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ABSTRACT

REPORTED IS AN ANALYSIS OF EIGHT CURRICULUM REVISION PROJECTS IN MATHEMATICS (1) THE BOSTON COLLEGE MATHEMATICS INSTITUTE (BCMI), (2) THE GREATER CLEVELAND MATHEMATICS PROGRAM (GCMP), (3) THE SYRACUSE UNIVERSITY-WEBSTER COLLEGE MADISON PROJECT, (4) THE UNIVERSITY OF MARYLAND MATHEMATICS PROJECTS (UMMAP), (5) THE ONTARIO MATHEMATICS COMMISSION (OMC), (6) THE SCHOOL MATHEMATICS STUDY GROUP (SMSG), (7) THE DEVELOPMENTAL PROJECT IN SECONDARY MATHEMATICS AT SOUTHERN ILLINOIS UNIVERSITY, AND (8) THE UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS (UICSM). THE ISSUES WHICH SEEMED TO BE CRUCIAL IN THE CHANGING MATHEMATICS PROGRAM AND ON WHICH THE COMMITTEE BASED ITS ANALYSIS WERE -- SOCIAL APPLICATION, PLACEMENT, STRUCTURE, VOCABULARY, METHODS, CONCEPTS VS. SKILLS, PROOF, AND EVALUATION. (RP)

an analysis of new mathematics programs

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

AN ANALYSIS OF NEW MATHEMATICS PROGRAMS

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
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Introduction

HOW THIS ANALYSIS WAS WRITTEN

It is a well known fact that mathematics programs in grades kindergarten through college are in a state of flux. This makes the task of program development extremely difficult for a school system and the individual teacher as well. Realizing these problems, the Board of Directors of the National Council of Teachers of Mathematics authorized in 1959 the establishment of the Committee on the Analysis of Experimental Mathematics Programs, which was charged with the responsibility of investigating ways in which the National Council of Teachers of Mathematics (NCTM) could assist teachers and school systems in their considerations of program changes. The committee consisted of five members as listed in the appendix.

After considerable thought, the committee established several basic assumptions from which it began its investigation:

1. The purpose of the committee is not to evaluate the various experimental programs in progress. Evaluation must be made in reference to certain conditions present in every situation where a particular program is to be used. Since these vary, no evaluation made by one group can be appropriate for all.
2. An analysis is valid only if the individuals responsible for making it have been carefully selected. They must be open-minded, alert to changes, and as objective as humanly possible.
3. An analysis must be made within a common frame of reference if the different programs and their positions are to be commonly understood.

The committee published notices in all mathematical journals, requesting reports of mathematics experimental programs in progress at the time. The criteria used to determine the inclusion of a program were (a) the consent of the director, (b) the availability of printed materials used by students, (c) materials that had not at that time been contracted for publication on a commercial basis. (Subsequently, some of the programs discussed in this report have had materials commercially distributed.) As a result, the programs of the following eight curriculum-revision projects* were analyzed:

1. The Boston College Mathematics Institute (BCMI)
2. The Greater Cleveland Mathematics Program (GCMP)

* Addresses given in the Appendix.

3. The Syracuse University-Webster College Madison Project
4. The University of Maryland Mathematics Project (UMMaP)
5. The Ontario Mathematics Commission (OMC)
6. The School Mathematics Study Group (SMSG)
7. The Developmental Project in Secondary Mathematics at Southern Illinois University
8. The University of Illinois Committee on School Mathematics (UICSM).

On the basis of the second hypothesis mentioned previously, the committee obtained from many members of the NCTM recommendations as to whom we might select as subcommittee members to help in the analysis. Since most of the programs provide a body of content for each of several levels, a subcommittee of three or more was selected for each level. The subcommittee members** represented persons from all levels of mathematics education and with a wide variety of teaching experiences.

Having studied the current literature and met in several conferences, the committee agreed upon a set of eight issues which seemed to be crucial in the changing mathematics programs. Each teacher will need to decide his position on these issues and thus make a valid evaluation of any particular program in view of his own needs. The issues on which the committee based its analysis were the following:

Social Applications

How much emphasis should be placed on the social applications of mathematics? What should be the purpose and nature of these applications? Positions vary from (a) only through social application can a clear concept be gained to (b) social application tends to cloud the clarity of a subject's structure. Some persons hold that the major purpose of mathematics is to serve as a tool in solving mankind's problems; others feel the building of the tool is more important. Experimental programs are cognizant of these two positions.

Placement

At a particular grade level, what topics can be most effectively developed and which are most appropriate? It has been fairly well established over the years that any one topic of mathematics can be studied successfully at any one of several levels. However, in view of pressures for time and efficiency, this is not enough. We must look for optimum placement. It should be borne in mind that the sequence in which topics are presented may affect their value as learning experiences.

** An alphabetical list of all participants appears in the Appendix.

Structure

What emphasis should be placed on the study of mathematical structures to bring about a better understanding and use of mathematics? In order to understand, appreciate, and use a body of knowledge is it necessary to know how its various components are put together? Is a subject better understood and appreciated when its structure is completely known? A related problem is the timing of the study of structure. Shall it be analyzed before, after, or during the study of the subject as a total body of knowledge?

Vocabulary

How rapidly should the student be led from the use of the general unsophisticated language of mathematics to the very precise and sophisticated use of it? Some persons believe there is only one acceptable way to state a fact, while others feel the statement must be adjusted to the listener. Experimental programs today are conscious of the vocabulary problems which have developed over the years. Part of the issue rests in determining what constitutes sophistication; part of it pertains to the changing meanings of non-changing words.

Methods

What is the relative merit of presenting a sequence of activities from which a student may independently come to recognize the desired knowledge as opposed to presenting the knowledge and helping students rationalize it? Does one learn best from demonstration, illustration, description—or from experimentation, observation, and generalization? No text material can *a priori* establish the method which will be used by any individual teacher. However, the presentation of context can be oriented to make a particular method more efficient than others.

Concepts *vs.* Skills

What relationship should exist in the mathematics programs between the function of developing concepts and that of developing skill in the manipulation of symbols? Some persons feel that a student can gain the full meaning of a concept only when he approaches the level of automatic response in his use of the concept. There is also the question: What level of skill is optimum for our present society as opposed to the changing concepts resulting from changing cultures?

Proof

At what level should proof be introduced and with what degree of rigor? How rapidly should a student be led to make proofs independently? At what level should he be aware of what mathematical proof is?

The term "proof" has many meanings and is, at best, qualified. Therefore, this issue is particularly difficult to resolve. Proof, with its many shades of meaning, must be carefully considered in all programs. However, the real problem is to adapt the level of proof to the particular level of study. There is also a fine line between careful rationalization and proof; this must be clarified for the individual.

Evaluation

Are there available measures of the changes taking place that can be applied at this time, and what provisions can be made for evaluating the same changes in the future? This issue is difficult because of the complex nature of measurements needed for valid evaluation. Likewise, is it possible to measure changes taking place in the student and at the same time not make evaluations of the program? Can valid measures of student progress be made while the program is current? Will the student be able to measure his success or failures only over long years ahead? This issue proved very difficult for the subcommittees to interpret in the light of materials studied. We have reported their findings, but we believe this issue was not sufficiently clear to warrant drawing conclusions on the basis of the reports. However, in this connection several rather comprehensive studies on the measurement of student growth are now in progress. The results will no doubt become available in the near future and will help the teacher decide his own position on the issue of evaluation.

The subcommittees were asked to analyze the text material in the light of the issues. Some of them, however, did visit situations where the text materials were being used. They were specifically requested to report positions and not decide whether the position was good or bad. All reports were returned to the parent committee and edited by it.

The edited reports were then sent to the director of each project with an invitation to present a short introductory statement and a short statement of comment if he so desired. The statements received appear in the body of the report as indicated. In several instances, the directors granted the committee the privilege of cutting their reports if necessary. This we have done but we have tried not to change the thought of the presentation.

HOW THIS REPORT CAN BE USED

One should not expect this report to give a predigested evaluation of the eight experimental programs analyzed. We do not believe this is possible, and we do believe each school must take the responsibility of making evaluations for its own needs.

We hope that this report will make the reader more aware of the need for careful analysis on the basis of real issues. This report was based on eight issues, although more could have been stated. We hope that as you study the position of an experimental program in relation to each issue you will be concerned about the appropriateness of that program for your school. We hope that you may be led to see how a given program, with proper changes, could more effectively meet your needs.

Naturally, in analyzing these eight programs, you will simultaneously consider your present program in the same frame of reference. It is hoped this will lead to a wise retention and deletion of items in your present program.

We also hope the study of this report may help you recognize the totality of your program. Do you now offer a program with a beginning and an end, with all phases of it locked in interrelationships? Or do you have only a coalition?

We hope this report will help you in arriving at your own decisions as to which type of experimental program you wish to follow.

We realize that today mathematics teachers are under pressure from various sources to adopt programs, buy equipment, and join experiments—all in the name of progress. Much of what is presented has some value, some has no value at all, and some is altogether detrimental to progress in mathematics. We hope our report will help the teacher faced with such decisions.

In establishing the Committee on the Analysis of Experimental Programs, the NCTM made it a standing committee so that reports could be made in some form as new experimental programs are developed. There are several programs now in existence that are not included in this report because they were just being formulated when our work started. You will hear about them at a later date.

Boston College Mathematics Institute

INTRODUCTORY STATEMENT

STANLEY J. BEZUSZKA

The Boston College Mathematics Institute began informally in September 1953 with a program of modern mathematics for mathematics majors (freshmen and sophomores) in the College of Liberal Arts and Sciences and the School of Education. After four years of experimentation on the college level, it became apparent that: (a) incoming freshmen from various high schools had very little if any background in modern mathematical terminology or in the basic concepts of mathematics, and (b) the new approach to mathematics as a structure and the emphasis on the concepts of mathematics were almost totally foreign to the thinking habits of the students.

In 1957, the Boston College Mathematics Institute became a formal organization whose goal was to be the re-education of high school teachers in the elements of contemporary mathematics.

For the first three years after its beginning, the objectives of the Boston College Mathematics Institute were the preparation and teaching of:

1. Modern mathematics courses which could become an integral part of a high school mathematics curriculum without drastic changes in the school teaching personnel or undue pressures on the administrative structure of the school system.
2. Modern mathematics courses which would increase and develop the professional competence of the teacher not only for direct teaching in the classroom but also for the direction of superior students in private study, extra-curricular projects, guidance, and future careers in mathematics.
3. Modern mathematics courses suitable for the teaching of modern physics since many teachers were responsible for both subjects either formally in class or informally in discussions with students.

During this time the Boston College Mathematics Institute began work on *Sets, Operations, and Patterns, a Course in Basic Mathematics,* Course 1. The text was written throughout the course of an inservice and a summer program with the aid, experimentation, and suggestions of the teachers in attendance and the reactions of a demonstration class. *Sets, Operations, and Patterns* was originally intended for the ninth grade, but after more than three years of experimental use the teachers are now convinced that it is eighth (and possibly seventh) grade material.

Course 1 covers most of the traditional first course in algebra material from a structural point of view; it also contains many of the concepts of contemporary mathematics. The text is meant for the average or better-than-average student. In several instances, however, teachers have reported that slow learners had become interested in and had worked effectively on many sections of the text due to the novelty of the ideas presented. Portions of *Sets, Operations, and Patterns* have been used successfully with children in grades 4, 5, and 6. To date, *Sets, Operations, and Patterns* has been used as a classroom text by more than 4,000 students.

COMMITTEE REPORT

Sets, Operations, and Patterns: A Course in Basic Mathematics

Social Applications

No attempt is made to analyze and solve problems similar to those encountered by adolescents and adults in their business pursuits or other everyday experiences. However, the student who masters the material will have understandings and insights about numbers which should enable him to apply them in everyday situations involving quantitative relationships. Since the text material does not provide opportunities for the student to apply the concepts developed in so-called practical applications, it may be assumed that the author does not consider these to be important.

Placement

This study has incorporated a wide range of topics into a ninth grade level book. Much of the first three chapters is review material. These chapters do give a comprehensive coverage of the cultural aspects of most of man's experiences with numbers as well as the number concepts involved in set theory. Chapters 4-7 include topics such as inequalities, the basic laws of addition and multiplication, factoring, and basic operations in algebra. These topics have been couched in the language of sets. Chapter 8 attempts to develop in a fairly rigorous manner the structure of mathematics. This is also done in the language of sets. By the end of Chapter 9, the student should have a grasp of fundamental algebra through quadratic equations.

Structure

The word used by the author in reference to structure is *patterns*. Since a structure (or pattern) is a set with certain characteristics, the question of structure cannot be considered in this text until after Chapter

3. However, after Chapter 3, the arrangement of material is too disjointed and, therefore, structure is not evident.

Vocabulary

The text attempts to develop a precise, mathematical language. The symbols of sets are introduced and used throughout. Careful explanations are given for each symbol, and specific rules are laid down for the correct use of the symbols. Old, familiar symbols are re-examined and given more precise meanings. Careful distinction is usually made between things and the symbols for these things: for example, numbers and numerals. The text is an example of a development of the whole numbers and operations with them, using language that is acceptable to the mathematician.

Methods

The text uses a combination of presenting knowledge to the student and helping him to rationalize it. This is done by presenting a sequence of activities from which the student may *discover* the desired concept or relationship. For example: the text tells the student what the union of two sets is, then has the student discover, by the working of examples with specific sets, that set union is commutative. The general impression one gets from examining the text is that it requires the student to be able to read with understanding. The text is a compromise between the "tell them" and the "let them discover" methods.

Concepts *vs.* Skills

Probably one of the most important aspects of the material in *Sets, Operations, and Patterns* is emphasis on the development of concepts. There are many problems and exercises to be worked throughout the text, but in some cases the practice material is limited. If a student understands all of the material in the text he will have acquired concepts as well as some facility in using them through basic skills.

Proof

The use of proof in the text is limited. The definition of proof is not wholly consistent with the usual definition of a proof in mathematics. However, when proof of theorems is used it is presented in an acceptable manner and does emphasize the importance of basic assumptions.

Evaluation

There are no evaluative devices in the text for the teacher's use.

COMMENTS

STANLEY J. BEZUSZKA

Social Applications

Sets, Operations, and Patterns will have a companion textbook for practical applications entitled *Concepts of the Physical Universe for Man the Observer, Man the Experimenter, and Man the Mathematician.* This volume will be available in the fall of 1963.

Placement

Sets, Operations, and Patterns was originally written as an eighth or ninth grade level book. In the last two years, we have found that students who begin the text in grade 7 can successfully complete the three parts of the book by the end of the eighth grade.

Structure

Structure in a mathematical system as used in the text consists of:

1. A set of elements
2. A set of operations defined on the elements
3. A set of assumptions (generalizations of the particular instances of the operations involving the elements).

The text develops each of these three concepts in succession.

The solution of linear equations illustrates applications of the set of elements, operations, and assumptions in a mathematical system.

Chapter 9 is an introduction to the concepts used in measurement and geometry.

Part 3 of *Sets, Operations, and Patterns* develops the mathematical system of natural fractions, integers, and rational numbers. This part will be available by the end of September 1963.

Greater Cleveland Mathematics Program

INTRODUCTORY STATEMENT

PREPARED BY THE GREATER CLEVELAND MATHEMATICS PROGRAM RESEARCH STAFF

Genesis and Purpose

The Educational Research Council of Greater Cleveland was created in 1959 by Dr. George H. Baird, the Council's Executive Director, as an answer to a recognized need for a dynamic and continuous effort to improve the quality of elementary and secondary education.

The Council is an independent, non-profit organization. Its permanent staff of subject matter and educational specialists devotes full time to educational research and implementation. With the help of local educators and nationally recognized authorities in such fields as mathematics, social science, and testing, the Council is working to develop modern school curricula which will enable all students to meet the ever-changing present-day needs and the ever-increasing undefinable needs of the future.

Responding to the concerted request of 21 superintendents from participating school districts in the Greater Cleveland area, the Curriculum Research Department of the Council undertook as its first project a research and implementation study in mathematics education. The purpose of the project was set forth: "... to develop a comprehensive, sequential mathematics program for *all* children in grades kindergarten through twelve, a program which is both mathematically correct and pedagogically sound." Surveys were made of existing arithmetic and mathematics programs and materials, of the best learning theories, and of the needs of the children and professional staffs of the Council schools. The Greater Cleveland Mathematics Program (GCMP) with its preparation of materials, teacher training, and evaluation design was the direct outgrowth of this initial investigation. The materials in this program were revised for use in 1962-63.

GCMP's Educational Philosophy

The Greater Cleveland Mathematcs Program is a concept-oriented modern mathematics program in which the primary emphasis has been placed upon thinking, reasoning, and understanding, rather than on purely mechanical responses to standard situations. The child is continuously encouraged to investigate how and why things happen in mathematics. He is led to make generalizations, to test these generalizations, and to find new applications for them.

Only a few years ago many of the mathematical concepts included in the GCMP were thought to be too difficult for children in the primary grades. Experience with more than 125,000 children, however, has shown that children are able not only to understand these concepts, but that each new step in the learning process takes on more meaning because of this understanding.

In developing GCMP, the recommendations of several nationally recognized groups were taken into consideration. These groups include the National Council of Teachers of Mathematics, the Mathematical Association of America, and the Commission on Mathematics of the College Entrance Examination Board. The recommendations of noted mathematicians, such as Dr. John G. Kemeny, chairman of the Department of Mathematics, Dartmouth College, were also carefully considered.

The GCMP is not an isolated educational experiment, but rather an organized effort not only to adapt the best of these national curriculum recommendations to local school needs but also to build a sound program in areas where no such program exists. It is designed to effect curricular change in a controlled way rather than in the unplanned, hazardous way in which other changes often have been effected in the past.

To accomplish its purpose effectively, the GCMP makes extensive use of both the logical structure of mathematics and the discovery approach to learning. Students participate in challenging and exciting phases of mathematics and are led to discover, through the aid of skillful teaching, the fundamental concepts that are part of the logical structure of mathematics. Problem situations and experiences are presented in such a manner that discovery has a good chance of taking place spontaneously. Then, students are led to the established symbolism. The logical structure of mathematics stimulates the imagination of children and leads to an appreciation of mathematics as a dynamic and meaningful study. Continuity and creativity are stressed throughout the program.

The GCMP has been guided by the belief that computational skills should be introduced only after the concepts necessary for understanding the particular operations have been developed and the children have demonstrated a grasp of them. The mathematics of the program is presented in a carefully integrated, sequential form in which there is a continuous flow of ideas. Old ideas are examined in the light of newly formed concepts, and the search for patterns and relationships is carefully stressed.

At each stage of the program the child works with challenging problems and is encouraged to consider all of the ways in which these problems might be attacked. He then tries original or unusual approaches in looking for a solution.

Mathematicians agree that the real power of mathematics is in isolating logical patterns which exist in many seemingly dissimilar situations, systematizing these patterns, and applying them to the solution of new problems. The discovery approach allows the child to participate in this process and to feel the thrill that comes from being a party to the creation of a mathematical principle or idea.

Teachers have found that, with proper guidance, most children can uncover the essential relationships and structural properties originally discovered by the mathematical geniuses of the past, and are fascinated and excited by the accomplishment. It should be remembered that the idea or principle being developed is *new* to the child, even though it may have been originally expressed thousands of years ago.

In summary, then, GCMP is a systematic effort to develop an articulated kindergarten-through-grade-12 mathematics curriculum, and is designed to take advantage of the experiences of all earlier movements.

The basic guidelines of the GCMP are:

1. The basic program must be suitable for use by *all* students.
2. The program must have a continuous and systematic flow of mathematical concept formation from grades K through 12.
3. The program must originate at the lowest level of instruction in kindergarten or first grade and be continued through to grade 12.
4. The teaching approach should make the greatest possible use of the discovery method of teaching and provide continuous challenge and stimulation to the student.
5. The program must be mathematically correct and pedagogically sound.

COMMITTEE REPORT

Social Applications

There are more than 200 problems related to social applications in the first grade material and more than 800 in the second and third grade material. Approximately 9 percent of the problem material in the first, second, and third grades is devoted primarily to social applications. Of the 179 statements of objectives, 28, or 16 percent, deal directly with social applications.

In addition to the material listed above, there are 120 story problems for the second grade and 228 story problems for the third grade dealing with concepts and skills taught in the units in which they are presented. The story problems represent approximately 3 percent of the total problem material.

Placement

Many topics are placed anywhere from 3 months to 16 months earlier than is usually the case for these grade levels. This may in part account for the test results reported in the section on evaluation.

Structure

In the *Elementary Mathematics Series,* Volumes I and II, the commutative property of addition is introduced to the first grade pupils midway in Volume I. There are 45 problems in Volume I and 32 more in Volume II. At the second grade level the property is explicit in 75 problems. By the third grade, students are expected to be familiar with the order property for addition, its name, and to use it in their reasoning.

The associative property of addition is introduced in the first grade one-third of the way through Volume II. One hundred twelve problems deal directly with this principle. Seventy more problems are devoted to it early in the second grade book, Volume I. Volume II has 27 more problems dealing with this principle. Students are assumed to be aware of it in column and row addition of more than two numbers.

The commutative property of multiplication is first introduced in Volume II for the second grade. Forty problems in the latter half of the book are intended to give an explicit introduction to this principle. It is reviewed in the third grade and used with other principles in the development of algorithms.

The distributive property for multiplication over addition is introduced briefly in the last section of Volume II for the second grade, where 40 problems are given as a preview of the work in the third grade. At the third grade level the distributive law is needed to rationalize the algorithms of multiplication. It is used in the development of these algorithms.

The associative property of multiplication is introduced in Volume II for grade three. Fifty-five problems in the first part of this volume are used to acquaint children with this principle, and 20 problems are used to review the principle later.

The inverse relationship of addition and subtraction is introduced in the first grade book, Volume I, with 158 problems. It is used throughout the first, second, and third grade materials. The inverse relationship of multiplication and division is introduced late in the second grade and used in the third grade materials.

The concept of inequalities is introduced in the first grade with 87 problems. It is reviewed in the second and third grade materials with 192 and 92 problems, respectively.

The total number of problems dealing directly with structure, not including the problem material used in the development of algorithms, is 975. This represents approximately 7 percent of the total number of problems.

Vocabulary

Sixty-three special terms are introduced in kindergarten, 68 additional terms are used in grade one, 78 additional terms in grade two, and an additional 41 terms in grade three. In kindergarten and the first three grades the children are expected to learn and use 250 terms with special meaning for the mathematics program.

Methods

The *Commentary for Teachers* outlines teaching procedures which move the student from concrete physical situations to their associated mathematical concepts. Principles are to be discovered by students through the use of the exercise material and other activities outlined in the commentary. Teachers are urged not to formulate principles or algorithms too early, but to allow students to formulate them. Principles and algorithms are developed with the material and eventually formulated and stated by the students and the teacher. After a concept has been developed and used, it is named. Students are then given story-problem situations in which they can apply new concepts and techniques that they have learned. The mastery levels are not specified.

Audio-visual aids to be used in each unit are listed together with suggestions for their use.

Concepts *vs.* Skills

There are 179 statements of objectives in the teachers' manuals for kindergarten through grade three. Each of these was classified as being oriented to (a) skills, (b) concepts, (c) evaluation, or (d) miscellaneous. This was done to compare the emphasis on skills and concepts as they are outlined for the teachers. The table below summarizes the results of this classification.

Statements of objectives	Oriented to skills	Oriented to concepts	Oriented to evaluation	Miscellaneous
179	76 or 42%	86 or 48%	11 or 6%	6 or 4%

Proof

Students are encouraged to reason out the processes they learn by building the development on basic properties. No effort is made to introduce or use in any form a formal proof.

Evaluation

A nationally standardized test (not specified) was given to students in 19 school districts using the GCMP and to a "carefully selected control group of 150 students per grade level not using the GCMP." The GCMP classes made higher mean scores and grade equivalents on both the computation and the problem solving and concepts portion of the test. Children using the GCMP showed a difference of two months for grade 1, and six months each for grades 2 and 3, according to literature released by the Educational Research Council of Greater Cleveland.

The above statements were taken from literature released by the GCMP. Without access to evaluation instruments, the statistical data, or an outline of the evaluation procedure, the committee felt that the information in this area was not adequate.

COMMENTS

GLENN D. BERKHEIMER

The Greater Cleveland Mathematics Program is a constantly evolving program whose primary objective is to provide good mathematics education for all pupils. In order to develop the maximum understanding of basic concepts at all levels, GCMP stresses the logic and structure of mathematics. To capitalize on the child's natural curiosity and assure greatest retention, the discovery approach to learning is used.

During the four years of its existence, the GCMP has undergone many changes. Initiated at the primary level, instructional materials were developed for pupils in grades kindergarten through 3. These materials have been field tested on approximately 10,000 pupils per grade level each year. Evaluation, both objective and subjective, of pupil progress has served as the basis for 11 revision of these materials.

It is interesting to note that the test results reported above were secured after pupils had only one full year of work with the GCMP. After using the 1961-62 materials (analyzed above) these children made scores on the Metropolitan Achievement Tests which are even more encouraging: they scored from 8 to 14 months above their actual grade levels. Subjective evaluation reports indicate a general increase in interest and participation in arithmetic activities and a new spirit of inquiry which often carries over into other areas.

The GCMP has recognized, from its inception, that while the development of good materials for pupils is vital to the improvement of mathematics education, the development of comprehensive inservice training materials is even more vital. Teachers' Guides have therefore been designed to give the teacher the necssary mathematical background and to provide a great variety of developmental and supplemental activities devised and tested by experienced teachers. A separate booklet containing a more extensive treatment of the mathematical concepts developed in the GCMP K-3 program has been written for inservice programs. In addition, a newly developed series of teacher-training films, stressing both content and teaching methods, have been produced.

The evolution of GCMP continues. Feedback from classrooms using the latest edition of pupil and teacher materials is now being appraised with an eye to further refinements.

The Syracuse University-Webster College Madison Project

INTRODUCTORY STATEMENT

ROBERT B. DAVIS

The Syracuse University-Webster College Madison Projest is concerned with modifying school mathematics through a several-phase approach. At the time of this writing (October 1962), the first-phase material is reasonably well developed. This material provides a supplementary program in algebra and coordinate geometry, with some applications in physical science. Although the material is not tied to specified school grade levels, it can be started at least as early as grade 2 (age: 7 years), and can be used at least as late as grade 8. However, most schools first introduce the material in grades 5, 6, and 7; as the school itself acquires more experience with the Madison Project materials, these materials begin to be used, in appropriate places, in earlier and in later grades.

Mathematical Content

From the point of view of mathematical content, the Phase I material is concerned mainly with:

1. Arithmetic of signed numbers
2. Variables, open sentences, and truth sets
3. Classification of statements as "True" and "False"
4. Functions, including functions obtained empirically
5. The number line and Cartesian coordinates
6. Graphs of functions and of truth sets, including linear functions and conic sections

7. Implication and contradiction
8. Identities (e.g., $3 + \square = \square + 3$, $\square + \square = 2 \times \square$, etc.)
9. The selection of a set of axioms (for algebra and arithmetic), and the process of deriving theorems systematically from these axioms
10. Matrices
11. The use of matrices in extending the rational number system to include complex numbers
12. Vectors, forces, and statics
13. Similar triangles
14. The trigonometric functions
15. The concepts of area, volume, and perimeter
16. Derivation of mensuration formulae for area, volume, and perimeter
17. Programming digital computers (the IBM 1620 and the IBM 650).

Needless to say, none of these topics is pursued to a level of very great depth; as will be seen in what follows, to develop these topics deeply at a first presentation would be contrary to the basic rationale of the Project's activities.

The School Program During Phase I

The traditional program provided the child, perhaps during grades K-3, with many experiences with whole numbers: children took attendance, counted how many were absent, collected and counted milk money, kept score in simple games, computed how many days until Christmas, and so on. This was not a matter of formal instruction at all—indeed, the "observe-and-practice" and the "memorize-and-repeat" kinds of formal instruction were markedly conspicuous by their absence.

It is a working hypothesis of the Madison Project that this kind of "experience-without-formal-instruction" should nearly always precede any formal instruction. Without it, only a few individuals having some special gift can fully grasp, understand, and make creative use of new formal abstract ideas and procedures. With it, a very large number of children can achieve a high degree of mastery, even with material which might seem far too abstract for them to comprehend.

Indeed, given enough experience-without-formal-instruction of a carefully chosen sort, it frequently turns out that the crowning formal instruction can be eliminated completely. The "prelude" has expanded into the entire *opus,* and nothing further is needed! The children have learned from "experience," without ever being "taught."

If we look at the traditional program, we find in most cases that such preliminary informal experience was very frequently absent. Children learned to add fractions, and even to divide fractions, but the informal experiences with fractions and with division that could have made this meaningful were entirely and sadly omitted.

The case of ninth grade algebra is noteworthy. Although some small amount of preliminary informal experience and exploration may have been included in the eighth grade—at least in theory—actually the practice was usually that the child began formal instruction in algebra without the advantage of very much previous informal exploration. As a result, much of the ninth grade algebra had to be re-taught in grade 11, and often again in the college freshman year.

This was nothing more than an illustration of a valid law of learning mathematics—the need for spiralling. To learn things exactly right—and completely—the first time around may sound desirable, but it is in fact the pursuit of an illusion.

To a first approximation, one can understand the general nature of the Madison Project materials by asking two questions: What background in basic algebraic experience would we like to see in students when they enter ninth grade algebra? And, secondly: What kinds of previous informal exploratory experiences would serve to build this background?

The Madison Project materials are conceived as an answer to these questions—at least to a first approximation.

Obviously, when one looks more closely, it becomes clear that the problem is somewhat more complicated than this. For one thing, the traditional ninth grade algebra could not be fully suitable for students with so extensive a background of previous experience, since it was designed for students without this background.

Furthermore, as David Page and others have pointed out, the traditional ninth grade algebra was an unsatisfactory course judged purely intrinsically—it usually dealt in memorized procedures for the manipulation of apparently incomprehensible symbols.

Even more important, given enough good preliminary experience in the exploration of algebraic ideas, the student emerges knowing algebra. He does not require to be subjected to the untasteful chicanery of the traditional ninth grade algebra course.

Consequently, the curriculum of a school using the Madison Project materials in Phase I would look generally like this:

Grades K-3: Mainly the school's original arithmetic program, hardly changed at all by the use of Madison Project materials. (Possibly Madison Project materials might find some small use here.)

Grades 4-7: Mainly the school's original arithmetic and science programs, but with a supplemental use of Madison Project materials—perhaps on a one-lesson-per-week basis.

A school using such a program would then need to follow this with a carefully designed course sequence for grades 8-12 that would take advantage of the stronger background of the children emerging from grade 7.

How Can I Learn About the Project?

Because of its many unusual features—its supplemental nature, its emphasis on student discovery, its major reliance upon motion picture films (for teachers, not for students) instead of written materials, its somewhat novel classroom dynamics, and so on—the Madison Project is not easy to learn about. The best approach seems to be to view the films *A Lesson with Second Graders* and *Graphing an Ellipse,* to read the pamphlet *The Madison Project—A Brief Description of Materials and Activities,* and then to proceed from there. Where possible, it is also advisable to visit one of the Project's experimental schools.

COMMITTEE REPORT

Social Applications

The Madison Project material has few social applications. For example: in *Discovery of Algebra,* Chapter 1, "Equations, Open Sentences, and Inequalities," there are only five examples; one of these topics is the theory of gravitation, which is unfamiliar to the pupils. Social applications dealing with the stock market, robbery, and stolen money are also poor. In all there are about 50 verbal problems, other than those which children are asked to make themselves.

Placement

The materials have little to offer pupils below the fifth grade except Chapter 1, "Equations, Open Sentences, and Inequalities"; Chapter 21, "Names for Numbers"; Chapter 43, "Area"; and Chapter 44, "How Many Squares?" Signed numbers are introduced with graphs, and ordered pairs with illustrations which show how they may be used in baseball scores and with good and bad checks. Although the lessons on implications and derivations are skillfully done, they are scattered throughout the book, and much other material comes between many of them. The material seems to be very abstract with little meaning in relation to children's experiences.

Structure

The Madison Project stresses the principles upon which number systems are based and develops these principles by a step-by-step process. The names of the principles are not revealed until almost the end of the course.

Vocabulary

The vocabulary is quite sophisticated. The words that are introduced are not defined, although the symbol is displayed and followed by a question about its meaning.

Methods

The Madison Project uses an informal, conversational approach to the discovery method. The pupils are not told the answers, but are given many illustrations of a concept and are encouraged to think out the solution for themselves. Instead of learning mathematical laws established by others, the pupils formulate their own. Although only a few topics are covered in the material, some teachers may feel that the pupils do not study one topic long enough to gain a reasonable understanding of the concept.

Concepts *vs.* Skills

The emphasis in this material is on concepts. After an idea has been once discovered, little time is spent in developing skill in its use. It is assumed that the student has the necessary skills or they are learned incidentally.

Proof

The Madison Project has much practice in trial and error discovery, but few mathematical generalizations are actually drawn from the data collected until near the end of the course. With this material pupils can discover certain number patterns. Pupils learn to transform identities in an intuitive way. Usually a few illustrations are used to establish proof.

COMMENTS

ROBERT B. DAVIS

Obviously, I would take exception to some of the analysis presented in the committee's report. Social applications were deliberately omitted from Project materials. All of the Project materials have been developed from classes with children, and none of the materials was observed to be inappropriate. The best evidence for this is contained in the Project's films which show actual classroom lessons. From these, the attitudes and actions of both teacher and students can be observed directly.

One matter of fact demands immediate correction: at no point in the Madison Project materials are illustrations ever used to establish proof. Illustrations are used to provide the children with a set of mental symbols which are suitable for internal mental manipulation—somewhat in the manner of Tolman's "cognitive maps." Where proof is concerned, the careful use of various examples is intended to show the student that generalizing from a non-exhaustive set of instances does not constitute a proof. As he starts the material, the child has available to him no reliable tool for inference other than generalization from instances. As he progresses through the Project materials, he becomes increasingly familiar with implication and contradiction, and he simultaneously becomes familiar with the limitations inherent in non-exhaustive generalization.

Anyone who wishes to understand how students in Project classes do, in fact, regard the matter of proof is referred to Project tape recording No. D-1, to the film *Graphing an Ellipse,* and to the various films on derivations.

University of Maryland Mathematics Project

INTRODUCTORY STATEMENT

JOHN R. MAYOR

The University of Maryland Mathematics Project (UMMaP) began its activities in the fall of 1957. The project has demonstrated that an environment could be created in which the university mathematician, the classroom teacher, the supervisor, and the administrator could cooperate successfully to produce significant course materials in mathematics for grades 7 and 8. By February 1961, a text for the seventh grade and a text for the eighth grade with accompanying teachers' manuals were being used in revised form in classrooms throughout the country. In a number of instances they have been also used with gifted sixth grade pupils. The texts are now available in hard covers.

Since the fall of 1960 the University of Maryland Mathematics Project has been concerned with demonstration projects in the mathematical preparation of elementary school teachers as well as studies in the learning of mathematics at the elementary and junior high school levels. It is anticipated that the seventh and eighth grade texts will be revised at some date in the future, not yet set, and probably not before 1964. In the meantime, the work of preparing materials in mathematics for elementary school teachers and studies in the learning of mathematics will be continued.

The University of Maryland Mathematics Project, since its beginning, has had the assistance of an advisory board consisting of representatives of the University of Maryland departments of mathematics, education, and psychology, and the College of Engineering. In addition, the advisory board consisted of mathematics supervisors in the four major public school systems in the Washington, D.C., area, U.S. Office of Education specialists in mathematics, and representatives from both the Maryland State Department of Education and the Mathematics Section of the Maryland State Teachers Association.

In the three years of project activities a seminar for junior high school teachers met weekly on the University of Maryland campus. Participants in the seminar were selected by the four major school systems in the area. Lectures in mathematics and psychology were given. During the second semester of the first year, the teachers started preparing units which they wanted to try out in seventh and eighth grade classes. These units became the basis of the organized courses for seventh and eighth grade. The courses were written by teachers and by members of the UMMaP staff working as a team, and were tried out over a two-year period by the teachers in the seminar and others in their school systems before the courses were put in their final form.

In the Maryland courses, language and mathematical structure have been stressed. It is recognized that one of the important problems in the teaching of mathematics is that of a language. Not only is mathematics a language itself, but language is necessary to communicate the concepts of mathematics. In writing the UMMaP courses a very serious attempt has been made to improve vocabulary. The distinction is carefully made between a symbol and that which the symbol represents. An important point which seems to aid in the comprehension of the nature of mathematics is that the meaning of mathematical symbols is merely a matter of agreement. Inequality symbols are used frequently throughout the course. There is stress on the idea of number sentences and the more modern interpretation of the term *equation*.

While much of the traditional content of seventh and eighth grade courses in mathematics is also a part of the Maryland courses, for these grades, the topics are approached from a new point of view. It is the purpose of the courses to assist students in achieving a great deal more than skill in manipulating with numbers. A junior high school student is given the opportunity to recognize that the number systems we use are structured like any mathematical system. Throughout the courses there is great emphasis on number systems, perhaps more than in any of the courses of other current curriculum projects.

A one-year evaluation of each course was made in response to the request of the school systems in the Washington area using these courses in grades 7 and 8. This evaluation consisted of a comparison of classes studying the Maryland materials and those studying traditional materials. The results in general show that (a) students who have studied the UMMaP materials have done as well on traditional tests as those who studied traditional courses, and (b) on tests on the new mathematics the UMMaP students have done much better than those who have used only traditional materials. These results seem quite satisfactory to the school systems. Indeed the school administrators apparently concluded that students of UMMaP courses were learning not only everything that students in traditional classes had learned but a great deal more besides. However, wide acceptance of the courses has been based more on the enthusiasm of the teachers who taught them and on the enthusiasm of the students who studied the courses than on any formal evaluation.

COMMITTEE REPORT

Mathematics for the Junior High School

Volumes I and II

Social Applications

The reviewers found little attempt on the part of the authors to apply the development of the mathematics to social problems or social situations. The few social problems included are found in connection with percent-type problems.

Placement

The UMMaP books include material generally found in traditional freshman algebra books, although it is presented in a different format. However, in addition to these, the following topics are introduced as new to the junior high school area: logic, the solution of triangles with trigonometry, irrational numbers, statistics, and probability.

Structure

The authors do not stress the idea of structure, *per se,* but in presenting number systems they are careful that each system builds from the previous system and that structure is evident from number system to number system.

Vocabulary

The authors are very careful in the use of precise definitions, and build from one concept to another utilizing the definitions. The language of logic and modern algebra is introduced early and used consistently throughout the course.

Methods

The ideas and materials are presented in such a manner that the student is led to form the generalizations desired. Inductive reasoning is used when principles are being formed; deductive reasoning is used when applicable.

Concepts *vs.* Skills

Far more emphasis is placed on the development of mathematical concepts than on the mathematical skills needed to use them. However, there are exercises which afford the student an opportunity to firm up the concepts through "drill."

Proof

The authors do not require students to do proofs in Book I, although proofs are used in the explanation of the mathematical concepts presented. Book II presents logic, and this is used in proofs which the student is expected to perform.

Evaluation

Although there are exercises throughout the book to augment the concepts being developed, the reviewers are not aware of any author-prepared tests to evaluate student progress. The reviewers are not familiar with statistical studies which compare traditionally taught students to students taught by this program.

COMMENTS

JOHN R. MAYOR

For a brief summary, the statements in general are very fair. Those responsible for the Maryland program do feel that there is somewhat greater emphasis on social problems and social applications than the report indicates.

During the first years of the tryout, the sample materials were used widely in Washington area school systems. Upon request of the school administrators, an evaluation of the new courses was conducted. The results of the evaluation show, in a way which is satisfactory to the school administrators, that students studying the new courses did as well as students studying traditional courses on traditional tests of achievement at the seventh and eighth grade levels, and that the students studying the new courses learned a great deal more besides. There was nothing in the evluation to cause us to be discouraged about the use of the University of Maryland materials with slower students.

The Ontario Mathematics Commission

INTRODUCTORY STATEMENT

THE ONTARIO MATHEMATICS COMMISSION

The initial encouragement, both moral and financial, for a thoroughgoing review of the Provincial Curriculuum in Mathematics was given by the Ontario Teachers' Federation. A Mathematics Committee appointed by the Federation developed in due course into a Federation Mathematics Commission with representation from various university, secondary school, and elementary school bodies. This group, in August 1959, in a workshop session held at Lakefield, Ontario, arrived at a sufficient measure of agreement regarding a desirable Secondary School Mathematics program to warrant the production of experimental teaching material beginning with grade nine.

By August 1960, conditions were ripe for the constitution of the Ontario Mathematics Commission as a representative independent body supported jointly by the Ontario Teachers' Federation and the Ontario Department of Education. In a 3-day session held at Queen's University, Kingston, a constitution was adopted, an executive and various standing committees elected, and curriculum outlines for the various secondary school grades discussed.

Both the composition of the Mathematics Commission and also its objectives are worthy of note. The broadly representative character of the Commission is evident from the following statement of the categories of its appointed members: 12 from the various Ontario universities, of which there are at present 11; 16 from the Ontario Teachers' Federation; 5 from the Ontario Department of Education; 2 from the Ontario College of Education; and 8 from various professional associations, together with a maximum of 3 members co-opted for a period of one year.

Among the objectives of the Commission as set forth in the Constitution are the following:

1. Encouraging the production of experimental teaching material in mathematics and seeking the cooperation of the Department of Education in testing such material in the schools of the Province.
2. Cooperating with the universities, the Ontario Teachers' Federation, and other appropriate bodies, in providing courses which will enable teachers to improve their qualifications and keep abreast of current experimentation in curriculum changes and in teaching techniques.

3. Undertaking curriculum research aimed at keeping the Province abreast of the best contemporary practice, and circulating such information to interested bodies.

It is evident that this constitution and these objectives imply certain convictions regarding the curriculum in mathematics, namely:

1. that it is a continuous curriculum and hence it is a matter of vital importance that a student's mathematical concepts, throughout their gradual development, be correct and clearly formulated even if, at first, necessarily incomplete;
2. that all persons having to do with instruction in mathematics, at whatever level, have a legitimate interest in what goes on at all levels;
3. that the successful adoption of desirable new approaches to the teaching of mathematics depends not only on the provision of suitable teaching material but on the adequate pedagogical preparation of the teacher through regular pre-service or special inservice training courses;
4. that the whole program of reviewing, up-grading, and maintaining at its highest possible level the entire mathematics curriculum is necessarily a long-term project demanding high standards of professional responsibility and effort on the part of all concerned.

The writer has no desire to give the impression that the activities promoted by the Ontario Mathematics Commission constitute the only significant indication of the current interest throughout this Province in the subject of mathematics and its teaching. For instance, the Association of the Teachers of Mathematics and Physics—a section of the Ontario Educational Association—has been largely responsible for the promotion during the past 2 or 3 years of seminars in mathematics attended by large numbers of secondary school teachers from the Toronto metropolitan area. In these seminar programs, several university professors have contributed timely and stimulating papers; many experienced high school teachers have led discussion groups and shared classroom experience.

Again, the Ontario Teachers' Federation has sponsored inservice courses and work shops in mathematics in various localities, to which a goodly number of teachers, particularly from the secondary schools, have gone to improve their acquaintance with recent developments in their subject.

From the considerations suggested earlier, however, it may be fair to say that the present program and the future possibilities of the Mathematics Commission warrant the belief that it has already exerted, and will continue to exert, a stimulating influence on the thinking of many Ontario teachers of mathematics.

Let us therefore take a brief look at the work of the Commission, particularly its activities during the past two years.

In the course of the Lakefield Conference of August 1959, the conviction grew that actual classroom teaching by able and experienced teachers provided with suitable experimental teaching material was fundamental to wise decisions regarding curriculum revision. Only a definite detailed exposition would clarify semantic misconceptions; only classroom trial would prove what innovations in approach and what attempts at a more orderly presentation of fundamental concepts were practicable at what grade levels and for what students.

It seemed evident, therefore, that a first requirement was the production of suitable experimental teaching material which would endeavor to incorporate the ideas for a change in approach which had found at least majority acceptance at Lakefield and which were subsequently criticized and modified at Kingston in the sessions of the Annual Meeting of August 1960. It was fully recognized that along with this writing activity must go the preparation of the teacher—that, in fact, the key to the success of any such revision of the traditional approach lay in this phase of the Commission's objectives. Fortunately, a corps of teachers able and willing to cooperate in the initial experiment was available. Hence it was decided that, in spite of the inadequacy of the time available, it was of first importance that a writing group should be recruited to prepare teaching material for trial in grade 9 during the school year 1960-61. This was done.

A first version of this material, known as *Mathematics 9*, was written under great pressure of time; schools and teachers were invited to participate in its classroom trial. Of those volunteering, between 35 and 40 schools and about 65 teachers were given permission by the Department of Education to take part. Chapter by chapter, these teachers sent in reports, many of them extremely detailed and helpful, on the degree of success they were experiencing in the use of *Mathematics 9* in the classroom. Comments were invited and given on the suitability of its language, the aptness of the explanatory material, and the adequacy of the illustrative examples and practice exercises. In addition, the participating teachers, divided into four regional groups, met in conference with one or more of the authors three times during the year. From these meetings there emerged a dependable consensus on the practicability at the grade 9 level of such innovations in approach and in maturity of mathematical concept as had been attempted by the writing group.

Many of the suggestions arising from this broad practical classroom experience were incorporated in a revision of *Mathematics 9* carried on continuously during the 1960-61 session. This appeared as *New Mathematics 9*, used, again with Departmental permission, by about 70 interested and capable teachers in some 40 schools during the school year 1961-62. Most of these schools had taken part in the previous year's trial.

Finally, a second thoroughgoing revision of the material, conducted during the 1961-62 school year, resulted in the commercial publication of a hard-cover text, *Mathematics 9* by Coleman, Del-Grande, Mulligan, Totton; this has been placed on the departmental list of approved text books for use in 1962-63. By mutual agreement between the Department and the Commission this concludes, for the present at least, the trial of Commission-sponsored material as far as grade 9 is concerned.

There can be no doubt that this procedure, involving university and secondary school cooperation in writing, extensive testing of the written material through the practical assistance of a very substantial number of able and experienced high school teachers, and painstaking revisions by the authors, based on frank, constructive criticism by those participating, has set a very promising pattern for the ground work of curricular revision in other subjects.

In the meantime, a second group of authors was occupied in writing experimental material for grade 10—both in algebra and in geometry. The algebra unit builds naturally on the foundation laid in grade 9, and uses the same terminology and deductive approach. The unit in geometry incorporates the same basic mathematical ideas and endeavors to build a logical deductive structure with assumptions as free from criticism as is feasible at this level. The material, known as *Mathematics 10*, was, with Departmental permission, tried out in about 35 classrooms in between 20 and 25 of the schools which the previous year had used the new approach in grade 9. Basically, the same pattern of teacher participation and criticism was followed. As was the case with grade 9, a revised version, based on this first year's trial of the grade 10 material, has been produced. This version will be tested in about 30 schools during 1962-63.

Meanwhile, work has been proceeding by still a third group of writers on the preparation of sequential experimental material in geometry and algebra for grade 11, while the Commission members, and in particular those of the Advisory Committee on Secondary Education, have been giving much thought to the formulation of a suitable sequence of topics for grades 12 and 13.

COMMITTEE REPORTS

Mathematics 9

Social Applications

In the revised edition of the Ontario material, considerable attention is given to the historical development of the decimal system. In the chapter on fractions, there is a section dealing with installment buying. An optional topic concerning motion is part of Chapter 3. Puzzle problems are prevalent. There are occasional references to science and spacemen.

However, the quantity of social problems found in some older texts is missing from the Ontario books. There are many applications but they are mathematical in nature. In the revised text, most applications are collected in Chapter 8 which is reserved for that purpose.

The authors, apparently, feel that little emphasis should be placed on social applications.

Placement

The authors of the Ontario mathematics text attach importance to fractions. Equations receive less attention than in most elementary algebra texts, either traditional or modern. There is one chapter dealing with rationals.

The topic of rationals is presented in two chapters. Discussion of mensuration is limited to the sections on area and volume.

Sets and sentences, natural numbers, integers, equations, exponents and factors, fractions, rationals and irrationals, geometry, area, volume, and the number plane form the revised list of chapter titles. The list contains most topics found in a traditional algebra course plus others thrown in for good measure.

Structure

The authors have been very careful to develop mathematical structure in the portions of the text which deal with algebra. The emphasis upon structure is not so noticeable in the material dealing with geometry.

In general, however, the authors show strong emphasis upon the structure of mathematics especially in the development of number systems and in factoring.

Vocabulary

The authors themselves have resolved this issue with statements which are quoted here from the preface of the revised edition. "In attempting to lighten the chapter on Fractions we saw more clearly why many students have difficulty with them. The traditional language confuses numbers, numerals, and number pairs. . . . The language has been simplified, yet has been kept sufficiently mature to stretch the student's command of English." The last sentence seems to summarize the issue. While the language of the Ontario text is not so precise as that found in some modern mathematics texts, it appears to suit the maturity of this age-level student to the extent that vocabulary, of itself, is not burdensome.

Methods

In general, material in the Ontario text is presented in traditional lecture-type manner and is followed by generous quantities of exercises. There are a few exceptions in the puzzle problems found at the end of certain chapters. The pattern of presenting knowledge and guiding students toward rationalization is followed. The method by which solution of equations is taught on pages 127 and 128 is a good illustration of this philosophy. Little attempt is made to use discovery methods.

The authors obviously consider that the presentation "of knowledge and helping students rationalize" has much more merit than the method of "presenting a sequence of activities from which a student may come to independently recognize the desired knowledge."

Concepts *vs.* Skills

There is no doubt about the importance the Ontario authors place upon the manipulation of symbols. Large quantities of manipulative problems appear in most topics of each chapter.

Concepts are also strongly emphasized in these materials. Many are clearly and prominently stated. They appear to be more dictated than developed. The authors seem to have used the "tell 'em and drill 'em" approach.

Obviously, the authors place more importance on manipulative skill than upon development of concepts.

Proof

The Ontario text for grade 9 uses proof in connection with deductive algebra in Chapter 2. Presumably some acquaintance with "the nature of proof" has been had earlier than grade 9. The authors include simple algebraic proofs which are well suited to the maturity of the bright students and are not too rigorous for them. Students are encouraged to write proofs of certain exercises in Chapter 2.

Evaluation

The original Ontario text, *Mathematics 9,* contained a series of improvement tests under the headings of "Arithmetical Computation" and "Algebraic Manipulation." There were six examinations at the end of the original text, some were of the essay type. Others required manipulation. The tests were dated Easter and June. Apparently they were intended to be given at those times. There are no tests in the revised version.

Information concerning reliable instruments of evaluation, as well as plans for evaluation in the future, were not made available with the texts themselves.

Mathematics 10

Social Applications

Chapter 11 in the algebra unit of the Ontario material is devoted entirely to the solution of problems involving social applications. In this chapter, the student translates word problems into equations and practices solving them. No new content is presented in this chapter. The variety and number of problems of this type seem to afford adequate experience in this field.

Placement

The tenth grade Ontario mathematics course is about equally divided between geometry and algebra, with the first half presenting geometry. The geometry includes topics such as similarity of triangles and the Pythagorean theorem. The geometry of circles is not included. Some of the topics presented in the algebra are: the real numbers with their properties, inequalities and absolute value, ordered pairs and Cartesian products, factoring, and rational expressions. Both the geometry and algebra are well adapted to the tenth grade level.

Structure

The geometry is generally developed with the use of sets to define terms; however, there is some lack of consistency. The different number systems previously presented are reviewed, and then the properties of real numbers are assumed, but no statement is made as to which system of numbers is to be used. The undefined terms are clearly stated. There are some loopholes in the geometry structure. In the algebra a consistent development is presented.

Vocabulary

There is precision of language used in the definitions. By stating the undefined terms, the student is led to the building of a vocabulary based on a sequence of terms. The presentation leads to the assumption that the student has become capable of realizing the characteristics of a good definition.

Methods

The material is presented with the assumption that the pupil is capable of proceeding with little direction from the teacher. Little opportunity is given for the student to discover new ideas and proofs. Many definitions are given for the first time in the context of problems that use the new terms for the first time.

Concepts vs. Skills

Generally, there is a proper balance between concepts and skills. In the algebra unit some concepts are developed with little drill material available; later, in other chapters considerable drill material is provided with no new concepts being emphasized.

Proof

In the illustration on page 10, which is taken from the ninth grade algebra to demonstrate a proof, no reasons are given for any of the statements made. This tenth grade text presupposes some previous knowledge of proof. The student is presented with the first theorem in Chapter 3 of the geometry unit. The algebra development is presented with reasons being given. The drill in algebra does not emphasize proof, however.

Evaluation

Apparently there are no specific measuring devices available to furnish data as to the learning of mathematics as presented in this project. No information was furnished as to the evaluation or plans for evaluating the new material.

School Mathematics Study Group

INTRODUCTORY STATEMENT

ADAPTED FROM PUBLISHED MATERIALS

In the spring of 1958, after consulting with the President of the National Council of Teachers of Mathematics and the Mathematical Association of America, the President of the American Mathematical Society appointed a small committee of educators and university mathematicians to organize a School Mathematics Study Group whose objective would be the improvement of the teaching of mathematics in the schools. Edward G. Begle was appointed director of the study group with headquarters at Yale University. The organizing committee also appointed an advisory committee, consisting of college and university mathematicians, high school teachers of mathematics, experts in education, and representatives of science and technology, to work with the director. The National Science Foundation, through a series of grants, has provided very substantial financial support for the work.

The world today demands more mathematical knowledge on the part of more people than the world of yesterday, and the world of tomorrow will make still greater demands. The number of our citizens skilled in mathematics must be greatly increased. An understanding of the role of mathematics in our society is now a prerequisite for intelligent citizenship. Since no one can predict with certainty his future profession, much less foretell which mathematical skills will be required in the future by a given profession, it is important that mathematics be so taught that students will be able in later life to learn the new mathematical skills which the future will surely demand of many of them.

To achieve this objective in the teaching of school mathematics, three things are required. First, we need an improved curriculum which will offer students not only the basic mathematical skills but also a deeper understanding of the basic concepts and structure of mathematics. Secondly, mathematics programs must attract and train more of those students who are capable of studying mathematics with profit. Finally, all help possible must be provided for teachers who are preparing themselves to teach those challenging and interesting courses. Each of the projects undertaken by SMSG is concerned with one or more of these three objectives.

The School Mathematics Study Group set up a number of projects for itself and from time to time has added to these. Its written text material and teachers' commentaries cover grades 4 through 12, with two levels of materials in grades 7, 8, and 9. The basic philosophy is that a greater substance should be introduced earlier in the mathematics sequence. There follows, of course, the opportunity to carry the sequence further than previously had been the case. The text materials were outlined and written by teams of scholars representing all areas interested in mathematics education.

The original editions were used widely in selected schools under area supervision of local people who had participated in their development. The teachers met in seminars and studied the materials, and at the same time provided much constructive criticism. These criticisms were received by the writing groups the following summer, and appropriate rewriting was carried out. It was recognized that teachers needed inservice training if they were to effectively teach the material; and consequently teacher materials and supplementary aides were provided.

There have been two major studies of results obtained with School Mathematics Study Group materials, and a third one is now in progress. The results as yet are not conclusive. But it should be noted that the studies are concerned with measurable gains in student performance over a one-year period only.

The question may be raised as to conditions under which this program might be introduced. A general frame of reference would appear to include: (a) the teacher should understand what precedes and follows the area of instruction for which he is responsible, (b) the focus is an understanding of mathematics, (c) time will be required for the teacher introducing this material for the first time, (d) inservice assistance is very beneficial if the teacher shares in this belief.

COMMITTEE REPORTS

Mathematics for the Elementary School
Grades 4-6

At the time this report was underway, the SMSG texts for grades 4, 5, and 6, which were used as a basis for the following analysis, were in their original experimental edition, and the revised texts were not available. Therefore, the comments of the analysis committee should be interpreted accordingly.

We considered omitting this analysis from the material but felt that teachers would want to know that there are experimental programs in grades 4, 5, and 6; for, on the basis of this information, the reader could then determine if he wished to obtain the revised materials for further consideration.

Your attention is also called to the comments of the director pertaining to grades 4, 5, and 6 for they will be helpful in your analysis.

Social Applications

Sections of the reviewed material give considerable attention to social application. This can be found in some of the exploration questions, in references to historical development, and certainly in the problems contained in the units involving the four basic operations. But in general, this material gives limited attention to social usage. The potential for social application was evident in much of the content, and attention could have been given to the aspect of social usage without sacrificing the mathematical integrity. Many of the problem exercises were highly suggestive of social application. However, it was clear that the problems posed were placed there primarily to demonstrate the application of a process or a concept. Relatively few of the problems posed were the kind which made the learner aware of the need for a new mathematical process and the concomitant conceptual development. Purpose, which becomes apparent to the user through the social application of his mathematical learnings, seemed often neglected. In the units on linear measurement, angle measurement, and recognition of common figures, some references to everyday usage were considered. But here, too, such consideration was sparse in terms of the development of the purely mathematical aspects. The three units devoted to the theory of sets were just that, and the situations contrived were concerned with operation within the theory rather than with social application.

Placement

It seems that each of the SMSG prepared materials for the elementary school is appropriately placed. Indeed, some of the ideas of almost every grade 4 unit probably should be given attention at the outset of any formal arithmetic program in the primary grades. Likewise, some of the ideas of the grade 5 and grade 6 units might be given attention at a lower grade.

Structure

The materials prepared by SMSG seriously attempt to reflect some of the structure of mathematics. Regarding structure, the materials pretty thoroughly accomplish these objectives: show the doing and undoing relatedness of addition and subtraction, and of multiplication and division; by using the number line, show the interrelatedness of the four basic operations; display to the learner the commutative, associative, and distributive laws as they apply to the fundamental operations; provide the learner with ideas and language related to number bases other than ten, factors, prime numbers, integers, and geometric concepts beyond what is usually found in an elementary mathematics program. But in tending to these objectives, there is neglect in tieing all these mathematical ideas together. For instance: the ideas and language of sets are rarely used in connection with the fundamental operations; the exercises involving number bases other than ten become manipulative tricks and do not shed much light on the base ten systems; because of the unit separations, there is really very little application of the four operations to anything except practice of the operations; the unit arrangement isolates, in a way, multiplication and division further from addition and subtraction than is done in many programs. It seems that the SMSG program gives more attention to mathematical structure than most elementary programs, then divides this structure into units that are often unrelated, or only incidentally connected.

Vocabulary

With only a few exceptions, we found the precise and sophisticated mathematical language of this program to be suitable for the children to whom this material might be presented. Generally, word designations for the concepts were clear and appropriate, and probably suitable for anyone intelligent enough to acquire the concepts involved. The precise mathematical vocabulary was generally sound throughout the program. It seems that this material may have gone too far in introducing a variety of new mathematical symbols. Letters of the alphabet, both capital and small, are used in profusion. Braces with sets and parentheses in sentences are used without much buildup. Then when these symbols, — $\cup$, $\cap$, $<$, $>$, $\angle$, $\neq$, $\longleftrightarrow$, $\underline{\overrightarrow{AB}}$, $*$, $\underline{m\overline{CD}}$, -(-(1)), 0/4, and 1 0/2 are introduced, the abstraction and intricacies of symbolization may interfere with the logical thinking of many children. We found the usage of the words "number" and "numeral" to be accurate throughout. The distinction was clear and concise.

Methods

In some sections of these materials, the learner's right to form his own generalization has been hindered by including declarative statements such as: "We know," "We can see," "We find," and "We could write this," rather than leading questions such as: "What do we find?" "What do we know?" "How could we write this?", which might enable the student to form his own generalizations. Ideas are used in one unit and developed in a later unit, and the units do not reach out to all of the mathematical ideas developed in previous ones. For example: "counting numbers" are used in unit EA1 and defined in unit EA2; the number line is used in unit EA2, while the properties of a line are defined in unit EA5.

Concepts *vs.* Skills

The program comes very close to providing a proper balance between concepts and skills. Suitable attention is given to the development of a concept, and then exercises and activities are given for practice. From time to time, through short exercises, children practice previously learned skills and apply concepts learned earlier. Often the increasing difficulty of manipulations is used in leading pupils to new and broader generalizations and concepts. In these materials we find ideas and manipulation progressing together, one often leading to the other and conversely.

Proof

The SMSG material does very little with proof. Many opportunities exist for justifying, or proving, an answer, but they are not taken advantage of or emphasized. The approach seems to be—"Now you see that, so what is this?" There are few questions such as: "How do you know?" "Would you get the same answer if—?" "Is there another way to get the answer?" "Are the answers the same?" "Why?" "Which answer is the better one?"

Evaluation

For our purpose, we must assume that program changes are aimed at improving the achievement of the following objectives: developing greater computational skill, developing the ability to think mathematically, developing insight into the structure of mathematics, developing favorable attitudes toward mathematics. The evaluation can then be made in the light of these objectives. Evaluation within the body of content was treated pretty much as the issue of proof. It was either overlooked or mastery was taken for granted.

Introduction to Secondary School Mathematics *Volume I* and *Volume II*

(These texts are on a different level from *Mathematics for Junior High School, Volume I* and *Volume II.*)

Social Applications

There seems to have been a real effort to make use of social applications to help the student picture what has gone on mathematically. The applications included topics such as family budgets, commissions, and discounts.

Placement

The level of difficulty of these books does not appear to be beyond the ability of students of this grade level. Some sections involve more difficult ideas; the students will need to spend some time with these, and the teacher will have to give a careful presentation of the material. However these are the very ideas which are of greatest value and which the student should enjoy working with most.

Structure

In the introduction of new material the book relies on both physical applications and mathematical properties. There is about the proper balance between these two ideas.

Vocabulary

The authors are careful in their use of words and make an attempt to see that the students understand such terms as number, numeral, rational number, etc. The authors have, however, not become unduly picky in the proper use of terms and symbolism.

Methods

The discovery method is not too much in evidence in this text in that the right questions are asked but the answer usually follows on the next line.

Concepts *vs.* Skills

There seems to be a good balance between time spent on concepts and time spent on skills.

Proof

The beginning ideas of proof are encountered in part two of *Volume I*. Most of the ideas are left to the teacher to develop.

Evaluation

The teachers' manual supplies sample test questions to further help the teacher understand the ideas and principles which are thought important by the authors, and shows ways of testing the teacher's understanding.

Mathematics for Junior High School
Volume I and *Volume II*

Social Applications

Applications are often used to introduce new ideas from which mathematical concepts are carefully developed. The establishment of mathematical concepts is then followed by numerous exercises pertaining to the physical world.

Placement

Many of the topics which are traditionally taught are treated in a somewhat unusual manner. Additional topics that the authors feel are appropriate for this grade level are also introduced. Some of these are: mathematical systems, statistical topics, elementary number theory, and scientific notation.

Structure

Mathematical structure is definitely emphasized, particularly in the development of the real numbers. With the exception of the chapter on mathematical systems, the idea of structure is largely developed intuitively.

Vocabulary

The authors use precise language as well as concise, natural definitions. The vocabulary used is understandable, and any new terms are clearly defined.

Methods

The text, in general, is organized on a conventional basis by introductory illustrations, explanation, and then exercises for reinforcement of the concept. This would fit naturally into a lecture-discussion method of teaching, but an instructor might, with some ingenuity, use the text in conjunction with discovery methods.

Concepts *vs.* Skills

The materials stress concepts over skills, although skills are not ignored. An ample number of exercises is given to develop skills.

Proof

There seems to be little emphasis on formal proof at this level, but students are often asked to give reasons for their action. Some simple deductive proofs are given in *Volume II.*

Evaluation

Since criteria for evaluation are not agreed upon at this time, these texts cannot be evaluated objectively.

The authors do give sample questions in the teachers' commentary at the end of each chapter to aid the instructor in evaluating the student's mastery of the subject matter included.

Introduction to Algebra

(This text is on a different level from *First Course in Algebra.*)

Social Applications

The reviewers feel that this material satisfies many of the demands made for the inclusion of social applications by giving a sufficient number and variety of word problems spaced throughout the text. The problems are random in regard to type, and thus the student is not pointed directly toward the solution. Although the authors have not come up with anything new in applications, they have succeeded in handling the traditional approach in a better-than-average fashion.

Placement

On the whole, the material and topics used in this text are appropriate and well placed. It is particularly gratifying to see inequalities presented with equalities. It is felt that the relation, and function, concept could be introduced much more firmly and certainly earlier.

Structure

The SMSG authors did not bring structure into play in this particular text. An experienced person would read the necessary structure into this material and a trained teacher could bring it out, but the general nature of any mathematical structure seems hidden from the student. In an apparent attempt to lead the student from the concrete to the abstract, the authors have fallen short of their goal: by failing to label a structure when it occurs; by not drawing attention to the unifying concepts of operations within sets; by not carefully distinguishing between postulates, definitions, and theorems; and by allowing many fundamental terms to remain undefined and intuitive.

Vocabulary

The vocabulary used in this material certainly does not tax the student. On the contrary, there are very few new terms introduced, considering this is a year's study. Such terms as might be considered new to the student are introduced casually and often without benefit of firm, decisive definition. It is felt that the authors did a very commendable job by making the material very readable for the student.

Methods

As has been noted, the material is quite readable and moves very well to lead the student by a series of concrete examples and illustrations. The student is given ample opportunity to draw conclusions and make generalizations. It is regretful that the material is not better spaced. Toward the end of each of the four parts, the authors seem to crowd many ideas into a relatively small space. In this regard it seems that the student is left dangling and is not given sufficient cause or opportunity to assimilate, round-out, and summarize the ideas and concepts developed.

Concepts *vs.* Skills

There is a good balance between the practice of manipulative skills and the realization of concepts, and certainly the extent of drill problems is sufficient to satisfy the most exacting taskmaster. The student would probably be able to develop as much skill with this material as with any other text.

Proof

The authors evidently felt that mathematical proofs of any nature were not appropriate. There are only a few instances of proof in the text (pp. 64, 65; Part 2).

Evaluation

This committee feels that, inasmuch as it has no evaluation devices, it cannot make a statement on evaluation. It should be noted that one member of the committee has had experience in teaching earlier forms of this material, and he feels that subjective evaluation has been sufficiently positive to warrant continuation of the material, and that this particular material, at the level for which it is intended, shows decided improvement and is very usable and rewarding.

First Course in Algebra

Social Applications

The SMSG *First Course in Algebra* attempts to get the student to think in terms of general structures and patterns so that he may apply them to specific social situations. The material stresses the development of the ability to make and use the tools of structural mathematics, not simply the ability to manipulate already developed tools. Problem sections *per se* are found in the SMSG material as in a traditional algebra text. As a particular mathematical tool is developed, social problems using it are introduced instead of presenting a type illustration followed by many problems.

Placement

The concepts of relation, ratio, and proportion are limited, and little material is included on statistics. Set operations such as union and intersection are not treated. Ordered pairs are introduced before work on relations leading into the concept of function. An introduction to proof is made in the ninth grade SMSG algebra with the introduction of some intuitive ideas of logic. Variables, order, relations, absolute value, sentences, and factoring are among the topics which are thoroughly covered.

Structure

By presenting leading questions and through verbalization, the SMSG material gives the student ample opportunity to discover the basic structure of algebra.

Vocabulary

Introductions to mathematical vocabulary throughout the SMSG *First Course in Algebra* are intuitive, but precise language is used after this first introduction. Extensive use is not made of symbols such as $\forall_x$, $\exists_x$, and ε.

Methods

The School Mathematics Study Group's ninth grade material develops most of these topics by using a discovery approach. Some of the exercises, however, are too independent of preceding problems, and there is little evolution of ideas through the sequence of some problem sets. The review exercises at the end of each chapter and the excellent "leading" discussions preceding the problem sets compensate for this deficiency. The spiralling of topics aids the student toward mastery of concepts and skills. The teacher's commentary helpfully provides background and warnings of possible pitfalls.

Concepts *vs.* Skills

The SMSG material maintains an interdependence between concepts and skills. The multiplication of polynomials is presented as an instance of the distributive principle which develops understanding and deeliminates the need for many special cases. Problems provide for developing skill of translation as well as for furthering understanding of the concepts involved. The units on addition and multiplication of the real numbers and exponents are developed through extensive use of rules.

Proof

The student is introduced to the number line in an intuitive development of the properties of the real number system, and he proves a few theorems. The student sees his first deductive proof of a theorem toward the end of Chapter 6, where he is also introduced to a proof by counterexample. In Chapter 7, all students are introduced to inductive and deductive reasoning, and the more capable student is given the opportunity to provide reasons for the steps in a deductive proof. Near the end of Chapter 8, some theorems are proved pertaining to inverses and their uniqueness, identity elements and their uniqueness, reciprocals and inequalities. The proofs of some other theorems pertaining to these concepts are left to the student as exercises.

Evaluation

There is a teacher's commentary which provides suggested test items and review problems for each chapter.

Geometry

Social Applications

There are few problems in SMSG *Geometry* (revised) that are of a social nature. They generally occur in the intuitive background sections, but occasionally a physical situation is described in a problem to illustrate the theorems being discussed. The reviewers feel the nature of the material is such that an occasional reference to this type of problem is sufficient and that the balance is properly maintained in this text.

Placement

To the extent that it profitably improves the development of the topics presented, SMSG *Geometry* (revised) develops the structure. It is easy to observe the gains in simple precise definitions and simplified correct structure. The changes from standard topics are, as stated, "well thought-out" and profitable to incorporate.

Structure

In the SMSG *Geometry* (revised) mathematical structure is emphasized, and the students are necessarily taught a better understanding of the philosophy and use of mathematics. The process is to examine how the mathematics develops, its formalization, and then its applications.

Vocabulary

Whenever a new area is introduced in the SMSG *Geometry* (revised), there is a small intuitive discussion of the terms, but it seems that the students should be familiar with some informal geometry. Then the definitions are formulated in precise terminology.

The authors have structured the system in a manner that allows precision in the definitions without too much length. For example their development of angles is refined and very interesting. After the formal definitions are presented, the students are expected to use the precise terminology.

Methods

Throughout the SMSG *Geometry* (revised) all the students are participants. The authors lead them through the intuitive processes that establish a conjecture, and then to the formal proof. Throughout the text, the problems are well selected to put the students into the system as opposed to presenting the system for the students to use.

Concepts *vs.* Skills

In geometry the main function is the development of concepts. SMSG *Geometry* (revised), with its metric approach, does contain considerably more algebra than the usual geometry text, but in no part is there emphasis on manipulation of symbols.

Proof

The authors assume that the students have previously been made aware of what a mathematical proof is, and that the students have had some experience with proofs. The proofs given in the text are generally complete and rigorous. When there is an occasional instance where the proof is too involved, the authors state this and then they explain the situation well in the *Teacher's Commentary*. There is no attempt to "disguise the gaps." The early statement that "proving of theorems is not a spectator sport" illustrates that the SMSG *Geometry* (revised) puts the students in a proof-construction situation as soon as possible. There are sections discussing the "if—, then—" relation and proof, but formal logic is not included.

Evaluation

A rough count of 440 review problems and 300 illustrative test items indicates that the SMSG *Geometry* (revised) has plenty of valid questions to measure the student's progress. The illustrative test items and answers are entirely contained in the *Teacher's Commentary*, so they can easily be employed to evaluate the changes taking place in the students.

The School Mathematics Study Group itself is constantly evaluating the text, its effect on students, and its effect on the curriculum.

Intermediate Mathematics

Social Applications

In terms of the total number of problems, the text contains few social applications. In some places in the discussion of equations of various kinds one finds a small number of problems related to the physical sciences, such as falling bodies, rates, and mixtures. The number of such problems is small in comparison with many of the current conventional texts. The one exception to this is in the work devoted to vector systems. Here there are numerous problems involving forces, velocities, work, etc. It would thus appear from a study of the text that the individuals who wrote the SMSG *Intermediate Mathematics* believed that applications, social or otherwise, were not sufficiently important for those students who would be taking this course to warrant the inclusion of many such problems.

Placement

The text includes essentially all topics ordinarily encountered in intermediate high school algebra and trigonometry texts. Also one finds much additional material. There is an extensive development of the structure of the natural numbers, the integers, and the rational numbers. The structuring and development of the real number system is not carried forward in the same detail; however, the spirit of approach is established. The extension is then made to complex numbers. The course also includes an extensive study of topics from analytic geometry and vector algebra. The authors apparently believe that students at this grade level are capable of and ready for a fairly detailed approach to the properties of the number systems—naturals, integers, rationals, reals and complex—vector algebra, algebraic structures, and analytic geometry, as well as the usual topics ordinarily included in a third year course in high school mathematics.

Structure

It is obvious that the study of mathematical structures would play a key role in a mathematics course based on this text. A major portion of the first chapter is devoted to re-examining the natural numbers, the integers, and the rational numbers from the point of view of their basic structural properties. Later in the chapter the real number system is characterized by its basic properties. In Chapter 5, the complex number system is developed. The chapter includes a short but illuminating section on the conditions required for the extension of a number system. The chapter on vector algebra approaches the topic from a structural point of view, including a discussion of the isomorphism between the system of vectors and forces, and between the system of vectors and the system of complex numbers. Finally, the last chapter includes discussions of commutative and non-commutative groups and fields. It appears that about 25 to 30 percent of the text deals specifically with structure and its application in the manipulation of algebraic expressions. The authors have made an attempt to build a course which emphasizes the structural concepts, particularly as they relate to the number systems and associated algebraic manipulations. Simplification of algebraic expressions and the solution of equations are founded on the properties of the number system under consideration.

Vocabulary

There is no evidence in the book that the authors regarded precise and sophisticated use of language on the part of the student as an objective of prime importance. This is not to imply that the book is written in a careless fashion; it is not. Rather, the implication is based on the fact that there is no discussion centering on the nature of a definition, no specifically planned opportunity for the student to produce definitions of his own, and no discussion of what is required in relation to correct use of language in mathematical exposition, for example, in the writing of a proof or an original exercise.

Methods

The writers seem to believe in presenting material to the student rather than having him participate in its development. The exposition is good, but is written in such a way that it would seem quite natural to have the student both read and be lectured to, and also to engage in discussion. It would, however, be difficult for a teacher to have students participate extensively in the development of new material. Rather, the format and style of this book are such as to advocate, if not urge, a teaching method in which the final product is presented to the students. Exceptions are the "Challenge Exercises" included at various places. These exercises are not included, however, for the purpose of developing basic material.

Concepts *vs.* Skills

The text clearly makes an attempt to develop both concepts and skills. It is also clear that the latter are given less emphasis. There is extensive exposition of each topic, followed by exercises to reinforce the mathematical concepts. It is only later in each section that exercises of a skill-building or maintaining type are given. There do not seem to be enough of these if the review exercises are not used. Apparently the philosophy of the writers is that concepts should be presented first with considerable rigor, and that related skills should be developed later.

Proof

Many proofs appear in the exposition throughout the book. A few are left as exercises. Many exercises call for proofs, although most of these call for little ingenuity since they follow the same patterns as those included in the expository material. The proofs in the exposition are for the most part fairly rigorous. In fact, they are more rigorous than those ordinarily found heretofore, for example, in college algebra and analytic geometry texts. Little space is devoted to discussing how to make a proof or the nature of proof. The logical basis for solving equations and inequalities is covered rather briefly. Little reference is made to this discussion in subsequent work. Apparently the writers expect that the students will be fairly competent in developing proofs when they begin this course, or that they will acquire the skill and an understanding of the nature of proof from the exposition and the exercises.

Evaluation

There is a teacher's commentary which provides suggested test items and review items for each chapter.

Elementary Functions

Social Applications

Examples from the sciences and other areas of knowledge are used quite effectively in the introductory paragraphs in most of SMSG's

Elementary Functions. Thus they serve to remind the student of the usefulness of the various topics under consideration here. The variety of situations mentioned is quite extensive. It includes not only some of the familiar maxima and minima problems, but such items as the law of cooling as applied to a cup of hot coffee and adequate presentations of the simple harmonic equations. As a result, the mathematical theory develops in a logical sequence and provides what appears to be a proper balance between theory and applications at this level of study. Verbal problems are not included in every set of exercises, but it would seem that the prior reference to application has served the purpose of providing the student with a better awareness of the usefulness of mathematics in a variety of situations.

Placement

The topics presented in *Elementary Functions* seem to be quite satisfactory for this level of study. Previous volumes in the SMSG series have laid the necessary ground work. Students who continue their study of mathematics will have received more than adequate preparation for the work that follows.

Structure

The structure of mathematics is recognized easily throughout this volume.

Vocabulary

New vocabulary is presented gradually. While most of the important terminology was introduced in earlier volumes, the language used here is much more sophisticated and precise. The vocabulary appears to be at the level which is being used in other academic subjects at this same grade level.

Methods

The student is led to recognize basic principles and to develop the necessary proofs in a way which is far superior to that of simply stating the principles and showing how to apply them to routine-type exercises. This approach may take longer in the matter of time spent on the subject, but as a result one should find much greater progress and understanding at future levels of instruction.

Concepts *vs.* Skills

Throughout this text, stress seems to be placed on developing the basic concepts; far less space than usual is devoted to the acquisition of certain skills. Whether this treatment is adequate is a question which can be answered best after further testing and evaluation of progress.

Proof

The authors assume that considerable attention has been given to proof in the earlier years. Thus it is possible to devote more space to rigorous proofs here. This emphasis is good for the student in that it should contribute considerably to his mathematical maturity and provide him with a growing understanding of proof as related to future study in various areas of mathematics.

Evaluation

As far as is known to members of the subcommittee, no valid measures have been found at this level of instruction and for this area of subject matter. The *Teacher's Commentary* contains some excellent test items at the end of each chapter. Later chapters give attempts to incorporate some of the concepts and principles developed in earlier chapters. The importance of these measures cannot be ignored. The suggested test items seem to provide an excellent step forward in the experiment at this point.

Introduction to Matrix Algebra

Social Applications

In the introduction of matrices at the beginning of this textbook, their social application is indicated through examples from the lives of the students (concerning baseball and TV); their use in industrial problems and elsewhere is also pointed out. The students are asked to find examples in their readings of such rectangular arrays. The introduction of the multiplication of matrices and the need for defining multiplication as it is defined is also shown through examples from the experiences of the students.

Once the point has been made that there are social applications, there is no further mention of direct social applications in the book, and rightly so. The work at hand becomes the development of an algebra that has many applications. However, throughout the book there are many indirect references to social applications, such as the use of matrices in solving systems of equations.

Placement

At the twelfth grade level, the topic of matrices is an excellent choice for starting the student in an easy way on the subject of linear algebras. He should at least be introduced to the subject before going to college. The topic is developed in such a way that the structure of the real number system can be studied at this time, if this has not been done before. If work with the real number system has been done before, it can be enriched and enhanced here. The need for such study becomes apparent in the development of matrix theory. The topics in this text are neither too difficult nor too abstract for a high school senior.

Structure

In this textbook structural knowledge of the real number system is strengthened through comparison with the characteristics of matrix multiplication and addition. The differences between a field and a ring are illustrated by showing the differences between the structure of the set of real numbers and that of the matrices, with their respective additions and multiplications. As a matter of fact, if a student had not been previously introduced to the structural properties of the real number system, it would be possible for him to gain that understanding through this course, with the additional motivation of the need for that understanding in order to prove properties for the matrices.

For increasing understanding of structural properties, this volume includes a number of examples of abstract sets with their operations. The student is asked to check these systems for various simple structural properties, such as closure, commutativity, associativity, etc., and in later exercises to determine whether they are rings or fields.

The structural ideas are introduced as they are needed in the development of the matrix algebra. For example, the concept of groups is not introduced until after the concepts of field and ring. This is due to the fact that the multiplicative inverse cannot be introduced as advantageously at an earlier point, and it is through the example of the 2 x 2 invertible matrices that the group concept is introduced. As is customary throughout this text, once the group concept is brought in by the use of matrix examples, it is followed by other examples, mostly from sets with which the student is already well acquainted.

Throughout the book there are many ties between structural systems with which the student has already had experience and those being introduced. Isomorphism is discussed in terms of the correspondence between the complex numbers and matrices.

After the basic structural ideas are thoroughly developed for the set of matrices with the operations of addition and multiplication, the structure of algebras in general is discussed.

One of the powerful things about this book is the fact that students can see the development of the simple 2 x 2 matrix theory lead to something as complex as vector spaces.

Vocabulary

The modern mathematical vocabulary used in this text is such that good students having studied SMSG, UICSM, or any of the other "modern" materials should be able to follow explanations, proofs, and problems without too much difficulty. If those seniors who have followed a traditional high school mathematics sequence of courses choose to study this text, some ground work with the vocabulary used in the "modern" programs would need to be considered. As a whole the vocabulary is not unsophisticated, and it seems to be fully intelligible to students properly placed in the course.

Methods

This text seems to have employed several methods by which to present the material at hand. The basic theorems, for the most part, are proved for the student. This sort of presentation leads to building confidence in the student when it comes to proving auxiliary theorems and in doing problems based on the theorems already proved. The text also provides the discovery approach. Many of the problems themselves seem to have imbedded in them opportunities to discover the ideas necessary or convenient in solving subsequent exercises. This approach is much appreciated by the high school senior whose interest in mathematics is already stimulated to the point where he is interested in and chooses to study matrix algebra.

Concepts *vs.* Skills

In high school, concepts need to be developed as far as skills which use these concepts can be taught and tested. That mathematics which in its purest, distilled form delights a mature mathematician will not be vitally interesting to a high school student for itself alone. The text seems to have struck a proper balance. The sets of exercises extend the material developed in the text by giving the student an opportunity to apply the methods of proof he has just seen to original problems. This keeps his attention focused on the logical structure and development of the subject. Simultaneously, however, he has exercises where he applies the theory "to get answers," and this is essential for the high school student.

Proof

Simple deductive proofs are demonstrated early in Chapter 1 after the student has been introduced to matrices in a thorough fashion and some intuitive developments have been made with them. Several techniques are used to help the student develop his own ability at making proofs. Parts of the proof are given for some theorems, with the remainder left to the student. Also, pitfalls which may occur in his development of matrix proofs are pointed out to him. For example, because he has been dealing for so long with the real numbers, where the property of commutativity for multiplication has been constantly called on, he is shown how the use of this property for matrices would lead to erroneous conclusions. Another common feature of this book is that many of the concepts which will be developed fairly rigorously in a subsequent chapter are presented in an intuitive fashion in the exercises of the preceding chapter; this helps the student discover ideas for later proofs.

The distinction between the proof of a theorem and the proof of its converse is made by setting up these proofs in two separate theorems, with the second theorem pointed out as the converse of the first.

Indirect proof is used, but rarely.

The distinction between analysis and synthesis of proofs is developed near the end of the book with the inclusion of examples.

Evaluation

It would probably be advantageous to the teaching and learning of this material if the authors had provided some reviews and self-administered tests at the end of at least every other chapter. Then in the *Teacher's Commentary* some suggested test questions could have been included. This procedure would have been a great help to both teachers and students.

CONCLUDING COMMENTS

E. G. BEGLE

These analyses of the SMSG texts for grades 7 through 12 will be very encouraging to the authors, since they indicate that the authors succeeded in doing what they set out to do.

Only one minor dissent is called for. The authors of the junior high school texts felt that the texts would be quite compatible with the discovery methods of teaching. Classroom experience seems to have borne them out.

On the other hand, the analysis of the SMSG texts for grades 4, 5, and 6 seems to be based on a fundamental misconception of the nature and purpose of the materials examined. In the first place, the analysis seems to have been based on the first (1960) experimental version of this material. The inevitable rough spots, only some of which are mentioned in the analysis, were discovered through classroom tryouts and smoothed out to the best of the writers' abilities in later revisions. On the other hand, some of the alleged rough spots proved in classroom use to be otherwise. For example, most children found mathematical symbols easier to use than the phrases they represent.

It was not the intention of the SMSG writers to prepare, in the preliminary version, a complete "program" for grades 4, 5, and 6; but, rather, to explore ways of developing mathematical ideas. Incorporation of these into a program, including decisions as to the emphasis to be placed on social applications, teaching methods, etc. is another matter.

Developmental Project in Secondary Mathematics at Southern Illinois University

COMMITTEE REPORTS

Elementary Concepts of Secondary School Mathematics
and
Concepts of Secondary School Mathematics, Book 1

Social Application

The material makes some attempt to include applications, but these are not emphasized. In light of the difficulty of providing good applications for ninth grade algebra, the applications in this course are as good as can be expected.

Placement

The two books (presumably designed for the ninth grade, though this is never specified) include the usual algebra through quadratic equations plus such new topics as elementary work with sets, different numeration systems, some discussion of the natural numbers, inequalities, simple number theory (primes, factors, even and odd numbers, a few proofs), and some statistics including mean, median, and mode. The students are given a sound intuitive feeling for conditions and composite conditions. There is a section on graphs early in the book—even the rather slow introduction does not detract greatly from the chapter on graphing.

Structure

It is unfair to say that this work is not structured. It is questionable whether the pupils will get a sense of structure from the material because of the way in which it is presented. For example, it is difficult to find any reference to grouping symbols or the associative law before page 692, although they were used throughout the text. Since there is no index, there may be some other reference to the associative law or grouping symbols but if so, it is not easily apparent.

Vocabulary

The vocabulary used in these books is quite sophisticated and is in fact radically different from that used in most ninth grade courses. The following examples of definitions will indicate the magnitude of the departure from common usage and the degree of sophistication:

Algebra is the study of variable sentences. (p. 72.)

Let m be a monomial. Then by V(m) we mean the factor of m containing only variables. By C(m) we mean the rational factor of m. Note that we call C(m) the coefficient of the monomial. (p. 476.)

Let m_1 and m_2 (read 'm sub 1 and m sub 2') be any two monomials. Then $m_1 \cdot m_2 = C(m_1) \cdot C(M_2) \cdot V(m_1) \cdot V(M_2)$. (p. 477.)

A binomial in one variable is a mathematical term of the form $ax + b$ where x is a variable and a and b are constants with b not zero. (p. 555.) (From the context, it is clear that the last *b* is a misprint and should read *a*.)

A trinomial is a mathematical expression of the form $ax^2 + bx + c$ where a, b, c are constants, and x is a variable.

From the last two definitions, as the text points out, x^2 is a trinomial and x is a binomial. The reason for these definitions is to avoid confusing an object and the name of the object—apparently the authors want the binomial not to be the symbol written on the paper, but rather the object which the symbol names. Thus in order to avoid calling $x + x$ binomial (because it equals $2x$, a monomial), they produce their definitions.

Method

The general method of these books is to make statements with no emphasis on discovery and relatively little attempt to help pupils rationalize. Apparent contradictions of the child's common sense often appear without sufficient explanation of the need for being careful. Many topics seem to be developed quickly, e.g., the concept and phrase, "completing the square" occurs for the first time, without explanation, in the derivation of the quadratic formula (p. 624). There is a tendency to use the English language rather than algebraic symbolism even when the latter would be more convenient.

Concepts *vs.* Skills

After a concept has been developed, drill does not seem to be present in as great a quantity as is common in most textbooks. For example, after the concept of multiplication of integers has been developed, there are ten oral exercises before a new concept (the distributive law) is introduced.

Proof

Some algebraic proofs are included, but these are somewhat different from usual. A typical proof that $17 + (-7) = 10$ is given below:

We Know	Therefore	We Can Conclude
$17 = 10 + 7$	————	$17 + (-7) = 10 + 7 + (-7)$
$7 + (-7) = 0$	————	$10 + 7 + (-7) = 10 + 0$
$x + 0 = x$	————	$10 + 0 = 10$

This is the first proof in the book, but similar proofs follow.

Evaluation

Presumably any text material should be evaluated in terms of the goals set by its authors. The goals of the authors can, in turn, be discussed and judged in light of the philosophy or prejudices of the evaluators. Since there is no teacher's edition or notes, the goals of the authors had to be determined from the text material itself.

Concepts of Secondary School Mathematics, Book II

It is extremely difficult to be fair in analyzing this book without examining the materials for an earlier level or those that are to follow. From the information received, the committee has examined the book as one for tenth grade.

Social Applications

With the exception of the section on shadow reckoning, applications are almost exclusively in the exercise material. The problems in algebra are traditional ones. There is slightly less emphasis on word problems (applications) than in usual texts; for example, the work on radicals is in no way related to the Pythagorean theorem and related applications. The purpose of the applications seems to be to provide practice with previously developed ideas.

Placement

There is no demonstrative geometry. The algebra topics are those usually associated with grade 9, or earlier in some programs. The treatment of mathematical logic is more formal and more condensed than that usually found in grade 10.

Structure

Attention is given to mathematical structure in the logic section, but consistent attention to structure is not given throughout the text. The book is a collection of isolated topics.

Vocabulary

There is an attempt at precision of vocabulary although it is sometimes unusual in nature. The vocabulary seems artificially sophisticated, and includes terms such as *integral quadratic surd* and *Modus Ponens*. It is also sometimes inconsistent. For example, a circle is viewed as a set of points such that each is the same distance from the center, while a triangle is viewed as the region formed by three lines. The transition from easy, ordinary language to precise language is often abrupt. For example, the section on proof dealing with the four logically equivalent statements of implication comes very quickly and the statements are given little explanation. Ideas of necessary and sufficient conditions are introduced with no explanation.

Methods

The method is predominantly one of exposition followed by exercises. Little attention is given to discovery except in selected exercises. In some cases pupils are asked to make generalizations on the basis of a very few specifics.

Concepts *vs.* Skills

In some cases, basic mathematical concepts are developed effectively. There are adequate and appropriate practice exercises and they seem to be of varying levels of difficulty. However, much of the work particularly on algebra, seems oriented towards attaining manipulative skills. In some places there are superficial or cursory explanations followed by long sets of exercises.

Proof

Proofs which the student does are limited to one or two steps. Perhaps the section on proof is intended for later use when more extensive attention is given to the topic.

Evaluation

At the end of each chapter there is a set of exercises. The exercises are divided into three groups, according to difficulty level. No end-of-year test is included.

Social Applications

Except for the entire chapter on "Coalitions," social applications receive very little emphasis. Evidently the intent of the author was to emphasize the theory of sets much more than social applications. All of the applications make use of the set theory, but only a small portion of the set theory developed is utilized in the applications. It is thus inferred that the author does not consider numerous applications to be necessary for practice in the newly learned concepts. Nor does it seem that he feels social applications need to be utilized heavily in developing problem solving ability.

Placement

The author presumably anticipates that the concept of set, as well as some uses of union and intersection of sets, have been introduced earlier in the student's program. This unit, designed for grade 11, is intended to be a thorough development and an amplification of the theory of sets. The development requires no prior knowledge of sets.

Structure

Opportunities abound in this unit to emphasize structure. For example, items are developed which are the same or similar in structure to other items previously experienced in mathematics. The similarity of these items is not pointed out. The important structural concept of forming a mathematical model of a physical or social phenomenon is developed in several places, both in general discussion and in examples.

Vocabulary

Technical mathematical vocabulary, especially that which pertains to the topic under study, is used with precision and is developed in accordance with predominant current practice.

Methods

In an over-all sense, preceding chapters lead the student to much of the content of Chapter 6, "An Axiomatic Development of Set Algebra." However, not nearly all of the particular axioms are developed through guided experience. Some specific concepts are developed, while others are stated. Some inductive developments are experiences in how mathematics is created, while some definitions are followed by examples which are not preceded by specific instances.

Concepts *vs.* Skills

Both concepts and skills receive attention. Probably there are not enough exercises to develop some of the specific skills such as performing the union and intersection operations on sets, applying the binomial theorem, and working with permutations.

Proof

Experience with fairly rigorous proof is possible in the study of sets, and the opportunity is exploited in this study. The student is asked to criticize some proofs as early as Chapter 1. He is asked to produce some of his own in Chapter 6 after having had many proofs presented to him. Thus, this study gradually leads the student into making proofs on his own. There is no indication in the text of a deliberate endeavor to make the student aware of the strategies of proof. It is not assumed that the author considers such awareness unimportant. It must then be concluded either that the author assumes the student already to have attained such awareness, or else that the author expects the student to become aware of the strategies of proof through exposure alone.

Evaluation

Changes to be anticipated in student behavior (the basis for evaluation) are largely those involving mathematical maturity. Such changes are recognized as being difficult to measure. No evidence of precise measures for evaluating student growth promoted by this course is found in the book. However, it is assumed that the teacher observing the student's performance would be able to make an evaluation of considerable reliability.

University of Illinois Committee on School Mathematics

INTRODUCTORY STATEMENT

MAX BEBERMAN

In December 1951 the Colleges of Education, Engineering, and Liberal Arts and Sciences established the University of Illinois Committee on School Mathematics to investigate problems concerning the content and teaching of high school mathematics in grades 9-12.

Since that time, our major concern has been with the development of instructional materials and their experimental trial in schools throughout the country. We have introduced some new content, rearranged some of the traditional content, and have developed many promising pedagogical techniques and approaches. To enable teachers to help us in the experimental trials of our teaching approaches, we have conducted summer training institutes which prepare teachers for the use of our materials. This training emphasizes both content and pedagogy.

Until 1958 the UICSM textbooks were not available for unrestricted classroom use. We distributed them in classroom lots only to teachers who had received special training in their use and who were willing to help us evaluate the texts. The materials are now generally available. Although we do not consider the present editions as experimental, we still recommend that they be used with caution and preferably only by teachers who have had an opportunity to study them under the supervision of a person who has had classroom experience in their use or who has made an intensive study of their content and the implicit pedagogy.

The present textbooks of the UICSM comprise just one component of a teaching program. They are designed to be used by students whose teachers have access to the detailed commentaries which are part of the teacher's editions of the texts. The commentaries contain the fruits of a decade of research on the use of the texts in thousands of classrooms. Suggestions are given for alternative approaches, for deletions or additions, and for procedures for adapting the materials to meet individual differences among students. The commentaries are kept up-to-date through the UICSM *Newsletters* which contain sample tests and articles on the handling of various topics in the textbooks.

The fact that the present units are bound separately should not mislead the reviewer into regarding the units as independent textbooks. In fact, the units should be looked at as separate chapters of one book entitled *High School Mathematics*. The units were written to form a thoroughly integrated and sequential course. For example, deductive proof is introduced in Unit 1 in a very informal way in connection with the derivation of specific numerical statements. The work on proof is carried on in Unit 2 in a more formal way as soon as students have learned the linguistic devices necessary for making general statements. It is not until Unit 6 that students look at the general problem of proof, and this occurs in a section on logic. Additional aspects of logic are brought into the course in Unit 7 as students learn about proofs by mathematical induction.

In view of the fact that the units are sequential, the problem of grade placement is resolved in a variety of ways by schools using this material. Most schools begin with Unit 1 in grade 9 and may or may not complete the course with Units 9, 10, and 11 in grade 12. Other schools start with Unit 1, during the second semester of grade 8. Still other schools begin Unit 1 even earlier in grade 8 or grade 7. The appropriateness of a particular topic in any unit must be judged in terms of the mathematical experiences students have had up to that point and not in terms of their grade level. Thus, some classes complete Units 1-6 by the end of the ninth grade and some complete these units by the end of the eleventh grade.

COMMITTEE REPORTS

High School Mathematics

Units 1–4

Social Applications

The UICSM course recognizes the value of social applications of mathematics by including numerous exercises on this subject in Unit 3. The applications cover many fields and tend to require critical thinking as well as careful analysis. The emphasis, however, is entirely upon mathematical principles, and the social applications serve to illustrate them. Mathematical principles and theorems are given significance by means of these exercises. The structure of the unit is such as to make it clear that the variety of applications is of secondary importance to the principles.

Placement

The material appropriate at the ninth grade level is resolved as follows: A detailed study of the real number system and the properties of real numbers; precise statement of principles and theorems and the proof of certain theorems; sets and set notation; solution of equations; graphing of equations and inequations; other topics bearing upon the preceding ideas. The UICSM materials are not greatly different from conventional elementary algebra in content, but the approach and techniques are substantially different.

Structure

The UICSM units were written with the structure of mathematics uppermost in mind. This is evidenced by the careful attention given to the development of the properties of rational numbers in Unit 1. Later, the emphasis on proof makes it clear that the logical structure of mathematics is paramount. The Unit 1 lays a careful foundation of vocabulary and fundamental concepts to permit substantial rigor in proofs in Unit 2.

Vocabulary

The UICSM courses are genuinely concerned with developing precision in the use of the language of mathematics. In the introduction to Unit 1 great care is taken to distinguish between the name of a thing and the thing itself. The distinction between numbers and their numerals is strictly maintained throughout all units. The distinction between things and their names is useful in work dealing with the use of letters in various expressions. This leads to the use of the term "pronumeral" as a place holder for a numeral. The use of the universal quantifier and quantifier phrases appear in Unit 2. The theorems and principles are stated in very precise terms, and the students themselves soon are using precise and sophisticated mathematical language in order to state generalizations. The UICSM believes that early in the course a student is ready and able to be led carefully from the general unsophisticated language of mathematics to a very precise and sophisticated use of it.

Methods

One of the fundamental concepts of this program seems to be the value which is attached to the principle of student discovery. Exploration exercises appear frequently, and these are very useful in encouraging and guiding the student in the discovery of generalizations. One of the first places where important generalizations are discovered is in the unit on the operations with signed numbers. The rules are not stated as such, but are discovered by the students after finding several interpretations of the symbols of signed numbers. Simple equations and inequations (inequalities) offer another topic in which discovery is of extreme importance. The students are led to discover their own methods for solving many kinds of equations. The pervading discovery techniques are intended to help the students when they are confronted with new problem situations and in the understanding of the mathematics developed. UICSM holds to the belief that the learning process is deepened by presenting a sequence of activities from which students may independently recognize some desired knowledge.

Concepts *vs.* Skills

Another distinguishing characteristic of the UICSM program is the great emphasis which is placed upon the development of concepts. The developing of concepts occupies a much more prominent role than the developing of manipulative skills. Acquiring skills and understanding basic concepts can certainly grow together. The development of skills is not neglected, however. A lengthy list of supplementary exercises appears at the end of each unit, these should certainly be more than adequate for the needs of most classes.

Proof

Deductive proof is introduced in UICSM materials in Unit 2. The if-then relationships are emphasized by the formalization of certain algebraic proofs. The pupils become aware that when premises are accepted, the logical conclusion must be accepted. Formal logic is not included in Units 1-4. There is considerable rigor in the proofs, especially in regard to the precision of statements and the need for authenticating each statement in a proof. The notation is at a fairly high level with use of quantifier and other special symbols. The addition rules are stated in advanced notation. The students make proofs independently by the end of the first semester of the ninth year, and they formulate simple proofs very early.

Evaluation

The unit tests are now fairly well validated and standardized. There are ten quizzes per unit, and final tests and pretests. The tests are completely in the terminology and notation of UICSM.

High School Mathematics

Unit 5

Social Applications

Functions and Relations, Unit 5 of the UICSM materials, does not put a strong emphasis on social applications. There are few places where a problem is presented and solved by means of the mathematics being developed in order to aid the subject development. New sections of study, however, are often introduced by means of an allusion to a non-mathematical field to show where this topic might be applied. The sciences are used extensively in this regard. In addition, the supplementary exercises contain enough thought problems to provide the necessary experience in solving this type of problem.

Placement

This unit provides an extremely smooth flow of development of the concepts of relation and function through the continuing use of the underlying concepts of sets and operations on them. Linear functions are introduced at this level. The development proceeds to the more complicated quadratic functions which are then studied in detail. The section that follows is the study of systems of equations and uses the student's familiarity with operation on sets. It is rather surprising to find that systems of quadratics are not discussed in this unit. It would seem that the machinery for such work has been developed, but for some reason it has not been used. The placement and resulting continuity is one of the strongest points of Unit 5.

Structure

Set theory and set theoretical concepts are used throughout Unit 5. Work with sets was begun early in the UICSM Program, but Unit 5 takes up a more formal presentation of set theory, including proofs of some theorems of complementations, union, and intersection. Methods of obtaining sets and the description of sets are developed. The basic structural concept of this unit is that of set and in particular a set of ordered pairs. The ideas of relations and functions are developed in terms of sets, and, as they apply, applications of these concepts and further topics of traditional second year algebra are interwoven. The unit is heavy in its emphasis on structure. This does not mean that Unit 5 is unbalanced in emphasis, but rather that the approach has been one of stressing a few underlying ideas (set, relation, function). After a careful exploration of the properties of these general ideas, applications to special situations are then studied.

Vocabulary

A precise language has been developed prior to Unit 5 by the University of Illinois Committee on School Mathematics and is continued in this unit. However, an attempt is made in this unit to use more of the conventional nomenclature. For example a numerical variable is introduced to replace pronumerals developed earlier in the program of the UICSM.

A careful exposition of notations, naming sets and definitions of set-theoretic operations, is developed. Mathematics requires precion in language for clarity and for exactness. Students are asked to prove some of the ideas introduced. By developing a precise vocabulary of basic set concepts and relations and functions, more involved topics such as functional compositions and intersection of relations and functions (simultaneous solution of conditions) can be exactly given and understood. A word of caution is in order here for the student or instructor not familiar with the notation and definitions of prior units of the UICSM. Not all the language is standard, and some acquaintance with Units 1-4 is occasionally advisable for checking and reference.

Methods

The writers of Unit 5 say many times in the *Teacher's Commentary* that the students may, by working certain problems, discover important principles. But should a student be expected to make a generalization on the basis of results from one or two examples? We are referring to the method of first stating a generalization and then asking the students to confirm it.

Concepts *vs.* Skills

Throughout the entire unit there is a large number of problems for each section. Most of this material is for the purpose of developing an understanding of the concepts being studied. In addition to this type of problem, there are three skill quizzes, in the early part of the unit, and an appendix of miscellaneous exercises. Since Unit 5 is one which deals almost exclusively with the development of concepts, the issue of concepts vs. skills does not particularly apply. For the few topics (e.g., proportions, quadratic equations, and systems of equations) where skill needs to be developed, there is an adequate supply of practice material.

Proof

In the first part, the set algebra proofs are well and rigorously done. Based on the assumption of ten principles of real numbers, they require more sophistication and awareness of proof techniques than the average junior or senior may be able to handle unless he has been trained in these methods. In the rest of the book, the student is expected to back up his answers to questions with reasons. So proof of a relatively unstructured form (informal) can be found throughout the book. Proof of the formal sort (statements for student demonstration) are not in the rest of the text. The gap between the thought required for a sequential series of supported statements of a long proof and that needed for supporting a single statement is wide. Much of this text ignores this need on the assumption that the student already has learned the basics in prior units. Generally the UICSM Program is strong on proof, but here the proof seems to be secondary. The early usage of formal proof indicates the assumption that the student knows how to prove. Concepts are developed at the sacrifice of further strengthening the proof techniques.

Teachability

It is the feeling of the reviewers that an analysis of Unit 5 would not be complete without some comment on its teachability. The material is approached from the point of view of sets and their operations, and draws heavily from modern mathematics. The students are presumed to have studied from Units 1-4 of the UICSM program. A definite vocabulary and philosophy have been developed previously and this background is essential to the student as well as to the teacher. Both students and teachers would be seriously handicapped without previous experience with the prior materials. Although the teacher's edition is extremely helpful, any teacher considering the use of these materials should be familiar with the more recent developments in the field of mathematics education as well as with the methods and vocabulary of this program.

High School Mathematics

Unit 6

Geometry, Unit 6 of the UICSM program, is designed to follow Units 1-5 and does not necessarily belong to any particular grade level. A critical review of the text indicates to us that UICSM has made the following decisions for whatever group the course is intended.

Social Application

For all practical purposes, UICSM does not believe in the value of social applications at this level. Applications are not used to motivate the study of the material. Rather, exercises of this type, and they are few in number, appear in a couple of pages after the theory has been developed.

Placement

The UICSM point of view seems to be that this material should not be taught unless the proper background in proof has been laid. There is no indication that geometry should be taught at a particular grade level but rather at an experience level. Furthermore, the emphasis is on proof rather than content and, although traditional content is used, it appears to be a vehicle rather than an essential element of content.

Structure

This material indicates that the way to a better understanding and use of mathematics is through conscious and continual study of structure.

Vocabulary

The text indicates a fundamental belief in the importance of sophisticated and precise language. However, the teacher's manual indicates that UICSM does not expect the teacher to require such precision in the classroom.

Methods

This text strongly emphasizes the presentation of "a sequence of activities from which a student may come to independently recognize the desired knowledge."

Concepts *vs.* Skills

The text places emphasis on the development of concepts. Manipulative skill in using these concepts is present. Skill in applying these concepts to other situations is essentially missing.

Proof

The text assumes experience in proof prior to the level at which this text is used. This implies that some understanding of what mathematical proof is should be arrived at earlier. The text also attempts to be as rigorous as possible, pointing out, at times, loopholes in the structure or points which are true but cannot be proved at this time. Early attempt to place the burden of proof on the pupil is a basic part of the course.

Evaluation

From the vantage point of our armchairs, it appears that the topics covered are well chosen. However, the proof of the pudding is in the eating, and we know of no measures for an objective evaluation of the material at this time.

High School Mathematics

Unit 7 and *Unit 8*

Social Applications

In *Unit 7, Mathematical Induction,* and in *Unit 8, Sequences,* the UICSM does not use social applications of mathematics to motivate the study of ideas, to develop basic principles, or to provide opportunities for applying the principles after they have been developed. The UICSM apparently does not consider social applications of mathematics as a major objective at this level.

Placement

The UICSM does not specify that the mathematical content of these units should be taught at a particular grade level, but rather at a particular experience level. The point of view seems to be that the mathematical content of these units is appropriate for the high school mathematics program, and that these ideas can be developed, provided the student has an adequate mathematical background.

Structure

Units 7 and 8 complete the development of the real number system which was begun in Unit 2, with the exception of a principle of completeness. The emphasis placed on the development of the real number system as a mathematical structure indicates the UICSM believes that the study of mathematical structures is essential if a better understanding and use of mathematics is to be achieved.

Vocabulary

From the beginning, both the students' texts and the teachers' guides use a precise and sophisticated language in conjunction with a compact mathematical symbolism to introduce and develop the ideas included in the materials. It is apparent that UICSM considers the use of precise and sophisticated language appropriate at this level.

Methods

These units include many "exploration" exercises to provide opportunities for the student to discover ideas independently. In addition, suggestions in the teachers' guides should aid teachers in framing and asking questions to extend and increase the effectiveness of the discovery approach. The UICSM seems to accept the discovery and heuristic approaches as highly desirable.

Concepts vs. Skills

The pattern of presentation used by the UICSM is the development of an idea followed by a large number of exercises to provide practice in applying the idea to theoretical mathematical problems. The UICSM seems to believe that a mathematics program should develop both the concepts and the necessary manipulative skills.

Proof

The concept of mathematical proof has been introduced prior to this experience level in the UICSM program. The concept of rigorous proof is central to the development of the ideas presented in these units. The logical development of the real number system and its subsets, and the extension of these ideas to include sequences seem to be the vehicle UICSM has chosen to develop the important idea of proof and to provide opportunities for the student to practice preparing and presenting mathematical proofs.

Evaluation

The materials contain no evaluative instruments, such as chapter tests, review tests, or other suggestions for evaluation. However, there are many graded review exercises that could be used effectively to obtain information concerning the changes taking place.

Appendix

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SOURCES FOR EXPERIMENTAL MATHEMATICS PROGRAM MATERIALS

Boston College Mathematics Institute
BCM Institute, Chestnut Hill 67, Massachusetts

Greater Cleveland Mathematics Program
Educational Research Council of Greater Cleveland, 75 Public Square, Cleveland 13, Ohio

Syracuse University-Webster College Madison Project
Syracuse University, Syracuse 10, New York

University of Maryland Mathematics Project
University of Maryland Bookstore, Princess Anne, Maryland

Ontario Mathematics Commission
Service and Smiles Distributors, 9 Mayfair Mews (off Bloor at Yonge), Toronto 5, Ontario, Canada

School Mathematics Study Group
Yale University Press, New Haven, Connecticut (Most of the SMSG materials are now available from A.C. Vroman, School Dept., 367 S. Pasadena Ave., Pasadena, Calif.)

Developmental Project in Secondary Mathematics at Southern Illinois University
Southern Illinois University, Carbondale, Illinois

University of Illinois Committee on School Mathematics
University of Illinois Press, Urbana, Illinois

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