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ABSTRACT

This booklet is a collection of the papers presented at a joint session of the Mathematical Association of America and the National Council of Teachers of Mathematics during the fiftieth annual meeting of the MAA. These papers were presented under the headings of "Geometry and School Mathematics," "High School Geometry," and "School Geometry and the Future." Eight papers are presented in their entirety. An abstract of a ninth paper, "Geometry: The Cambridge Conference View" by Edwin E. Moise is also included. Authors of the individual papers do not act as spokesmen for the MAA or NCTM but express personal viewpoints. (FL)

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GEOMETRY

IN THE SECONDARY SCHOOL

A compendium of papers presented in Houston, Texas

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FOREWORD

The National Council of Teachers of Mathematics is pleased to make the papers in this booklet available to those who were unable to attend the sessions at which they were originally presented: a program presented by the Mathematical Association of America, at its fiftieth annual meeting, with the cosponsorship of the NCTM.

The papers were presented at three sessions, under the headings of "Geometry and School Mathematics," "High School Geometry," and "School Geometry of the Future"; but they are, of course, interrelated. The authors speak as individuals, not as official spokesmen for either sponsoring organization, and the viewpoints expressed are even more stimulating and thought-provoking in that they are not always in agreement.

The NCTM is grateful to the contributors and to the MAA for permission to publish the material presented here.

W. K. McNabb
Program Coordinator and Editor

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GEOMETRY IN THE UNITED STATES

By *Bruce E. Meserve*
University of Vermont
Burlington, Vermont

The changing role of geometry is a source of confusion to many teachers and administrators, a challenge to all who are alert to the needs of their students. In the few minutes available I shall try (a) to identify some of the changes that have taken, or are taking, place, and (b) to suggest some of the implications of these changes for the future of geometry in mathematics education in the United States. Since several of you could probably undertake these tasks more effectively than I can, it is with considerable trepidation and humility that I make this initial presentation at our joint MAA-NCTM meeting concerned with geometry. As a basis for discussion, I present my concept of the problem. Other speakers will speak to various aspects of the problem. You will have opportunities to question us individually here and at future meetings.

What Is Going On?

Solid geometry as a separate course has almost disappeared. However, paradoxically, we seem to feel that the geometry of space deserves greater emphasis than it has received in the past.

The traditional plane geometry course has lost its "sacred cow" status and, from the point of view of some mathematicians, has been defiled by the injection of the use of algebra, as in

Coordinate systems
Distance and midpoint formulas
Slope
Equations of common curves and surfaces
Use of inequalities to represent regions on a plane and in space

The geometry course that is evolving from the traditional plane geometry course has been overwhelmed with topics proposed for inclusion:

Coordinate approaches to geometry
Transformation groups
Vector approaches to geometry
Solid geometry
Emphasis upon postulational systems
Emphasis upon logical structure
Study of other geometries such as spherical geometry, finite projective geometries, topology, and the non-Euclidean geometries

Sometimes it seems that for each course in geometry at a university there is a corresponding suggestion for including related concepts in school geometry.

It seems to me that the formal course in secondary school geometry is currently floundering as an over-encumbered giant circumscribed by drastically sharpened standards for definitions, postulational systems, uses of logical concepts, and the use of mathematical terminology.

Consider the postulates for secondary school geometry. Except for the works of a very few individualists such as Swenson, Veblen, Birkhoff, and Beatley, teachers formerly had some security in a very stable (even though imperfect) system of postulates. Today the exceptions of the past provide guidelines for a wide variety of postulational systems. Mathematicians, almost universally, take an independent "I'll define it as I please and be consistent with myself" attitude. I appreciate and, to a considerable extent, share this attitude. However, we need to let others understand the reasons behind some of our proposed changes. We especially need to allow and encourage secondary school teachers to comprehend what we are trying to accomplish by our independent approaches to geometry. Only with this background can teachers enter into the true spirit of the teaching of geometry.

Let us look briefly into the background for some of our "idealistic whims" and see why we are changing the ground rules for secondary school geometry. Each of our changes is intended to enhance the student's understanding and appreciation of geometry as a mathematical system and, in particular, as a system concerned with points, lines, planes, relations among figures, and the representations of figures in the world around us.

1. *Existence.*—We used to assume the existence of any geometric figure that could be constructed. Even though Euclid's postulates were probably based upon Plato's postulates for constructions, we were very casual about the possibility of making constructions. In our present emphasis upon geometry as a postulational system independent of physical representations (models), we now include postulates of existence in our mathematical system. For example:

There exists at least one line.

If $\overleftrightarrow{AB}$ is a line, then there exists at least one point C that is not a point of $\overleftrightarrow{AB}$.

If ABC is a plane, then there exists at least one point D that is not a point of ABC.

2. *Equality versus congruence.*—We used to say,

"Base angles of an isosceles triangle are equal."

Now we reserve the equality relation for different names given to the same number, figure, or other element. We use the congruence relation to indicate that two figures have the same measure. Thus we now say,

"Base angles of an isosceles triangle are congruent."

Secondary school teachers rapidly grasp the distinction that we are trying to make, when we take the trouble to tell them what we are trying to do.

3. *Figure versus picture.*—Many secondary school textbooks used to have a postulate such as this:

"A line may be extended to any required length in either direction or in both directions."

We now consider a line to be an undefined element. We do not extend lines; rather, we extend pictures (models, representations) of lines. We need to help teachers understand that we have a situation in geometry that is analogous to the number-numeral problem in algebra and arithmetic.

4. *Generality.*—We used to select definitions so that there was as little overlapping as possible. For example, an isosceles triangle had exactly two congruent sides; a trapezoid had exactly two parallel sides. Now, with our emphasis upon generality and interrelations among figures (rather than just in naming them), we include an equilateral triangle as a special case of an isosceles triangle, a parallelogram as a special case of a trapezoid, and a cube as a special case of a rectangular parallelepiped. Other such examples could be cited.

5. *Varied uses of terms.*—Differences in the usage of terms in different textbooks provide awkward and embarrassing pitfalls for many teachers. For example, are all polygons simple curves? All polygons convex? All quadrilaterals convex? All measures numbers? Is a line parallel to itself? I am strongly in favor of the independence of authors in such situations; but I think we need to be very explicit about our assumptions, and we should not hesitate to remind our audience periodically of our assumptions.

Other changes and the reasons for them could be cited, but let us return to our original charge. Each of the items that I have mentioned seems to me to be adding to the confusion and congestion in secondary school geometry courses. What are we doing to reduce that confusion?

We might, as some schools do for their slower students, simply reduce the number of proofs and present a course based upon applications of rules without concern for "discovery" techniques, developmental-type exercises, or proofs based upon extensive analyses of problems. However, such an approach is definitely contrary to the pedagogically sound trend toward student discoveries and the development of skills in analyzing problems.

Most geometry curricula appear to be eliminating constructions with straightedge and compasses whenever an analysis of the problem is needed. While sympathetic to the need for reducing the bulk of the geometry course, I personally consider the elimination of advanced constructions as a misguided effort that is out of keeping with our desire to develop the reasoning abilities of our students.

Where Are We Headed?

The changes in geometry are part of the basic changes that are taking place in mathematics curricula from kindergarten through college. Geometry in the United States is in the process of assuming a basic role throughout a comprehensive mathematics curriculum, in place of maintaining its previous isolated position.

One of the most encouraging trends of our time is the rapidly expanding informal treatment of geometry in elementary and junior high schools. Common geometric figures are identified, defined, and used in the development of the student's concept of mathematics. Constructions with straightedge and compasses are considered while most students are still anxious to work with their hands, are interested in developing geometric patterns, and are curious about a wide variety of things.

Another encouraging trend is the recognition that reasoning should be a part of all branches of mathematics. Thus, simple implications are considered in elementary school, and the logical steps that are used in algebra are specifically recognized.

Each trend that I have mentioned, along with many other aspects of the teaching of geometry, is being extended and explored in a wide variety of experimental programs. Such explorations seem to me to indicate a very healthy situation. We need divergent views. We need to explore drastically different approaches and methods. Above all, we need to maintain a flexibility in our curriculum which allows us to modify our courses to take advantage of changes for students who can profit from them, without placing all students in a straightjacket of either new or old curricular material.

Geometry and algebra should be expected to continue their evolution as two interrelated approaches to the study of mathematics. Each approach reinforces the other. Geometric concepts help students understand algebraic concepts, and vice versa. We should encourage students to use either an algebraic or a geometric approach (and sometimes both) in solving problems. Neither approach should dominate or exclude the other.

In conclusion, let us look at the formal course in geometry that has traditionally been a course in synthetic plane geometry. The informal treatment of geometric figures and elementary constructions in earlier grades and the recognition of patterns of logical reasoning in algebra reduce some of the pressures on this course. However, it is still not realistic for most students to attempt thorough treatments of plane and solid geometry from both synthetic and coordinate points of view in a single course that also includes the recognition of other geometries. There are many pedagogical advantages to considering selected properties of three-dimensional figures at the time that corresponding properties of plane figures are studied. These advantages provide the basis for the inclusion of both plane and solid geometry in a single course. However, the treatment cannot be as thorough as many of us would like it to be, and the time allotment of a full year for the geometry course cannot be cut unless geometric concepts are considered extensively in several other courses, possibly from other points of view. Experimental programs are beginning to consider drastic revisions of the structure of secondary school mathematics so that both geometric and algebraic concepts can be emphasized in the study of mathematics at each grade level.

I believe that the paths along which we are headed and which are the most promising as we search for solutions to the many problems before us are these:

Informal development of geometric concepts throughout the elementary and junior high school mathematics curriculum

Increased use of geometric concepts in the development of arithmetic and algebraic concepts

Introduction of, and explorations with, constructions with straightedge and compasses before the formal study of geometry

Emphasis upon visualization of plane and space figures at all grade levels

Inclusion of mensuration of geometric figures at several grade levels

Increased emphasis upon coordinate geometry of two dimensions and of three dimensions in the last two years of high school

A full-year course with a primary emphasis upon geometry based upon a substantial introduction of synthetic postulational geometry and including a coordinate approach to many of the problems considered for both two- and three-dimensional figures.

THE ONTARIO K-13 GEOMETRY REPORT

By *H. S. M. Coxeter*
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In the prevalent desire for single courses in mathematics rather than separate courses in its various branches, there has been a tendency for geometry to be squeezed out. This tendency is not only regrettable but unreasonable. Geometry has interactions with other branches of mathematics and should be taught alongside them, not before or after; but it should by no means be neglected. The purpose of the Ontario K-13 Geometry Report is to outline a program whereby this downgrading of geometry can be avoided.

Synthetic geometry develops and refines spatial intuition. In the physics of crystals or the chemistry of complicated organic compounds, many significant geometric relationships are revealed. There are also increasingly complicated problems of architecture and the subtle intricacies of the space-time continuum.

We believe that visual and intuitive work is indispensable at every level of mathematics and science, both as an aid to clarification of particular problems and as a source of inspiration. Great care should be taken to encourage the mental constructions of the student, whether rudimentary or advanced. Self-reliance goes hand in hand with the cultivation of intuitive judgment and artistic taste.

By dealing with geometry informally, by plausible reasoning rather than by strict proof, it is possible to reach interesting and surprising results much more quickly: the student does not spend a whole hour on a proof of something he regards as obvious. Thus several theorems can be covered in one lesson, and students can be given a bird's-eye view of the subject as a whole.

The good and bad features of our geometrical tradition must be carefully disentangled. In the hands of a good teacher, who does not take the textbook too seriously, a geometry lesson can be a stimulating experience. One Ontario student said, in an essay, that geometry lessons were a revelation to him, because in arithmetic and algebra he was told what to do, but in geometry there was discussion and reasons were given.

In geometry, perhaps more than in the other subjects, a student can exercise originality and ingenuity in devising a construction or seeking a proof.

In former times, the wholesale adoption of Euclid's axiomatic method as an authority, and as a model to be emulated, presented an enormously difficult program for most students. Instead of the axiomatic approach, with rules and definitions, we recommend the intuitive "interest" approach through problems significant to the student. Certain properties of simple figures are assumed. These lead to short chains of easy deductions. Later a more ambitious use of assumptions can be made, so that a wider range of problems is accessible, and some of the old tentative assumptions become theorems. This method minimizes the laying down of authority and the making of apparently arbitrary rules at the outset.

It has been well remarked that, in general, a *definition* sums up an experience and should not precede it.

The systematic use of axioms in geometry is admissible only after the students have already had several years of experience with simple deductions. Actually, for exercises in deductive reasoning, algebra is probably more suitable than geometry. Geometry should be taught rather for its interesting results and as an exercise in *informal* reasoning. After all, the work that culminated in the discovery of non-Euclidean geometry occurred *before* the logical gaps in Euclid had been noticed. Neither Bolyai nor Lobachevsky lived to see a proof of the relative consistency of hyperbolic geometry. Incidentally, as the discovery of the non-Euclidean geometries is the most significant development in the whole field of geometry since Euclid, neglect of this development would hardly be compatible with the position of geometry in contemporary liberal education. As Felix Klein once remarked, non-Euclidean geometry "forms one of the few parts of mathematics which is talked about in wide circles, so that any teacher may be asked about it at any moment."

In the primary school, children should become familiar with simple objects that illustrate the ideas of shape, size, and measurement. Solids such as spheres, cylinders, cones, pyramids, prisms, antiprisms, and other polyhedrons can be appreciated at an earlier age than their two-dimensional counterparts: circles, triangles, squares, rectangles, parallelograms, pentagons, hexagons, and other polygons. The square first arises as a face of a cube!

As soon as a child has become familiar with two- and three-dimensional figures, he should begin to make *patterns*. He will soon see that some shapes, such as triangles, regular hexagons, and cubes, can be repeated to fill and cover the plane or space, whereas other shapes, such as regular pentagons, circles, and spheres, cannot. When a child is ready to handle a straightedge and a pair of compasses, he should be encouraged to invent and color patterns of his own liking, and to construct models by such means as straws and pipe cleaners, soft wire, sticks with glue, plasticene, cardboard, and ready-made polygons. Pairs of plane mirrors can be used to study reflection, rotation, translation, and the simplest notions of symmetry (as in the Minnemath film *Dihedral Kaleidoscopes*). The teacher should draw attention to the fact that the rim of a lampshade may cast shadows that are circles, ellipses, parabolas, or hyperbolas.

The comparison of size of similar figures and of angles can be considered at an early age, in preparation for the idea of measuring volume, area, length, and angle, and for the use of instruments such as set square, parallel ruler, compasses, and protractor for making scale models and maps.

A good informal treatment of mensuration is illustrated by the problem of finding the volume of a *pyramid* of height z based on a rectangle $2x \times 2y$ where x, y, z can have any convenient values, such as 3, 4, 5 (inches or centimeters). A cuboid $x \times y \times z$ is dissected into six pieces ("orthoschemes") by planes joining one pair of opposite vertices to each of the other three pairs in turn. It is very plausible that these six pieces all have the same volume $\frac{xyz}{6}$. (Those pairs of pieces which are congruent are not *directly* congruent but *oppositely* congruent, like a pair of shoes or an arbitrary solid and its mirror image.) The whole construction is then repeated so as to produce twelve pieces. (These can be mixed up. One child is asked to choose a piece, and another to find a mate for it, either directly congruent or oppositely congruent.) Finally, eight of the twelve pieces are reassembled to form the desired pyramid, whose volume is thus seen to be $\frac{4xyz}{3}$, that is, one third the base times the height.

The remaining four pieces, with four of the eight, make amother pyramid (with base $2y \times 2z$ and height x, or base $2z \times 2x$ and height y).

We follow the British and Russians in recommending the introduction of geometric transformations (or "motion geometry") as early as possible, not only as a tricky way to prove theorems but as a means of inculcating a feeling for space. This idea is closely related to symmetry and thus appeals to the artistic side of children. A child is aware of the symmetry of a butterfly before the concept of distance has become fully clarified. He will enjoy making his own "butterfly" by folding a sheet of paper with a wet spot of ink near the crease. The classification of frieze patterns according to their seven symmetry groups can be appreciated by children in Grades 4 through 6, and still holds interest for much older students.

Motion geometry includes the concept of *translation*, another name for which is *vector*, and for the first time children meet the plus sign in a nonarithmetical context. In expressing a translation as the "sum" of two half-turns, they obtain a first taste of a noncommutative algebra.

Cartesian coordinates can easily be introduced in Grade 6 or Grade 7, so as to provide a stimulating synthesis of geometry, arithmetic, algebra, and trigonometry. Vectors, having previously appeared as translations, can be represented by pairs or triples of numbers, and then by forces or velocities, thus linking pure and applied mathematics.

The algebraic aspect of motion geometry can be developed by using *matrices* (of two rows and two columns) to represent translations, half-turns, quarter-turns, reflections (in the coordinate axes or their angle bisectors), and dilatations (from the origin). Trigonometry, which many children used to dislike, can be enlivened by judicious use of vectors and of polar coordinates. The dull routine of numerical solution of triangles should be replaced by a taste of the most elegant trigonometric identities.

Any child old enough to handle a pair of compasses can appreciate the idea of *coaxal circles*, leading naturally to *inversion*: the simplest example of a transformation that changes shape (more precisely, a conformal transformation that is not merely a similarity). From this it is an easy step to continuous transformations and an informal introduction to *topology*: the sphere, torus, and Möbius strip; maps and coloring problems; topological networks or "graphs"; the unsolved problem of classifying knots; and Euler's formula connecting the numbers of vertices, edges, and faces of a polyhedron.

Exceptionally able students should be encouraged to study quadric surfaces, the geometry of complex numbers, and the possibility of using various sets of axioms so as to replace Euclidean geometry by other geometries: affine, inversive, projective, absolute, spherical, hyperbolic. Such excursions will give the correct impression of geometry as a subject that is still developing in a lively manner.

GEOMETRY IN GREAT BRITAIN

By *Andrew Elliott*
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When any program of school mathematics in Great Britain is discussed, one must always remember that great variety is possible. In theory, at least, and to a large extent in practice also, each elementary school principal and each high school head of department has complete authority to design and implement his own curriculum. However, once we bear this in mind, we can discuss general trends and programs in the country.

At present, apart from the traditional arithmetic in the elementary schools (five- to eleven-year age group, approximately Grades 1 to 6), there are two main new programs. The first is the Nuffield Mathematics Teaching Project, under Dr. Geoffrey Mathews, which undoubtedly owes something of its inspiration in methodology to Miss Edith Biggs of the Ministry of Education. The second is the Leicestershire Experiment, which is based largely on the proposals of Dienes.

At the high school level (ages eleven to eighteen, approximately Grades 6 to 12 or 13) there are several programs. In Scotland, the Scottish Mathematics Group is essentially alone. In England the School Mathematics Project, directed by Professor Bryan Thwaites of Southampton University, is perhaps dominant in the public (private) schools and academic grammer schools. The Midlands Mathematics Experiment, directed by Cyril Hope of Worcester, is active in the modern secondary, technical-commercial, and academic high schools of the national (public) school system. A third group is the Mathematics and Industry Conference, which includes some public and national high schools. In all these programs it is safe to say that the traditional Euclidean geometry course is nonexistent and that relatively little emphasis is placed on analytic geometry either.

But let us look first at geometry at the elementary school levels. Here the Nuffield project has even gone so far as to avoid almost deliberately the word "geometry." The leaders claim that so many teachers in elementary schools have a profound fear of the Euclidean deductive high school geometry that they are repelled by the word "geometry." In its whole program the Nuffield project makes a special plea that teachers must teach the *enjoyment* of learning mathematics rather than the fear of the subject. For this reason, also, their methodology tries to emphasize intrinsic motivation rather than external motivation by either reward or punishment through prizes, marks, or extra drill. In common with proposals developed in the last three years in Ontario, they insist that geometry must be a continuous experience from kindergarten onwards. They also insist that the geometry of the real world is three-dimensional, so that it should receive almost equal emphasis with two-dimensional work throughout the program. Their present publications *Shape and Size*, 1 and 2, carry the program only to the Grade 3 or Grade 4 level, but they indicate very clearly that symmetry, tessellations (tiling patterns), and classification of two- and three-dimensional shapes play a major role. Formal drawing and construction is minimized at this stage; for example, right angles are formed by paper folding, and string and thumbtacks (or ropes and pegs, out-of-doors) are used to show that the shortest distance from a point to a line is along the perpendicular. Line symmetries and reflections are studied with mirrors, blot patterns, and folding; congruencies by direct superposition of cutout models and later by the transfer of tracings; and similarity by matching of corners and by visual projections. Tessellations are formed at first from gummed paper cutouts and later by tracing round card models. The problem of covering the plane by geometric figures is introduced in this way, as is the study of translations, rotations, and reflections in interrelation. In three dimensions as well as two, model building is stressed. The triangle is realized as a *strong* figure that is undeformable. The tetrahedron is discovered as a space quadrilateral with its stiffening diagonals. The decomposition of shapes, skeletons as well as polygons, into triangles and tetrahedra arises informally in such experimental studies. The covering of irregular areas by unit squares and triangles leads to area measurement; the computation of area by multiplication follows later. Volume measure is studied in a similar way.

An outline written for internal circulation in the project makes it obvious, even if these initial publications did not, that the final aim of the program (up to the age of thirteen) is to form a firm basis for a high school geometry of vectors, translations, rotations, and reflections, with coordinate methods as one way of writing these topics. Intimately bound in with this approach is the concept of the invariance of geometrical properties under the transformations, and the group and other properties of the transformations themselves. This is indicated clearly in the "Conspectus of Ideas" section of *Shape and Size, 2*.

All the high school projects mentioned treat geometry mainly as a study in transformations, and they use vectors, group properties, coordinate systems, and matrices as required. The differences arise mainly in emphasis and in the order of topics.

The Midland Mathematics Experiment, for example, begins the first book (for Grades 6-7) with a treatment of navigation problems by drawing on a coordinate background. Then parallels are discussed informally in association with parallelogram tessellations. Modular arithmetic patterns on circles lead to paper-folding studies of polygons (including the regular pentagon); towards the end of the book vectors are introduced on a coordinate system, using eastings and northings from navigation. In the following book (for Grades 7-8) vectors are restudied as line segments and on a Cartesian coordinate background in two dimensions, including the scalar product. The study includes proofs of some of the usual geometrical properties of triangles and parallelograms, but with no formal theorems. Transformations are introduced with matrices and include rotations and various deformations. They are studied first by the detailed direct plotting of image points on a coordinate lattice. Later the rotations are studied in more detail, and the symmetry rotation groups for the equilateral triangle, rectangle, and square are examined in some detail.

Incidentally, sine and cosine are introduced at this level. In the final book before O-level (Grades 9-10) the study of transformations continues, but vectors and matrices are carried to n-vectors and the algebra of matrices rather than being used in geometry. The weaknesses of this program are the paucity of three-dimensional work and an overemphasis on the coordinate background.

On the whole, the School Mathematics Project pursues a coordinate independent approach, beginning with line and rotation symmetries and translations in Grades 6-8. Again tessellations are used to study parallels and translations. Some elementary topological ideas of sidedness, convexity, connectibility, and networks are studied from the experimental viewpoint. The traditional defining properties of the various special triangles and parallelograms are developed through, and associated with, their symmetry properties, which are regarded as more fundamental. Similar triangles are introduced, and the proportionality of the sides is discovered by means of a tessellation with triangles. In the later books, Grades 9-10, great emphasis is placed on the reflection transformation; and, in fact, a rotation is treated as a two-reflection process in nonparallel mirrors, while a translation is treated as the result of two reflections in parallel mirrors. This gives a very interesting slant to geometry, but, unfortunately, it is confined to two dimensions. Vectors are introduced at this level as directed line segments indicating translations and, together with matrices, are used in a more general study of transformations in a coordinate system. However, vector methods are also used to study the appropriate properties of triangles. In this program, again, three-dimensional studies are rather slight except for an early chapter on the construction of regular polyhedra and later a good chapter on plans and elevations.

The Scottish program strongly emphasizes the experimental approach to the group-symmetry properties of geometrical figures, in both two and three dimensions. In the first book cubes and various rectangular prisms are fitted into corresponding boxes, and the different ways in which this can be done are noted and used to classify the properties of the solids. This type of work is repeated with models of squares, rectangles, and right-angled isosceles and equilateral triangles. Coordinates are used sparingly to assist in the study of tessellations and isosceles and equilateral triangles. In the second book the work is extended to more complex figures, such as rhombuses, kites, and parallelograms, and to their association with the standard constructions in geometry. It is uncertain from these publications how far this group will go in the use of vectors and matrices in later years, and the work in three dimensions is again rather slight.

It should be remenbered that all these high school programs are based on a traditional, almost nonexistent, study of geometry in the elementary school (Grades 1-5).

The other vital point in all these programs, in contrast to approaches in the United States, is that no attempt is made in Grades 1-10 to introduce any axiomatic deductive system, nor is much work done along the traditional lines of ruler-and-compass construction. Indeed, one might almost say that the topics are deliberately avoided. The first topic is avoided because British opinion is, and has been for many years, that axiomatic deductive geometry is suitable only for about the top 5 percent or less of the student body. The second topic is reduced because it is relatively useless and irrelevant in realistic geometrical, architectural, engineering, or artistic drawing. Geometry is presented to the mass of pupils as a useful tool and a thing of artistic beauty, but it is presented in such a way that it forms a good base which the mathematics specialist can use as a foundation for modern higher mathematics in vector algebras, transformation theories, and topology, as well as in projective, affine, and inversive geometries, and in Euclidean deductive geometry also.

AFFINE GEOMETRY

By *G. P. Johnson*
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I assume we are here today out of at least two convictions: that the Great Society needs a better geometry course and that the war on Euclid needs to be escalated. In order that I might try to advance both of these causes I would like you to allow me a couple of other assumptions, which are not quite as easy. The first is that there is merit in basing introductory geometry on algebra, or at least in putting a lot of algebra into it. The second is that, other things being more or less equal, the fewer axioms one has to inflict on a mathematical system, the better—and an introduction to axiomatics is no exception. Like students, axioms are essential in our business, but they can become burdensome in large numbers.

There is not the time now—nor, I am sure, a need—for me to argue at great length the case for a fusion of algebra and geometry at the secondary school level. Klein was doing so at the turn of the century, and he has had a good deal of very respectable company since then, including, in recent years, the authors of the SMSG texts. The main point is that the concept of number is now the predominant notion in mathematics education and, for that matter, in much of the rest of mathematics. Number comes first in a student's education and gets the bulk of his attention in the courses he takes before tenth grade. It is therefore unnatural and undesirable to isolate the first formal geometry from numerical considerations to the extent that traditional courses have. Students entering tenth grade are well prepared to accept, for example, the number line as a prototype for the lines of formal geometry; and it seems plausible that they should find a geometry so based —that is, a coordinate geometry—to be at least as natural as the customary development and to be no more formidable. What is even more important and, I think, no less plausible is that they should also find such a course more relevant to the mathematics they subsequently study. Is it not reasonable to believe that the notion of proof should carry over more readily to quite different kinds of mathematics? If geometry were really doing its job, it ought to be not quite as painful as it is to teach undergraduates what a proof is in, say, analysis.

The question of axiom counting, which is a dubious game on several counts, is of less importance. I am aware that my concern about it comes in part from the fact that in the early stages of the writing project I shall describe shortly, at a time when things were not going very well, we frequently consoled ourselves that if our system had no other virtues, at least it had fewer axioms than anyone else's system we could think of! It is, of course, pleasant to a mathematician not to have a cumbersome logical structure. But elegance is not of much point in a case such as this, where the main concerns are pedagogical. Nevertheless, I believe there is a point. In his first encounter with an axiomatic system a student is likely to have trouble keeping straight all the rules of the game—that is, the axioms—if there are a great many of them. Just what has been assumed and what has been proved gets more than a little hard to remember. Furthermore, especially if he is perceptive, he may feel that too much is being assumed, and much of the axiomatic method's power fails to be revealed to him. It certainly is not a very good game if it appears that many of the important results are postulated, particularly if one has to admit that some redundant axioms have been introduced for the purpose of avoiding hard proofs.

Now there is conceptually an easy way around these problems, and that is to define the Euclidean plane to be the Cartesian product of the reals with themselves. Lines are the loci of linear equations, and distance is defined in the obvious way. The geometric structure of the plane can then be developed, first strictly by analytic methods and soon thereafter by synthetic methods as well. There is no lack of analysis in the process and, if the reals are assumed, no need whatsoever for axioms. This last is rather too much; for we are, after all, trying to inculcate an appreciation for axiomatics. Also, a good deal of the spirit of geometry, in either an historical or a modern sense, seems to get lost, at least at the outset. That spirit is not easy to define; but, whatever it is, most of us feel it is of considerable value to the student of mathematics.

The development of geometry I shall describe today bears resemblance to the Cartesian product development, as does any coordinate geometry, and shares some of its drawbacks. Yet it retains a pronouncedly geometric flavor. In general terms, the axioms that are used provide for a metric development of plane Euclidean geometry. However, these axioms are such that Euclidean structure is by no means immediately apparent in them. It takes, in fact, quite a bit of analysis—not very difficult but lengthy—to show that it is possible to define distance and perpendicularity appropriately. Once this has been done, one has Euclidean structure displayed explicitly and Euclidean geometry can proceed normally. But in the meantime an impressive amount of geometry can be done, and it makes sense to do it. Thus the axioms are first taken at face value to develop the incidence properties and parallelism—that is, the affine structure—that exists in the plane. Along with preliminaries and much discussion to convince the student that in terms of his background in informal geometry the axioms are at least reasonable requirements—but not to convince him as yet of their efficacy—this affine geometry makes up not quite half the course. The rest is Euclidean geometry.

The work is self-contained, except for its assumption of the real numbers and their immediate properties. Even here there is an admission that some properties are more immediate, at least more familiar, than some others; and material appears that deals with inequalities and pairs of linear equations, those parts of the algebraic machinery used which are most likely to be unfamiliar. A fair indication of the level of sophistication required is that which goes with an understanding of the function concept. Functions not only appear; they pervade. A coordinate system, for example, is a mapping from a line to the reals; a congruence is a restriction of an isometry. Again, all the necessary background appears in the text, but undoubtedly a student will do better in the course if he has become facile with very simple functions beforehand.

Before going into further detail, I should like to discuss the project's history. The basic ideas for the development are contained in the book *Foundations of Geometry and Trigonometry* (Prentice-Hall, 1956 and 1960) by Howard Levi. So far as I know, the first high school use of material in this form was made during 1958/59 by S. S. Willoughby, who has reported his observations in a mathematics education note appearing in the June-July 1966 issue of the *American Mathematical Monthly*. Another such experiment was conducted in 1961/62 by a group of six teachers headed by Harry Sitomer, and it is because of favorable reaction to this experiment and, in no small measure, Sitomer's untiring efforts over several years, that the writing project came into being. In the summer of 1964 a group of high school and university people, directed by R. A. Rosenbaum and sponsored by Wesleyan University and the National Science Foundation, wrote a text and commentaries for teachers. These were used during the following year in some thirty-five classes in various parts of the country and were revised by the writing group the next summer. Additional testing has gone on since then, and after further but minor revision the text, *Modern Coordinate Geometry*, now published by Wesleyan, will be published commercially.

The content, in general terms, is as follows. Immediately following introductory material, the axioms for a line appear. A line is a set with coordinate systems—that is, one-to-one correspondences from the set to the reals—such that two points determine exactly one coordinate sustem and any two coordinate systems are related affinely. There then follows a short development of the line and of isomorphisms between lines, after which the axioms are completed by introducing those for the plane. A plane is a collection of at least two lines, the points of these lines being the points of the plane such that any pair of points determines exactly one line, Euclid's parallel postulate holds, and parallel projection from one line to another is an isomorphism of the lines. Planar coordinate systems now fall out directly, and we are in business to do simple analytic geometry and the affine geometry I mentioned. This includes treatment of triangles, quadrilaterals, and polygons generally; Desargues's theorem; plane separation; convex sets; and linear programming in the plane. The main purpose of the last, admittedly, is to glamorize the content in the eyes of the students, and they do seem to like it. The remainder of the affine development has likewise proved very satisfying; for a great deal gets accomplished, and most of it comes out rather easily once the necessary machinery has been set up.

The move to Euclidean structure is not as nice; indeed, in my opinion, it is the least attractive point in the course. The problem, in one form, is to show that one can introduce planar distance and perpendicularity so that they make sense and are compatible with the affine structure that has gone before. There are two difficulties: first, it is hard to explain to the student what the problem is, and, second, each of the two methods we have used in the two versions of the text is long. Either one might well be considered cruel and unusual punishment.

Once this hurdle is past—and a number of teachers have found a fairly satisfactory way to get by it, namely, skipping parts of it—things go comfortably again through such standard topics as perpendicular projection and bisector, similarity and congruence, and the circle. Then another problem arises, although a lesser one. Up to this point the course has been logically complete, at no unreasonable expense in time or in effort. But we want now to treat arc length, angle measure, and area; to do so calls for explicit use of the completeness of the reals; and this in its full-blown glory is not reasonable at the tenth-grade level. A compromise is called for, and we have chosen to discuss the role of completeness at some length but to sketch proofs in most cases where the analysis required is of calculus nature. Actually, students and teachers are especially fond of these chapters, the final ones of the text. I'm not sure whether this means that the topics are especially appealing, or that the students have had enough of logical completeness, or that they are just glad to be done.

In the course as a whole, virtually all the classical results of Euclidean geometry are treated, with one notable exception. There is no discussion of solid geometry. This was omitted for reasons of time and also because the algebra required is enough more complicated to restrain us from asking it of tenth graders. To me it does not seem a serious loss, but not everyone agrees with me.

In the note I cited, Willoughby asks the two essential questions on evaluation of new materials: Can they be taught, and should they be taught? The evidence that has been gathered on this project provides a substantially convincing answer to the first. There is no real doubt on the part of teachers or those from universities who assisted them that an acceptable proportion of the students involved, including those of average ability, learned the content satisfactorily. That same evidence, however, does not provide much for the second question. To be sure, the reaction of the users has been enthusiastic, and the view is widely held among them that the course does impart important and relevant understanding. But the warm feelings for the undertaking in any of us involved with it, including the students, is likely to have come in some degree from the novelty of the approach, its elegance, and some, perhaps subconscious, allegiance to the project. It is hard to say to what extent this is so. Also, some disaffection has been voiced, mostly in terms of the demands put upon the students' preparation in algebra and in terms of text's sophistication. Thus, although it would be ungracious and illogical to see other than encouragement in the testing, I feel compelled at this point to rely mainly on my own intuition and prejudices. To me it is doubtful the course will work well with fair-to-middling college-bound students. But I believe that with moderately select groups it provides, on all the considerations I have discussed with you, a very attractive alternative to the traditional course. I shall not try to defend this belief further, for all that I could add would be platitudes. The real defense must wait on additional experience.

VECTOR GEOMETRY

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This report concerns a two-year secondary school mathematics course which is now in the final stage of development by the University of Illinois Committee on School Mathematics and of testing in a number of schools in various parts of the country. This work is supported by a National Science Foundation grant.

Very briefly, three-dimensional Euclidean geometry is developed as the theory of an inner product space T—the set of translations—acting on a set E of points—the points of Euclidean space.

The original version of the course was written in 1962/63, and a revised version of the first year's material was taught in 1963/64 to two classes of tenth-year students in University High School. These students continued studying the second year's material in 1964/65, and additional students began their study of a second revision of the course. In 1965/66 four other schools (in New Jersey, Missouri, Nevada, and Washington) began to use the materials. As one result of an institute held during the summer of 1966, two more schools (in Pennsylvania and California) were added to those in which the course was being taught. At present the text for the first year is being revised into final form for publication. There will be a teacher's edition which will include much background material. It is expected that the final version of the text and commentary for the second year will be completed during 1967/68.

The course is intended for rather bright students who have had a beginning algebra course in which some attention has been given to the deductive organization of their knowledge of the real numbers and, so, to some of the techniques of deduction. Since the course bases geometry on mappings, some knowledge of functions is highly desirable. Nevertheless, the course is being taught with apparent success to students of varied abilities and backgrounds. As with any organization of geometry, those students will profit most who have had a continuing experience with intuitive geometry throughout their previous schooling.

The course has been described in more detail than is possible here in an article by Steven Szabo in the March 1966 issue of the *Mathematics Teacher*. Hence I shall confine my remarks to some general comments on its relation to other treatments of geometry and on its notation and postulates.

To begin with, it is worth stressing that the course deals with translations—as members of a vector space—*operating on points*. The subject matter is actually Euclidean (metric) geometry, as contrasted with the centered Euclidean geometry of a vector space. Since "vectors" are mappings—and points are points—there is no danger of confusing the two fundamental types of objects.

In the second place, the introduction of translations per se bypasses the conceptual difficulties which are inherent in defining vectors as equivalence classes. (These difficulties would not be serious for students who had had the experience with equivalence relations which, hopefully, students in future years will have had. But, at present, few students at this level have had such experience.) Intuition concerning translations is readily developed through the use of various mechanical devices and provides the basis for postulates and definitions and also for conjecturing theorems.

As mappings, translations can be composed with one another, and, being one-to-one, they have inverses. Also, as mappings, translations act on points. The first postulates describe how translations act on points and how they react with each other under composition and inversion. In describing these matters, capital letters are used as variables over the set $\mathcal{E}$ of points, and lowercase arrow-letters (see below) are used as variables over the set $\mathcal{T}$ of translations. Later, lowercase letters (without arrows) are used as variables over the set $\mathcal{R}$ of real numbers.

A fundamental peculiarity of translations is that a translation is completely determined once one knows the image under it of any single point. This fact is made use of by speaking of *the translation from A to B* and, for this phrase, introducing the notation '$B - A$'. With this notation it is convenient to use '$A + \vec{a}$' [rather than '$\vec{a}(A)$'] in referring to the image of a point A under a translation $\vec{a}$. The first two postulates introduce this notation and formulate this two-point property of translations:

1. (a) $B - A \in T$ (b) $A + \vec{a} \in E$
2. (a) $A + (B - A) = B$ (b) $\vec{a} = (A + \vec{a}) - A$

(Postulate 2 amounts to saying that $A + \vec{a} = B$ if and only if $\vec{a} = B - A$.) Since a resultant of translations is a translation, the translation from A to B followed by the translation from B to C is the translation from A to C. In connection with the subtraction-addition notation already introduced, it turns out to be convenient to refer to function-composition—as well as to function-application [see Postulate 1(b)]—as addition. Specifically, '$\vec{a} + \vec{b}$' is used in place of the more usual '$\vec{b} \circ \vec{a}$'. (In practice, it turns out that the two uses of '+' give rise to no confusion.) The third postulate formulates the closure of under function-composition:

3. $(B - A) + (C - B) = C - A$

Postulates 1-3 tell how translations act on points and also imply the associativity of addition in T. The remaining postulate, 4 (which is built up gradually), gives additional information as to how translations react with each other and how they are acted on by real numbers. To begin with, composition of translations is commutative, and it follows from this and the preceding postulates that

4'''. T is a commutative group with respect to addition.

(This, rather than merely the commutativity of addition in T, is adopted as a provisional postulate. It will be strengthened as time goes on, becoming, successively, 4'', 4', and, finally, 4.)

At this point the advantages of the notation become evident. Given a sentence—say:

$$A + (B - C) = B + (A - C)$$

—of the language developed up to this point, there is an easy way for students to check whether or not this sentence is a consequence of 1-4'''. Any such sentence has a real number analogue obtained by replacing all variables by (distinct) real number variables. In the example in question, the sentence:

$$a + (b - c) = b + (a - c)$$

is such an analogue. It can be proved that the given sentence follows from 1-4''' if and only if its real number analogue is a consequence of the additive group postulates for real numbers. So, with the exercise of a little care, students can carry out symbolic manipulations at this stage with the same skills they have mastered during their study of real number algebra. [The care that is needed is sufficiently illustrated by noting that since addition of points is not defined, '$(A + B) - C = (B + A) - C$' is not a theorem even though its real number analogue is a theorem.] Of course, skill in manipulating symbols is merely an essential for an easy life. Students (and readers) are encouraged to investigate the content of theorems by drawing figures.

The next step in the development of Postulate 4 is to note that since, intuitively, a translation moves all points the same distance and in the same sense, multiplication of translations by real numbers can be introduced. There is no need, here, to go into details. The usual postulates for this operation are easily motivated in the usual ways, and 4''' is strengthened to:

4''. $\mathcal{T}$ is a vector over $\mathcal{R}$.

After a study of linear dependence, leading to definitions of line, segment, plane, etc., and the proof of various incidence properties, 4'' is strengthened to:

4'. $\mathcal{T}$ is a three-dimensional vector over $\mathcal{R}$.

At this point—and even with the provisional Postulate 4''—affine geometry is present and many conventionally "geometric" theorems can be proved. Also, it is a trivial matter to introduce coordinates and, when desired, to use analytic methods of proof.

To arrive at Euclidean geometry, one needs distance and perpendicularity. At this point intuitive notions concerning orthogonal projection are developed, and these suggest the postulational introduction of an inner product with the usual properties. The final version of the fourth postulate is, then:

4. T is a three-dimensional inner product space over R.

Postulates 1, 2, 3, and 4 (together with the implicit postulate that R is a complete ordered field) constitute the basis for the entire course. (Completeness of R is not needed until rather late.)

Distance and perpendicularity are defined in terms of the inner product [for example, $d(A, B) = ||B - A||$]; the cosine and sine functions are defined; and congruence theorems, for example, are consequences of now-trivial algebraic identities such as the law of cosines. Incidentally, the definition of the sine function shows the importance of orientation—an important subject usually neglected in the high school curriculum. (The cosine and sine referred to above have sensed angles for arguments. This is sufficient for much of geometry, but the usual circular functions are needed also, and their definition and that of angle measure depends on the completeness of R.)

Two concluding remarks are in order.

First, the preceding outline has probably given the impression that the course is purely algebraic and departs undesirably from the spirit of geometry. Fortunately, this is not so. From the beginning the algebra is "translated" into pictures of geometric figures. Moreover, once some basic geometric theorems have been proved, others are deduced from them by the usual synthetic methods. This gives ample scope for the development of "geometrical thinking," and the fundamental use of translations furnishes additional purely geometric tools for the solution of geometric problems.

Second, it has been noted earlier that analytic methods involving the use of coordinates are at hand quite early for the use of those who delight in them. As one might suspect, however, the algebra proper to the course not only is simpler but has the advantage of dealing with geometric objects rather than with a "coding" procedure. So, except as a very useful vehicle for practice in real number algebra, analytic methods are of little use in this course. (They must be [and are] introduced, of course, as preparation for later courses.)

GEOMETRIC TRANSFORMATIONS

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During the past several years, there has developed something of a stir concerning the introduction of geometric transformations into the high school program. This stir is due partly to the influence of certain European programs and partly to our own "growing up"—the result of a decade or more of curriculum reform. Certainly, SMSG's publication of Yaglom's book *Geometric Transformations*, in the "New Mathematical Library" series, helped give prominence to the subject in the minds of many school people.

There are several indications of the increased interest in geometric transformations: three recently published high school textbooks (that I know of) introduce transformations; virtually all of the recent textbooks for prospective teachers treat geometric transformations; during the past year, there appeared an excellent pamphlet by Walter Prenowitz and Henry Swain for in-service teachers concerning a study of congruence by means of transformations; there are now at least three translations of foreign works dealing with the subject at a moderately elementary level; and, finally, the most significant indication is that my orders for today clearly stated that I was to talk on "Geometric Transformations."

I take it that my responsibility is to tell you why I think that this stir is a good thing and, in particular, why I believe that the transformation viewpoint belongs in high school geometry.

Congruence

The reformers of high school geometry began their efforts with an attempt to give a mathematically sound treatment of the Euclidean geometry that is now taught mainly in the tenth grade. The primary problem was to come to grips with the question of order in geometry: the order of points on a line, betweenness, inside and outside of a figure, and so on. Another order of business was to give honest, but not overly complicated, proofs of theorems that depend upon completeness of the real line. The third problem—the one I wish to dwell on—was the problem of congruence.

Virtually all the high school textbooks up to 1955 discussed congruence in terms of superposition, as did Euclid. Superposition is unsatisfactory as a logical basis for the notion of congruence. First of all, it is a physical notion and not a mathematical one. There is nothing in the usual axiom systems that allows one to lift up a triangle and place it somewhere else. Of course, almost any idea can be introduced as an axiom, but then one has to give a careful description of the procedure—and draw only conclusions implied by the axiom. But even if this were done, the idea of superposition would still be unsatisfactory, if only for the reason that it doesn't generalize. It might be relatively simple and quite satisfactory if axioms for superposition were set forth to treat two-dimensional problems. But what happens in three dimensions? You cannot superimpose one sphere on another, and this means that you cannot prove that two spheres having equal radii are congruent. (For a more detailed discussion of these matters, see *Congruence and Motion in Geometry* by Walter Prenowitz and Henry Swain, D. C. Heath and Company.) Now, there are various ways in which the theory of congruence can be given a satisfactory basis from a mathematical and logical point of view. However, I want to make a pitch for using the idea of an isometry, a distance-preserving transformation.

Let me digress with some examples of transformations. The particular transformations encountered in elementary Euclidean geometry of the plane are one-to-one mappings of the plane onto itself. If α is such a transformation, then α associates with each point P a point P', called the *image* of P under α. This is often written as

$$\alpha : P \longrightarrow P' .$$

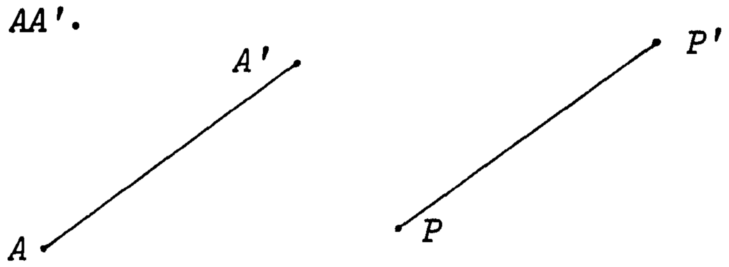
"One-to-one" means that α establishes a one-to-one correspondence between the points P and their images P'; "onto" means that every point in the plane is the image of some point.

Example 1: Translation (alias "shift")

With a given directed segment AA', we associate a translation T, defined as follows. If

$$T: P \longrightarrow P',$$

then the directed segment PP' has the same length and direction as AA'.



Example 2: Reflection

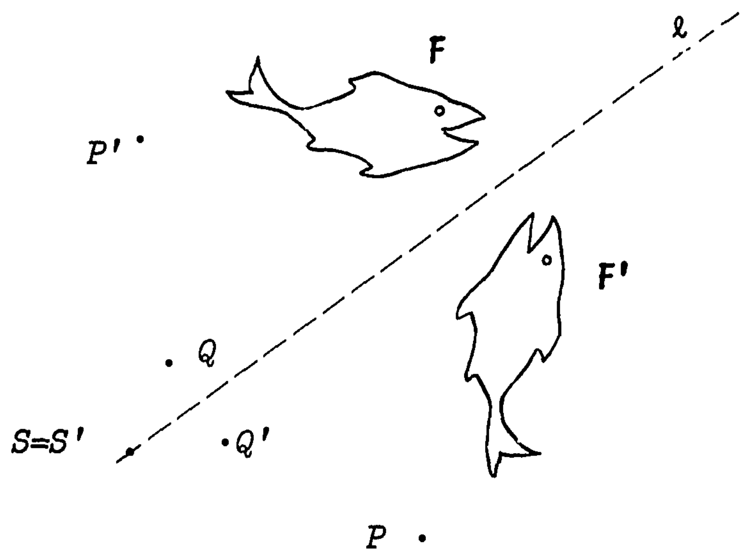
With a given line ℓ we associate a reflection ρ in the line ℓ.

$$\rho: P \longrightarrow P'$$

defined as follows:

(i) If point P is not on ℓ, then $P' \neq P$ and ℓ is the perpendicular bisector of segment PP'.

(ii) If P is on ℓ, then $P' = P$. That is, each point of ℓ is invariant. The line ℓ is called the *axis* or *mirror* of the reflection.



Example 3: Rotation

With a given point O and any given real number θ, we associate a rotation β,

$$\beta : P \longrightarrow P',$$

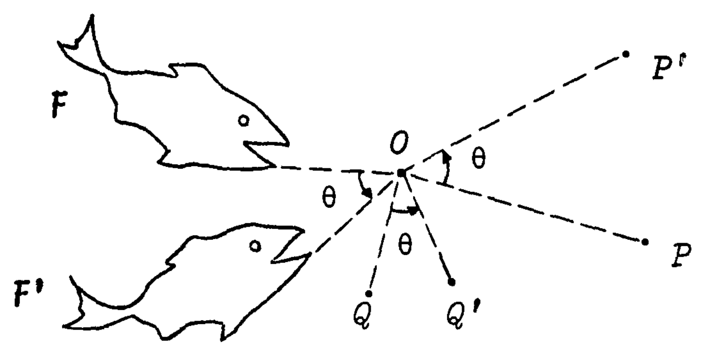
defined as follows:

(i) If $P = O$, then $P' = P$. That is,

$$\beta : O \longrightarrow O,$$

so O is invariant.

(ii) If $P \neq O$, then segments OP and OP' are equal in length and the counterclockwise measure of $\angle POP'$ is equal to θ.



The transformation β is called a rotation with *center* O through O radians.

The three examples are of transformations that preserve distances. (The image of a segment is a segment of equal length; it follows that the image of an angle is an angle of equal measure.) Therefore, these are examples of isometries.

Very briefly, *congruence* is defined in terms of isometries by stating that two figures F and F' are congruent if there exists an isometry α under which F maps into F'; that is,

$$\alpha : F \longrightarrow F'.$$

Again, I refer you to *Congruence and Motion in Geometry* by Prenowitz and Swain for a more extensive treatment of this subject.

The notion of an isometry is a mathematical abstraction of the idea of superposition, but as a mathematical abstraction it has many advantages. First of all, it has logical cleanliness. Secondly, it is general, so that the same notion holds for zero-dimensional, one-dimensional, two-dimensional, three-dimensional and n-dimensional configurations. And, thirdly, it is general in the sense that the congruence of general figures is dealt with in a single idea. Whereas many logically satisfactory developments of congruence are based on axioms and definitions for, first of all, the congruence of triangles, and then later of polygons and polyhedra—building from segments on upward—the theory of congruence based on isometries has the advantage of dealing with all figures simultaneously. The two figures F and F' are general geometric configurations—point sets, if you like. The mathematical economy is obvious, and I believe the method has aesthetic appeal as well.

Functions and Other Themes

No one argues with the fact that the idea of *function* is one of the most important ideas in all of mathematics, if not *the* most important. The first reforms in collegiate mathematics that took place after World War II took cognizance of this and worked toward giving students a clearer idea of what a function is, in order to provide them with a more solid foundation on which to build their mathematical education. These efforts had overtones that soon affected the high school programs and now affect some of the elementary school programs.

Increasing numbers of students are being exposed to the ideas, including the language and notation, of functions. But no matter what a student's elementary program was like, his eighth- and ninth-grade algebra have certainly made explicit use of functions and functional notation. Then along comes tenth-grade geometry, which appears an anomaly if it is only slightly related to previously studied mathematics. I expect we all agree that this is a bad thing and that the more relationships and interrelationships we can show between mathematical ideas in various subjects, the better our curriculum will be. I therefore assert that we should bring into geometry the all-important idea of function in its most natural geometric form, namely, as a geometric transformation.

The "spiral approach" to curriculum seems to be favored by many in recent years. For those who prefer the spiral approach, here is a golden opportunity to reiterate an important theme (functions) and to do so in a most natural way.

If the study of groups is already part of the curriculum, then transformations present still another golden opportunity for reinforcement. The transformation groups are nontrivial yet quite accessible, especially because they are geometric. If the students haven't encountered groups, then the geometric transformations, serving as motivation, may furnish an excellent entrée. (In fact, it is the study of groups of transformations that enabled Felix Klein to give his famous classification of geometries which, in some sense, answers the question: What is a geometry?)

Still another tie-up with high school work is with the subject of linear algebra. While the number of schools that teach linear algebra is still small, that number is increasing with the availability of excellent books like the SMSG *Introduction to Matrix Algebra* and Philip Davis' book entitled *The Mathematics of Matrices*, published by the Blaisdell Publishing Company. Since matrices have their most natural interpretation as geometric transformations, any school that intends teaching linear algebra would be wise to consider introducing transformations synthetically in the geometry course—before the linear algebra. Among other advantages is the fact that the linear course will be provided with built-in drama, since it will be giving students an algebraic interpretation of a familiar topic along with deeper insights that result from the power of the algebra.

Symmetry

If we think of where our students are likely to go and what kind of use they will make of the mathematics we are teaching, then we are likely to come up with a list that includes as subjects of future interest: physics, chemistry, engineering, biology, economics, and mathematics (pure and applied).

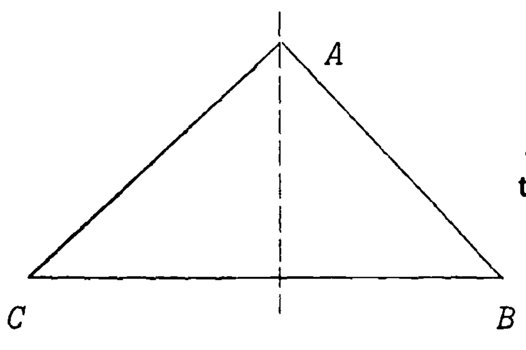
The notions of symmetry are of extreme inportance in all of these fields. There is no time to go into very many ways in which symmetry comes up in the different academic disciplines, so I'll confine myself to mentioning only one specific application, but a pervasive one: X-ray crystallography. This is the subject that deals with ascertaining the molecular structure of a compound in crystal form. It is important in physics, chemistry, and molecular biology. Its importance to physics and chemistry is obvious; as for biology, suffice it to say that the structure of the DNA molecule was determined by the methods of X-ray crystallography. The entire subject of crystallography rests on the notions of geometric symmetry and the group theory used to classify the symmetry groups of the plane and space.

What does all of this have to do with geometric transformations? The answer is simple. Transformations provide the key to understanding symmetry. Symmetry cannot be defined or understood without the notion of a transformation. Artists and many lay people use the work "symmetry" as part of their normal vocabulary, but very few could give a precise definition that would be useful in a scientific sense.

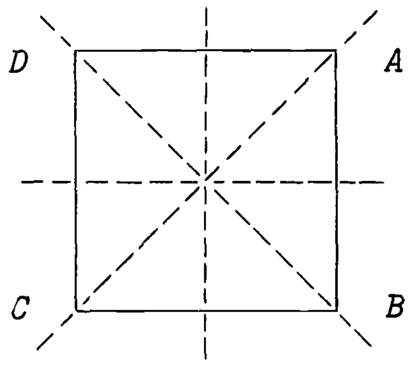
If a definition of symmetry is to serve science, it should enable one to distinguish, with precision, figures that are symmetric from those that are not; the judgment of whether or not a figure has symmetry must be taken out of the realm of the subjective. The scientists' definition of symmetry can be stated as follows: If figure F is invariant under a transformation α, that is, if

$$\alpha : F \longrightarrow F,$$

then α is called a *symmetry operation* for F, and F is said to have α-symmetry.



$\triangle$ *ABC* has reflective symmetry (= bilateral symmetry).



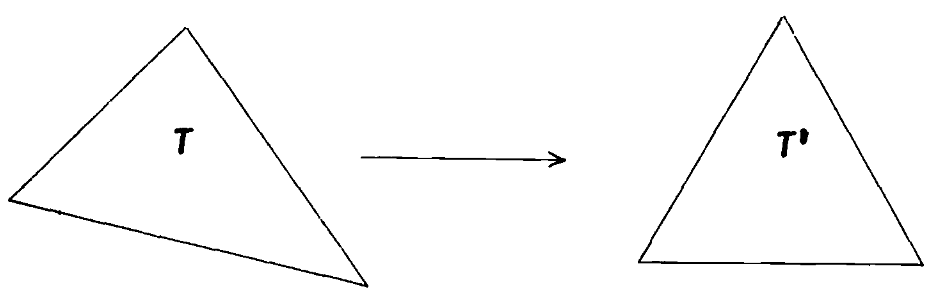
□ *ABCD* has several kinds of reflective symmetry and also rotational symmetry.

Extensions of the idea of symmetry are prevalent in physical science. Since *invariance* is a basic ingredient, every *conservation law* (which says that some quantity is invariant) is a statement of symmetry. This has convinced physicists—especially those who work in particle theory—that the fundamental laws of the universe are symmetry laws. For an excellent discussion of this topic, I suggest *The World of Elementary Particles* by Kenneth W. Ford, especially Chapter 4.

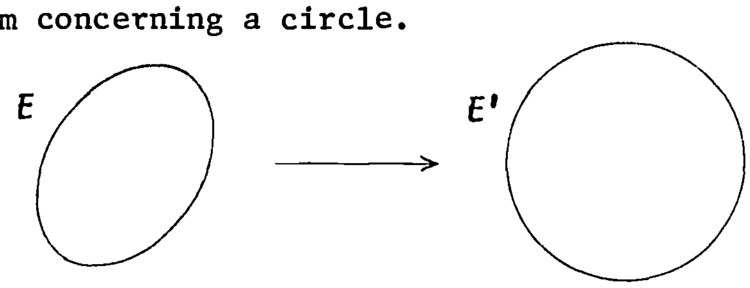
Problem Solving

Transformations are problem-solving instruments because they often provide a method for transforming a hard problem into an easy one. After solving the easy problem, you obtain a solution to the original hard one by applying the inverse of the first transformation.

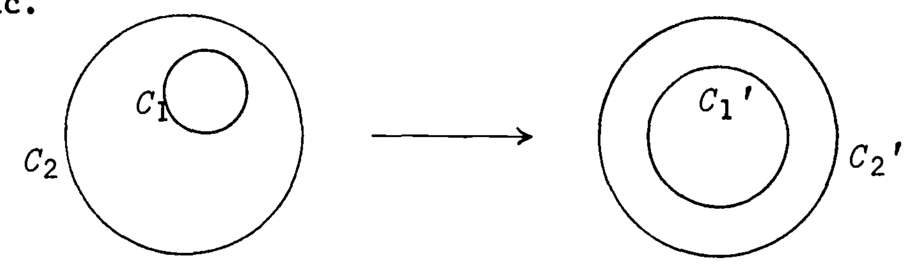
A problem concerning a general triangle might be solved by transforming the general triangle into an equilateral triangle.



A problem concerning an ellipse can be converted into a problem concerning a circle.



A problem concerning a pair of nonintersecting circles can be transformed into a problem with the circles concentric.



Again, there is no time to discuss specific problems, so I refer you to *Introduction to Geometry* by H. S. M. Coxeter, where you will find many such problems strewn throughout the book. I suggest, also, a paper by M. S. Klamkin and D. J. Newman entitled "The Philosophy and Applications of Transform Theory," *SIAM Review*, January 1961, pages 10-36. This paper is devoted to a discussion of problem solving via transformations; the problems range from the elementary to the esoteric. Another good reference to problem solving by means of transformations is the earlier-mentioned book by Yaglom.

The application of function theory to engineering problems relies principally on applying conformal transformations. Boundary-value problems can often be solved by transforming the problem with an impossible boundary condition into one that is just intolerable.

So, one again, the argument is that by introducing a student to geometric transformations we are contributing to the mainstream of his mathematical education and providing him with an instrument of lifelong value.

Finally, I would like to close with two points that might temper some of my earlier remarks.

First, I am not proposing that tenth-grade geometry be reorganized so that it be developed around the notion of a geometric transformation. Perhaps this extreme kind of reorganization is appropriate, but I have not looked at the problem carefully enough, nor do I have sufficient experience with high school geometry, to assert that this is the most advisable way to go.

Second, I want to plead for maintaining a sense of perspective with regard to the geometry program at the tenth-grade and, for that matter, at any level. To be explicit, let me present the following point of view. Geometry is a subject that deals with things such as points, lines, surfaces, area, volume, etc. There are various mathematical tools that are used to study these entities and to solve problems involving them. There are the synthetic method, the vector method, the analytic method, methods of calculus, etc. I don't think that any one method is preferable to all others. A problem may admit of a simple synthetic solution while the analytic solution may be vicious; the opposite will be true of another problem. At different stages in his career a student should be acquainted with new techniques and new approaches. And if any one method is to be adopted for a segment of a student's life, it should be adopted for only a finite time and should not be treated as sacrosanct. Maintaining this kind of perspective will help to inculcate in students the freedom so essential to do mathematics and to enjoy it.

GEOMETRY: THE CAMBRIDGE CONFERENCE VIEW (Abstract)

By *Edwin E. Moise*
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There were two Cambridge Conference views described in the CCSM report, one algebraic and "modern," the other more classical. In support of the classical (minority) view, the following points were made.

1. In a choice of content and style, at a given teaching level, a fundamental criterion is the availability of challenging and workable problems. At low maturity levels, calculus and classical geometry outrank linear algebra and transformation groups under this criterion.

2. A vectorial conception of geometry is on a high level of abstraction because one cannot significantly say of a point that it is a vector; the significant statement is that a space (considered as an entity in itself) is a vector space.

TOPOLOGY

By *Gail S. Young*
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I would like to begin by making quite clear the tentative nature of my proposal in topology for the schools. The process of curriculum development in education is a good deal like having the pieces for four jigsaw puzzles put into one box and attempting to make the most attractive picture out of the box. There are in mathematics many more things that are both interesting and important than can possibly be taught in the schools. The process of selecting and ordering the topics in a curriculum is a fascinating and deep intellectual exercise.

Many intellectual solutions are possible. But in my mind these solutions are all inadequate because they do not—in fact they cannot—take proper account of the psychology of mathematical learning. The reason they neglect this psychology is that our knowledge of it hardly exists. We have the brilliant intuition of Piaget and very little else.

When we properly understand the psychology of mathematical learning, it could turn out that topological ideas will replace much of current geometry, or it could turn out that all such ideas will be postponed to college; and I have no idea, myself, which way this will go. In the meantime I will describe some things that, it seems to me, topology can do in the schools.

First of all, topology has already entered the curriculum, particularly in the elementary schools. There are now many elementary school students who can talk about simple closed curves and tell you that simple closed curves divide the plane into two pieces.

There is room for somewhat deeper ideas based on such intuitive concepts. For example, the reader will recall the water-light-gas problem: Given two sets of three points in the plane, join each point of one set to each point of the other set by arcs that meet only at end points.

If one accepts the Jordan curve theorem as intuitive, then it is almost as easy to accept the theta-curve theorem, that a figure homeomorphic to the Greek letter theta divides the plane into three connecting pieces, in the obvious way. With that result, it is easy to see that the water-light-gas problem has no solution. I must say that it makes me nervous to see such arguments presented as mathematics when I know that there are formidable technical difficulties that are not being faced; and I am not convinced of its desirability. However, quite fascinating things can be done, based on this problem. For example, ask the child if it is possible to solve the problem on the surface of a doughnut, a torus. He will find that it is. Then ask him about the water-light-gas-telephone-sewerage problem on a torus, and he will decide that he can't do that. Ask him if it would be possible to do it on the surface of a doughnut with two holes.

I would also discuss unicursal problems, like the problem of whether it is possible to draw a square plus its diagonals without lifting the pencil off the paper or retracing steps. The argument that wherever one begins, one must end at both the first and the second vertex with an odd number of edges coming in, is rather easy.

The two types of problems together permit one to bring out early the difference between intrinsic properties of the figure, those that do not depend on whatever space it is located in, and the extrinsic properties of the figure, those that depend on whether it is supposed to be located is a plane or a torus or in three-space. All these are problems about linear graphs, which are becoming increasingly important, at least as a language, in many types of applications of mathematics.

To my mind, one of the fascinating things about contemporary mathematics is the rapidity with which it is becoming unified again, particularly by the category-functor concept. If people in other fields are to understand what mathematics is doing, they too must learn this language and see how many things are simplified by it.

One of the major standardizations in this unification is the introduction of the language and methods of general topology. Thus one could introduce open and closed sets, limit points, closure, connectedness, etc., much earlier than the graduate school and, in a program of the future, quite conceivably in the high school. As soon as one knows the notion of open set, one can use the topologist's definition of continuous function: that is, a function f from a space A to a space B is continuous if the inverse of every open subset of B is open in A. With this definition, elementary theorems of general topology give simple proofs of what have been rather complicated existence proofs in elementary real variables. The theorem that a continuous function on the line which is positive at one point and negative at another must have a zero at some intermediate point follows immediately from the easily proved theorem that the continuous image of a connected space is connected. The fact that a continuous function on a closed interval assumes a maximum and a minimum follows, equally easily, from the somewhat deeper theorem that the continuous image of a compact space is compact. The key theorems of elementary real variables are that intervals are connected and that closed intervals are compact. One might as well let the students in on these facts as early as possible.

The tremendously powerful existence methods based on fixed points can be brought in much earlier. We are always talking about finding the zeros of functions. But every problem concerning a zero is equivalent to a problem concerning existence of a fixed point. Every zero of a function $f(x)$ is a fixed point of the function $f(x) + x$. [If $f(x_0) + x_0 = x_0$, then $f(x_0) = 0$, and vice versa.] It is easy to talk about the Brouwer fixed-point theorem in lower dimensions. One can prove it quite rigorously for mappings of the closed interval into itself, though I suspect a general proof will never enter the high school curriculum, at least until there are many more biological mutations. Suppose, in fact, "let $f:i \rightarrow i$" is a continuous function mapping the closed interval (0, 1) into itself. If $f(0) = 0$, then 0 is a fixed point. Otherwise $f(0)$ must be a positive number. If $f(1)$ is not equal to 1, then $f(1)$ is a nonnegative number less than 1. Consider the function g defined by $g(x) = f(x) - x$. This is continuous, since it is the difference of two continuous functions. We have $g(0) = f(0) - 0 > 0$; $g(1) = f(1) - 1 < 0$, and from the intermediate-value theorem we see that therefore there must be a number $g(x_0)$ such that $g(x_0) = 0$.

But that implies that $f(x_0) = x_0$, so that we have a fixed point.

I think it is rather easy to introduce the ideas of a metric space and of a contraction mapping. Suppose we have a metric space M and a function $f{:}M \to M$. The mapping f is a contraction mapping if there is a constant $k < 1$ such that for each two points x and y, $d[f(x), f(y)] < d(x, y)$, that is, if each two points are moved closer together by the mapping with a certain maximal ratio of distances. If the metric space is complete, it is not hard to show the existence of a unique fixed point. This takes a great deal of time, at an elementary level, but the number of problems that reduce to the contraction principle is astonishing. I will not take time here to go into that. One excellent place to see them is the book of Kolmogoroff and Fomin, *Elements of the Theory of Functions*. This principle is not merely theoretical; it is the basis for many practical numerical methods.

Closely related to the notion of a fixed point are the concepts of index of a curve and degree of a mapping. If we have a parametrized curve in the plane (by which I mean not only the point set but a description of how one moves over the point set), one has the notion of index of the curve at a point which is, intuitively, the number of times that the curve passes around the point. It is rather easy to prove that the index of a curve is a homotopy invariant: that is, if we deform one curve into another but at no time in the deformation pass through a point p, then the indices of both curves around p are the same. With this it is, for example, rather easy to prove the fundamental theorem of algebra. Suppose there is a polynomial "$f(z) = a_0 z^n + \dots$" that has no zero. Then $f(0) \neq 0$, and the index of the curve described by f acting on a small circle around the origin must be zero. But by moving continuously from the small circle to a large circle, it follows that the index of the curve given by f on the large circle must also be zero. However, on a large circle we can deform the mapping f to the mapping "$g(z) = a_0 z^n$" without passing through the origin. One can compute directly that the degree of g with respect to the origin is always n, which is not zero, and this gives a contradiction. This is a typical example of an index argument as used in advanced mathematics. In particular, the notion of deforming the mapping to a simple one where it is possible to compute the index is basic.

School geometry seems to me to be a very good place to introduce the elementary concepts of what used to be called combinatorial topology. The notion of a complex, in the form of a figure made up of a number of elementary building blocks, points, intervals, triangles, tetrahedra, is quite simple and intuitive. One can ask questions like this: Given that we will permit triangles to intersect only in vertices or edges, what is the least number of triangles needed to build a torus, acting as though the triangles were made of rubber? One can introduce the Euler characteristics and prove topologically the fact that there are only five regular polyhedra. This is the place to talk about orientability of surfaces, and one can at least state the theorem that two closed surfaces are homeomorphic if and only if they have the same characteristics and are both orientable or are both nonorientable. This introduces the concept of invariants, another of the unifying themes of contemporary mathematics, and the notion of a set of invariants characterizing a space. One finds himself also faced at once with questions (such as the real "existence" of Klein bottles or projective planes) which lead quite naturally into discussion of higher dimensional spaces, in a fashion that seems to me more concrete than the approach by linear algebra, and which to my mind greatly reinforce the student's motive for studying linear algebra.

In summary: Topology can, first of all, provide early some opportunities for creative work. In the overall development of mathematics, the sooner topological ideas can be introduced, the less time is wasted on successive generalizations and the sooner the student understands what is "really" going on. And, lastly, very powerful tools in mathematics have elementary expressions in rather elementary parts of topology. How much of all this can or should be done in the schools must be left to experimentation and further study. But there is no a priori reason why the topics I have mentioned cannot be introduced.

AN APPROACH TO ANALYTIC PROJECTIVE GEOMETRY

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In the summer of 1966 the author gave a series of 32 lectures on analytic projective geometry to a class of 40 high school students selected from 19 states. This was part of a program sponsored by the National Science Foundation at the University of Oklahoma. The academic achievement of the students in this course on projective geometry and their apparent enjoyment of the study were sufficiently gratifying to suggest a repetition of the course for another group in the summer of 1967.

The material covered in this course consisted of the first three chapters of Harold Dorwart's *The Geometry of Incidence* (Prentice-Hall, 1966) and of selected topics and exercises from the author's *Geometry and Analysis of Projective Spaces* (W. H. Freeman & Co., 1964). Following an introductory treatment of homographies on a line, the concepts of "join," "intersection," and "incidence in the plane" were covered, with a view toward solution of problems on linkages, point loci, and line envelopes. The geometric interpretation of invariants under projectivities on the line and in the plane constituted an important segment of the course. An attempt was made to impress the fact that Euclidean geometry is a specialization of a more general geometry. Further, it was emphasized that linear algebra and projective geometry lend richness to each other.

The principal aim of this paper is to suggest an approach to homogeneous point and line coordinates as a basis for the study of projective geometry in the high school.

Point and Line Coordinates in $E^{(2)}$

It is assumed that the student has become familiar with the set of points in the Cartesian plane $E^{(2)}$ whose coordinates (x, y) satisfy the equation

$$(1) \qquad Ax + By + C = 0,$$

where A, B, C are given real constants. The equation

$$(2) \qquad (kA)x + (kB)y + (kC) = 0, \qquad (k \neq 0)$$

represents the same set of points as that given by Equation 1. Therefore, the line represented by Equation 1 is determined by the ratios $[A:B:C]$. The singly infinite set of ordered triples $[kA:kB:kC]$ is equivalent to the one representative triple $[A:B:C]$ for which $k = 1$. Equations 1 and 2 are said to be *dependent*. Every choice of an ordered triple of numbers for $A:B:C$ in (1), with A and B not both zero, yields a line. Hence, one calls A, B, C coordinates of the line, written as $[A:B:C]$, with square brackets. Of course, by Equation 2, the coordinates may be written as $[kA:kB:kC]$ with $k \neq 0$. The choice $k = 0$ would not determine a line, so the triple $[0:0:0]$ is excluded. The totality of lines in the Cartesian plane $E^{(2)}$ is given by the totality of ordered triples $[A:B:C]$, with A and B not both zero.

Examples: The line given by "$2x - 3y + 6 = 0$" has line coordinates $[2:-3:6]$. The line "$y = 0$" has line coordinates $[0:1:0]$, and the line "$x = 0$" has line coordinates $[1:0:0]$.

The triple $[0:0:C]$, with $C \neq 0$, is apparently excluded at this stage of the development because the equation $0x + 0y + C = 0$ gives $C = 0$, an apparent contradiction. The simple representative of the triples $[0:0:C]$ is $[0:0:1]$, which is not allowable at present. Note that if A and B are not both zero, then C may be zero; that is, $[0:1:0]$ and $[1:0:0]$ are acceptable as line coordinates. Of all nonzero triples, only $[0:0:1]$ is now excluded, but a way will be found to include it also.

Equation 1 is an incidence relation. It states that a line with given coordinates $[A:B:C]$ is incident with a point with Cartesian coordinates (x, y). Here, A, B, C are fixed, and the variable point (x, y) is on the line $[A:B:C]$. It appears one-sided and awkward to have three coordinates for a line and only two coordinates for a point. One could write the point (x, y) as $(x, y, 1)$, or as (x, y, z) with z always regarded as 1. In this case the incidence equation would be

(3) $$Ax + By + Cz = 0.$$

It is clear that Equation 3 is equivalent to

(4) $$A(kx) + B(ky) + C(kz) = 0, \qquad k \neq 0;$$

that is, if (x, y, z) satisfies Equation 3, then (kx, ky, kz) also satisfies that equation. This means that the third coordinate in the triple $(x:y:z)$ is arbitrary, subject to the restriction that $z \neq 0$ if $\left(\frac{x}{z}, \frac{y}{z}, 1\right)$ is to be the point $\left(\frac{x}{z}, \frac{y}{z}\right)$ in the Cartesian plane $E^{(2)}$. The triple $(x;y;z)$, with parentheses, is a set of homogeneous point coordinates of the point $\left(\frac{x}{z}, \frac{y}{z}\right)$. The triple $(a:b:0)$ does not represent a point in the Cartesian plane.

Examples: The point (0, 0) has homogeneous coordinates $(0:0:z)$, which is equivalent to (0:0:1). The point (0, 1) has homogeneous coordinates (0:1:1). The point $\left(\frac{2}{3}, -\frac{5}{3}\right)$ corresponds to (2:-5:3), and the point $\left(-\frac{3}{4}, \frac{2}{5}\right)$ may be written as (-15:8:20), or equivalently as (15:-8:-20).

It is apparent that one advantage of homogeneous coordinates is that a point with rational coordinates $\left(\frac{p}{q}, \frac{r}{s}\right)$ can always be designated by a triple of integers $(sp:rq:qs)$.

The left-hand member of Equation 3 is called the *inner product* of the triples $[A:B:C]$ and $(x:y:z)$.

Examples: The line [1:2:3] is incident with the point (-3:0:1) because the inner product (1)(-3) + (2)(0) + (3)(1) = 0. The point (1:-2:1) is also incident with the line [1:2:3] because the inner product (1)(1) + (-2)(2) + (1)(3) = 0.

A Model for the Affine Plane

It is useful at this point to show that Equation 3, with A, B, C as parameters, represents all planes through the origin of Cartesian coordinates in Euclidean 3-space $E^{(3)}$.

(In accord with the Cambridge Report, it is assumed at this stage that the student will have studied some elementary analytic geometry of three-dimensional Euclidean space $E^{(3)}$. This would include equations of lines and planes, and conditions for orthogonality of lines and of planes.)

A fixed set of ratios $[A:B:C]$ determines the normal to a plane in $E^{(3)}$ with (3) as its equation. The incidence relation (3) is now an orthogonality condition under which the normal $[A:B:C]$ is orthogonal to all lines OP from the origin O to any point $P(x:y:z)$ in the plane with Equation 3. On the other hand, for a fixed set of ratios $P(x:y:z)$ and variable $[A:B:C]$, Equation 3 is satisfied by the normals to all the planes that contain the line OP. One notes that a given set $[A:B:C]$ determines a plane through O, and a given set $(x:y:z)$ determines a line through O. All triples $[A:B:C]$ and $(x:y:z)$ are allowable [except the set $(0:0:0)$]. In particular, the point triple $(a:b:0)$, which was not allowed in the Cartesian plane $E^{(2)}$, is now used to determine the line from O to any other point $(a, b, 0)$ in the xy-plane of $E^{(3)}$. For instance, the point triple $(1:0:0)$ determines the x-axis. The line triple $[0:0:1]$, which was excluded as a line in the plane $E^{(2)}$, now determines a unique plane with the equation $z = 0$ in $E^{(3)}$. It is possible to establish a one-to-one correspondence between the lines (and planes) through the origin in $E^{(3)}$ and the points (and lines) in the plane $E^{(2)}$. However, if points with coordinates $(a:b:0)$ and a line with coordinates $[0:0:1]$ are adjoined $E^{(2)}$ to obtain the extended Euclidean plane, then the correspondence is complete. Note that for arbitrary a and b (not both zero), all points of the set $(a:b:0)$ are incident with the line $[0:0:1]$. Let this line be named the *ideal* line. With the addition of this ideal line to the Euclidean plane $E^{(2)}$, one obtains the *affine* plane $A^{(2)}$. It must be borne in mind that the plane $z = 0$ in $E^{(3)}$ plays a special role in the bundle of lines through the origin O in $E^{(3)}$. Under a mapping in $E^{(3)}$ given by

$$
\begin{aligned}
\bar{x} &= A_1x + B_1y + C_1z, \\
\bar{y} &= A_2x + B_2y + C_2z, \qquad (5) \\
\bar{z} &= A_3x + B_3y + C_3z,
\end{aligned}
$$

it is required that the point $(a{:}b{:}0)$ go into another point $(\bar{a}{:}\bar{b}{:}0)$ in the plane $z = 0$. This means that lines of the plane $z = 0$ map into lines of the same plane. A consequence is that $A \ = B \ = 0$. Equations 5 determine either a change of coordinates in $E^{(3)}$ or a line-to-line mapping, and the same equations determine either a change of coordinates in $A^{(2)}$ or a point-to-point mapping in the affine plane. The geometry of the affine plane $A^{(2)}$ is completely equivalent to the geometry of the lines in a bundle in $E^{(3)}$, in which one plane is fixed. The bundle of lines with one plane invariant is a *model* for the affine plane. In the affine plane $A^{(2)}$, the ideal line $z = 0$ maps into itself.

If the restriction that lines in the plane $z = 0$ of $E^{(3)}$ map into lines of the same plane be dropped—that is, if the plane $z = 0$ is treated as any other plane of the sheaf of planes through the origin in $E^{(3)}$—then the ideal line in $A^{(2)}$ is not distinguished from any other line in the plane. With this generalization, the affine plane $A^{(2)}$ becomes the *projective* plane $P^{(2)}$. The geometry of the projective plane is completely equivalent to the geometry of the lines of a bundle. The lines of the bundle correspond to the points of $P^{(2)}$, and the planes of the sheaf correspond to the lines of $P^{(2)}$. In the projective plane, any line may be given the equation $z = 0$. The three coordinates $(x{:}y{:}z)$ are on a par. Any three nonconcurrent lines in $P^{(2)}$ can be given the equations $x = 0$, $y = 0$, $z = 0$, that is, the line coordinates [1:0:0], [0:1:0], [0:0:1].

Join and Intersection in $P^{(2)}$

Let $[A:B:C]$ be the join of the points $P_1(x_1:y_1:z_1)$ and $P_2(x_2:y_2:z_2)$ in the projective plane $P^{(2)}$. By the incidence relation (3) one has

$$(6)\qquad \begin{aligned} Ax_1 + By_1 + Cz_1 &= 0, \\ Ax_2 + By_2 + Cz_2 &= 0. \end{aligned}$$

The solution for these equations may be written in the form

$$A = \frac{\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} C, \qquad B = \frac{\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} C,$$

so that

$$(7)\qquad A:B:C = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} : \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} : \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

It is seen that A, B, C are proportional to the three determinants of the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

obtained by leaving out a column at a time and maintaining the proper cyclical order. The triple $[A:B:C]$ is called the *cross product* of the triples $(x_1:y_1:z_1)$ and $(x_2:y_2:z_2)$.

Example: The join $[A:B:C]$ for the points (3:-1:2) and (1:2:4) is found from the matrix

$$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$$

to be [-8:-10:7], which is equivalent to [8:10:-7]. All points $(x:y:z)$ on the join satisfy the equation

$$8x + 10y - 7z = 0.$$

From Equation 7 it is seen that the join of the points $P_1(x_1:y_1:z_1)$ and $P_2(x_2:y_2:z_2)$ has the equation

$$x \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} + y \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} + z \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0,$$

which can be written in the determinantal form

$$\text{(8)} \qquad \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

This is the point equation of the *join* of P_1 and P_2.

The point $(x:y:z)$ which is incident with both of the lines $L_1\left[A_1:B_1:C_1\right]$ and $L_2\left[A_2:B_2:C_2\right]$ is called the *intersection* of the lines. Because of the incidence conditions, one has

$$\text{(9)} \qquad \begin{aligned} A_1x + B_1y + C_1z &= 0, \\ A_2x + B_2y + C_2z &= 0; \end{aligned}$$

and the ratios $x:y:z$ are found (by the same algebraic procedure as that used for the join) to be given by the cross-product components selected from the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}.$$

Thus,

$$x:y:z = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} : \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} : \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

and a line $[A:B:C]$ is incident with this point if and only if

$$A \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} + B \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} + C \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0,$$

which can be written in the form

$$(10) \qquad \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0.$$

This is the line equation of the intersection of lines L_1 and L_2.

Note that if $C_1 = C_2 = 0$, but $A_1{:}B_1 \neq A_2{:}B_2$, then the lines L_1 and L_2 are distinct, and both are incident with the origin of the Cartesian plane $E^{(2)}$. From Equation 10 it follows that $C = 0$ is the line equation of the origin. (Remember that $z \neq 0$ in $E^{(2)}$.) The *dual* of the last statement is obtained from Equation 8 as follows: If $z_1 = z_2 = 0$, but $x_1{:}y_1 \neq x_2{:}y_2$, then the join of P_1 and P_2 is the ideal line with the point equation $z = 0$ in $A^{(2)}$. Note that the dual exists in $A^{(2)}$ but not in $E^{(2)}$.

The incidence condition

$$Ax + By + Cz = 0$$

can be interpreted in two ways. If A, B, C are fixed, it is the point equation of a *line* of points. If x, y, z are fixed, it is the line equation of a *pencil* of lines. A pencil of lines is the *dual* of a line of points. One notes that after the concepts of join, intersection, and incidence are introduced, one has a built-in duality principle by virtue of the algebraic framework. Also, one deduces that complete duality is not attained in $E^{(2)}$, whereas it is in $A^{(2)}$ and $P^{(2)}$.

Next, let the use of line coordinates be illustrated by finding the intersection of two lines in $E^{(2)}$.

Example: Find the intersection of the lines

$$x + 2y - 4 = 0,$$
$$3x - 5y + 11 = 0.$$

Solution: One needs to write only

$$\begin{bmatrix} 1 & 2 & -4 \\ 3 & -5 & 11 \end{bmatrix} \rightarrow (2:-23:-11)$$

to obtain the Cartesian point coordinates $\left(-\frac{2}{11}, \frac{23}{11}\right)$ of the intersection. The equation of the intersection is $2A - 23B - 11C = 0$.

The next example shows how the dual works.

Example: Find the join of the points $\left(\frac{2}{3}, -\frac{3}{4}\right)$ and $\left(\frac{5}{7}, \frac{1}{3}\right)$.

Solution: The homogeneous coordinates of these points are (8:-9:12) and (15:7:21). Write

$$\begin{pmatrix} 8 & -9 & 12 \\ 15 & 7 & 21 \end{pmatrix} \rightarrow [-273:12:191].$$

The equation of the join is $273x - 12y - 191z = 0$.

The final example indicates a situation in the affine plane $A^{(2)}$.

Example: Find the intersection of the lines [1:2:-4] and [2:4:5].

Solution:

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & 5 \end{bmatrix} \rightarrow (26:-13:0) = (2:-1:0).$$

The result is a point on the line $z = 0$ in the affine plane. This point has no image in the Euclidean plane. There is no solution for the equations $x + 2y - 4 = 0$ and $2x + 4y + 5 = 0$ in the Euclidean plane.

Continuation

One would introduce next the necessary concepts of linear dependence of three distinct points on a line, and of three distinct lines through a point. A basic set of points $P_1(1:0:0)$, $P_2(0:1:0)$, $P_3(0:0:1)$, with a unit point $U(1:1:1)$, or, dually, a basic set of lines $\ell_1[1:0:0]$, $\ell_2[0:1:0]$, $\ell_3[0:0:1]$, with a unit line $u[1:1:1]$, should be studied next. With this background the student is ready to have fun by solving many linkage problems, including the Theorem of Pappus and the Theorem of Desargues on perspective triangles in the plane.