (4 \times 1) = 800 - 4 = 796.$$

If you analyze the shortcuts you or your students use for mental computation, you will run into many other uses of the field properties.

13. CONCLUSION

In the preceding pages, you have been made aware of the important role of the field properties in all areas of elementary arithmetic. We hope that this analysis of the role of the field properties has provided you with additional insight into arithmetic. We further hope that you see the importance of these properties and why modern mathematics curricula lay such great stress upon them.

Above all, we hope that these properties no longer seem to you as just a collection of names and ideas which have been imposed on the arithmetic of the elementary school; that you see that real understanding of arithmetic requires an understanding of these properties; that learning them is not just a fancy frosting on the cake, but that these properties determine the structure of our systems of numbers.

It would be fruitless to hope that we have listed all possible uses of the field properties. By reading this unit, perhaps you will be a bit more cognizant of these properties in the future, and you will be more alert to point out their uses to your students. Probably you will discover still more places where these properties play a useful and important role--this is as it should be. Mathematics is a subject in which one continually gets deeper insights.

Index and Appendix

The charts which conclude this report summarize the data and suggestions contained in earlier pages. The titles of the sections referred to in these charts are:

PLACES IN THE UNIT WHERE SPECIFIC FIELD PROPERTIES ARE USED

Section	Closure Property of Add.	Closure Property of Mult.	Commutative Property of Add.	Commutative Property of Mult.	Associative Property of Add.	Associative Property of Mult.	Additive Identity	Multiplicative Identity	Additive Inverse	Multiplicative Inverse (Reciprocal)	Distributive Property (x over +)	Other Distributive Properties
2.1			X		X							
2.2					X							
2.3			X		X							
2.4			X		X							
2.5	X		X				X					
2.6			X									
2.7			X		X							
3.1											X	
3.2				X		X					X	
3.3											X	
3.4				X		X						
3.5								X			X	
3.6				X		X						
3.7		X		X				X				
4.1											X	
4.2												X
4.3												X
4.4						X						
5	X	X							X	X		
6								X				
7			X	X	X			X			X	X
8				X	X	X		X			X	

PLACES IN THE UNIT WHERE SPECIFIC FIELD PROPERTIES ARE USED (Continued)

Section	Closure Property of Add.	Closure Property of Mult.	Commutative Property of Add.	Commutative Property of Mult.	Associative Property of Add.	Associative Property of Mult.	Additive Identity	Multiplicative Identity	Additive Inverse	Multiplicative Inverse (Reciprocal)	Distributive Property (x over +)	Other Distributive Properties
9.1										X		
9.2				X		X		X		X		
9.3								X		X		
9.4								X		X		
10.1											X	X
10.2				X		X						
10.3								X				
11.1							X		X			
11.2			X		X		X		X			
11.3							X		X		X	
11.4								X		X		
12.1					X	X						
12.2												
12.3						X						
12.4						X					X	X

2

GRADE RANGE WHERE TOPICS ARE STUDIED

Section	Primary Grades (K - 3)	Upper Elementary (4 - 6)
2.1	I	
2.2	I	
2.3	I	M
2.4	I	M
2.5	I	M
2.6	I & M	M
2.7	I & M	M
3.1	I & M	M
3.2		I & M
3.3		I
3.4		I & M
3.5	I	M
3.6	I	
3.7	I	M
4.1	I	M
4.2		I
4.3		I (Occasionally)
4.4		I & M
5	I	M
6	I	M
7	I	I & M
8	I (Occasionally)	I & M
9.1		I & M
9.2		I & M
9.3		I & M
9.4		I & M
10.1		I & M
10.2		I & M
10.3		I & M
11.1		I & M
11.2		I & M
11.3		I
11.4		I
12.1	I	I & M
12.2		I
12.3	I & M	I & M
12.4	I & M	I & M

I - Introduced

M - Maintained