DOCUMENT RESUME

ED 035 574 24 SE 007 912

AUTHOR Dixon, Lyle J.
TITLE An Analysis of Certain Aspects of Approximation Theory as Related to Mathematical Instruction in Algebra. Final Report.
INSTITUTION Kansas State Univ., Manhattan.
SPONS AGENCY Office of Education (DHEW), Washington, D.C. Bureau of Research.
BUREAU NO BR-8-F-030
PUB DATE Aug 69
GRANT OEG-6-9-008030-0035(057)
NOTE 59p.

EDRS PRICE EDRS Price MF-$0.50 HC-$3.05
DESCRIPTORS *Algebra, *Instruction, Mathematical Concepts, *Mathematics, Mathematics Education, *Secondary School Mathematics

ABSTRACT

This study is concerned with certain aspects of approximation theory which can be introduced into the mathematics curriculum at the secondary school level. The investigation examines existing literature in mathematics which relates to this subject in an effort to determine what is available in the way of mathematical concepts pertinent to this study. As a result of the literature review the material collected has been arranged in a structured mathematical form and existing mathematical theory has been extended to make the material useful to instructional problems in high school algebra. The results of the study are found in the expository material which comprises the major portion of this report. This material contains ideas of approximation theory which relate to elementary algebra. Also, included is a collection of references for this material. The report concludes that certain techniques of approximating a root of a polynomial by finding roots of a derived polynomial can be presented in a manner which is suitable for courses in elementary algebra. (FL)

BR 8-F-030
PA 24

FINAL REPORT

Project No. 8-F-030

Grant No. OEG-6-9-008030-0035(057)

AN ANALYSIS OF CERTAIN ASPECTS OF APPROXIMATION THEORY AS RELATED TO MATHEMATICAL INSTRUCTION IN ALGEBRA

Lyle J. Dixon
Kansas State University
Manhattan, Kansas 66502

August 31, 1969

U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE

Office of Education
Bureau of Research

FINAL REPORT

Project No. 8-F-030

Grant No. OEG-6-9-008030-0035(057)

AN ANALYSIS OF CERTAIN ASPECTS OF APPROXIMATION THEORY AS RELATED TO MATHEMATICAL INSTRUCTION IN ALGEBRA

Lyle J. Dixon

Kansas State University

Manhattan, Kansas 66502

August 31, 1969

The research reported herein was performed pursuant to a grant with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

U.S. DEPARTMENT OF

HEALTH, EDUCATION, AND WELFARE

Office of Education

Bureau of Research

TABLE OF CONTENTS

SUMMARY

It is a well-known mathematical principle that certain solutions to an algebraic equation may be approximated by considering an equation derived from the original equation by "ignoring" certain terms of the equation. Estimates of the errors introduced by this sort of procedure are given in advanced courses in mathematics. At present, mathematical instruction at the secondary level does not include such principles. This is true even though other areas of science instruction have need for the basic concepts involved in this principle.

The fundamental question for this investigation is: Is there a systematic body of mathematical knowledge which covers this approximation technique and is it possible to present this knowledge in some convenient form whereby a program of instruction can be established in courses in elementary algebra at the secondary school level? The basic task is to answer this question.

In order to answer this question it was first necessary to make a rather extensive examination of the existing literature in mathematics to find out what was already available in the way of mathematical concepts pertinent to the study. A search was made of educational literature to determine if an attempt had been made to relate these approximation techniques to elementary algebra at the secondary school level.

The search of the literature revealed some sources of mathematical information but no reference was found in the educational literature on the subject of this study. The mathematical concepts were classified by areas and those pertinent to the study were put in a convenient structured mathematical form for future use.

One significant observation was made about the level of mathematical concepts found in the literature. Almost all were at an advanced level and those which might be pertinent to the study were found in research journals in the period from 1900 to 1930. The trend in mathematical research in recent years has not included the area of this specific study. As a result of this, many of the ideas presented in the expository material in Appendix A had to be developed and/or extended as original research.

The criteria used to determine if the concepts could be used in elementary algebra and if these concepts were to be included in the expository material were the following: Is it pertinent approximation theory? Is it good mathematics which is readable by the "average" high school mathematics teacher? Is it related to high school algebra in the sense it depends upon strictly algebraic and/or geometric concepts? Is the material presented in the expository section at a level that a high school teacher can understand? Is it adaptable to what a high school student knows about mathematics?

The results of the study are found in the expository material in

Appendix A. In this material one can find the pertinent ideas of approximation theory as related to elementary algebra. It is clear that the basic question of this study can be answered and in the affirmative. Certain aspects of approximation theory can be presented with some rigor and/or in an intuitive manner to make an additional contribution to mathematics education at the secondary school level. This contribution has two significant aspects. One, it broadens the range of mathematical concepts which are available at a given experience level. Secondly, it provides the kind of mathematical experience which the applied sciences can use and provides it early enough to be relevant to the first courses in these sciences at the secondary school level.

A by-product of the study has been the additional insight into mathematics obtained by the principle investigator during the course of the investigation. There is no quantitative way to measure this, but it has been significant.

The principle recommendation which should be made as the result of the findings is the continuing of this work to the last and ultimate test of its value. Some teaching units should be prepared and be used in an experimental situation to determine the teachability of the concepts outlined in the expository material. The expository material is essentially background for the units and should not be used as teaching material at the secondary school level. It would be appropriate for prospective mathematics teachers.

I. PROBLEM UNDER CONSIDERATION

Certain equations which arise from physical problems prove to be difficult to solve because of the numerical computations which have to be completed. Sometimes an approximation for a solution of the original equation can be obtained by ignoring a given term of the equation and solving the "reduced" equation. For example:

An equation such as

$$(1) \qquad \sqrt{\frac{d}{16}} + \frac{d}{1100} = 3.1$$

can be transformed into the equation

$$(2) \qquad \frac{d}{16} + \frac{6.2d}{1100} - \frac{d^2}{1,210,000} = 9.61$$

A solution to (2) is approximated by solving the equation

$$(3) \qquad \frac{d}{16} + \frac{6.2d}{1100} = 9.61 .$$

Equation (3) is obtained from (2) by "ignoring" or deleting the term in d^2. The solution of (3) is approximately 141 and this comes reasonably close to satisfying (2) and, hence, satisfying (1).

One might be tempted to conclude that this is an "accident" of the equation involved but this is not the case. Physicists use the procedure just described quite frequently and chemists are known to use a similar procedure in the solution of some of their equations. Moreover, there appears to be a desire on the part of teachers in these sciences that some instruction take place in the mathematics curriculum on precisely these procedures.

The fundamental question which should be answered is:

Is there a systematic body of mathematical knowledge which covers such examples as just described (and consequently others) and is it possible to present this knowledge in some convenient form whereby a program of instruction can be established in courses in elementary algebra at the secondary level?

The last half of the question depends upon answers to the first half of the question and several other factors which do not readily lend themselves to research and analysis. These other factors, such as teachers, curriculum, etc., are not necessarily included in this study. However, a significant portion of the fundamental question can be answered if the specific objectives of investigation are:

1(a) to determine if any significant body of knowledge exists in approximation theory which could be made available for secondary mathematics instruction, and

1(b) if such information is available, to systemize this information in order that secondary teachers of science and mathematics might have a significant and structured elementary mathematical model to use in relating mathematics and science,

2. to extend this information in whatever manner necessary to make the mathematical model more convenient for use at the secondary level and

3. to present the various aspects of the findings of the first three objectives in an expository form which would be readable by the "average" secondary mathematics teacher.

One rather clear limiting factor imposed within the objectives just noted is the relevancy of the material to be investigated to the secondary school level. This means, in essence, that no mathematical concepts should enter into the results of the investigation which do not have a basis within high school Algebra or Geometry. Any approximation techniques which depend upon the Calculus are automatically excluded from the investigation. For a similar reason infinite series are also excluded even if there might be a partial justification for including this approach on the grounds that one or two high school textbooks include some discussion of the subject. What should be clear is that the results of the investigation depend upon the "usual" (and perhaps traditional) content of the subjects of Algebra and Geometry. This limitation has particularly significant implications for educational applications of the results of this study.

II. METHODOLOGY FOR THE INVESTIGATION

The procedures followed in this study were essentially the same as those of any scholarly investigation. These procedures are given here by phases with appropriate comments on each phase.

Phase 1. An extensive examination of existing literature related to the general area of approximation theory was made. This was done in order to determine what is already known in the area and what might be appropriately related to this investigation. Library resources, such as the Educational Index, were examined for references to the topic. Books in the areas of numerical techniques, engineering applications, computer sciences and any area which might make use of approximation techniques were examined for suitable materials and for references to other possible sources. The mathematical journals, both pure and applied, were examined for papers and references on approximations. Members of the mathematical community were consulted for sources of information and the free exchange of ideas with their members has been most helpful during the investigation.

Phase 2. After the material and sources of literature had been collected, it was necessary to examine each reference found to determine if it was related to the problem and if it might be useable material. This was clearly a judgment decision determined by the previously mentioned specific objectives of the study. It was not always clear, at that point, that some source was necessary for future use. This was minimized by the establishment of certain criteria for selection of materials. These criteria were established before the investigation actually began and are enumerated here. In many instances during the search of the literature a source was rejected at that point because it obviously failed to meet these criteria.

The criteria used ask the following questions:

(a) Is it pertinent? Does it really belong in approximation theory? (Some material was related, but quite secondary in nature.)

(b) Is it good sound mathematics? Is it readable by the "average" high school mathematics teacher? (Much periodical literature in mathematics is presented so abstractly only another research mathematician can read it.)

(c) Is it related to high school Algebra? Does it depend upon strictly algebraic and/or geometric concepts (deducible from these subjects at the high school level)? (Some methods depend upon Calculus and were not specifically suited for this project.)

(d) Is this material at a level that a high school teacher can understand? (This means making a decision on what most teachers know and what their training has been. It is clear from various studies that this is not as high as desired and for this study to have any impact or application it must produce something which most teachers can study and, hence, learn.)

(e) Is it adaptable to what a high school student knows about mathematics? This follows naturally from (d).

(f) If more than one technique is available to accomplish some approximation procedure, which one best lends itself to teaching and learning theory?

These criteria were adequate for the job intended, except the last one was never actually used. The reasons for this are given in the findings.

Phase 3. The next step was to order the material by areas of application. As it turned out this was essentially done concurrently with Phase 2, and frequently as early as Phase 1. For instance, to order or group desirable information on procedures to obtain approximations to roots of an algebraic polynomial into one area called for all material on Horner's rule, Newton's method and others to be classed together as a certain general approximation concept. In contrast, the approximation of a solution of a quadratic equation by solving a linear equation obtained from the quadratic constitutes an entirely different order of approximation concepts. The areas of grouping were somewhat arbitrary but, generally speaking, were determined by what the investigators found in the literature.

Phase 4. The next phase consisted of arranging the accumulated information in some conveniently structured mathematical form. This was essentially a matter of developing the mathematical aspects of existing theory into a logically coherent system which can be used by teachers of secondary school mathematics. In some instances this was easily done due to the circumstances but in others this was possible only as a result of work done during Phase 5.

Phase 5. Existing mathematical theory was extended, wherever possible and where needed, in a manner to make the system of Phase 4 more complete and useful in application to instructional problems in Algebra. Some very large gaps were noted during Phase 4 in certain areas. While it was not anticipated that significant mathematical research would be needed, it turned out that the literature left significant gaps in the mathematics (which may be filled by information known or assumed by others and not available to this investigation). These gaps were filled by mathematics generated by the principal investigator.

Some aspects of this phase began as early as Phase 4 and extended into Phase 6, with a great deal of this kind of activity in the last phase.

Phase 6. The last step of the investigation was the preparation of an expository presentation with adequate reference to explicit mathematical development which covered the structured system developed and expanded in Phase 5. The presentation was prepared in a form consistent with the objectives of the study and special emphasis was placed upon the level of abstractness of the material. The expository development is found in Appendix A of this report.

III. FINDINGS AND RESULTS

Since the major portion of this investigation is the preparation of some expository material, it would be proper to include that material at this point. However, due to the structure of the expository material, it will be found in Appendix A, along with the collection of references for this material. However, there are several observations which are pertinent to the investigation as a whole and these are made at this time.

A review of the literature reveals two rather significant facts. First, there is a sizeable body of knowledge available in various places about approximation theory. Nearly all of this is in advanced texts, journal articles and technical reports. All of these sources are available to mathematicians who regularly require their use and are versed in th. general area. A good deal of the recent information is frequently linked to numerical analysis and the computer. One also finds the application sources are frequently the best sources of readable mathematics in this area. This suggests that most mathematics is so abstractly written and presented that only a relatively small segment of the population has an opportunity to digest and use the information presented in such form.

The second significant fact obtained from the review of the literature is that there is no source on approximation theory which presents in an elementary fashion those aspects of approximation theory which are used in moderate amounts by the various scientific disciplines found in secondary school courses of study. This was somewhat anticipated and was a motivation factor of this investigation. However, it was anticipated that some sources, maybe obscure, would be found. This was not the case. In fact, the search for this kind of material continues, even as the final report is being prepared.

Another related observation was the recognization that mathematics changes rather rapidly. Prior to 1935 a great deal of attention was devoted to algebraic polynomials and their solutions. By 1940 there are almost no references in periodical literature to such matters and fewer references are to be found since then. The general area of mathematics which had its central theme in courses called Theory of Equations has essentially disappeared or been absorbed by other areas of mathematics. This shift in interest by mathematicians probably contributed to the shortage of useable material on certain topics in approximation theory and forced the creation of certain theorems to fill gaps left by this shift. Their theorems may not be as original as some but no references have been found for them. The theorems are identified in the expository material by a D in the theorem number, such as Theorem D2.

IV. CONCLUSIONS AND RECOMMENDATIONS

The conclusions of the investigation are to be found in the expository material as this was to be the major task of the study. As has been previously indicated the material is not as extensive as had been anticipated and there might be an aspect which could have been included, but the expository material is representative and respectable mathematics.

There are some recommendations which should be made as a result of this study and all are within the capability of educational endeavors. First, and not necessarily dependent upon this study but obviously recognized in carrying out the investigation, is the need for expository materials in mathematics which translate some phases of the subject into a form which high school teachers can use. This project has attempted to do this for an area of mathematics (applied in nature) and there are many and perhaps more important areas which need the same treatment. For instance, the social sciences are becoming more mathematical as these areas develop the science aspect of their endeavors. The present high school mathematics is almost totally mathematical or physical science orientated. If the other areas are important to our society, there should be some literature which describes these relationships and which suggests how it might be adapted to existing patterns of curriculum and instruction. Examples of problems from economics, psychology, and other disciplines are appropriately a part of mathematics instruction and expository material would seem to be the best avenue to get these ideas into the hands of teachers.

Second, a natural consequence of this study would be the preparation of suitable material for lessons for high school students to use. This is essentially the adaption of this report to a practical situation. This set of lessons should be tested in a classroom situation to determine the success of the whole idea. Mathematically, there is no reason why it cannot be done. Perhaps there are non-mathematical reasons why approximation theory of any sort has no place in secondary school mathematics, but none are now known.

Finally, the investigation revealed the lack of material at any level on the general area of "theory of equations." It would seem that in our rush into "modern mathematics" we may have left out significant educational concepts which ought to be restored. It is unlikely that the college and university mathematics courses will include these ideas and perhaps a major portion of it can go into high school mathematics. It is mathematically feasible and has already been done on an experimental basis.

Appendix A

Expository Material on Approximation Theory

I. Introduction.

The general area of approximation theory is quite broad, too broad to be adequately covered in a publication of this size. However, there are aspects of this general theory which are quite pertinent to secondary school mathematics and on more than one occasion (for example, see Bowen [4]) science teachers have suggested that some portions of this material might be quite appropriate for the curriculum in mathematics and at an early stage in the secondary schools. The question of what portions are available and appropriate is essentially an unanswered question as far as the literature is concerned. Moreover, if there are portions which are appropriate, can these portions be organized in a manner which makes good mathematical sense and which is strictly dependent upon a bare minimum of secondary school mathematics?

The purpose of this material is to provide some affirmative answers to these questions and to present some aspects of approximation theory which might be appropriate for secondary school mathematics. This we shall do in the following sections. We begin by examining some history of mathematics and some types of approximation. Following this we will present the mathematics of one of these types by citing pertinent theorems and facts. Most of the theorems are already in the literature someplace and they only need to be assembled into a meaningful presentation. There are included certain theorems which were not found but were needed to fill gaps in the area.

One very important principle was kept in mind in the presentation of this material. It should be material which the "average" high school teacher can use with relative ease and it should be based upon elementary enough mathematics so that it could be adapted to a classroom situation. The latter restriction essentially means a year of algebra and maybe a reasonably good acquaintance with graphing.

II. Historical Background on Solving Equations.

In this section we will give a very brief review of the historical development of the solving of equations and in particular, the solving of the quadratic and cubic equations. The purpose in reviewing the background of these equations is to recognize that approximations played an important part in the early history of solving of equations, and to recognize that this process was a long, evolutionary one. We shall not be complete in our presentation, but will touch upon only the significant highlights of that history. This will be accomplished by first treating the quadratic, then the cubic and finally all other degrees of algebraic polynomials.

1. The Quadratic Equation. The earliest solutions of problems involving equations were obtained by trial. At least no written record has been found to indicate otherwise. By 1500 B.C. the Egyptians (as described in in the Ahmes Papyrus) were able to make a trial guess and adjust the results of the trial guess in a manner which gave the correct solution of

the equation under consideration. This technique was called the "Rule of False Position" and the procedure was used by subsequent students of mathematics as late as the 1700's.

A similar technique called the "Rule of Double False Position" was used rather extensively for many centuries. This technique, as well as the other one, were subsequently replaced by our modern techniques but only after the algebraic symbolism was developed by Vieta and others.

A study of the history of early mathematics (before 1600) reveals that most solving of algebraic type equations was done by trial and error with the initial step being to make a guess or approximation of the solution and make subsequent adjustments on the basis of the first approximation. This procedure was applied to all degrees of polynomial equations which arose from the problems to be solved.

In the Berlin Papyrus (c. 2160-1700 B.C.) one finds the first known solution of a quadratic equation. The problem reduces to solving the equations

$$x^2 + y^2 = 100 \text{ and } y = \frac{3}{4} x \text{ .}$$

The solution technique reduces to a simple case of the "Rule of False Position."

The Greeks were able to systematically solve quadratic equations by geometric means. Euclid (c. 300 B.C.) gave solutions and procedures to solve problems such as

$$xy = k \text{ and } x - y = a$$

and $a(a - x) = x^2$.

The Hindus may have been able to solve quadratic equations by 500 B.C. although no record of the method of solution has been found. By 500 A.D. we do find a rule, relating to the sum of geometric series, which shows that the solution of the quadratic equation was known, but we have no rule for the solution of the equation itself. About 628 A.D. Brahmagupta gave a definite rule for solving a quadratic. Smith [19] gives the steps of solution for an example of a quadratic ($x^2 - 10x = -9$) which turns out to be, in modern terminology,

$$x = \frac{-9.1 + (-5)^2 - (-5)}{1} = 9 \text{ .}$$

One might note the similarity of the solution to the quadratic formula except he found only one solution, i.e., the positive one. This was quite typical of all early attempts to solve equations in that only posi- solutions were acceptable.

Mahavira (c. 850) had a way of solving quadratics for a positive root, but did not write it out. However, an analysis of his solution leads one to believe he knew substantially the modern rule for finding a positive root of a quadratic.

The Hindu Rule was first given about 1025 and was widely used in India.

Al-Khowarismi (c. 825) gave two general methods of solving quadratics of the form

$$x^2 + px = q .$$

Negative roots were neglected. Omar Khayyam (c. 1100) also gave a rule for solving the equation

$$x^2 + px = q .$$

The techniques of solution mentioned above are typical of ways of solving quadratics until about 1600. Solutions for other higher degree equations were essentially based upon an approximation and a subsequent adjustment, as in "Rule of False Position".

In 1631 Harriot gave the first important presentation of the solution of a quadratic (and other equations) by factoring.

Vieta (c. 1600) replaced the geometric method of solving quadratics by an analytic method. The solution of the equation $x^2 + ax + b = 0$ was given as

$$x = -\frac{1}{2} a \pm \frac{1}{2} \sqrt{a^2 - 4b} .$$

The symbolism introduced by Vieta and others of his time made possible such a representation and a closeness to the complete solution of any quadratic. It is easy to see that the trial and error stage is being replaced by a systematic and complete method of solving a quadratic.

2. The Cubic Equation. The earliest known (c. 350 B.C.) cubic equation was of the form $x^3 = h$ although some Babylonian tablets give tables of cubes about two thousand years earlier. The problem of the duplication of the cube (said to have been known by Hipprocrates (c. 460 B.C.)) depends upon the finding of two mean proportionals between two given lines. This means to find x and y in the equations

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b} .$$

Archimedes referred to a problem of cutting a sphere by a plane so that the two segments shall have a given ratio. The problem reduces to the proportion

$$\frac{c - x}{b} = \frac{c^2}{x^2}$$

and this produces the cubic $x^3 + c^2b = cx^2$. Eutorius (c. 560) tells how Archimedes solved the problem by finding the intersection of two conics. Diophantus apparently solved the equation $x^3 + x = 4x^2 + 4$, possibly by noting that $x(x^2 + 1) = 4(x^2 + 1)$.

Several Arabic mathematicians, notably Almahani (c. 860), Tabit ibn Qorra (c. 870), Abu Ja far al-Khayin (c. 960) and Alhazen (c. 1000) discussed the cubic and at least one found solutions to certain equations in a manner similar to Archimedes.

Omar Khayyam (c. 1100) specificed thirteen forms of the cubic that had positive roots. This is a distinct advance in general theory but a long way from complete solutions for any cubic. However, he did expand the number of cubics of certain forms which could be solved by the "intersection of conics" techniques. Most Arab writers believed that the cubic equation, in general, was impossible to solve.

From about 1200 to 1500 several writers in western civilization mentioned the cubic but were unable to solve all cubics, although several special cubics were solved by special techniques, such as factoring.

The real interest in the cubic lies in the work of Cardan and Tartaglia. History does not clearly describe who should receive the credit and the stories surrounding the Cardan-Tartaglia controversy make interesting reading. It is clear that the solution for the cubic $x^3 + ax^2 = c$ was obtained by 1535 and very shortly after that a method of solution for the cubic $x^3 + bx = c$ was found. Cardan in his Ars Magna (1545) showed how to transform the first form into the second and succeeded in transforming the general cubic $x^3 + ax^2 + bx + c = 0$ into a reduced equation type $x^3 + px = q$. This he already knew how to solve and, therefore, every cubic was solvable. Cardan's examples showed positive and negative roots, the latter being included for the first time.

Vieta (1615) subsequently found the transformation which reduced the general cubic to the form $y^3 + 3by = 2c$ and, by a second transformation or substitution, arrived at a form $z^6 + 2cz^3 = b^2$ and the latter was solved as a quadratic in z^3.

Again history reveals the struggle to solve an equation and the subsequent success some two thousand years after the known existence of the equation. The cubic equation can be solved although some of the solutions involve complicated radicals and these are frequently approximated for ease of computation. The application of a cubic to a physical problem almost necessitates the approximation in order to have a suitable answer to use in a physical sense.

3. The Quartic Equation. One should probably suspect that any consideration of a quartic equation would be minor during the time the cubic was unsolved. History reveals this to be the case. One can find only slight mention of the Quartic prior to the time of Cardan. The first serious consideration of the Quartic came in 1540, almost immediately on the heels of the solution of the cubic.

A young Italian student by the name of Ferrari solved the problem by reducing the equation to an equation involving a cubic. Cardan did a great deal to clarify the process and spread the knowledge of the method in his book Ars Magna.

Not all solutions were necessarily included as negative ones were frequently ignored. Vieta (c. 1590) and Descartes (1637) improved the system of solution and the modern form is due to Simpson (1745).

4. The Quintic Equation. Euler found a method different from Ferrari's for reducing the solution of the general quartic equation to that of a cubic equation. He attempted to apply this technique to the Quintic in hopes of reducing the solution of it to the solution of the quartic and thereby solve the quintic. However, he failed as did other notables in the world of mathematics. These failures prompted Ruffini (1803, 1805) to try and show that the quintic could not be solved by such means.

Abel and Falois eventually resolved the question. Galois (1846) essentially answered the total question in his posthumously published works. Abel had earlier shown (1824) that the roots of the general quintic cannot be expressed in terms of its coefficients by means of radicals. This we are able to do for the quadratic, cubic and quartic. Intuitively it would seem possible to be able to do the same for the quintic, but it is not.

5. Solutions of other equations. The question of higher degree equations does not seem to have interested mathematicians of western civilization until quite late. The examination of the equations of higher order takes two major directions. The first is to find out from the equation all you can about the roots without actually solving for them. (We often do this for the quadratic by looking at the value of its discriminant.) Cardan may have known about the Rule of Signs which tells how many positive roots were possible. Harriot (1621) may have made the formulation of the Rule of Signs and Descartes (1637) certainly gave such a Rule.

The second direction for the solution of equation of higher order was to approximate solutions and by some process of iteration subsequently obtain better approximations. The Chinese scholars of the 13th and 14th centuries were outstanding in this area and this seems to be China's particular contribution to mathematics. In 1247 Ch'in Kiu-shao's writings reflect a high degree of perfection in this aspect of equation solving and equivalent form of Horner's method (1819) is given.

Fibonacci (1225) attempted some improvements as did Vieta (1600).

Newton (1669) simplified Vieta's work and perfected an approximation technique. Horner (1819) further simplified Newton's work.

It is interesting to note that Newton's method was replaced by Horner's method (as can be observed in any of the Theory of Equations books) because of the ease of hand computation. However, with the advent of computers, some of the earlier methods are better ones to use because they adapt easier to machine language. Progress does not always make old things obsolete.

6. The Fundamental Theorem of Algebra. Perhaps the most interesting problem related to the solution of equations was the apparent disregard of negative solutions long after techniques were available to find them. It is not too difficult to understand the ignoring of complex solutions, as concepts of complex numbers had not been fully developed. However, there came a time (c. 1608) when mathematicians began to assert that an equation of nth degree has exactly n roots. It was restated at various times without proof and Gauss (1799) gave the first rigorous demonstration of the theorem.

The theorem, together with the techniques of the solving of linear, quadratic, cubic and quartic, completes the major theory of equation solving. For the quintic and others of higher order, the theory is not quite as complete, but complete as mathematically possible.

7. Review and Summary of Historical Background. We have just sketched the basic highlights of the solution of certain types of equations. It should be noted that all early attempts at any particular type of equation were essentially based upon approximations (and, in some instances, adjustments on these approximations). After rather lengthy periods of time, solutions were found for just one, then another and then another type of equation. Only after the symbolism became fully developed does one find significant contributions in this area.

For those equations which no techniques of solution were found, mathematicians seemed to be able to approximate roots of equations with some degree of accuracy. Subsequently, certain techniques were developed to improve a given approximation, usually by an iteration process.

In recent times the users of mathematics, i.e., physicists, chemists, etc., have begun to solve certain equations by deleting some term of the equation and solving the resulting equation. This is frequently done even when formulas are available to solve the starting equation. This is essentially an approximating technique and tends to indicate a cyclic treatment of the subject as far as history is concerned. It also suggests that a second stage should follow at some future time and a general theory be founded to justify the procedures and approximations now being used. In what follows we partially provide an indication of what this second stage should contain.

III. Types of Approximations.

When one begins to examine approximation theory it becomes clear that there are different types of approximations which may arise from or be related to different circumstances. In this section we will examine a few different types which will serve to illustrate different aspects of approximation theory. Some are elementary and have already been seen even by the beginning high school student.

1. Approximating a number by another number. In the elementary school arithmetic one finds the establishment of a decimal representation of the rational (and sometimes irrational) numbers by giving decimal equivalents to certain fractions. Thus

$$\frac{1}{2} = .5$$

$$\frac{1}{4} = .25$$

$$\frac{1}{8} = .125$$

The list is usually limited to the more common (in usage) fractions. However, we may see statements like this:

$$\frac{1}{3} = .333$$

$$\frac{2}{3} = .666$$

or, in some instances,

$$\pi = \frac{22}{7} \quad \text{or} \quad \pi = 3.14 \; .$$

As every good student of mathematics knows, these last statements are not actually correct. The fraction $\frac{1}{3}$ is not .333, and π is neither $\frac{22}{7}$ nor 3.14.

What should be properly written is a statement that $\frac{1}{3}$ is approximately .333 or that π is approximately $\frac{22}{7}$ or 3.14. This is sometimes written with a variety of symbols for the equality but a $\sim$ or $\approx$ distinguishes these examples from the first ones. Thus we may write

$$\frac{1}{3} \approx .333 \; ,$$

$$\text{and } \pi \approx \frac{22}{7} \text{ or } \pi \approx 3.14 \; .$$

What we really mean in these instances is that .333 is a decimal whose value is reasonably close to the value of $\frac{1}{3}$ and that 3.14 is reasonably close to value of π. This is usually done in computational situations either because we do not know the exact value of some number (π for instance) or because we are willing to settle for an answer reasonably close to some number. For example the circumference or area of a circle with radius of 10 can be precisely given by the formula

$$C = 2\pi r$$

$$\text{and} \quad A = \pi r^2$$

which in this case become

$$C = 20\pi$$

$$\text{and} \quad A = 100\pi \ .$$

If, because the problem depends upon physical data or because we want to know approximately what C and A are, we may use an approximation for π and obtain

$$C \approx 20(3.14) \quad \text{or} \quad C \approx 62.80$$

$$\text{and} \quad A \approx 100(3.14) \quad \text{or} \quad A \approx 314$$

Thus 62.80 and 314 become approximations for the exact values 20π and 100π.

When public officials say 160 million Americans watched the recent walk on the moon they are approximating (actually an estimate or guess) the actual number who did watch. When an engineering student counts the number of steps between two points and multiplies this by 3 he determines approximately the number of feet between the two points. These are but two instances in which the actual number involved is not known but is approximated by a number, the closeness of the approximation being determined by the actual amount of information exactly known. Each step of the engineering student is not exactly three feet in length and the computed distance varies as the length of the step. Sometimes the exact value of a number could be stated, but we are satisfied with something which approximates it. It may have rained .475 of an inch but most people would probably say it rained one-half of an inch, being satisfied with the message one-half conveyed in the conversation.

In all these instances we see an actual value of a number being replaced by a number which is approximately correct. For the fraction $\frac{1}{3}$ we might use .3 or .33 or .3333, depending on precisely how close we wish to approximate $\frac{1}{3}$.

2. Approximating a root of an equation. The concept of finding a

number which approximates another number is also found in other circumstances. If the second number is the root of an equation, i.e., a number which satisfies some mathematical expression, it is frequently possible to use the equation or parts of it to determine an approximation for one of its roots. This is similar to finding an approximation for, say $\frac{1}{3}$, but the significant difference is that we have an equation and the equation becomes the source of approximations whereas one must guess or develop a source of approximations for $\frac{1}{3}$. There is no mathematical equation which would determine how many people watched the walk on the moon. In the case of the equation, we have a basis for making an approximation. In section II we observed how early attempts to solve equations began with guesses and the guess was refined by making use of the equation. In the case of the Rule of False Position a refinement produced the correct number and not just an approximation. The equation itself was used to determine what the second approximation should be. In the case of Horner's Method (which we will describe shortly) the first approximation for the root of an equation, together with the equation, produce a second and better approximation. There are several techniques which are similar and the computer makes it possible to repeat the process as long as one wishes and one can refine the sequence of approximations to obtain as close an approximation as one wishes.

We give a description of Horner's Method as a device to approximate a root of algebraic equation*. First, a real root of the algebraic equation $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ must be isolated, that is, it must be determined that a root lies between a and b, $a < b$. This can be done by several techniques, but the most elementary way is to find (guess at) two values a and b such that $P(a)$ is opposite in sign from $P(b)$. If this be true then there exists c, where $a < c < b$ such that $P(c) = 0$. Hence, c is a root of $P(x) = 0$.

Second, the equation

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0 \tag{1}$$

is now transformed by replacing x by $x - a$, and this diminishes the roots by a. We now have a new equation.

$$P_1(x) = a_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n = 0 \tag{2}$$

with a root of this equation between 0 and $(b - a)$. We now determine two more values c and d such that $0 < c < d < (b - a)$ and $P_1(c)$ is opposite in sign from $P_1(d)$. Now repeat the transformation again. Each new

*We assume for all discussions that the algebraic equations are polynomial equations with real coefficients.

step of the sequence, $0 < c < x < d < (b - a) = 1$ produces numbers c and d which are closer to the root of $P_1(x) = 0$ than previous approximations, and, if $b - a = 1$ in the beginning, the approximations get better much quicker in this process. Fortunately, this process can be done by using synthetic division and the computational work is considerably reduced by this procedure.

Newton's Method (1676) is a similar procedure and it has an advantage over Horner's Method in that it can also be applied to trigonometric, or logarithmic, or other simple functions. Horner's method is strictly reserved for algebraic polynomials with real coefficients.

The technique of successive approximations where each new approximation is a refinement of the preceding approximation is an excellent technique to apply to the task of finding a root of an equation. Bernoulli's method of approximating the largest root of the equation

$$x^3 = ax^2 + bx + c$$

with real coefficients is one such example. The method uses a set of recursion formula such that successive values for a variable $A_n(a,b,c)$ when compared with the preceding value $A_{n-1}(a,b,c)$ gave an approximation for the absolute value of the largest root of the cubic, if that root were real.

T. A. Pierce [13] gave a recursive technique for finding the least root of a cubic by developing Bernoulli-type formulas for approximating the absolute value of that root, if it is real. The formulas are strictly algebraic in nature, but rather complicated to deduce. Moreover, other techniques seem to be more elementary and cover more types of equations than just the cubics.

Other techniques employing the method of successive approximations can be found which use the equation and various derivatives of the polynomial function. These techniques employ mathematics beyond the secondary school level and, therefore, are not appropriate for this investigation. As an example of this type involving the derivative, see Ford [7].

Other techniques were developed in the twenties and thirties for solving certain polynomials. For example, Kennedy [9] showed an algebraic treatment for polynomial equations of a certain kind via a logarithmic process to find approximations for the roots of those certain polynomials. Running [14] gave techniques for finding real roots of cubics and quartics from graphs of certain straight lines which depended upon the discriminant of the equation. Grant [8] gave still more information on how to solve quartic equations by graphical means.

This brief summary of some pertinent papers on roots of polynomials shows the search for various techniques to simplify or avoid laborious calculations necessary in finding approximations to roots of an equation by methods such as Horner's Method. An entire period of mathematical work has been devoted to this kind of research. In some ways, the computer has made some of this work obsolete, while making other aspects of it even more valuable.

3. Approximating a function by another function. One of the most widely used approximation procedures in analysis and applied mathematics is the procedure of approximating some function by another function. This technique is particularly useful if the original function is complicated and the approximating function is simpler. We are not mathematically able to find the values of certain functions, but we are able to use these functions because we can approximate them by ones which we can evaluate.

A classic example of this sort of a procedure is given by a very important theorem from analysis. The first Weierstrass Approximation Theorem [1] asserts that

> if $f(x)$ is a function which is continuous in the finite interval $[a,b]$, then for every $\epsilon > 0$ there exists a polynomial $P_n(x)$ of degree $n = n(\epsilon)$ such that the inequality
>
> $$|f(x) - P_n(x)| \leq \epsilon$$
>
> holds throughout the interval $[a,b]$.

The theorem is really very powerful for two reasons. First, any continuous functions on a closed interval can be approximated by a polynomial of degree n, where the degree n depends only upon how close one wishes to approximate the function. When we say any continuous function, this includes an extremely large collection of functions which are not polynomials. It would include, except for certain instances or for certain levels of definition, the logarithmic, trigonometric, hyperbolic and many others.
The function $f(x) = \frac{1}{1-x}$ on $[0,.9]$ can be approximated by a polynomial in
The functions which can be approximated seem endless, except they are all continuous on $[a,b]$.

Second, these functions are approximated by a single kind of function, namely the polynomial. The polynomial function is also a continuous function and, perhaps is the most used class of continuous functions. It is clearly simple, and it is the first kind of function discussed in secondary school mathematics. It is a well-defined class of functions with many facts already established and well-known. There is no general class of continuous functions which is simpler than the polynomial class, and at the same time, has so much known about it.

Another theorem which illustrates this type of approximation is

Weierstrass's Second Theorem [1] and it says

If $F(t)$ is a continuous function of period 2π, then for every $\epsilon > 0$ there exists a trigonometric sum

$$S_n(t) = a_o + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

where $n = n(\epsilon)$, such that

$$|F(t) - S_n(t)| \leq \epsilon$$

for all t .

This theorem is not as general as the first theorem but illustrates a similar kind of procedure. Here periodic functions are approximated by a function of sines and cosines. The two observations about the first Weierstrass Approximation Theorem would also seem to be correspondingly appropriate for the Second Weierstrass Approximation Theorem.

Both theorems are existence theorems in the sense that an nth degree polynomial exists which approximate an $f(x)$ on $[a,b]$. It may be very difficult to actually find such a polynomial. However, it turns out that Bernstein [3] provided a subclass of polynomials which would do the job. The Bernstein theorem states that

If $f(x)$ is continuous on $[0,1]$, its Bernstein polynomials B_n, where

$$B_n(x) \equiv \sum_{0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \cdot x^k(1 - x)^{n-k}, \ \binom{n}{k} = \frac{n!}{k!(n - k)!}$$

converge uniformly to it on $[0,1]$ as $n \to \infty$.

Thus we have a specific collection of polynomials which are known and which can be used to approximate any continuous function on $[0,1]$.

4. Approximating Roots of One Function by Roots of Another Related Function. The last type of approximation technique to be mentioned is not obvious and, as we shall see later, has its difficulties and limitations. We shall describe the technique by using the example mentioned in section I. Suppose we drop a stone into a well and measure the time from releasing the stone until the sound of the stone hitting the water reaches our ears. Suppose this time is 3.1 seconds. Then the time for the stone to fall can be written as $t_1 = \sqrt{\frac{d}{16}}$ and the time for the sound to return to the ear as $t_2 = \frac{d}{1100}$ where 1100 ft. per sec. is

the assumed velocity of sound in air. Thus the total time elapsed is

$$t_1 + t_2 = \sqrt{\frac{d}{16}} + \frac{d}{1100} = 3.1 \tag{1}$$

If we want to know the depth of the well, we must solve this equation for d . To do so, we use the technique of squaring both sides to eliminate the square root and we obtain

$$\frac{d}{16} + \frac{6.2d}{1100} - \frac{d^2}{1,210,000} = 9.61 \ . \tag{2}$$

This is a quadratic in d and can be solved by the quadratic formula but the computations involved are going to be lengthy. So we look for a way out. We observe that the well could not be very deep because the two times, t_1 and t_2 have to be small and, therefore, d/1100 is relatively small. As a consequence $d^2/1,210,000$ would be very small relative to the other terms of (2). If we ignore this term, or consider only

$$\frac{d}{16} + \frac{6.2d}{1100} = 9.61 \tag{3}$$

we find a new equation which can be solved for d . Incidentally, the solution of (3) for d certainly requires less computational work than solving (2) for d .

Solving (3), we find d = 141. Checking this in (2) one can verify that it approximately satisfies (2) and therefore, can be verified as satisfying (1). So a solution to our original problem has been obtained.

The basic technique used in the solving of this problem is an approximation. We found equation (2) which needed to be solved. We obtained equation (3) from equation (2) by deleting a certain term. The solution of equation (3) is an approximation for a solution of equation (2). Thus we are actually approximating the roots of one function by the roots of another related function. The related function must in some way depend upon the original function, or in the case of polynomial equations, depend upon the coefficients of the polynomial.

It would appear, at first glance, that this technique might be questionable. In fact, it is not above a blemish or two, but it works in a surprisingly large number of cases. We shall explore this kind of an approximation in some depth in later sections. It is this procedure which the science teachers would like to see developed in our courses in algebra.

5. Comments. If one looks at these four types of approximations one observes that they all involve finding one number which approximates

another number. The basic difference is where does one get the information to make an approximation. In the last three types described there is given something about a desired number, i.e., root of an equation, etc., and the equation has been the basic source of information for the approximations. In sections 2 and 4 we did quite different things with the same information, thus providing different approximation procedures.

IV. Some Theorems Related to Approximations.

1. Introduction. In this section we shall cite some theorems which have some bearing upon roots of polynomial equations. These are not to be considered all that could be cited here, but are representative of those already available. A few will be essential for later sections and many illustrate related kinds of information. Some are too advanced to be considered at the secondary level, but those which can be proven by elementary algebraic means are proven. The proofs are included merely to illustrate their adaptability to secondary school mathematics.

2. Locations of Roots of $P(x) = 0$. A short reference to the location of roots of a polynomial equation was given in the historical background. We would like to expand on this here by citing several different types of locating theorems and concepts. At least one is elementary enough to be intuitively justified at the secondary level. In all the examples selected one determines something about the roots by examining the coefficients of the polynomial or by examining certain computations entirely dependent upon these coefficients.

Mathematically, we are justified in doing this because the coefficients of any polynomial equation can be expressed in terms of the roots of the equation. For example, a quadratic equation with roots a and b can be written as

$$(x - a)(x - b) = 0$$

Multiplying and collecting terms we have

$$x^2 - (a + b)x + ab = 0 .$$

Thus any quadratic of the form $x^2 - px + q = 0$ has its coefficients uniquely determined by the roots of that quadratic. All quadratic equations may be transformed to this form by dividing by the coefficient of x^2, if it is not already 1.

A similar argument for a cubic equation produces a form

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$$

where a, b and c are the roots of the cubic. Likewise the nth degree polynomial equation has as its corresponding form

$$x^n - (a_1 + a_2 + a_3 + \dots a_n)x^{n-1} + \dots \pm a_1 \cdot a_2 \cdot a_3 \dots a_n = 0$$

where the ± signs depend upon n being odd or even.

Thus, every coefficient of any polynomial $P(x) = 0$ is a function of the roots of that equation and the coefficients should determine something about the location of the roots of $P(x) = 0$. In what follows the equation $P(x) = 0$ shall be understood to be

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0 \tag{1}$$

unless otherwise stated with a_i being real numbers.

We take as a first example of location of roots the Routh-Hurwitz criterion. The Routh-Hurwitz criterion provides an algebraic manipulation of the coefficients of a polynomial to find out if the roots are to the right or left of the y-axis. Given the polynomial equated to zero

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$$

form the Hurwitz determinants $D_1, D_2, \dots, D_n$ given by

$$D_k = \begin{vmatrix} a_1 & a_3 & a_5 & \dots & a_{2k-1} \\ 1 & a_2 & a_4 & \dots & a_{2k-2} \\ 0 & a_1 & a_3 & \dots & a_{2k-3} \\ 0 & 1 & a_2 & \dots & a_{2k-4} \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & 0 & & 0 \end{vmatrix}$$

$$k = 1,2,\dots,n$$

where the coefficients with indexes larger than n or with negative indexes are replaced by zeros.

A necessary and sufficient condition that there be no zeros of $P(x) = 0$ to the right of the y-axis is that all the D_k's be positive. For each determinant with a negative sign there is a root to the right of the y-axis. Further, if D_n is the only one which is negative, there is a single root to the right of the y-axis, and this root must lie on the x-axis. If D_n is zero, the single root is at the origin. It is clear from this statement, which is given without proof that one can determine something about the character of the roots of $P(x) = 0$ without being able to actually find the roots.

Van Vleck [15] proved a theorem which further aids in the location of the roots of $P(x) = 0$ if the theorem is applied along with the Routh-Hurwitz criterion. The theorem is as follows:

If c_i is real and the terms of the sequence

$$c_0 \,, \quad \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix} \,, \quad \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix} \,, \ldots, \Lambda_n = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & & \cdots & c_{n+1} \\ \cdot & \cdot & & \cdots & \cdot \\ \cdot & \cdot & & \cdots & \cdot \\ \cdot & \cdot & & \cdots & \cdot \\ c_n & c_{n+1} & & \cdots & c_{2n+2} \end{vmatrix}$$

are positive, all the roots of the equation

$$c_o + c_1 x + c_2 x^2 + \ldots + c_{2n} x^{2n} = 0$$

are imaginary, and all but one of the roots of the equation

$$c_o + c_1 x + c_2 x^2 + \ldots + c_{2n+1} x^{2n+1} = 0$$

are imaginary.

Descartes' Rule of Sign is a similar kind of criterion for determining the character of the roots of $P(x) = 0$. The rule states that the number of positive real roots of $P(x) = 0$ is equal to the number of changes of sign in the coefficients (when taken from x^n down to x^o) diminished by $2k$, where k is an integer with minimum value of zero. The rule simply describes the maximum possible number of real positive roots and not actually how many exists. Descartes' Rule of Sign applied to $P(-x) = 0$ gives the maximum possible number of real positive roots of $P(-x) = 0$ and hence, the maximum possible number of real negative roots of $P(x) = 0$.

These criteria are helpful in determining some information about $P(x) = 0$ but in no way aid in solving $P(x) = 0$ or in approximating any roots of $P(x) = 0$.

Consider the example

$$x^3 + 6x^2 + 11x + 6 = 0 \,, \tag{2}$$

the Hurwitz determinants may be computed and they give

$$D_1 = 6 \qquad D_2 = 60 \qquad D_3 = 360$$

and the Routh-Hurwitz criterion applied to this case reveals no roots to the right of the y-axis.

The Descartes Rule of Signs applied to this example indicates no changes in sign so there are zero positive real roots. If we replace x by $-x$ in the example, we obtain

$$-x^3 + 6x^2 - 11x + 6 = 0 \tag{3}$$

and the Descartes Rule of Signs shows 3 sign changes so equation (3) has at most 3 positive roots or possibly only 1. Hence, the original equation (2) has 3 or 1 negative roots. The Routh-Hurwitz criterion applied to (3) indicates 3 roots to the right of the y-axis. Hence, (2) has 3 real negative roots. The Descartes Rule of Signs gave 3 as a maximum number but we could not know that 3 was actually the right number. In this sense, the Routh-Hurwitz criterion is better than the Descartes Rule of Signs, but it is also more complicated to apply (in the computational sense).

A second, (and frequently useful) piece of information about the character of the roots of $P(x) = 0$ can be found if one can determine the limits or bounds upon the roots themselves. By this, we mean, is it possible to say that all roots of $P(x) = 0$ are numerically smaller than some M, and M being a function of the coefficients of $P(x)$? A great deal of attention was directed to this kind of a mathematical problem during the early 1900's and much of it is quite appropriate to this discussion. We cite several of these to illustrate this technique.

If we refer to a theorem from complex variables we note that the region in which all roots of $P(x) = 0$ lie is a circle with center at the origin and a radius of

$$1 + |a_k| \text{ max.} \tag{4}$$

The $|a_k|$ max means to use the absolute value of the largest coefficient of $P(x)$. For example all roots of (2) would be within 12 of the origin. This is true because the roots are -1, -2, and -3. For this example the radius is too large for it to be really effective or to be used as an approximation for the largest root. However, it is very easy to compute.

There are other estimates and approximations for the location of roots of $P(x) = 0$. We give three such expressions to illustrate further this kind of investigation into bounds on the roots.

The expression in (4) can be replaced by other, yet similar, expressions giving the values of the radius of a circle containing all the roots of $P(x) = 0$. Walsh [16] gave the following limits on roots.

1. All roots of $x^2 + a_1 x = 0$ lie in or on the circle with center at origin and a radius of $|a_1|$.

2. All roots of $x^2 + a_1x + a_2 = 0$ lie in or on a circle with radius of $|a_1| + \sqrt{|a_2|}$. (Center is always at the origin.)

3. All roots of $x^3 + a_1x^2 + a_2x + a_3 = 0$ lie in or on a circle with radius $|a_1| + \sqrt{|a_2|} + \sqrt[3]{|a_3|}$.

For the example (2) this radius would be

$$|6| + \sqrt{|11|} + \sqrt[3]{6}$$

or approximately

$$6 + 3.3166 + 1.8171 = 11.1337$$

which is some better, but not much. This sequence of values for the radius finally can be expressed as

4. All roots of $P(x) = x^n + a_1x^{n-1} + \ldots + a_n = 0$ lie in or on a circle at the origin and with a radius of $|a_1| + \sqrt{|a_2|} + \sqrt[3]{|a_3|} + \ldots + \sqrt[n]{|a_n|}$.

Carmichael and Mason [5] proved that all roots of $P(x) = 0$ are in absolute value less than or equal to

$$\sqrt{1 + |a_1|^2 + |a_2|^2 + \ldots + |a_n|^2}$$

For example (2) this would be $\sqrt{194}$ which is actually larger in value than the value of 12 which we obtained earlier.

Williams [18] later showed that the absolute value of any root of $P(x) = 0$ was less than

$$\sqrt{1 + |a_1 - 1|^2 + |a_2 - a_1|^2 + \ldots + |a_n - a_{n-1}|^2 + |a_n|^2}$$

Again for our example this would be $\sqrt{112}$, or better than the other estimates.

From the literature on this subject one is notable to determine, except for special equations $P(x) = 0$, when these values stand a chance of being a reasonably good approximation for the largest root of $P(x) = 0$. There are several sources which give expressions for the least root of $P(x) = 0$ and we cite a few at this time.

Landau [10] was able to show that every equation of the form $ax^n + x + 1 = 0$ has a root, the absolute value of which is not greater than 2, and that every equation of the form $ax^n + bx^m + x + 1 = 0$ has a root, the absolute value of which is not greater than 8. These two observations were to establish a chain of discoveries and refinements which are excellent illustrations of the creative process in mathematics.

Allardice [2] was able to generalize these results to obtain the following theorems:

Theorem. The equation $ax^n + x + 1 = 0$ has a root, the absolute value of which is not greater than $n/n-1$, which is the value of equal roots.

Theorem. The equation $ax^n + bx^m + x + 1$ always has a root whose absolute value is not greater than $\frac{n \cdot m}{(n-1)(m-1)}$.

Theorem. The equation $ax^n + bx^m + cx + \ldots + a_1x + a_o = 0$ has a root whose absolute value is not greater than

$$\frac{a_o}{a_1} \cdot \frac{n}{n-1} \cdot \frac{m}{m-1} \cdot \frac{\ell}{\ell-1} \cdots$$

regardless of the other coefficients of the equation.

Applying Allardice's approximation to example (2) we find this value to be $\frac{18}{11}$, with $|-1|$ less than this, a surprisingly good approximation. Landau's condition would have been a root whose absolute value was less than 8 and not a good approximation.

About the same time Fejér [6] also generalized Landau's results by a very elegant theorem. We state the theorem for algebraic polynomials with integral exponents although the theorem is more general than this case.

Let

$$a_o + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

be an equation of n+1 terms, $a_o \neq 0$, $a_1 \neq 0$. Let p be the root of this equation of least absolute value. Then

$$p \leq n \frac{a_0}{a_1}$$

The value $n\left|\frac{a_0}{a_1}\right|$ gives both Landau's value for $ax^n + x + 1$ and Fejér's value for the same equations. For our example (2) we see

$$p \leq 3 \cdot \left|\frac{6}{11}\right| = \frac{18}{11}$$

Carmichael and Mason [5], Montel [11] and others were able to provide a slightly generalized statement on the last root. It is easy to intuitively verify these results for the quadratic equation and should make interesting enrichment material for some secondary school students.

3. Some Approximation Theorems.

In this section we will note some theorems which have previously been proven and are related to our discussion. One should note that there are various and diverse techniques for isolated instances of root finding. In the following theorems we shall be concerned with the nth degree polynomial equation of the form

(1) $$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$$

If $P(x) = 0$ has a numerically large real root it is possible to find an approximation for this root by solving the equation

$$x - a_1 = 0$$

The reasoning for this is that the coefficient a_1 in equation (1) is the sum of all roots of $P(x) = 0$. If α_1 is this numerically large real root, then

$$a_1 = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n$$

where $\alpha_2, \alpha_3, \dots, \alpha_n$ are the other roots of $P(x) = 0$. a_1 is approximately α_1 if α_1 is large in comparison to $\alpha_2 + \alpha_3 + \dots + \alpha_n$. However, this approximation idea has limited application because α_1 must be so large with respect to other roots.

There is, however, a technique for finding this large root and it is stated in the following theorem.

Theorem 1.* If α_1 is a numerically large root of $P(x) = 0$, where

*The Theorem is said to be well-known by Oldenburger [12] and we give a similar proof.

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

Then α_1 approximately satisfies the equation

$$x^2 + a_1x + a_2 = 0 .$$

Proof: Let σ_1 be the sum of the other roots $\alpha_2, \alpha_3, \dots, \alpha_n$ of (1). It follows that $a_1 = -(\alpha_1 + \sigma_1)$. Let σ_2 be the sum of the products of $\alpha_2, \alpha_3, \dots, \alpha_n$ in pairs (that is, $\sigma_2 = \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 + \dots$). From general knowledge about theory of equations the coefficient a_2 can be written as

$$a_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \dots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 + \dots + \alpha_{n-1}\alpha_n.$$

Hence,

$$a_2 = \alpha_1 \cdot (\alpha_2 + \alpha_3 + \alpha_4 + \dots + \alpha_n) + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 + \dots$$

or

$$a_2 = \alpha_1\sigma_1 + \sigma_2$$

Since $\alpha_1\sigma_1$ is large compared to σ_2 (since σ_2 does not contain a term with α_1 in it), then we can say a_2 is approximately $\alpha_1\sigma_1$. The equation $x^2 + a_1x + a_2 = 0$ can now be written as

$$x^2 - (\alpha_1 + \sigma_1)x + \alpha_1\sigma_1 = 0$$

and α_1 is a root of this equation.

There are equations for which $x^2 + a_1x + a_2 = 0$ will not give a good approximation for α_1; in which case we increase the degree of the equation by one to obtain

$$x^3 + a_1x^2 + a_2x + a_3 = 0$$

A similar argument can be given to show that α_1, if large, satisfies this equation. One must also assume α_1 is the <u>only</u> large root, as one tacitly assumed in Theorem 1.

In fact, one may generalize this to produce the following theorem:

Theorem 2. If α_1 is a numerically large root of $P(x) = 0$, then α_1 approximately satisfies the equation

$$x^{n-1} + a_1x^{n-2} + a_2x^{n-3} + \dots + a_{n-1} = 0 .$$

The proof is similar to the proof of Theorem 1.

A word of caution is necessary here. By a numerically large root, we mean either positive or negative. The proof essentially rests upon absolute values and this is clear if we go back to equation (2) of section IV, 2. The equation $x^3 + 6x^2 + 11x + 6 = 0$ has as its largest root $|-3|$. If $x^2 + 6x + 11 = 0$ is used as an approximating equation, the roots of it are

$$x = \frac{-6 \pm \sqrt{-8}}{2} .$$

The absolute value of x is $\left|\frac{-6 \pm \sqrt{-8}}{2}\right| = \frac{17}{4}$, a reasonably good approximation for $|-3|$ considering that $|-3|$ is not relatively large with respect to the other roots. If one takes the equation

$$x^3 - 9x^2 - 12x + 20 = 0$$

which has roots of 10, 1 and -2, the approximating equation becomes

$$x^2 - 9x - 12 = 0$$

and its largest root is $\frac{9 + \sqrt{129}}{2}$ or approximately 10.18, which is a very good approximation for the largest roots which was 10.

We mentioned previously in section III the Weierstrass Approximation Theorem where a function is approximated by a polynomial under a certain set of conditions. We restate the theorem for immediate reference.

Weierstrass Approximation Theorem: If $f(x)$ is a continuous function on the closed interval $[a,b]$, then for $\epsilon > 0$ there exists an $n = n(\epsilon)$ and a polynomial $P_n(x)$ of degree n such that

$$|f(x) - P_n(x)| < \epsilon \quad \text{for } a \le x \le b . \tag{2}$$

The polynomial $P_n(x)$ and n are not unique. That is, one may find a second polynomial of degree m which would satisfy (2) for the same ϵ. Thus, polynomial approximations to $f(x)$ exist which have a predetermined accuracy on $[a,b]$. The theorem says the polynomials exist, but does not give any clue as to how to find them.

One class of functions $f(x)$ which are continuous is the class of polynomials. Thus $P(x)$, a polynomial of degree m, is a continuous function. In fact, it is continuous for every real number x, and therefore, would be continuous for any closed interval $[a,b]$. The Weierstrass Approximation Theorem applied to this continuous function $P_m(x)$ over some interval $[a,b]$ states there exists a polynomial of degree n which is within a certain ϵ's distance of $P_m(x)$. It is clear that one of the polynomials which approximates $P_m(x)$ is the polynomial $P_m(x)$ itself. However, this is not necessarily the only one as we observed in the preceding paragraph. Immediately, two questions which are of importance to this study need to be answered. First, are there polynomials of degree n, where $n \leq m$, which approximate $P_m(x)$ to a desired accuracy? Second, if there is a polynomial $P_n(x)$, $n \leq m$ which approximates $P_m(x)$ on some interval, do the roots of $P_n(x)$ approximate the roots of $P_m(x)$?

What we really would like to find is a polynomial $P_n(x)$, $n < m$, which approximates $P_m(x)$ and where the coefficients of $P_n(x)$ are obtained in some fashion from the coefficients of $P_m(x)$. If the roots of $P_n(x)$ could approximate the roots of $P_m(x)$ whereever possible, then we could solve for the roots of $P_n(x) = 0$ and obtain approximations for the roots of $P_m(x) = 0$.

In what follows we shall establish some conditions under which the answers to the two questions just raised are affirmative, and, in particular, which allow us to do what we expressed in the last paragraph as a desire to be able to do. The following theorems, some of which were not found in the literature, will outline some of these conditions. We shall be much more specific in the section on quadratics.

Let $P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ with single roots. Let $Q(x)$ be composed of all terms of $P(x)$ except that one term, say b_jx^{n-j}, $1 \leq j \leq n-1$, is different from the corresponding term a_jx^{n-j} by an arbitrarily small amount less than ϵ over the interval $[0,1]$. Let 0 be a lower bound and 1 be an upper bound on the roots of $P(x) = 0$. Then we have

Theorem D3. If x_2 is a root of $Q(x) = 0$, then it is an approximation of a root x_1 of $P(x) = 0$.

Since $P(x)$ is continuous and $Q(x)$ is arbitrarily close to $P(x)$ over an interval $[0,1]$ the Weierstrass Approximation Theorem holds. We know that

$$|P(x) - Q(x)| = |b_jx^{n-j} - a_jx^{n-j}| < \epsilon \text{ for some fixed } j\text{ , } 1 \leq j \leq n-1.$$

Suppose x_2 is a root of $Q(x) = 0$ and x_1 is the corresponding root of

$P(x) = 0$. We know that

$$|P(x_1) - Q(x_1) + P(x_2) - Q(x_2)| \le |P(x_1) - Q(x_1)| + |P(x_2) - Q(x_2)|$$

$$< \epsilon + \epsilon = 2\epsilon$$

But $Q(x_2) = 0$ and $P(x_1) = 0$ so

$$|P(x_2) - Q(x_1)| < 2\epsilon$$

This can be written as

$$|a_j x_2^{n-j} - b_j x_1^{n-j}| < 2\epsilon$$

Since x_2^{n-j} and x_1^{n-j} are positive, and we lose no generality assuming $a_j > b_j$, then we have

$$|a_j x_2^{n-j} - a_j x_1^{n-j}| \le |a_j x_2^{n-j} - b_j x_1^{n-j}| < 2\epsilon$$

Therefore

$$|a_j| \cdot |x_2^{n-j} - x_1^{n-j}| < 2\epsilon$$

or

$$|x_2^{n-j} - x_1^{n-j}| < \frac{2\epsilon}{|a_j|}$$

but

$$x_2^{n-j} - x_1^{n-j} = (x_2 - x_1)(x_2^{n-j-1} + x_2^{n-j-2} \cdot x_1 + \dots + x_1^{n-j-1})$$

We have

$$|x_2 - x_1| \cdot |x_2^{n-j-1} + x_2^{n-j-2} \cdot x_1 + \dots + x_1^{n-j-1}| < \frac{2\epsilon}{|a_j|}$$

or

$$|x_2 - x_1| < \frac{2\epsilon}{|a_j| \cdot |M|}$$

where

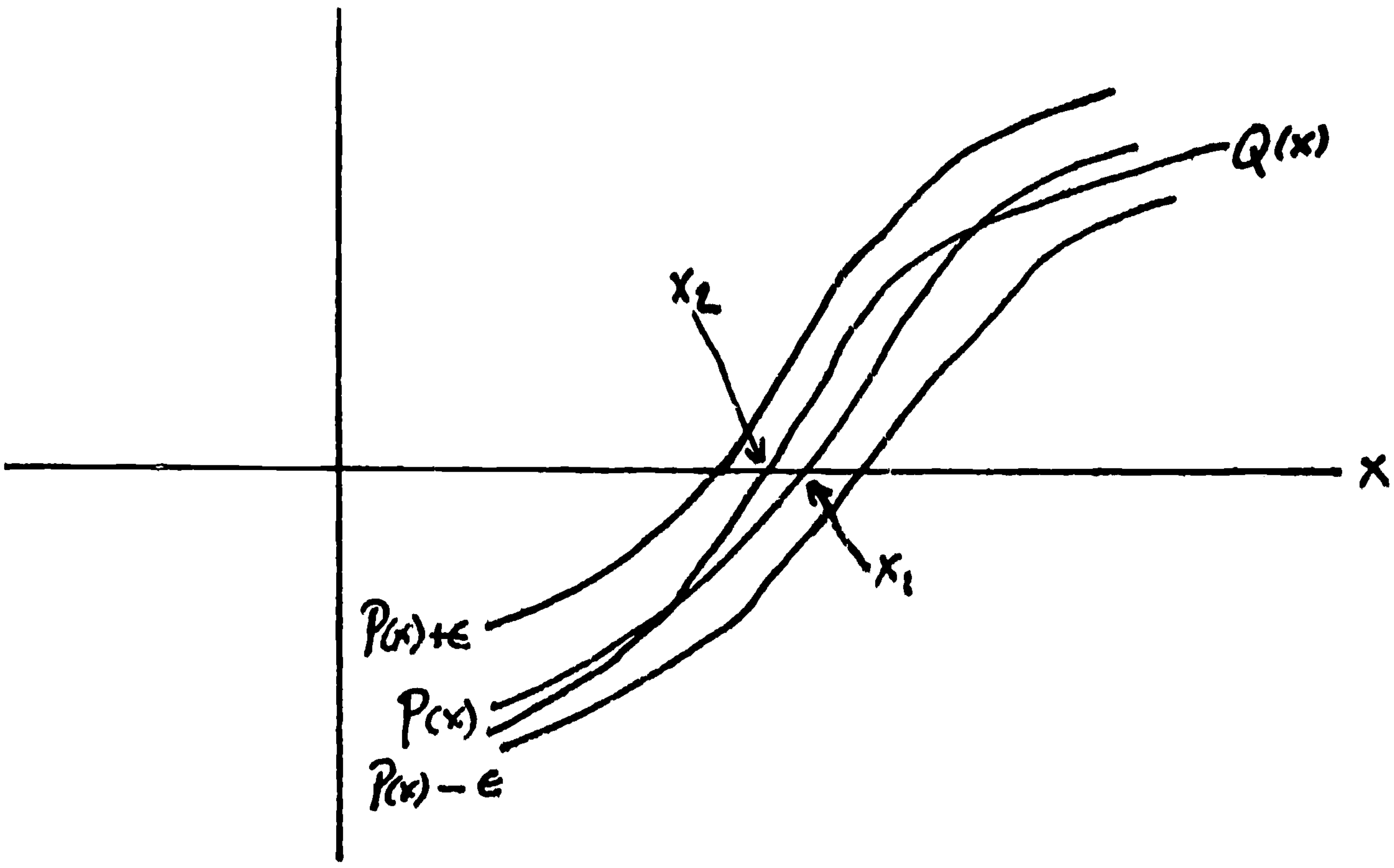
$$|M| = |x_2^{n-j-1} + x_2^{n-j-1} \cdot x_1 + \dots + x_1^{n-j-1}|$$

Since $|a_j|$ and $|M|$ are finite, we have

$$|x_2 - x_1| < \epsilon_1$$

Thus x_2 and x_1 are arbitrarily close, depending upon the closeness of $P(x)$ and $Q(x)$. Hence, a root of $Q(x) = 0$ is an approximation to a root of $P(x) = 0$.

This theorem is intuitively obvious if one constructs a graph of $P(x)$ and looks at where $Q(x)$ must go.



Since $Q(x)$ must lie between $P(x) + \epsilon$ and $P(x) - \epsilon$ it must also cross the x-axis in some small interval about x_1. Hence an approximation for x_1 is obtained at x_2.

It is also intuitively obvious that not all roots of $P_n(x) = 0$ can be roots of any polynomial of degree n-1. The following theorems show this to be true and under what circumstances.

Lemma D1. If $b_1, b_2, b_3, \ldots, b_n$ are the roots of the nth degree polynomial equation

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

and if $a_n \neq 0$, then $b_i \neq 0$, $i = 1,2,\ldots,n$.

Proof: Let b_i be a root of $P(x) = 0$. Since $a_n \neq 0$, then $b_i \neq 0$. For if $b_i = 0$, then

$$(b_i)^n + a_1(b_i)^{n-1} + \ldots + a_n = 0$$

and every term containing a b_i is also zero and this would mean a_n would also have to be zero for b_i to be a root of $P(x) = 0$. Since $a_n \neq 0$, then $b_i \neq 0$.

Theorem D4. Let $b_1, b_2, b_3, \ldots, b_n$ be the roots of the nth degree polynomial equation

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

Further, let a_n be non-zero. Then no root of $P(x) = 0$ can be a root of $Q(x) = 0$, where $Q(x)$ is obtained from $P(x)$ by deleting the x^n term.

Proof: Assume b_i also is a root of $Q(x) = 0$. Then $Q(b_i) = 0$ and we already know $P(b_i) = 0$; but $P(x) = x^n + Q(x)$ for all x, and therefore,

$$P(b_i) = (b_i)^n + Q(b_i)$$

or

$$0 = (b_i)^n + 0$$

The only way this last equation can hold is for $b_i = 0$. But by the lemma $b_i \neq 0$, therefore, b_i cannot be a root of $Q(x) = 0$.

Theorem D5. Let b_i, $(i = 1, \ldots, n)$ be the roots of $P(x) = 0$. Let $Q(x) = P(x) - a_jx^{n-j}$, $j = 1, 2, \ldots, (n-1)$, $a_j \neq 0$, $a_n \neq 0$. Then no root of $P(x) = 0$ can be a root of $Q(x) = 0$.

Proof: Since $a_n \neq 0$, then $b_i \neq 0$, Assume b_i is a root of $Q(x) = 0$. Then we have $Q(b_i) = P(b_i) - a_j(b_i)^{n-j}$ or $0 = 0 - a_j(b_i)^{n-j}$. Since $a_j \neq 0$ and $(b_i)^{n-j} \neq 0$, then b_i cannot be a root of $Q(x) = 0$.

In both of these theorems the condition $a_n \neq 0$ holds and the theorems hold for large collections of polynomial equations, but not all. For example, if $P(x) = x^4 - x^3 + x^2 - x = 0$, a root of this equation satisfies the equation

$$Q(x) = -x^3 + x^2 - x = 0$$

or

$$Q(x) = x^4 - x^3 - x = 0 .$$

It is clear that the root which does this is the zero root. We will now examine the cases in which $a_n = 0$ and see if we can enlarge the collection of polynomial equations which satisfy Theorems D4 and D5. If $a_n = 0$ then the polynomial equation $P(x) = 0$ can be written in the following manner:

$$P(x) = x^k \cdot R(x) \quad \text{where } k = 1,2,\ldots,n .$$

If a_n is the only zero constant of $P(x)$ then $k = 1$. $R(x)$ is a polynomial of degree n-k with the constant a_{n-k} being nonzero. This expression for $P(x)$ is obtained for any $P(x)$ with $a_n = 0$ by factoring the highest power of x from $P(x)$ and $R(x)$ is the remaining factor. One may deduce two interesting relationships between the roots of $P(x) = 0$ and $R(x) = 0$.

Theorem D6. If $a_n = 0$, $b_1 = 0$ and $P(b_1) = 0$, then $R(b_1) \neq 0$.

Proof: Let

$$P(x) = x^k \cdot R(x)$$

or

$$P(x) = x^k \cdot (x^{n-k} + a_1 x^{n-k-1} + \ldots + a_{n-k})$$

where

$$a_{n-k} \neq 0 .$$

(If a_{n-k} were equal to zero, an additional factor of x could be removed from $R(x)$ and some other nonzero coefficient would be the constant term of a new $R(x)$.) Then consider

$$R(b_1) = b_1^{n-k} + a_1(b_1)^{n-k-1} + \ldots + a_{n-k} .$$

Since $b_1 = 0$ we have

$$R(b_1) = R(0) = 0^{n-k} + a_1(0)^{n-k-1} + \ldots + a_{n-k}$$

and

$$R(0) = a_{n-k} .$$

Hence, the zero root of $P(x) = 0$ is not a root of $R(x) = 0$.

Theorem D7. If $a_n = 0$, and b_i is a nonzero root of $P(x) = 0$, then $R(b_i) = 0$.

Proof: Let $P(x) = x^k \cdot R(x)$. If $b_i \neq 0$ and is a root of $P(x) = 0$, we have

$$P(b_i) = 0 = (b_i)^k \cdot R(b_i) .$$

Since $b_i \neq 0$, $(b_i)^k \neq 0$ and consequently, $R(b_i)$ must be zero. Hence b_i is a root of $R(x) = 0$. These two theorems together say that the nonzero roots of $P(x) = 0$ (where $a_n = 0$) are the nonzero roots of $R(x) = 0$, and any zero roots of $P(x) = 0$ are not roots of $R(x) = 0$. Lemma 1 further says that $R(x) = 0$ has no zero roots. We shall now reconsider theorems D4 and D5 in a broader sense.

Theorem D8. Let $b_1, b_2, \ldots, b_n$ be roots of the nth degree polynomial equation

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

Then no nonzero root of $P(x) = 0$ can be a root of $Q(x) = 0$ where $Q(x)$ is obtained from $P(x)$ by deleting the x^n term.

Proof: If $a_n \neq 0$, then D4 holds. If $a_n = 0$, then $P(x) = 0$ takes on the form

$$P(x) = x^kR(x) = x^k(x^{n-k} + a_1x^{n-k-1} + \ldots + a_{n-k}) = 0$$

Then $Q(x) = P(x) - x^n$. If $b_i \neq 0$, then

$$Q(b_i) = P(b_i) - b_i^n .$$

Since, by hypotheses $P(b_i) = 0$ and $(b_i)^n \neq 0$, we have

$$Q(b_i) = (b_i)^n \neq 0$$

and b_i is not a root of $Q(x) = 0$.

Theorem D9. Let b_i, $i = 1,2,\ldots,n$ be the roots of $P(x) = 0$ (where $P(x)$ is the polynomial of Theorem D8). Let $Q(x) = P(x) - a_jx^{n-j}$ where $a_j \neq 0$. Then no nonzero root of $P(x) = 0$ can be a root of $Q(x) = 0$.

Proof: If $a_n \neq 0$, then Theorem D5 holds. If $a_n = 0$, then we can write $P(x) = 0$ as $x^k \cdot R(x) = 0$ as we did in Theorem D8. Then $Q(x) = x^k \cdot R(x) - a_jx^{n-j}$. If $b_i \neq 0$, then we have

$$Q(b_i) = (b_i)^k \cdot R(x) - a_j(b_i)^{n-j}$$

but, by Theorem D7, if $P(b_i) = 0$, $P(x) = x^k \cdot R(x)$, and $b_i \neq 0$, then $R(b_i) = 0$, and

$$Q(b_i) = b_i^k \cdot 0 - a_j(b_i)^{n-j}$$

or

$$Q(b_i) = -a_j(b_i)^{n-j} \neq 0 .$$

Hence b_i cannot be a root of $Q(x) = 0$. The sequence of theorems have shown that a nonzero root of a polynomial equation $P(x) = 0$ cannot be a root of a polynomial equation obtained by deleting some nonzero term of the original polynomial equation.

If we examine Theorems D3 and D9 together, we may observe one very interesting possibility. Theorem D9 says no nonzero root of $P(x) = 0$ can be a root of $Q(x) = P(x) - a_jx^{n-j} = 0$. Theorem D3 says if $P(x)$ and $Q(x)$ are alike except for some term a_jx^{n-j} and these terms are arbitrarily close, then a root of $Q(x) = 0$ is an approximation to a root of $P(x) = 0$. The roots cannot be the same but can be close to being the same. Theorem D3 can be used on the $P(x)$ and $Q(x)$ of Theorem D9 if a_j is relatively small with respect to the other coefficients and $Q(x)$ then can be taken to have no term a_jx^{n-j} in it. In this case we have the following theorem.

Theorem D10. If $P(x) = 0$ has a term a_jx^{n-j} with a_j relatively small with respect to the other coefficients of $P(x)$, then the polynomial $Q(x) = P(x) - a_jx^{n-j} = 0$ has a root which approximates a root of $P(x) = 0$.

If a_jx^{n-j} is relatively small on some interval then, by the Weierstrass Approximation Theorem, $Q(x)$ is an approximation for $P(x)$ over this interval. Hence, by Theorem D3 $Q(x) = 0$ has a root which is an approximation for a root of $P(x) = 0$, and the theorem is proved.

In the section on the quadratic equation we shall see the implications of this theorem in application.

V. Some Approximation Techniques for Quadratic Equations.

1. Introduction. We have already presented some general theorems on approximations of polynomials of nth degree. Now we will direct our attention to the quadratic equation and the approximating equations for the quadratic. The quadratic is especially important because it is the most elementary polynomial, besides the linear one, which is found in the algebra taught at the secondary school level. Moreover, we can be more specific for this polynomial in several ways and this makes the approximations more valuable.

We will not consider the case where we approximate the roots of a quadratic by the roots of another quadratic. Since we would have to solve the second quadratic anyway, the effort involved might just as well be directed toward solving the first quadratic. It is also true that, for elementary problems, this is the least interesting case. We will consider approximating the roots of a quadratic by deleting a term of the quadratic and solving the resulting equation for its roots. This will be done by considering the various possible types of roots of the quadratic equation and the various terms which can be deleted for each type of root considered. We will be concerned with the deletion of only one term from the quadratic, and there will be a total of six cases to be considered.

2. The Quadratic Equation with Real Roots. Write the quadratic equation $P(x) = 0$ in the form

$$P(x) = x^2 - (a+b)x + ab = 0$$

where a and b are the real roots of $P(x) = 0$ and $a \neq b$. We now construct $Q(x) = 0$ from $P(x) = 0$ by deleting a term of $P(x) = 0$.

Case A. The first term we shall delete is x^2. Let

$$Q(x) = P(x) - x^2 = 0 .$$

Then

$$Q(x) = -(a+b)x + ab = 0 .$$

We wish to know how the root of $Q(x) = 0$ is related to either root of $P(x) = 0$. Solve $Q(x) = 0$ for x. Since

$$Q(x) = -(a+b)x + ab = 0$$

we have

$$x = \frac{ab}{a+b} .$$

Note that the expression $\frac{ab}{a+b}$ is a symmetric one in a and b, so whatever we might say relative to a could be said relative to b. Since $a \neq b$, let a be greater than b. If a is large relative to b, the expression $\frac{ab}{a+b}$ becomes relatively close to b, i.e., $\frac{ab}{a+b}$ is an approximation for b if a is large relative to b. But this means that $\frac{ab}{a+b}$ approximates a root of $P(x) = 0$ under this set of conditions. The related question of just how good an approximation one has can be determined by looking at the following table.

Table I - Values of $\frac{ab}{a+b}$

a	a = 2b	a = 4b	a = 8b	a = 20b	a = 100b
b	b	b	b	b	b
$\frac{ab}{a+b}$	$\frac{2}{3}$ b	$\frac{4}{5}$ b	$\frac{8}{9}$ b	$\frac{20}{21}$ b	$\frac{100}{101}$ b

As one observes in the table the expression $\frac{ab}{a+b}$ takes on values which get closer and closer to b as a becomes larger (relative to b). In fact, the limit of $\frac{ab}{a+b}$ is b, as a increases without bound. Moreover, from the conditions described for this particular case, the approximation obtained is always for the smallest of the real roots.

If a is sufficiently large in the negative direction, i.e., $|a| > b$, the same sort of approximation is obtained. Examine the table below.

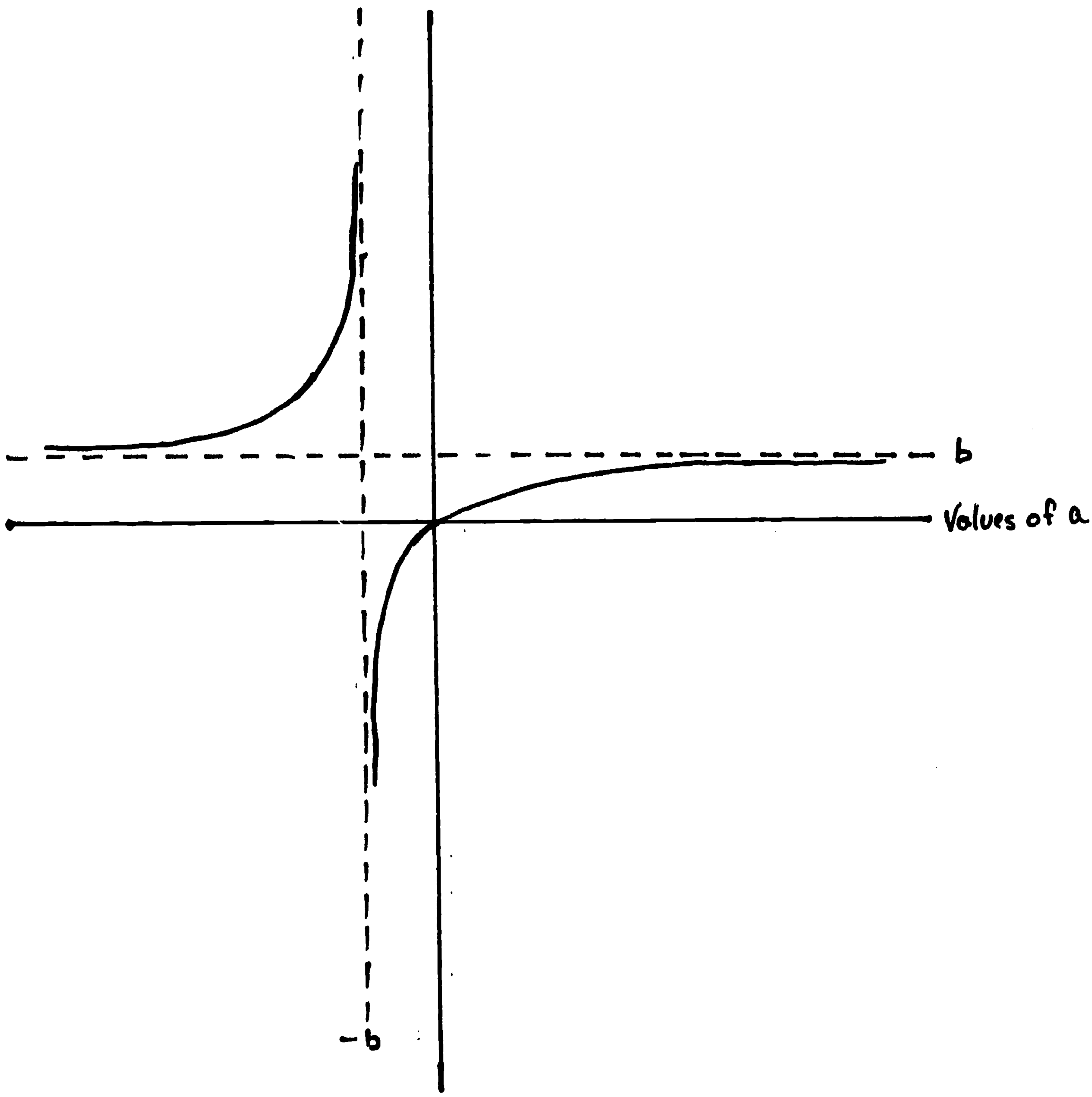
Table II - More Values of $\frac{ab}{a+b}$

a =	-2b	-4b	-8b	-10b	-100b
b =	b	b	b	b	b
$\frac{ab}{a+b}$ =	+2b	$\frac{4}{3}$	$\frac{8b}{7}$	$\frac{10b}{9}$	$\frac{100b}{99}$

Thus $\frac{ab}{a+b}$ is an approximation for b if $|a|$ is greater than b. The closeness of the approximation, for a given b, depends only on the value of a.

If one takes the values found in Tables I and II and puts this in graphical form one obtains the equilateral hyperbola for a graph, provided one fixes b at some value. Table III is one such graph for a fixed b. The graph reflects the two observations made about $\frac{ab}{a+b}$ being an approximation for b, provided $|a|$ is relatively large. One should also note that $\frac{ab}{a+b}$ does not approximate b in the central region of the graph where a is close to -b. In the event a is close to -b in value we observe (a+b) is very small and the coefficient of x is small with respect to the other terms of the quadratic. This gives

Table III - Graph of $\frac{ab}{a+b}$, for fixed b .



us a clue as to when we can depend upon $\frac{ab}{a+b}$ being a good approximation for a root of $P(x) = 0$. We formalize this in the following theorem.

Theorem D11. If $P(x) = x^2(a+b)x + ab = 0$, with a,b real roots of $P(x) = 0$, $a \neq 0$, $b \neq 0$ and $|a| > b$, then the equation

$$Q(x) = P(x) - x^2 = -(a+b)x + ab = 0$$

has a root, $x = \frac{ab}{a+b}$, which is an approximation for the smallest root of $P(x) = 0$, provided the coefficient of x^2 is relatively small in comparison to the coefficient of x and the constant term $a \cdot b$.

Comment 1. Note that if either root is zero, we have a special quadratic which need not be solved by approximation.

Comment 2. Note that the key as to when one can depend upon $\frac{ab}{a+b}$ to be an approximation of a root of $P(x) = 0$ is the coefficients of the polynomial $P(x)$. We have written $P(x) = 0$ in the form $x^2 - (a+b)x + ab = 0$ so as to reduce the number of unknown coefficients. If the coefficient of x^2, i.e., 1, is small relative to $|-(a+b)|$ and $|ab|$ then $\frac{ab}{a+b}$ provides an approximation to a root of $P(x) = 0$.

Comment 3. The theorem does not say so, but since $|a| > b$, the root approximated is, numerically, the smallest.

Comment 4. The error of approximation is likewise not given in the theorem, but it is easily determined by computing the following:

$$\text{Error} = b - \frac{ab}{a+b} .$$

However, this requires a knowledge of what b is and if we must know b in order to compute the error then we already know a root of $P(x) = 0$ and all this approximation business is needless. However, the Tables I and II suggest a means to determine relative error and Table III gives enough of a graphic picture to approximate the relative error. Error and relative error are strictly functions of two variables, a and b, and, therefore, for any fixed equation $P(x) = 0$ the amount of error is previously fixed by the roots of the equation.

Comment 5. This theorem justifies the procedure used in the introduction to solve a certain well problem and puts the procedure on a sound mathematical basis.

Case B. Consider how $Q(x) = P(x) - (a+b)x = 0$; that is, take $P(x) = 0$ and delete the x term. We are still assuming a and b are real roots of $P(x)$ and $a \neq b$. Then

$$Q(x) = x^2 + ab = 0$$

and

$$x^2 = -ab$$

for $x^2 = -ab$ to have real roots, either a or b must be negative. Assume a is. Then b is positive and -ab is positive. Hence $x = \pm\sqrt{-ab}$ and the latter is a real number. But $\sqrt{-ab}$ can be an approximation for b only when $a = -b$. In either case the expression $x = \sqrt{-ab}$ can approximate a root of $P(x) = 0$ only if a is approximately equal to -b. When this happens the coefficient of x, i.e., $-(a+b)$ becomes very small and approximately zero. Thus the term being deleted from $P(x) = 0$ has a relatively small coefficient in comparison to the other coefficients in the equation.

For example, the equation

$$x^2 + .01x = (12)(11.99) = 0$$

has a root which can be approximated by a root of $x^2 - (12)(11.99) = 0$.

$$\text{i.e., } x = \pm\sqrt{12(11.99)}$$

In fact, we have approximations for both roots. These observations can be stated in the following theorem.

Theorem D12. Let $P(x) = x^2 - (a+b)x + ab = 0$, with a and b real, $a \neq 0$, $b \neq 0$. Let $|-(a+b)|$ be relatively small as compared to 1 and $|ab|$. Then, both roots of $P(x) = 0$ are approximated by the roots of $Q(x) = x^2 + ab = 0$.

Comment 1. If $-(a+b)$ is approximately zero, then a is approximately the negative of b. Thus a and b must be real roots, otherwise ab would be complex or numerically smaller than a+b and neither are to be allowed in the theorem.

Case C. Now we consider the case where $Q(x) = P(x) - ab = 0$ or $Q(x) = x^2 - (a+b)x = 0$. We still assume a and b are real roots of $P(x) = 0$, $a \neq b$. Solving $Q(x) = x^2 - (a+b)x = 0$, we obtain

$$x = 0 \quad \text{and} \quad x = (a+b) .$$

If ab is to be relatively small with respect to 1 and $|-(a+b)|$, then either a is close to zero or b is close to zero. If a is approximately zero then $|-(a+b)|$ is approximately b. In this case the two roots of $Q(x) = 0$ give approximations for the two roots of $P(x) = 0$. Similarly, if b is approximately zero, $|-(a+b)|$ is approximately a and again we have two approximations for the roots of $P(x) = 0$.

We have, therefore, the following theorem.

Theorem D13. Let $P(x) = x^2 - (a+b)x + ab = 0$, with a and b real,

$a \neq 0$, $b \neq 0$, $a > b$. Let $|ab|$ be relatively small as compared to 1 and $|-(a+b)|$. Then the two roots of $P(x) = 0$ are approximated by the roots of $Q(x) = x^2 - (a+b)x = 0$.

3. The Quadratic with Complex Roots. We consider now the cases where the roots are complex. Suppose

$$x^2 - a_1x + a_2 = 0$$

has complex roots. Let a+bi and a-bi be these roots. [We know complex roots of $P(x) = 0$ occur in pairs and are conjugate pairs.] Thus the quadratic becomes

$$(1) \qquad x^2 - (2a)x + (a^2+b^2) = 0$$

Now consider the various equations obtained from (1) by the deletion of a term.

Case A. We begin by considering the deletion of x^2. Then

$$Q(x) = -2ax + (a^2+b^2) = 0 .$$

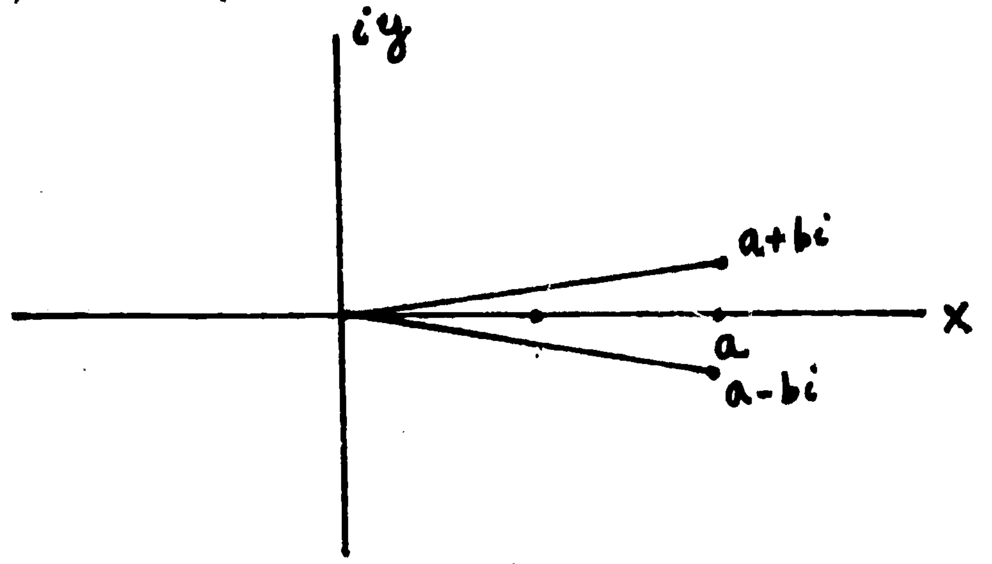
Solving for x we have

$$x = \frac{a^2+b^2}{2a}$$

or

$$x = \frac{a}{2} + \frac{b^2}{2a} .$$

If a is relatively large and b is relatively small, the value for x becomes approximately $a/2$. If b is not small, then $b^2/2a$ becomes a significant part of the value for x and x may be much larger than $a/2$. But if b is small in comparison to a, then the root of $Q(x) = 0$ enables us to find an approximation to a root of $P(x) = 0$. This seems to be a contradiction for x is a real number from $Q(x) = 0$ and the roots of $P(x)$ are complex. This contradiction is no longer a contradiction if one locates these values in the complex plane. The roots of $P(x) = 0$ and $Q(x) = 0$ are put on the graph as follows:



Since b is relatively small the roots of $P(x) = 0$ lie close to the x-axis. The root of $Q(x) = 0$ is $a/2$ and it does not approximate either root of $P(x) = 0$, but a does! Thus, for this case, $Q(x) = 0$ gives a value which can be multiplied by 2 to get an approximation of <u>either</u> root of $P(x) = 0$. Therefore, we may state the following theorem.

Theorem D14. If $P(x) = x^2 - a_1x + a_2 = 0$ has complex roots $a \pm bi$, and a is relatively large and b is relatively small, then the root of $Q(x) = -a_1x + a_2 = 0$ (or $Q(x) = P(x) - x^2 = 0$), when doubled, becomes an approximation for the roots of $P(x) = 0$.

Case B. Instead of deleting x^2 from $P(x) = 0$, consider the $Q(x)$ obtained by deleting the x term. We have

$$Q(x) = x^2 + (a^2+b^2) = 0$$

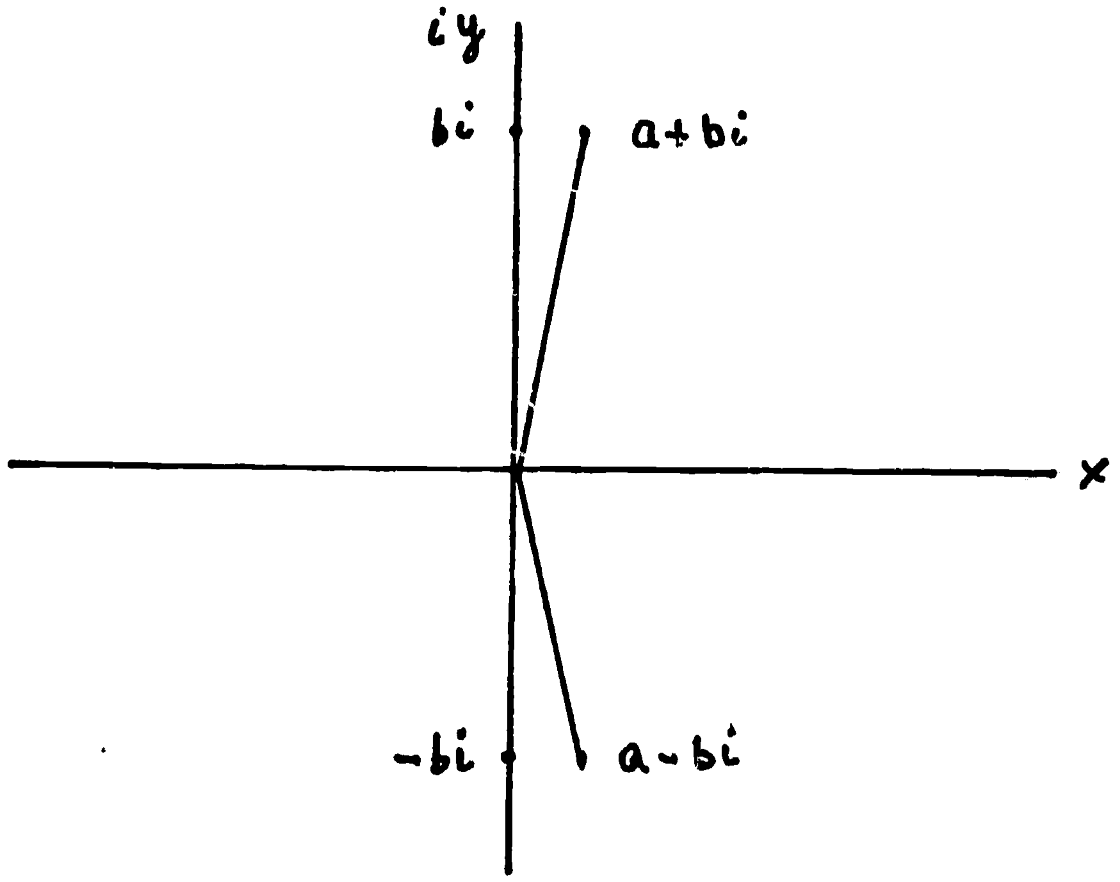
Solving for x we obtain

$$x = \pm \sqrt{-(a^2+b^2)}$$

or

$$x = \pm i \sqrt{a^2+b^2}$$

If a is relatively small, then the value of (a^2+b^2) is approximately b^2, and thus x is approximately $\pm i \cdot b$. Hence, if a is small, the solutions of $Q(x) = 0$ are approximations to the solutions of $P(x) = 0$. Again this becomes obvious if we locate these values on the complex plane.



Thus we have the following theorem.

Theorem D15. If $P(x) = x^2 - a_1x + a_2 = 0$ has complex roots $a \pm bi$, and a is relatively small and b is relatively large, then the roots of $Q(x) = x^2 + (a^2+b^2) = 0$ (where $a_2 = a^2+b^2$), then the roots of $Q(x) = 0$ are approximations for the roots of $P(x) = 0$.

Case C. If we delete the constant term we know that $Q(x) = 0$ is defined to be $P(x) - a_2 = 0$ or

$$Q(x) = x^2 - 2ax = 0 .$$

The roots of $Q(x) = 0$ are $x = 0$ and $x = 2a$. If the roots of $P(x) = 0$ are $a \pm bi$, then the only way for the roots of (x) to approximate the roots of $P(x) = 0$ are for the roots to be close to zero or close to a. In the first instance, a and b would both be close to zero and so would $a \pm bi$. Thus both roots of (4) are approximations to the roots of $P(x) = 0$. In the second instance, if b is relatively small then 1/2 of the root $x = 2a$ will give an approximation to $a \pm bi$.

From this analysis we can state the following theorems:

Theorem D16(a). If $P(x) = x^2 - a_1x + a_2 = 0$ has complex roots $a \pm bi$ and a is relatively large and b is relatively small, then the nonzero root of $Q(x) = x^2 - a_1x = 0$ multiplied by 1/2 is an approximation for either root of $P(x) = 0$.

Comment: The roots of $P(x) = 0$ lie close to the x-axis on a graph and thus a is an approximation for either root. Notice the similarity of this case to Case A considered under the section on complex roots.

We also have one other theorem we may state at this time.

Theorem D16(b). If a is also relatively small in Theorem D16(a), then both roots of $Q(x) = 0$ are approximations to the roots of $P(x) = 0$.

This must be true for if both a and b are very small, then $a \pm bi$ both lie close to the origin. Hence either 0 or $2a$ will be approximations for $a \pm bi$.

It is interesting to observe that in all cases of complex roots the amount of error of an approximation to a root of $P(x) = 0$ obtained from a $Q(x) = 0$ depends almost entirely upon just one part of the complex root, that is, either the real part or the imaginary part. The graphs associated with the various theorems of this section show this rather clearly.

4. Comments and Observations. In every case considered we were able to deduce conditions under which $Q(x) = 0$ produced roots which were approximations to $P(x) = 0$. In each case the condition contained some stipulation upon the roots in order to insure that approximations were possible. If we do not know what the roots are, then we cannot insure whether the

theorem applies or not. We could make some assumptions however and proceed on the basis of the assumptions. This is not necessarily safe in the game of mathematics. There is, however, an observation about the six cases we have considered. They are mutually exclusive in character. This means we can use each theorem under just certain conditions and no others.

Take an example. Suppose $P(x) = x^2 - 7x + 9 = 0$. Looking over the 6 theorems we conclude the only ones which could possibly apply (without) knowing the roots of $P(x) = 0$ are D11 and D14. These two are the only ones in which the coefficient of x^2 is the numerically smallest coefficient. In both theorems we used $Q(x) = -7x + 9 = 0$ to approximate a root of $P(x) = 0$. Thus $x = \frac{9}{7}$ is an approximation to the smallest root of $P(x) = 0$ regardless of whether $P(x) = 0$ has real or complex roots!

If $P(x) = 0$ has complex roots, we know that $\left(\frac{9}{7}\right)^2 + b^2 = 9$ must hold and $b = \sqrt{9 - \left(\frac{9}{7}\right)^2}$. But for Theorem D14 to apply b must be relatively small with respect to a and this is not true in this case. Hence, D14 is out, leaving only D11.

If $P(x) = 0$ has real roots (and it has) then $-(a+b) = -7$ and one root is approximately $\frac{9}{7}$ so the other must be $+7 - \frac{9}{7}$ or $+\frac{40}{7}$, and we observe that we were justified in using theorem D11.

If one really wished to insure the right selection, check the coefficients of $P(x) = 0$ and compute the discriminant. Not only does the discriminant of $P(x)$ tell if the roots are real or complex, but the size of the discriminant when it is negative determines when the complex roots may lie close to the real or imaginary axis.

It turns out then that we do not really need to know the roots to choose the appropriate approximation theorem because the coefficients of $P(x) = 0$ can really tell us this. This is due to the fact that the coefficients of $P(x) = 0$ are specific functions of the roots and, hence, tell us as much as the roots do if we but know how to read them.

One last observation which can be made about quadratics has to do with two quadratic functions which differ by a small amount.

Let $P(x) = x^2 - (a+b)x + ab$, where a and b are roots of $P(x) = 0$. Let $Q(x) = P(x) - \epsilon$, where $Q(x)$ is a quadratic which differs from $P(x)$ by exactly ϵ. From the Weierstrass Approximation Theorem and Theorem D10 we know the roots of $Q(x) = 0$ approximate the roots of $P(x) = 0$. In this case we have

$$Q(x) = x^2 - (a+b)x + ab - \epsilon = 0$$

Solving by quadratic formula we have

$$x = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4.1(ab-\epsilon)}}{2}$$

$$= \frac{(a+b) \pm \sqrt{(a-b)^2 + 4\epsilon}}{2}$$

$$\leq \frac{(a+b) \pm [\sqrt{(a-b)^2} + \sqrt{4\epsilon}]}{2}$$

(1) $$x \leq \frac{(a+b) + (a-b)}{2} + \sqrt{\epsilon} = a + \sqrt{\epsilon}$$

and

(2) $$x \leq \frac{(a+b) - (a-b)}{2} - \sqrt{\epsilon} = b - \sqrt{\epsilon}$$

Thus the roots of $Q(x) = 0$ differ from the roots of $P(x) = 0$ by at most $\sqrt{\epsilon}$. When $Q(x)$ differs from $P(x)$ by exactly ϵ. A similar statement can be made regarding a pair of cubic functions which differ by exactly ϵ , except the roots of one cannot differ from the roots of the other by more than $K \cdot \sqrt[3]{\epsilon}$, where K is a specific function of the coefficients of the cubic $P(x)$. It is probably true that a corresponding statement can be made for the nth degree polynomial.

VI. Approximations and Other Equations.

On the basis of what we have already seen, one would hope that the concepts of sections IV and V could be extended to other polynomial equations of higher degree. The problem of such an extension is that we cannot always solve an nth degree equation and, therefore, don't know if approximations to the roots of one equation can be actually computed from a derived equation. The Weierstrass Approximation Theorem and Theorem D10 state that approximations do exist but finding them is another matter. Even the cubic equation presents some serious problems as far as applying the concepts of section V. We shall examine this one very briefly.

Suppose $P(x) = x^3 + a_1x^2 + a_2x + a_3 = 0$. If we delete certain terms one by one we have the following derived equations:

$$Q_1(x) = a_1x^2 + a_2x + a_3 = 0$$

$$Q_2(x) = x^3 + a_2x + a_3 = 0$$

$$Q_3(x) = x^3 + a_1x^2 + a_3 = 0$$

and

$$Q_4(x) = x^3 + a_1x^2 + a_2x = 0$$

Equations $Q_2(x) = 0$ and $Q_3(x) = 0$ are both cubics which would have to be solved to obtain solutions which approximated solutions for $P(x) = 0$. There is really no advantage in solving these as precisely the same kind of effort would have to be exerted as would have been necessary to solve the original equation, which is also a cubic. One might just as well solve $P(x) = 0$ in these cases and obtain the actual roots.

In the case of $Q_4(x)$ we are assuming one root of $P(x) = 0$ is approximately zero and the other roots are readily obtained by solving a quadratic equation. One could, if the theorems applied, obtain solutions of the quadratic via the theorems of section V but this would be finding approximations to approximations to the actual roots and this could be unsatisfactory as a process.

The only derived equation which shows any promise is $Q_1(x) = 0$, since it is a quadratic derived from $P(x) = 0$ on the condition that a_1, a_2 and a_3 are all relatively large with respect to the coefficient of x^3, which is 1. Without going into all the possibilities (which are really beyond the level intended in this study) we will cite some examples to show how the approximations turn out.

Example 1. If $P(x) = x^3 - 9x^2 - 12x + 20 = 0$, then $Q_1(x) = -9x^2 - 12x + 20 = 0$ and we have

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-9)(20)}}{2(-9)}$$

or

$$x = \frac{12 \pm \sqrt{864}}{-18}$$

Taking $\sqrt{864}$ to be 29.39 we find x to be approximately $\frac{-41.39}{18}$ and $\frac{17.39}{18}$. The three roots of $P(x) = 0$ are 10, -2 and 1. Thus the roots of $Q_1(x)$ give approximations to the two smaller roots of $P(x) = 0$. The third root of $P(x) = 0$ can be approximated by the expression

$$20 \div \left(\frac{-41.39}{18}\right)\left(\frac{17.39}{18}\right)$$

This is true because 20 is the negative of the product of the three roots.

Example 2. If $P(x) = x^3 - 10x^2 - 6x + 200 = 0$ then $Q_1(x) = -10x^2 - 6x + 200 = 0$ and

$$x = \frac{6 \pm \sqrt{8036}}{-20} .$$

Taking $\sqrt{8036}$ to be 89.6 we find x to be -4.73 and 4.18. The roots of $P(x) = 0$ are $7 \pm i$ and -4 so one root of $Q_1(x)$ approximates the real root of $P(x) = 0$.

Example 3. If $P(x) = x^3 + 13x^2 + 32x + 20 = 0$ then $Q_1(x) = 13x^2 + 32x + 20 = 0$. But

$$x = \frac{-32 \pm \sqrt{-16}}{26}$$

and $P(x) = 0$ has only real roots of -10, -2 and -1. If we consider

$$|x| = \left|\frac{-32 \pm \sqrt{-16}}{26}\right|$$

we find

$$|x| = \frac{31.7}{26}$$

and this is numerically an approximation for the smallest root.

These examples are not intended to take the place of the theory which could be developed, but merely serve to show that, in one case, the concepts of section V on the quadratic are extendable, with modifications and additional conditions, to the cubic equation. Since the solutions of the general cubic equation are not a part of the algebraic materials at the secondary school level, there seems to be little need in developing the approximation theory for this presentation.

It should be noted that approximations to solutions of an nth degree equation can be obtained by methods such as Horner's method or others of a similar nature. These methods are adequately described in any of the books on theory of equations (for example see Dickson, Theory of Equations).

VII. Implications for Secondary School Mathematics.

In this last section we wish to relate the previous sections to the business of teaching secondary school mathematics and indicate some implications for the future. We do this in the context of answering the original basic question which motivated this entire investigation.

We wished, if possible, to determine if there was a systematic body of mathematical knowledge which covered the technique of approximating a root to a polynomial by finding roots of a derived polynomial and if it were possible to present this knowledge in some convenient form whereby a program of instruction could be established in courses in elementary algebra at the secondary school level. We believe the answer to this two part question is in the affirmative for both parts and we elaborate by giving our reasons for this belief.

In reviewing section III of this Appendix, we note the four types of approximations listed include three for which an extensive amount of structured mathematics exists. The three are

1. Approximating a number by another number
2. Approximating a root of an equation
3. Approximating a function by another function

Of these three, the first two are quite old techniques, which we hope the historical material of section II amply indicates. The techniques involved in these two range from simple arithmetic computations to such complicated computations as finding Sterling's formula or others of a similar nature from the theory of numbers area in mathematics. Approximating roots of an equation can be as simple as guessing or as complicated as using recursive formulas depending upon the derivative of the polynomial equations. In either case, these procedures are well founded and are an integral part of the mathematical structure although they may occur at various levels within the structure. In fact, the first educational implication might very well be stated at this point.

In the process of modernizing college and university level mathematics, we have, for the most part, discontinued several courses as a main portion of the mathematics required of those wishing to be mathematicians (and I include teachers in this group). Of particular importance to this study and most closely related of all such courses was the **course** offered by the title of Theory of Equations. This type of course has nearly disappeared from the college catalog. There are probably good reasons for this. Much of the content of such courses apparently has been absorbed into other courses, most notably the abstract algebra. However, if one examines these other courses one finds the so-called elementary concepts have frequently been excluded because they are too elementary. One may find some of these elementary concepts in a College Algebra course, but not all. With more and more students coming to college with College Algebra as a background in mathematics, it seems that the algebraic content of the secondary school must include, in a significant way, the basic concepts from the theory of equations. The key here is the phrase, "in a significant way". This would mean that techniques for finding approximations to roots of equations, such as Horner's method, would be an appropriate part of the content of secondary school mathematics. It would seem that this would be an essential part of any accelerated program at that level. The mathematics is not too difficult for the secondary school. We need only find the proper kind of presentation to make such a program not only feasible, but practical in implementation. In fact, the major accomplishment in mathematics education for the next generation might very well be the creating of elegant but simple ways to present the more complex mathematics at an earlier stage of instruction than at present. This would be only one of many possible areas in which this could be done.

The third type of approximation referred to in section III, approximating a function by another function, is also a part of an existing mathematical structure. In fact, it is a foundation principle for the area in analysis which has made possible much of our current mathematics, particularly the applied mathematics. Much of this mathematics depends upon the calculus and, therefore, is not adaptable to the secondary school curriculum. Moreover, there appears to be no mathematical reason or need to suggest the necessity for such an adaptation.

With respect to the fourth type of approximation given in section III, it is possible to make more positive statements of implication for secondary school mathematics. This type of approximation, the approximating of roots of one function by roots of another related function, was examined in some detail with various relations spelled out in the theorems of sections IV and V. The following observations seem to be pertinent to these sections.

1. There is a mathematical structure available for this type of approximation technique. It is not as clearly delineated in the literature as the other types and, for the purposes of this presentation, several gaps were found.

2. It is possible to establish certain theorems, based essentially upon concepts from the theory of equations and the elementary algebra which show the connections necessary to fill the gaps noted.

3. The essential features of the mathematical structure are noted in these two sections, first in a general way and then specifically for the quadratic equation. This expository presentation was not intended to be complete in every detail and perhaps there is a more elegant presentation, but the features presented show that it is possible to put some systematic order to this type of approximation.

4. Most of the concepts presented are dependent upon very elementary mathematics of the secondary school level and, hence, are adaptable to this level. By implication, this means much of the material presented herein can be made a part of mathematics instruction at the secondary level. The material presented in this Appendix is intended for the high school teacher who will have to make the necessary adaptation to the classroom situation. There does not appear to be any source available which has done this for the teacher.

5. The fact that it appears to be feasible to adapt these concepts to secondary school mathematics and the fact that there appears to be at least a suggested need for this material seems to imply that such an adaptation should be attempted. The basic suggestion here is that a portion of time and effort be directed to making the high school student's mathematics more relevant to his science instruction. (The same implication could be made relative to the social sciences, but from other facts not necessarily related to approximation theory.)

6. The presentation of this approximation theory should probably be more intuitive than rigorous. This is essentially a psychological question or observation and we have ample evidence from groups such as the School Mathematics Study Group that such an approach is feasible and that it actually works in the classroom on concepts in mathematics.

In summary, there is a significant area of mathematics in approximation theory which is elementary enough to be adaptable to secondary school mathematics. It is mathematically possible to make such an adaptation, and, if properly presented, this adaptation could make a contribution to mathematics and to science instruction at that level.

Appendix B: References

[1] Achieser, N. I. Theory of Approximations, Frederick Ungar Publishing Co., New York, 1956.

This book contains almost all of the fundamental notions of approximating one function by another function. It is well written and one of the better sources for this type of approximation theory.

[2] Allardice, R. E. "On a limit of the roots of an equation that is independent of all but two coefficients", Bulletin of the American Mathematical Society, 13(1906-07), 443-447.

[3] Bernstein, S. N., "Sur l'approximation des fonctions continues par des polynomes", Comptes Rendus de l'Academie des Sciences, (Paris), 152(1911)

[4] Bowen, John J., "Mathematics and the teaching of science", The Mathematics Teacher, LIX(1966), 536-42.

[5] Carmichael, R. D. and Mason, T. E., "Note on the roots of algebraic equations", Bulletin of the American Mathematical Society, 21(1914), 14-22.

[6] Fejér, L., "Ueber die Wurzel vom kleinsten absoluten Betrage einer algebraischen Gleichung", Mathematische Annalen, 65(1908), 413-23.

[7] Ford, L. R., "One solution of equations by the method of successive approximations", American Mathematical Monthly, 32(1925), 272-87.

[8] Grant, J. D., "A graphical solution of the quartic", American Mathematical Monthly, 40(1933), 31-32.

[9] Kennedy, E. C., "Concerning the solution of certain types of equations", American Mathematical Monthly, 39(1932), 535-37.

[10] Landau, E., "Sur quelques generalisations du théorème de M. Picard", Annales de L'Ecole Normale Supérieure, (3), 24(1907), 179-201.

[11] Montel, Paul, "Sur un théorème d'algebre", Comptes Rendus, 174(1922), 850-52. (See also pp. 1220-1222 in same volume.)

[12] Oldenburger, R., "Practical computational methods in the solution of equations", American Mathematical Monthly, 55(1948), 335-42.

[13] Pierce, T. A., "An approximation to the least root of a cubic equation with application to the determination of units in pure cubic fields", Bulletin of the American Mathematical Society, 32(1926), 263-68.

[14] Running, T. R., "Graphical solutions of the quadratic, cubic and biquadratic equations", American Mathematical Monthly, 28(1921) 415-23.

[15] Van Vleck, E. B., "A sufficient condition for the maximum number of imaginary roots of an equation of nth degree", Annals of Mathematics, (2) 4(1903), 191-92.

[16] Walsh, J. W., "An inequality for the roots of an algebraic equation", Annals of Mathematics, (2) 25(1924), 285-86.

[17] Weierstrass, K., "Uber die analytische Dars ellbarheit sogenaunter willkurlicher Funktionen einer reellen Veranderlichen", Berliner Berichte, 1885.

[18] Williams, K. P., "Note concerning the roots of an equation", Bulletin, American Mathematical Society, 28(1922), 394-96.

The best single source for a history of equations and their solutions is

[19] Smith, D. E., History of Mathematics, Vol. II, Ginn and Company, New York, 1925.

2

Appendix C: Bibliography

Achieser, N. I., "On the asymptotic expressions of the best approximation of certain rational functions through polynomials", Proceedings of the Kharkov Mathematical Society, 1932. (In Russian)

Bernstein, S. N., "Sur une propriété des fonctions entiéres", Comptes Rendus, 176(1923),

Conkwright, N. B., Introduction to Theory of Equations, Ginn and Company, Boston, 1941.

Dickson, L. E., New First Course in the Theory of Equations, John Wiley & Sons, Inc., New York, 1939.

Gummer, C. F., "The relative distribution of the real roots of a system of polynomials", Transactions of the American Mathematical Society, 23(1922), 265-82

Kazarinoff, N. D., Analytic Inequalities, Holt, Rinehart and Winston, Inc., New York, 1961.

Kellogg, O. D., "A necessary condition that all the roots of an algebraic equation be real", Annals of Math., (2), 9(1907-8), 97-98.

Scarborough, J. B., Numerical Mathematical Analysis, 6th edition, The John Hopkins Press, Baltimore, 1966.

This book contains a good presentation of various numerical techniques associated with approximation theory.

Van Vleck, E. B., "On the location of roots of polynomials and entire functions", Bulletin of the American Mathematical Society, 35(1929), 648-83.

This article is an expository presentation of material related to approximations and is an excellent summary of previous research. It contains a very lengthy bibliography (144 entries) on this related material, the location of roots, inequalities involving roots, upper and lower bounds on roots, etc. Van Vleck did not discuss the finding of roots of a polynomial by approximation from roots of a derived polynomial.

Van Vleck, E. B., "On limits to the absolute value of the roots of a polynomial", Bulletin de la Societe Mathematique, 53(1925), 105-25.