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Abstract

In the winter of 1965, an experimental course in Elementary Number Theory was presented to a 6th grade class in the Hosmer School, Watertown, Massachusetts. Prior to the introduction of the present material, students had been exposed in class to such topics from the University of Illinois Arithmetic Project as lattices, number lines, frame equations, and linear affine transformations. The present materials are concerned with such mathematical concepts as (1) fundamental operations involving integers, (2) division of integers which included remainders, factorization, and the Sieve of Eratosthenes, and (3) number systems in bases 2, 7, 10, and 12. Teacher and student materials that were used for a period of 14 weeks are included. [Not available in hardcopy due to marginal legibility of original document.] (RP)

CCSM
Feasibility Study No. 35

Mary Jacqueline Hatch
28 March 1965

Experimental Course in Elementary Number Theory (A continuation of work begun at the 1964 Cambridge Conference by Peter Hilton (Cornell University) and Brian Griffiths, Southampton University).

Preface:

A greatly expanded version of the course, first presented to twelve students at the 1964 Morse School summer session, is being tested with a 6th grade class at the Hosmer School in Watertown, Massachusetts. The teacher is (Mrs.) Mary Jacquelin Hatch, who observed the summer school experiment and who is now a staff member of the University of Illinois Arithmetic Project at Educational Services Incorporated.

The students being used in the current phase of the experiment are in a regular 6th grade class in the Watertown Public schools. Heterogenecusly grouped, the twenty-three children range from a very high aptitude in mathematics to a few whose previous performance in basic arithmetic has been consistently very poor. There are thirteen boys and ten girls. Because they are segregated on two separate sides of the room it is very noticeable that the boys are much more verbal in class than the girls, although performance on tests seems to spread evenly among both sexes.

Before the course on Elementary Number Theory was begun, Mrs Hatch had taught the class daily for three months, including such topics from the University of Illinois Arithmetic Project as lattices, number lines, frame equations, and linear affine transformations. Thus the students were already familiar with negative numbers and readily accept new symbols and concepts.

Outline and Observations:

I. Operations on the Integers. (Jan. 4 - Jan. 14)
Rather extensive tables were made by each student for addition, subtraction, multiplication and division of the integers. Each was arranged like a coordinate plane with the numbers being operated or placed along two perpendicicular number lines intersecting at zero. Significant patterns on the tables were noted and described to familiarize the students with rows, columns, symmetry, diagonals, quadrants, etc. This was the first time in their experience that the students had organized such a vast array of mathematical information for study and instant referral. Being so accustomed to rote learning in arithmetic, the possibility of being allowed to use tables as a short cut to tedious computation came as an unexpected surprise to the youngsters. The sudden

Outline and Observations (continued)

appearance of fractions on the division table drove home the idea that necessity is indeed the mother of invention.

II. Division of Integers (Jan 15 - Feb. 1)

1. Remainders - whole numbers including zero.
2. Factors of N, Solutions to □ x △ = N
   a. Proper factors
   b. Smallest Possible Factors
3. The Sieve of Eratosthenes

In traditional arithmetic remainders in division appear only after years of experience with carefully contrived situations where the quotient is always a whole number. The students had no trouble "reverting" to expressing remainders as a natural number rather than a fractional part of the quotient. However, consideration of zero as a legitimate remainder was a more difficult hurdle. Factors were defined as divisors which give zero as a remainder and prime factorization was approached as the list of smallest possible whole numbers which, when multiplied together, have the given number as a product. The uniqueness of prime factorization was noted long before the word "prime" was introduced.

Prime numbers were hinted at by looking for all possible factors of several consecutive numbers (i.e. 45, 46, 47, 48, 49, 50) as larger numbers were considered, various tests of divisibility were explored for divisors 2, 3, 4, 5, 6, 8, 9, and 10. (#7 left out on purpose). Three different arrangements of the natural numbers were used as Sieves of Eratosthenes, and primes were first defined as numbers not crossed out on the Sieve and latter as numbers having no proper factors. Some unsolved problems of history about primes were mentioned to show the students that the primes which they had just met for the first time have fascintated and confounded man for thousands of years. The distribution of perfect squares on a spiral array of natural numbers led into consideration of the sum of consecutive odd number series as a challenge for the more able students. The concept of Relative Prime pairs was introduced and included in a fresh look at lowest common denominators of unlike fractions. Euler's ϕ function was briefly explored for numbers 2 through 20.

III. Number systems of bases 10, 7, 2, 12 (Feb. 2 - March 5) includes a week's vacation.

The fundamental structure of our base 10 number system was laid bare as bases 7,2, and 12 were explored in great detail. Translations <u>from</u> base 10 proved more difficult than translations <u>to</u> base 10. Numberlines

were used and negative numbers and fractions as well as counting numbers were considered in the various bases. The students were encouraged to try to "think" in the new base and to check computations by translating the entire problem into base 10.

Russian Peasant's multiplication was used as an introduction to base 2 and was of great fascination to the students. Vertical repeated division by 2 was rotated 90° to provide a less confusing method for the students translate from base 10 into base 2.

	1	1	0	0	1	0	0
0	1	3	6	12	25	50	100
	2	2	2	2	2	2	2

Remainders: Base 2 notation.

This same method was then used for other base translation by some of the children. In doing some rather long multiplication with base 2 numbers the students had their first experience in carrying over more than one column. The need for two additional digits in base 12 notation was easily accepted after considerable familiarily with bases 7 and 2.

There was a great deal of class discussion about base in general. Some tables were prepared and compared to base 10 tables. Perfect squares, even and odd numbers, and the behavior of the digit (b-1) for base b were explored for all cases.

IV. Arithmetic Mod 10 (March 8 - March 18)
The importance of the units digit in regular addition, subtraction and multiplication was noted. A very large table of addition Mod 10 was made so that the students would see the repetition of the 100 digit array of information. The introduction of several new symbols =10, + 10, X,o, and $\frac{\ }{10}$ (mod 10) was taken in stride by most of the students as they made mod 10 tables for both addition and multiplication and carefully studied the resulting patterns. Equivalence classes were extended into the negative numbers and solutions to the congruence $a\,\square \equiv b$ (mod 10) gave meaning to certain fractions with denominators relatively prime to 10. The pattern of successive powers of integers mod 10 was explored in great detail, looking first at rows of 1st, 2nd, 3rd and 4th powers before writing 5th powers

and discovering them to be the same as the 1st power row.

V. Arithmetic Mod 7 (March 19 - March 26)
The concept of arithmetic Mod 7 was suggested by comparison to a calendar. Coincidentally March 1st fell on a Monday this year and gave a convenient and timely array of numbers following into Mod 7 equivalence classes. It was decided that we could arbitrarily call the days of April by numbers 32,33,34... and a student volunteered the suggestion that then Feb. 28, Feb. 27, etc. could be thought of as 0, (-1),(-2,....With the equivalence classes (Mod 7) written in this calendar form on the blackboard the children said it looked just like a seven-fold lattice,only upside down.

The class was able to predict correctly the pattern of the 49 digits of the Mod 7 addition table and for the first time this cylic permutation was verbalized as "all seven digits, always in the right order, but each time starting at a different place." The class quickly and efficiently filled in the Mod 7 multiplication table with no unnecessary computation. The CAP laws were tested on arithmetic Mod 7 as they had been on Mod 10. Similarities and difference between the tables for the two mods have been noted, especially the shuffles (permutations) of digits on every row of mod 7 multiplication table, though no specific explanation has been sought out at this stage. Solutions of linear congruences Mod 7 will be explored, along with the accompanying equivalent fractions.

VI Topics to be Completed. (April)
In the coming weeks Mods 2,3,4,5,6,8,9,11,and 12 will be quickly introduced and compared to Mod 7 and Mod 10. Explanations for casting out 9's, 8's and 11's will be discerned from equivalence classes to these moduli. Finally, the importance of prime moduli will be culminated in a presentation of Fermat's theorem $X^{(p-1)} \equiv 1$ (Mod p) and Wilson's theorem $(p-1)! \equiv -1$ (Mod p).

General Impressions:

The material included in this experiment has without question captivated the natural curiosity of these students. With relatively few exceptions they have kept up an active interest in the continuing exploration and have in fact communicated much of the content to friends who are learning only traditional 6th grade arithmetic. Most involuntarily take notes whenever a new concept is being introduced with its new symbolism and terminology, and all have systematically kept their growing collection of tables. Of particular interest is their lively verbal participation (more noticeable from the boys however) and their keenness and enthusiasm in seeking out patterns in new data.

Their homeroom teacher gives some work (mostly word problems) every day in arithmetic along more traditional lines. He has noticed a considerable difference in both attitude and approach in the students as the year has progressed.

The children all took the Stanford Achievement Tests this month (March 1965). This test of course does not include any of the experimental material these students have been working with this year, so their scores can be considered only as an indication of speed and skill in traditional arithmetic content. The scores of the children in this particular class ranged from a high of 11.1 (first month in eleventh grade) to a low of 3.9 with a median of 7.0. When compared to scores of the other three 6th grade classrooms in the same school, this particular group had lower scores on computation, about the same on application, and higher scores on concepts.

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ELEMENTARY NUMBER THEORY

Experimental Class with 6th Grade Children - 1965

Introduction:

In the winter of 1965 an experimental course in Elementary Number Theory was presented to a 6th grade class in the Hosmer School, Watertown, Massachusetts. The group of twenty-three students (13 boys and 10 girls) was widely heterogeneous with respect to mathematical ability, but all had had only traditional arithmetic in their previous years of schooling. It was a characteristic of this particular "set" of children that the boys were more verbal than the girls, but the performance in written work showed a broad span of interest and ability in both sexes. The work in number theory represents a daily hour of classroom time over a period of fourteen weeks.

The following is a summary of notes taken during the classes on number theory. For the sake of readability, much of the classroom discussion has been condensed into what would seem to be more of a lecture by the teacher than was actually the case. Verbatim dialogue is included only when it shows the development of an idea of particular significance. In this case questions, answers, or comments made by the students are indicated by quotation marks. A few explanatory notes added by the writer are inserted from time to time and are shown in brackets.

I. The CAD Laws of Arithmetic

[It is assumed that by the time a unit in Elementary Number Theory is presented to a group of students, they would already be familiar with the laws of arithmetic; at least in an informal way. The laws were restated in this lesson both as a review and so that they could be referred to later on as criteria for developing modular arithmetic. They might be explored in greater detail with some classes or omitted entirely with others, depending upon previous exposure.]

Suppose for some reason you suddenly needed to know the product 7 x 8 and, for the life of you, you could not remember what it was. We all have spent a lot of time memorizing the multiplication tables, but there are moments when for no apparent reason we forget bits of information that usually we know very well.

What is a long sure way we could go about finding the answer to 7 x 8 (which for the moment we have forgotten)?

"You could add 7 eight times."

"Or add 8 seven times."

Would the answer be the same in both cases? Can we always depend on this, even if we use larger messy numbers like 6358 x 719, or fractions like 15 3/4 x 26 1/2 or negative numbers like (-17) x (-53)? Of course we know a short way of finding the product of 6358 x 719 without too much work but we could also take the very long route and add 719 6358 times or 6358 719 times. Notice that even in using the short way of finding the product of 6358 x 719 we still must know the product of 7 x 8 and this is just exactly the thing which we can't remember today.

Often we can find out something we've forgotten by starting with something we're sure we remember. Now everybody should know what 5 x 8 is 7 x 8 is two more 8's than 5 x 8 and we certainly all know what 2 x 8 is. So if we put 40 and 16 together by adding we get 56 which is the number we were trying to remember.

9 x 12 is another one of the things that people often forget. What is a quick way to think this one through?

10 x 12 is easy, but that's one 12 too many so we have to subtract 12 from 120 to get the answer we really wanted.

We can think of many specific problems, but it saves us a lot of time and energy if we can write some general statements about these ideas.

We all know that 3 + 2 and 2 + 3 both equal 5, but even more important is that there is a true principle here no matter what two numbers we want to add together. Again, 3 x 2 and 2 x 3 both have the same answer and it is always true that the product of any two numbers is the same no matter which of the two numbers we happen to write first. In symbols we can quickly say all this by writing

$$\square + \triangle = \triangle + \square, \quad \square \times \triangle = \triangle \times \square$$

In words we can simply say that addition and multiplication are both <u>commutative</u>.

Sometimes we have more than two numbers to add. Does it make any difference which two we group together first? We often check additions of several numbers by adding from bottom to top instead of the other way around.

$$\begin{array}{r} 7 \\ 5 \\ 6 \\ +\ 9 \\ \hline \end{array} \quad \begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} 7 \\ 5 \end{array}\right\} 12 \\ 6 \end{array}\right\} 18 \\ 9 \end{array}\right\} 27 \end{array} \qquad \begin{array}{r} 7 \\ 5 \\ 6 \\ +\ 9 \\ \hline \end{array} \quad \begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} 7 \\ 5 \\ 6 \end{array}\right\} 15 \\ 9 \end{array}\right\} 20 \end{array} \Bigg\} 27$$

The sums of course should come out the same both ways, but because we have grouped the numbers differently we get different partial sums along the way and have a good check that we probably added correctly.

This same kind of thing is also true when we have more than two numbers

to multiply $(3 \times 7) \times 5 = 3 \times (7 \times 5)$

$21 \times 5 = 3 \times 35$ (notice the different grouping)

$105 = 105$

Because these are ideas that are true for all addition and all multiplication in regular arithmetic we once again can write this in symbolic form as

$(\square + \triangle) + \nabla = \square + (\triangle + \nabla)$

$(\square \times \triangle) \times \nabla = \square \times (\triangle \times \nabla)$

In words we can say that both addition and multiplication are <u>associative</u>.

In our original attempt to remember the product of 7×8 we took a route where we combined multiplication and addition in our thinking. We broke up the multiplication problem whose answer we could not remember into two smaller ones and then added the two products together. Remember,

$$7 \times 8 = (5 \times 8) + (2 \times 8) = 40 + 16 = 56$$

Since this works for any two numbers we wish to multiply we can write it symbolically as

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

and in words we say that multiplication is <u>distributive</u> over addition.

Can we also say that addition distributes over multiplication? Is it certain that for any numbers we want to put in the three shapes the following will always be true?

$$\square + (\triangle \times \nabla) = (\square + \triangle) \times (\square + \nabla)$$

Try several numbers. What do you find? Are there any numbers for which this <u>is</u> true?

The short cut we commonly use in multiplying two numbers of more than one digit works because it is merely using the distributive law several times. For in multiplying we actually break up each number into parts which are quick and easy

for us to work with.

```
    6 7  =        60 + 7        35
  x 3 5  =      x 30 + 5       300
  -------                      210
  3 3 5                       1800
2 0 1                         ----
-------                       2345
2 3 4 5
```

Having memorized the products of pairs of one digit numbers enables us to find the product of 67 and 35 by four easy multiplications and some addition.

THE LAWS OF ARITHMETIC

For all number chosen completely at random, the following properties always hold true. Because these properties are true for any numbers we wish to use (whole numbers or fractions, positive numbers, negative numbers and zero) they are called the Laws of Arithmetic.

$\square + \triangle = \triangle + \square$ Commutative law of addition

$\square \times \triangle = \triangle \times \square$ Commutative law of multiplication

$(\square + \triangle) + \nabla = \square + (\triangle + \nabla)$ Associative law of addition

$(\square \times \triangle) \times \nabla = \square \times (\triangle \times \nabla)$ Associative law of multiplication

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

The distributive law of multiplication over addition.

II. Tables for Addition, Subtraction, Multiplication and Division of Integers.

Much of your time in arithmetic in the early grades was spent in memorizing the basic facts (as they are called) of addition and multiplication. You probably remember writing them out in long tables or reciting them aloud.

1 x 1 = 1	1 x 2 = 2	1 x 3 = 3	1 x 4 = 4
2 x 1 = 2	2 x 2 = 4	2 x 3 = 6	2 x 4 = 8
3 x 1 = 3	3 x 2 = 6	3 x 3 = 9	3 x 4 = 12
⋮	⋮	⋮	⋮

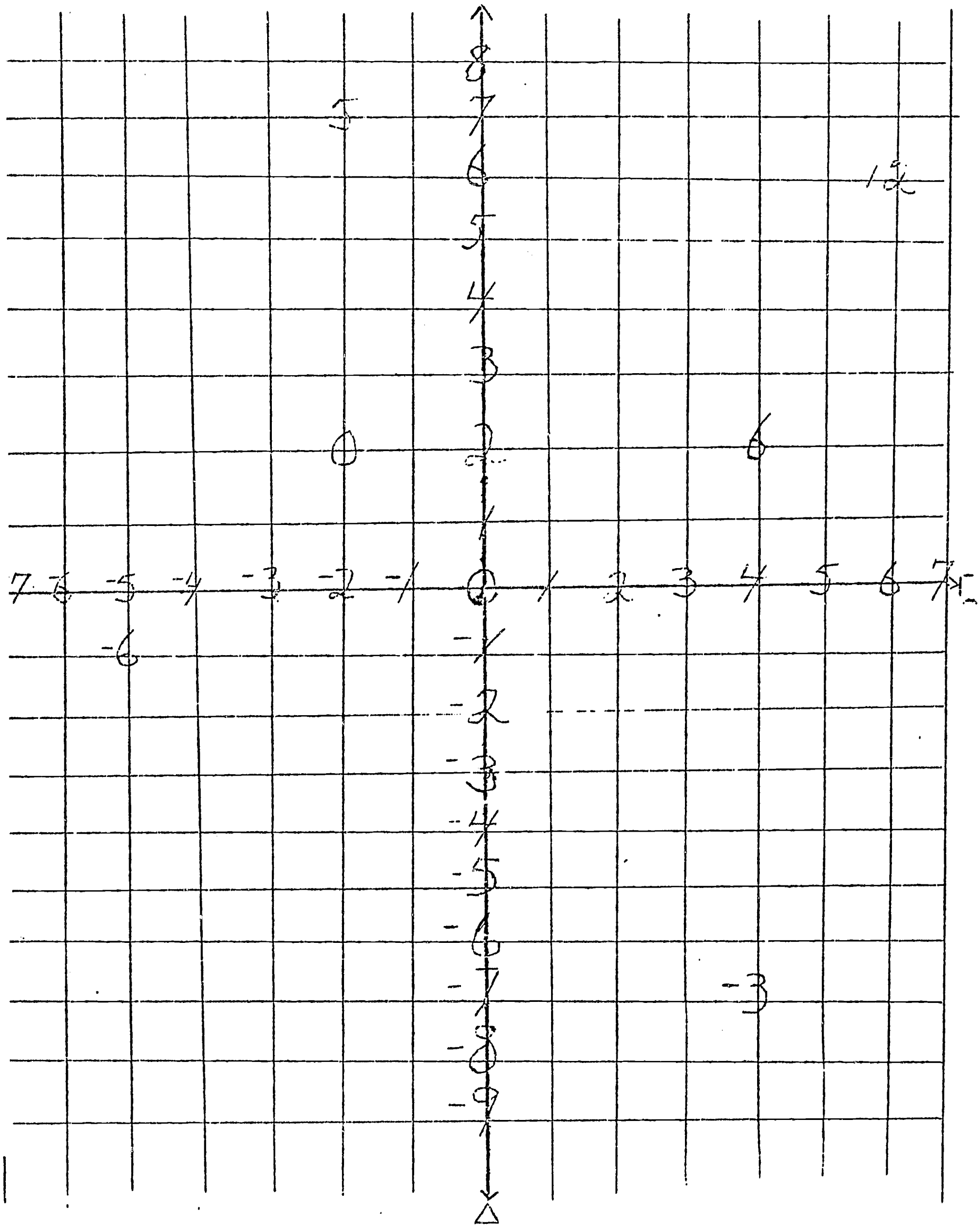
Writing tables this way takes up a lot of space. We could say the seven's table quickly simply by counting by sevens. 7, 14, 21, 28, etc.

Once we have learned these facts by memory we usually don't refer to the entire tables again but there are other important uses for tables of this kind. Sometimes it is necessary to use data that is too complicated for most people to memorize. If all the information can be collected and arranged ahead of time in some orderly way then it can be referred to quickly whenever needed. Another very important useof tables in mathematics is to study the special patterns an array of information makes. In this way we can often learn a great deal without doing a lot of unnecessary work. I think you will see later what I mean about both of these uses of tables as we put together many tables of our own.

Take a number line (Illustration 1) and mark off on it the whole numbers as far as you wish to go. We can draw a second number line, crossing the first one at 0 and perpendicular to it. Where the two lines meet will also be 0 on the vertical number line, and like a thermometer, we shall put the positive integers above and the negative numbers blow the 0. [The tables were begun in this fashion in order to provide a first exposure to the concept of a cartesian plane.]

In order to tell the two number lines apart, we'll mark the one going left and right with a □ and the one going up and down with a △. If we draw

(Addition table on Cartesian Plane) (Illustration 1)



lots of lighter lines parallel to both number lines we have filled the space with lots of little squares and each light line goes through a number on one of the number lines. (See Illustration 1)

Suppose we think now of the mathematical sentence □ + △ = ⬡ . Each time we choose a pair of numbers to put in the □ and △ there is only one number that can be put in the ⬡ to make the sentence true. Using the grid we have just made we can begin to record the values for ⬡ as we change the numbers in □ and △.

For instance 4 + 2 = 6 so where the 4 □ line meets the 2 △ line we will write 6, the sum of 4 and 2. What numbers then shall I record here, and here, and here?

We can of course use both positive and negative numbers in □ and △. What is $\boxed{+4}$ + $\triangle$ -7 ? What is $\boxed{-5}$ + $\triangle$ -1 ?

Each child was then given a sheet of ½" graph paper and drew the two in intersecting number lines in order to make his own table for addition of integers. No short cuts for filling in the sums were mentioned but after a short while most children discovered an efficient method for recording data without further computation. Those few who used haphazard values for □ and △ would have gone a long time before noticing an emerging pattern and in fact these same students made a sufficient number of errors in calculation that the pattern was obscured, especially in the 2nd, 3rd, and 4th quadrants.

Observations of the table □ + △ = ⬡ (Illustration 2)

1. "Wherever you start you just count up or down."

2. "The rows all look like number lines."
   "So do the columns"

3. "The diagonal line from upper left to lower right is all zeros. Then all positive ones above, negative ones below, and so on."

4. "How about the diagonals going the other way? They count by two's. Either all odd numbers or all even numbers on any one of the diagonals going from lower left to upper right."

ADDITION

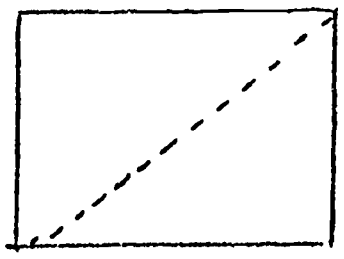
2nd Quadrant — 1st Quadrant

							0							
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
	Diagonal	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
main	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	
-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2

Diagonal

3rd Quadrant — 4th Quadrant

5. Here is another thing that you probably would not notice on your own. What happens if you fold that table in half along this diagonal?



Each number folds on to a number just like itself, so the upper left half and lower right half are exactly the same. This diagonal is called the main diagonal and when numbers fold on themselves in this way we say that they reflect across the main diagonal. This happens because addition is commutative, because

□ + △ = △ + □

6. What happens if we fold the table along the other (secondary) diagonal? Each number folds on to another just like it but of opposite sign.

Homework: Make a similar table for subtraction.

□ − △ = ⬡ Be careful always to think along the number line first and do think twice when using negatives.

[Although the subtraction table begins harmlessly enough, it is full of booby-traps for the unwary student. There was a great tendency among a few students to always subtract the smaller number from the larger one thereby avoiding negative differences in the first quadrant. Handling one or more negative integers for □ and △ resulted in so many errors for these same few children that it was more successful to point out the pattern of the table to enable them to fill it out accurately and quickly.]

[The importance of using two colors of chalk on the board and two colors of ink and/or pencil on the students' tables became quickly apparent with the subtraction table. Most had not noticed or thought unimportant an unusual occurrence along the vertical number line.]

Look for a minute at your addition table. What happens in □ + △ = ⬡ if either □ or △ is zero? □ + 0 = □ 0 + △ = △. So where we would have recorded the sum, the number we needed was already there, on the numberline itself.

Now, in subtraction, if △ is 0 we have □ − 0 = □ and once again the number on the □ number line is the same as the difference we would

-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2
-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Main Diagonal

otherwise record. But what if □ is zero? 0 - △ = -△ and so for the difference we need to write the negative of the number already there. Just putting in a negative sign before each number on this vertical number line will make things very confusing because then the number line looks upside down from what we want it to be. The easiest way out of this difficulty is to use two different colors, one for the values of □ and △ along the number lines and a second color for the differences or values of ⬡ we record on the table. Then along the vertical number line itself we have two numbers, one of each color. These numbers will always look alike (have the same absolute value) but will be opposite in sign.

Observations of the Subtraction Table: □ - △ = ⬡ (Illustration 3)

1. "It looks sort of like the addition table but it's turned around part way." (rotated 90° clockwise)

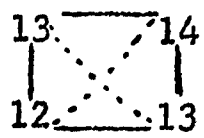
2. "The zeros all go the other way." (are along the main diagonal instead of the secondary one)

3. "Each row is still a number line, but each column is now like an upside down thermometer with the below zero numbers on the top."

4. Will this table fold or reflect along the main diagonal? "Not really." You get the same number but with opposite sign which is the same thing as saying □ - △ = -(△ - □)

With numbers we can quickly see that this is indeed true. 8 - 2 = 6 but 2 - 8 = -6. Subtraction then is not commutative and the order of the numbers cannot be changed without changing their difference except in the trivial case when □ and △ are exactly alike. The addition and subtraction tables as we have written them are concerned only with sums and differences of whole numbers. But we can use them to help us see what happens if □ or △ or both happens to be fractions.

Suppose □ = 7 3/5 and △ = 5 6/7 . Now $7 < 7\ 3/5 < 8$ and $5 < 5\ 6/7 < 6$. We can see quickly by glancing at the table that 7 3/5 + 5 6/7 must be somewhere between 12 and 14. Because the sum must fall somewhere in the square bounded by



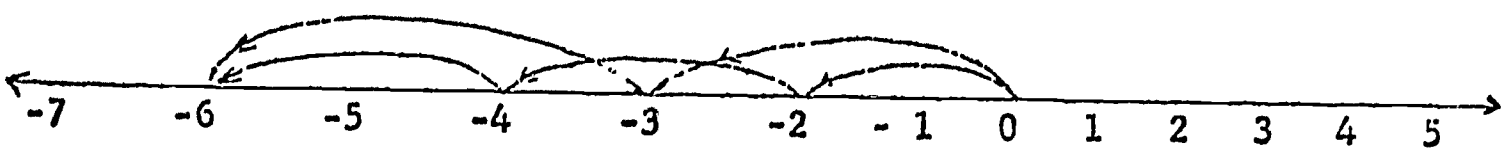
What if we had 7 1/2 + 5 1/2 . Here we can quickly see that the horizontal and vertical lines through these two numbers will meet on the diagonal that connects all the 13's on our table.

We can of course make a table not only for whole numbers but also for some fractions as well, but there is always the problem of deciding which fractions to include and which to leave out.

With a new Sheet of graph paper let's make a table for multiplication of the integers. This means thinking of products □ x △ and recording these just as we did for sums and differences.

[For the benefit of some in the class the products of signed numbers was reviewed at this point although it had been covered previously.]

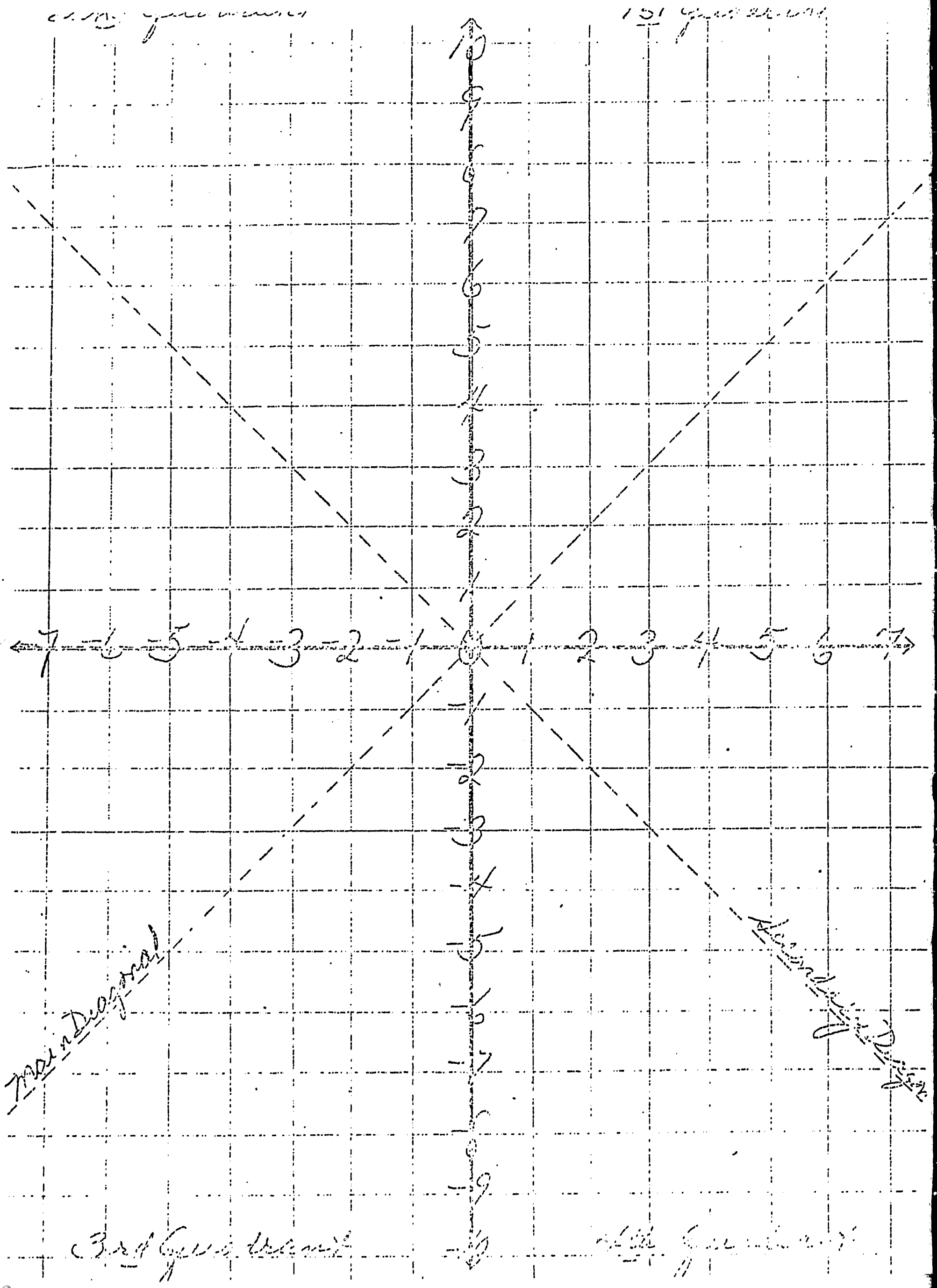
From the number line it is easy to tell that a positive number times a negative number is negative.



3 x (-2) = (-6) and 2 x (-3) = (-6) = (-3) x 2

It is more difficult to see why the product of two negative numbers must be positive. But if we insist that the laws of arithmetic must hold true in all cases then we can show a rather simple proof.

We know for instance that any number times 0 is 0.


Main Diagonal
Secondary Diagonal

$(-7) \times 0 = 0$	But 0 can be written in many different ways.
$(-7) \times (5-5) = 0$	Because (5-5) is another name for 0.
$(-7 \times 5) + ((-7) \times (-5)) = 0$	Because multiplication distributes over addition.
$(-35) + ((-7) \times (-5)) = 0$	Since we already know that (-7 x 5) is (-35) and that there is a zero on the right hand side we know that ((-7) x (-5)) must be the number that, when added to (-35) will give 0.
$(-35) + (+35) = 0$	There is only one such number and it is of course +35.

This then proves that the product of two negative numbers is positive. Now go ahead and fill in the table for □ x △ = ⬡ keeping in mind whether the product is positive or negative. Look for your own short cuts in order to save yourself time and energy.

Since we are especially interested in the general patterns these tables make it is helpful to have names for the various parts. The two crossed number lines, called axes, divide the entire surface into four quarters which are called quadrants, and each of these has a number so we can quickly tell them apart. (Illustration 4) Starting in the upper right hand corner is the first quadrant. Then we move counter-clockwise into the second, third, and fourth quadrants. The main diagonal passes through the 1st and 3rd quadrants and going the opposite way, in the 2nd and 4th quadrants, is the secondary diagonal.

Observations of the Multiplication aTable: (Illustration 5) □ x △ = ⬡

1. "The 1st and 3rd quadrants are all positive."
   "The 2nd and 4th quadrants are all negative."

2. "There are zeros all along both number lines."

3. How about the main diagonal. Do these numbers look familiar?

0	1	4	9	16	25 ...
(0 x 0)	(1 x 1)	(2 x 2)	(3 x 3)	(4 x 4)	(5 x 5)...

Michael: "You add 1 to 0 to get 1. You add 3 to 1 to get 4. You add 5 to 4 to get 9, and this keeps on going like that."

Let's see if Michael is right.

Main diagonal: 0, 1, 4, 9, 16, 25, 36, 49, 64 ...

Differences: 1, 3, 5, 7, 9, 11, 13, 15, ...

This is a very nice pattern and what Michael has noticed is true no matter how far out we go on this diagonal. Last fall when we made up some machines that did things to numbers (operated on them) we had a special name for the machine that took whatever number we put in and multiplied it by itself. We called it a "squarer" and this list of numbers is called the list of perfect squares. You should become so familiar with these numbers that you immediately recognize them as perfect squares.

The same list of numbers is on the lower part of the main diagonal and the same numbers, only negative, on the secondary diagonal.

4. What about the rows and columns on this table? We can call any particular row or column by the number that it passes through on the number line.

"The 1 row (and column) are just the counting numbers.
The 2 row (and column) is counting by 2's.
The 7 row (and column) is counting by 7's.
The 53 row (and column) would be counting by 53's."

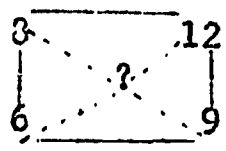
[This came as a surprise to some of the students, indicating a somewhat hazy understanding of what multiplication really is.]

5. Is it always true that $\square \times \triangle = \triangle \times \square$? If so, then this means that we should be able to fold the entire table along the main diagonal and have every number land on another just like it. Because multiplication is commutative the table does reflect across this diagonal.

What about fractions? Suppose we wanted to multiply 2 1/2 x 3 1/2 .

-70	-60	-50	-40	-30	-20	-10	0	10	20	30	40	50	60	70
-63	-54	-45	-36	-27	-18	-9	0	9	18	27	36	45	54	63
-56	-48	-40	-32	-24	-16	-8	0	8	16	24	32	40	48	56
-49	-42	-35	-28	-21	-14	-7	0	7	14	21	28	35	42	49
-42	-36	-30	-24	-18	-12	-6	0	6	12	18	24	30	36	42
-35	-30	-25	-20	-15	-10	-5	0	5	10	15	20	25	30	35
-28	-24	-20	-16	-12	-8	-4	0	4	8	12	16	20	24	28
-21	-18	-15	-12	-9	-6	-3	0	3	6	9	12	15	18	21
-14	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	14
-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7
14	12	10	8	6	4	2	0	-2	-4	-6	-8	-10	-12	-14
21	18	15	12	9	6	3	0	-3	-6	-9	-12	-15	-18	-21
28	24	20	16	12	8	4	0	-4	-8	-12	-16	-20	-24	-28
35	30		20	15	10	5	0	-5	-10	-15	-20		-30	-35
42		30	24	18	12	6	0	-6	-12	-18	-24	-30		-42
49	42	35	28	21	14	7	0	-7	-14	-21	-28	-35	-42	
56	48	40	32	24	16	8	0	-8	-16	-24	-32	-40	-48	-56
63	54	45	36	27	18	9	0	-9	-18	-27	-36	-45	-54	-63
							0							

Looking at the table what limits are there on the product? $2 \times 3 = 6$ and $3 \times 4 = 12$ so $6 < (2\ 1/2 \times 3\ 1/2) < 12$. In fact the lines going through 2 1/2 and 3 1/2 would meet right in the center of the square which has these numbers at the corners.



If we look at the diagonals, the difference between any two numbers along a diagonal is not always the same. Half way between 8 and 9 would be 8 1/2, but half way between 6 and 12 would be 9. Only one number could be at this exact spot so lets figure out exactly what 2 1/2 x 3 1/2 is.

$$2\ 1/2 \times 3\ 1/2 = (2 + 1/2) \times (3 + 1/2) = (2 \times 3) + (2 \times 1/2) + (1/2 \times 3) + (1/2 \times 1/2)$$

$$8\ 3/4 = 6 + 1 + 1\ 1/2 + 1/4$$

8 3/4 happens to be the average between 8 1/2 and 9.

[Much more time could have been spent in interpolation, but since the work in number theory would be concerned almost entirely with integers, fractions were not pursued any further.]

The division table is a little bit trickier than the others so I think it is best we begin it together in class. Once again we make the intersecting number lines and then begin to fill in the quotients when $\square$ is divided by $\triangle$. It is important that we keep in mind which of the two numbers is the divisor. We can say $\square \div \triangle$ or $\frac{\square}{\triangle}$, but in any case we must think of the $\square$ number first and divide it by the $\triangle$ number, no matter which is bigger.

$8 \div 1 = 8$
$8 \div 2 = 4$
$8 \div 3 = 2\ 2/3$
$8 \div 8 = 1$
$8 \div 9 = 8/9$

For the first time we find that not all of the data on this table will be integers. When a number is not whole we shall agree on writing it as a mixed number if possible and you can reduce a fraction to lowest terms if you wish.

$\frac{7}{2}$	$\frac{5}{3}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{4}{7}$	$-\frac{5}{7}$	$-\frac{6}{7}$
$\frac{8}{7}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{4}{7}$	$-\frac{5}{7}$	$-\frac{6}{7}$
	$\frac{6}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{4}{7}$	$-\frac{5}{7}$	$-\frac{6}{7}$
		$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{3}{6}$	$-\frac{4}{6}$	$-\frac{5}{6}$	
			$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$		
$\frac{7}{5}$	$\frac{6}{5}$		1	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$	-1
$\frac{7}{4}$	$\frac{6}{4}$	$\frac{5}{4}$	1	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{2}{4}$	$-\frac{3}{4}$	-1	$-\frac{5}{4}$	$-\frac{6}{4}$
$\frac{7}{3}$	2	$\frac{5}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1	$-\frac{4}{3}$	$-\frac{5}{3}$	-2
$\frac{7}{2}$	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3
7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6

$-\frac{7}{5}$	$-\frac{6}{5}$	$-\frac{5}{5}$	$-\frac{4}{5}$	$-\frac{3}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$
$-\frac{6}{4}$	$-\frac{5}{4}$	$-\frac{4}{4}$	$-\frac{3}{4}$	-1	$-\frac{2}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1	$\frac{4}{4}$	$\frac{5}{4}$
$-\frac{7}{3}$	-2	$-\frac{5}{3}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$-\frac{7}{2}$	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6

$\frac{7}{5}$	$\frac{6}{5}$	$\frac{5}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{5}{5}$	$-\frac{6}{5}$
$\frac{7}{4}$	$\frac{6}{4}$	$\frac{5}{4}$	$\frac{4}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{2}{4}$	$-\frac{3}{4}$	$-\frac{4}{4}$	$-\frac{5}{4}$	$-\frac{6}{4}$
$\frac{7}{3}$	$\frac{6}{3}$	$\frac{5}{3}$	$\frac{4}{3}$	$\frac{3}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{3}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{6}{3}$
$\frac{7}{2}$	$\frac{6}{2}$	$\frac{5}{2}$	$\frac{4}{2}$	$\frac{3}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{2}{2}$	$-\frac{3}{2}$	$-\frac{4}{2}$	$-\frac{5}{2}$	$-\frac{6}{2}$
$\frac{7}{1}$	$\frac{6}{1}$	$\frac{5}{1}$	$\frac{4}{1}$	$\frac{3}{1}$	$\frac{2}{1}$	$\frac{1}{1}$	0	$-\frac{1}{1}$	$-\frac{2}{1}$	$-\frac{3}{1}$	$-\frac{4}{1}$	$-\frac{5}{1}$	$-\frac{6}{1}$
$\frac{7}{0}$	$\frac{6}{0}$	$\frac{5}{0}$	$\frac{4}{0}$	$\frac{3}{0}$	$\frac{2}{0}$	$\frac{1}{0}$	0	$-\frac{1}{0}$	$-\frac{2}{0}$	$-\frac{3}{0}$	$-\frac{4}{0}$	$-\frac{5}{0}$	$-\frac{6}{0}$
$\frac{7}{-1}$	$\frac{6}{-1}$	$\frac{5}{-1}$	$\frac{4}{-1}$	$\frac{3}{-1}$	$\frac{2}{-1}$	$\frac{1}{-1}$	0	$-\frac{1}{-1}$	$-\frac{2}{-1}$	$-\frac{3}{-1}$	$-\frac{4}{-1}$	$-\frac{5}{-1}$	$-\frac{6}{-1}$
$\frac{7}{-2}$	$\frac{6}{-2}$	$\frac{5}{-2}$	$\frac{4}{-2}$	$\frac{3}{-2}$	$\frac{2}{-2}$	$\frac{1}{-2}$	0	$-\frac{1}{-2}$	$-\frac{2}{-2}$	$-\frac{3}{-2}$	$-\frac{4}{-2}$	$-\frac{5}{-2}$	$-\frac{6}{-2}$
$\frac{7}{-3}$	$\frac{6}{-3}$	$\frac{5}{-3}$	$\frac{4}{-3}$	$\frac{3}{-3}$	$\frac{2}{-3}$	$\frac{1}{-3}$	0	$-\frac{1}{-3}$	$-\frac{2}{-3}$	$-\frac{3}{-3}$	$-\frac{4}{-3}$	$-\frac{5}{-3}$	$-\frac{6}{-3}$
$\frac{7}{-4}$	$\frac{6}{-4}$	$\frac{5}{-4}$	$\frac{4}{-4}$	$\frac{3}{-4}$	$\frac{2}{-4}$	$\frac{1}{-4}$	0	$-\frac{1}{-4}$	$-\frac{2}{-4}$	$-\frac{3}{-4}$	$-\frac{4}{-4}$	$-\frac{5}{-4}$	$-\frac{6}{-4}$
$\frac{7}{-5}$	$\frac{6}{-5}$	$\frac{5}{-5}$	$\frac{4}{-5}$	$\frac{3}{-5}$	$\frac{2}{-5}$	$\frac{1}{-5}$	0	$-\frac{1}{-5}$	$-\frac{2}{-5}$	$-\frac{3}{-5}$	$-\frac{4}{-5}$	$-\frac{5}{-5}$	$-\frac{6}{-5}$
$\frac{7}{-6}$	$\frac{6}{-6}$	$\frac{5}{-6}$	$\frac{4}{-6}$	$\frac{3}{-6}$	$\frac{2}{-6}$	$\frac{1}{-6}$	0	$-\frac{1}{-6}$	$-\frac{2}{-6}$	$-\frac{3}{-6}$	$-\frac{4}{-6}$	$-\frac{5}{-6}$	$-\frac{6}{-6}$
$\frac{7}{-7}$	$\frac{6}{-7}$	$\frac{5}{-7}$	$\frac{4}{-7}$	$\frac{3}{-7}$	$\frac{2}{-7}$	$\frac{1}{-7}$	0	$-\frac{1}{-7}$	$-\frac{2}{-7}$	$-\frac{3}{-7}$	$-\frac{4}{-7}$	$-\frac{5}{-7}$	$-\frac{6}{-7}$

Go ahead now and fill in the first quadrant of the table and as usual look for a short cut to save yourself unnecessary work.

[Many of the children soon noticed a pattern which enabled them to fill in the first quadrant with little calculation. They were obviously delighted with their discovery. For a few students division is such a stumbling block that filling in this table was a terrible chore. They were unable to do even the simplest divisions without writing out the traditional algorithm ($2\overline{)8}$) and had a great tendency to always use the smaller number as divisor thus greatly distorting the pattern. Even when the pattern of the table was discussed by the class they were not able to proceed on their own to correct their errors, so that it was necessary to spend an extra hour working with them alone to be sure their division tables were filled in correctly.]

Observations of the Division Table $\square \div \triangle = \bigcirc$ (Illustration 6)

1. "It has lots of fractions on it."

2. "There are ones all along the main diagonal."

3. "The 1 row is just like the numberline."

4. "On the 2 row you count by 1/2's
   " " 3 " " " " 1/3's
   etc.

5. "The 1 column goes 1, 1/2, 1/3, 1/4, 1/5, ...
   " 2 " " 2, 2/2, 2/3, 2/4, 2/5, ...
   etc.

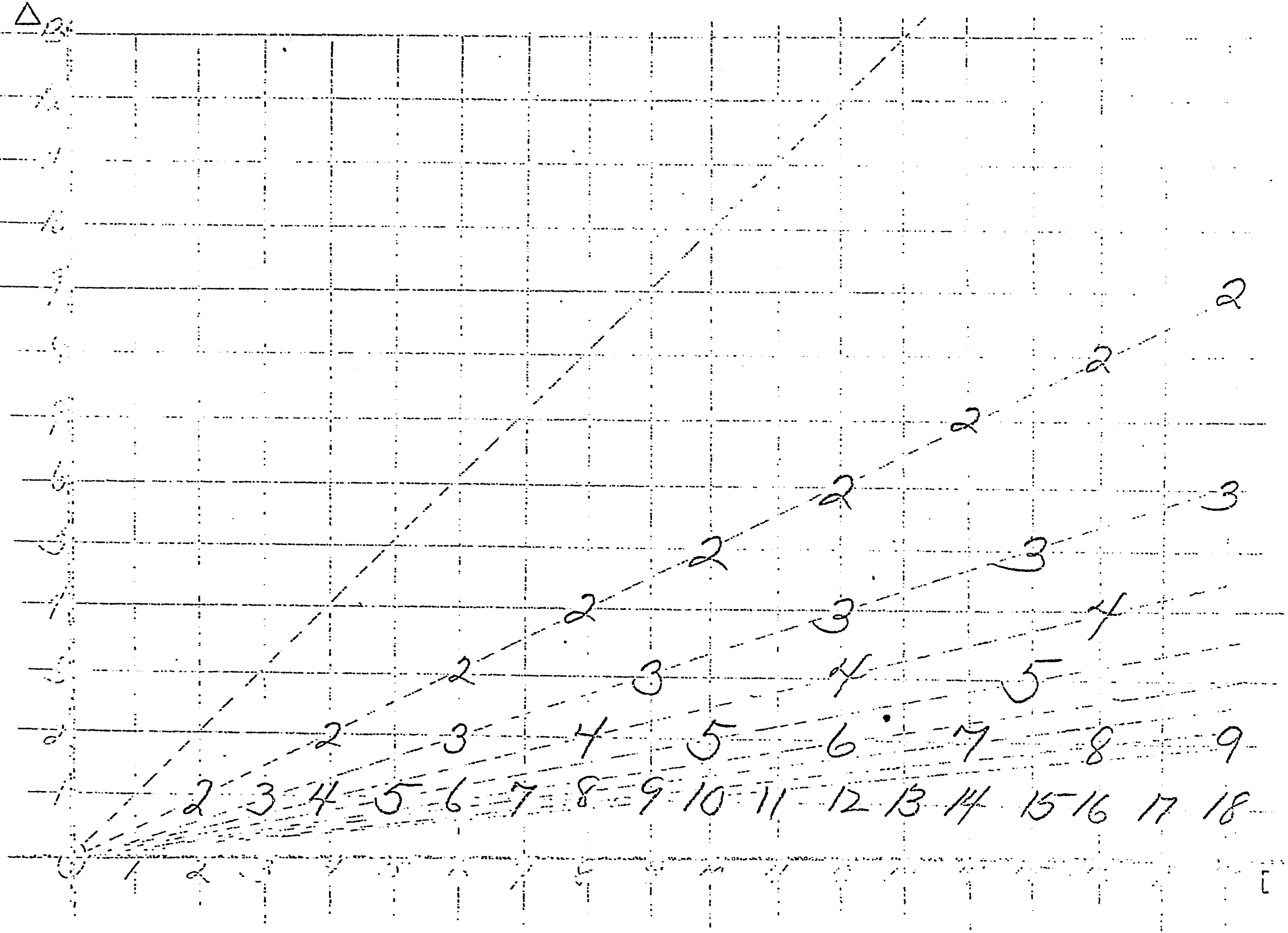
[As the third quadrant was filled in a child noted, "It's just like the 1st one, sort of turned over." Following this, the 2nd and 4th quadrants were completed with the comment that "these would be the same only negative."]

More Observations of the Division Table

1. "There are mostly fractions with a few whole numbers."

2. "The main diagonal is all +1 and the secondary diagonal all -1."

3. Where are all the twos? "In a straight line that slants a little below the ones, but they are further apart." How about the 3's, 4's, 5's, etc.?



Distribution of Integers on division table (1st quadrant)

4. "All the numbers above the line of ones are proper fractions."

5. Where are the zeros? [Most had not put any answers down for $0 \div \triangle$.] The fractions here are getting smaller and smaller. As soon as we cross the vertical number line from the first to the second quadrant we are into negative fractions. Somewhere along the way there must be a zero. "It would be right on the number line because 0 divided by any number is 0. (Use two colors)

[A child started to put 0's also along the horizontal number line.]

Wait, is it true that $\square \div 0 = 0$? When we say that $8 \div 2 = 4$ then it must also be true that $4 \times 2 = 8$, because division and multiplication are closely interrelated. Then if we say that $\square \div 0 = 0$ (let's use 8 in the $\square$) $8 \div 0 = 0$ then it must also be true that $0 \times 0 = 8$. Now other rules we have tell us that $0 \times 0 = 0$ and so we know then that $8 \div 0$ cannot be 0. What about $0 \div 0 = 0$? This seems to work because it is true that $0 \times 0 = 0$. But this problem breaks another law we have which says that any number divided by itself is always 1. $\square \div \square = 1$. So if I obey this rule $\square$ must equal 1 not 0. Division by zero leads us to so many contradictory ideas that we just avoid the whole thing by saying that division by zero is not allowed. So on our table we won't put any answers along the $\square$ number line; in fact I have marked mine with a wiggly line just to try to show that there are no answers there.

Michael: "What if we wrote the answers to division in decimals instead of regular fractions"? That would be all right, although it might be harder to see the patterns because some of the decimals (like that for 1/3) don't come out nicely so it would get rather messy.

More than a week has been devoted to making the tables and discussing their patterns. Tomorrow there will be a test. You may use all of the tables to help you with the test in any way you like so be sure to have them with you.

[There was an attempt in this first test at discussion questions which might draw verbal generalizations from all of the children about material that should

have been long familiar to them in bits and pieces. Answers to this type of question ranged from thorough descriptions to complete blanks. The tables spread out before them did not prove to be much help to those already weak in computational skills or still insecure with manipulation of negative numbers. Perhaps with a much slower pace and more classroom practice in using tables, these few children could have developed more faith than fear.]

TEST

Name_____

I. In the following problems use the word positive or negative in the blanks to make these sentences true.

(1) A negative number plus a negative number is a _____number.

(2) A negative number times a negative number is a _____number.

(3) (-11) - (7) is a _____ _____number.

(4) (-11) - (-7) is a _____number.

(5) A negative number divided by a positive number is a _____number.

(6) (-10) subtracted from (-3) is a _____number.

(7) A negative number divided by a negative number is a _____number.

(8) (-107) times (+33) is a _____number.

(9) (-5672½) divided by (+17 1/3) is a _____number.

(10) 13,576 - 31,576 is a _____number.

II. Write true or false beside each of these sentences.

(1) 0 x (-7) = 0 _____

(2) (-5) - (-5) = (-10)_____

(3) 56½ x (-1) = 56½_____

(4) 8 x (8-8) = 0 _____

(5) 0 ÷ 0 = 1 _____

(6) 0 ÷ 9 = 0 _____

(7) 9 ÷ 0 = 0 _____

(8) 0 - (-14) = 14 _____

(9) (-4) ÷ ((-8) ÷ 4) = 0_____

(10) (-8) - ((-4) ÷ 4) = 0 _____

III. Find the numbers to put in the boxes which will make these sentences true.

(1) 7 x 9 = □

(2) □ x 4 = -24

(3) □ ÷ 3 = 2 1/3

(4) 54 ÷ □ = 9

(5) 8 1/3 ÷ 5 2/5 = □

(6) (-4) - □ = -11

(7) □ ÷ (-5_ = (-9)

(8) 2/3 x 1/2 = □

(9) 4 1/2 x 7 = □

(10) 9 1/2 ÷ 4 = □

IV. Here are some problems in old fashioned arithmetic. Use your tables to help you or to check your work.

(1) 7536
x89

(2) 27/4928634

(3) 6 3/4 x 9 1/3 =

(4) 5678599
- 9794

(5) 67
103
58
75
449
+ 72

(6) 75/15

(7) 7 3/8 - 4 1/4 =

(8) 857.5 ÷ 1366.49 =

(9) 17.375 - 14.25

(10) 10 1/2 ÷ 1/4 =

V.

(1) Suppose you were trying to describe the addition table (□ x △) to someone without actually showing him the table itself. What things would you say about it to help him picture it in his mind?

(2) Now do the same thing by describing the multiplication table (□ x △).

III. Factoring: Remainders of 0 (factors).

The tables we have made for addition, subtraction, multiplication and division have included both positive and negative numbers. Although they are concerned mostly with whole numbers (integers) we have seen that they could be useful in working with fractions as well. For a while we are going to concentrate on positive whole numbers only (sometimes called the counting or natural numbers) and so we shall be thinking mainly of the first quadrant in these various tables.

Look for instance at the 12 row and the 12 column on the division table.

0/12, 1/12, 2/12, 3/12, 4/12, 5/12, 6/12, 7/12, 8/12, 9/12, 10/12, 11/12, 12/12, 13/12, . . .

0, 1/12, 1/6, 1/4, 1/3, 5/12, 1/2, 7/12, 2/3, 3/4, 5/6, 11/12, 1, 1 1/12, . . .

"The denominator is always 12, but the numerator get one bigger every time."

Are you sure this will keep on this way, no matter how large we make the table?

What number will be in the numerator if we go all the way out to where $\square$ = 113?

"Some of the fractions can be reduced and some are really whole numbers, like 12/12 ."

Look at the 12 column, starting from the bottom.

12/1, 12/2, 12/3, 12/4, 12/5, 12/6, 12/7, 12/8, 12/9, 12/10, 12/11, 12/12, 12/13,...

12, 6, 4, 3, 2 2/5, 2, 1 5/7, 1 1/2, 1 1/3, 1 1/5, 1 1/11, 1, 12/13, ...

"This time the numerator is always 12, but the denominators get one bigger the higher up we go."

As the denominators keep getting larger and the numerators always stay 12, what happens to the numbers themselves?

"They are getting smaller and smaller. After you pass 12/12 or 1 you won't ever get any more whole numbers because from then on the denominator will always be larger than the numerator."

Suppose that we had a table which was long enough to go out to the 24 column. Then we are talking about 24 ÷ △ = ⬡ and this time, I shall write the remainder as a whole number instead of as a fraction.

24 ÷ 1 = 24 r 0	24 ÷ 9 = 2 r 6	24 ÷ 17 = 1 r 7
24 ÷ 2 = 12 r 0	24 ÷ 10 = 2 r 4	24 ÷ 18 = 1 r 6
24 ÷ 3 = 8 r 0	24 ÷ 11 = 2 r 2	24 ÷ 19 = 1 r 5
24 ÷ 4 = 6 r 0	24 ÷ 12 = 2 r 0	24 ÷ 20 = 1 r 4
24 ÷ 5 = 4 r 4	24 ÷ 13 = 1 r 11	24 ÷ 21 = 1 r 3
24 ÷ 6 = 4 r 0	24 ÷ 14 = 1 r 10	24 ÷ 22 = 1 r 2
24 ÷ 7 = 3 r 3	24 ÷ 15 = 1 r 9	24 ÷ 23 = 1 r 1
24 ÷ 8 = 3 r 0	24 ÷ 16 = 1 r 8	24 ÷ 24 = 1 r 0

[The pattern of this particular table should evoke a variety of comments from the students.]

What whole numbers can I put in this sentence to make it true? □ x △ = 24

"Any of the pairs in the list above where the division came out even (had a zero remainder)"

How about □ x △ = 96?

96 x 1 = 96
48 x 2 = 96
24 x 4 = 96
12 x 8 = 96
6 x16 = 96
3 x32 = 96

[After a few pairs of factors were suggested at random, Michael said "You can just divid by 2 and multiply by 2."]

How about □ x △ = 162 and □ x △ = 185?

Homework: Find all the whole numbers that will work for □ x △ = ⬡ when ⬡ is 45, 46, 47, 48, 49, 50.

The numbers from 45 through 50 are all nearly the same size but when we look for whole numbers which will make □ x △ = ⬡ true we find out that 48 has several pairs of numbers which will work, 47 has only one pair, and so forth.

FACTORS

45 = 45 x 1 = 15 x 3 = 5 x 9

46 = 46 x 1 = 23 x 2

47 = 47 x 1

48 = 48 x 1 = 24 x 2= 16 x 3 = 12 x 4 = 8 x 6

49 = 49 x 1 = 7 x 7

50 = 50 x 1 = 25 x 2 = 5 x 10

The size of the number does not seem to be of much importance here. The whole numbers which will divide exactly (leaving 0 for a remainder) into a number are called the factors of that number. Every number seems to have at least two factors, itself and 1, and we found that 47 had only these two while 48 just a little bit larger had many other factors as well.

Let's go back to these same numbers 45 - 50 and on a piece of paper I have written out the possible pairs of factors for each. Sometimes a factor itself can be factored into two even smaller numbers. Notice that I said smaller numbers and so this rules out the trivial pair (□, 1). If we keep on doing this as far as we can go, what do you notice?

"We always end up with the same small numbers to multiply together, no matter which pair of factors we start with."

All of these numbers are factors of our original number and we can make some generalizations.

1. Every number has at least two factors, itself and 1.

2. Some numbers have other factors as well, which are called proper factors (excluding 1 and the number itself).

3. For every number there is a unique list of smallest possible numbers (not counting 1) which can be multiplied together to give that num , but for a few numbers there is only one number in this list (the number itself).

[Primeness was not elaborated on or called by name at this point because it would be developed later on. For the same reason prime factors were called instead "smallest possible factors".]

Let's take another list of numbers that are all near each other on the number line and look for their proper factors. [Students did these at the board.]

120	121	122	123	124	125
5, 24	11	2	3	2	25
6, 20		61	41	62	5
2, 60				4	
4, 30				31	
10, 12					
3, 40					
8, 15					

[Michael who had been working with 124, went on to find its prime factors.]

$$124 = 2 \times 62 = 2 \times (2 \times 31)$$
$$= 4 \times 31 = (2 \times 2) \times 31$$

[The class observed the unexpected occurrence of 11, 61, 41, and 31 among the proper factors, listed above, so these numbers and others of similar nature were explored still further.]

Which of these numbers has proper factors?

11 = 11	41 = 41	71 = 71
21 = 3 x 7	51 = 3 x 17	81 = 9 x 9 = 3 x 27
31 = 31	61 = 61	91 = 7 x 13

[The factors for 51 and 91 were of course not readily apparent and unfortunately the curious appearance of 3, 7, 13, and 17 were not pursued.]

Homework: Find the list of smallest possible factors (prime factors) of:

408 480 804 840

IV. Divisibility - Primes

How can we quickly tell if a number is exactly divisible by 2?

"If it ends in 0, 2, 4, 6, or 8."

Then looking only at the units digit tells us immediately what the remainder is when we divided by 2. If the number is even the remainder is 0. If it is odd the remainder is 1.

2 is an even number and, as we have seen before, we can think of E for even, O for odd, and write the following table.

E x E = E E x O = E O x E = E O x O = O	So if the number we are considering is even, it must be the product of 2 and some other number, which could be either odd or even.

Does anyone know a quick way to tell if a number can be divided exactly by 3?

[The class had already been shown the test of adding digits by a previous teacher, although they had no explanation for this procedure at this time.]

"You have to add up all the numbers (digits). If the answer (sum) can be divided by 3, so can the original number."

What about 4?

"If you divide by 2 and the number is still even, it can be divided by 4."

[Pythagoras called this "evenly even".] Are there any other ways that might be even faster For instance, if we are going to bother to divide a very large number (one of many digits) by 2, it might be just about as fast to divide by 4 in the first place. Is 32 divisible by 4? How about 232, 732, 55532, etc.? Is 86 divisible by 4? 386, or 2286, or 100,000,086? Then a quick test can be to look only at the first two digits on the right. If this small number is divisible by 4, then so will be the entire number regardless of what all the other digits are. Can anyone explain why this is so? Is 100 divisible by 4? Is 700? Is 331500?

Mark: "Any number which ends in hundreds can be divided by 4."

So then all we have to bother about is whether the tens and units digits part of the number is divisible by 4.

How about telling if a number can be divided by 5?

"If it ends in 0 or 5."

That's an easy one, of course, and suggests the test for divisibility by 10.

Let's make a chart of all the information we have so far.

Divisibility by		Test
1		Any number.
2		If the units digit is 0, 2, 4, 6, 8.
3		If the sum of the digits is a multiple of 3.
4		If the first 2 digits on the right form a number divisible by 4.
5		If the units digit is 0 or 5.
6	Filled in later	If it passes the test for 2 and for 3.
7		No easy test.
8		If the first three digits is a multiple of 8.
9		If the sum of the digits is divisible by 9.
10		If the units digit is 0.

We still have 6, 7, 8, and 9 to do

"You can cast out 9's for the 9 one."

"Can you cast out 8's too?"

We can try. 88 certainly works, but 96 does not. So I guess we have to find some other way.

"You can divide by 4 and then if its still even, it could be divided by 8."

But this is nearly as much work again as dividing by 8 from the start. This idea might give us a hint though for a shorter way. Look at the test for 4.

"Do you mean if the first two digits can be divided by 8, it works?"

Let's try. How about 132? "No." 264? "No."

Michael: "If the first <u>three</u> digits can be divided by 8 it works, because thousands are always divisible by 8."

How about 6? What other numbers must it also be divisible by?

"2 and 3 - divide by 2 first and then see if you can still divide by 3."

Do we have to bother to actually divide? Can't we just see if the tests for 2 and for 3 both work? I think we had better leave 7 blank for now. There are some tests but they are complicated to remember and nearly as much work as dividing by 7 at the outset.

Let's look at some large numbers and practice using these divisibility tests.

24730596
1047240

[A 10 x 10 array of numbers (with 1 omitted) was passed out to each student.] (Illustration 8)

More than two thousand years ago a man named Eratosthenes did a great deal of thinking about divisibility and came up with a clever scheme. You all know what a sieve or strainer is. Eratosthenes thought of an imaginary sieve which would be of many levels, with holes spaced evenly apart. He thought of all the counting numbers stretched out in one long continuous row and dropping through each level of the sieve. The first level would have holes at every other number holding back all even numbers, but letting the odd numbers fall through. The next level of the sieve would hold back all the numbers exactly divisible by 3, the next level by 5, (not 4 because the even numbers had already been caught by the first level) and so on. At each screening a few more numbers would be held back.

Although the idea is for all whole numbers, we shall try it only on the first 100 of them. On the sheet I gave you start with the first number 2 and then cross out all other numbers that can be divided exactly by 2.

"Why isn't there any 1?"

What would happen if we crossed out all numbers that can be exactly divided by 1?

	2.	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	35	39	40
41	42	43	44	45	46	47	48	49	50
57	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

"All the numbers would be crossed out."

[Most children immediately saw the short cut for crossing out all even numbered columns.] What is the smallest number now that is not crossed out? "3". Then this time cross out everything that is left which can be divided by 3. How about 4?

"The even numbers are already crossed out."

So the next number left is 5, and so on. Finish it up at home tonight.

What numbers have you left on your chart that did not get crossed out? These are the same numbers that would be the first in each level of Eratosthenes' imaginary sieve. These numbers cannot be divided exactly by any other numbers except themselves and 1. We might also say that they cannot be written as the product of <u>two smaller</u> numbers.

There is a special name for numbers of this kind - prime.

"2 is the only even one."

"Except for 2 and 5 they all have 1, 3, 7, or 9 in the units digit."

This is one reason for arranging the chart this way. Will they continue to be so if we go on to numbers bigger than 100?

If we write them out another way, just in a long row, we might notice something else.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,
37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79,
83, 89, 97, . . .

Several of them are only two numbers apart. These are called prime twins. It looks from this chart that maybe there are no more pairs of prime twins after 73, but if you made the sieve longer you will see that there are many more. Another very famous ancient Greek named Euclid (who lived about 300 B.C.) showed a very elegant and simple proof that prime numbers go on forever. It is also believed that although they do become sparser, the prime twins are also infinite in number. No one has ever proved that this is so, however.

The study of prime numbers has been a source of fascination for thousands of years and there are many unsolved problems concerning them. For many centuries it was thought that any number of the form $(2^p - 1)$ was prime where 2 was raised to a prime power p and 1 is then subtracted from the answer. For instance:

$p = 2$	$2^2 - 1 = 4 - 1 = 3$	
$p = 3$	$2^3 - 1 = 8 - 1 = 7$	(3, 7, 31 and 127
$p = 5$	$2^5 - 1 = 32 - 1 = 31$	are all prime)
$p = 7$	$2^7 - 1 = 128 - 1 = 127$	

But in 1903 a professor at Columbia University in New York found that $(2^{67}-1)$ is not prime but is the product of two very large primes. $(2^{67}-1) = (193707721 \times 761838257287)$. If we multiply two 1 digit numbers we usually get a 2 digit number. If we multiply two 2 digit numbers we get a number of 3 or 4 digits. With two 3 digit numbers the product has 5 or 6 digits and so on. So you can imagine how very large the product will be when a 9 digit and 12 digit number are multiplied.

[Very large numbers seem to intrigue students of this age. Even those who have difficulty with multiplication of 2 and 3 digit numbers were somehow entranced by the very thought of numbers of a far greater magnitude. Once presented with the idea of $2^{67}-1)$ many set about to compute this by repeated powers of 2 or by multiplying its two factors.]

This enormous calculation is obviously not prescribed as homework, but the following is.

Find the prime factors of the following numbers:

(369), (396), (693), (639), (963), (936)

We have already seen two different arrangements of the infinite set of counting numbers and the prime numbers that they contain. In one, the numbers were placed in an endless chain like a number line; in the other in an infinite

PRIME NUMBER PATTERN

(Discovered accidentally in 1963 by Stanislaw-Ulam, Los Alamos Scientific Laboratory)

(Color lightly the squares that contain a prime number)

100	99	98	97	96	95	94	93	92	91
65	64	63	62	61	60	59	58	57	90
66	37	36	35	34	33	32	31	56	89
67	38	17	16	15	14	13	30	55	88
68	39	18	5	4	3	12	29	54	87
69	40	19	6	1	2	11	28	53	86
70	41	20	7	8	9	10	27	52	85
71	42	21	22	23	24	25	26	51	84
72	43	44	45	46	47	48	49	50	83
73	74	75	76	77	78	79	80	91	82

number of rows with 10 numbers in each row like a ten-fold lattice. There are many other possible arrangements or patterns for picturing the natural numbers. Here are two more.

The first of these (Illustration9) places all of the counting numbers (1 is included this time) in a continuous spiral. The heavy line marks the path of the spiral as it unwinds from the center. The numbers shown here from 1 to 100 form a 10 x 10 square which has quite a different pattern than the first 10 x 10 arrangement we looked at.

For a moment, looking at the lattice array, instead of considering prime numbers, think even - odd. If we shaded in all the even numbers what kind of pattern would we have?

"Stripes. All the columns starting with 2, 4, 6, 8, 10 are even. The others (1, 3, 5, 7, 9) are odd."

Suppose on this spiral array we shaded in all the even numbers. What would the pattern be now?

"It will look like a checkerboard."

What about the perfect squares on this arrangement? They have a very specia pattern along a diagonal which makes a job at the center.

Here is an interesting homework problem for those of you who want to give it some real thought. Suppose we started out to add a long list of consecutive odd numbers, beginning with 1. (1 + 3 + 5 + 7 + 9 + 11 + ...) and we kept on going until we had 50 numbers to add. Can you find a short cut for quickly giving the sum of the first 50 odd numbers?

[Several of the children had attempted the sum by actually adding the 50 numbers. All of these had made one or more errors in their calculations so that the results hovered around the exact sum. Michael was the only one who had the

correct solution and he explained his method.]

"I tried a few and found a way that worked. 50 divided by 10 is 5. 5 x 5 = 25 and then I multiplied this by 100 to get 2500."

Suppose then I asked for the sum of the first 60 consecutive odd numbers.

"60 divided by 10 is 6. 6 x 6 = 36, times 100 equals 3600."

Then how about the first 78 consecutive odd numbers?

"Then my method won't work because 78 cannot be divided (exactly) by 10."

Michael, your method is on the right track. But you put in an unnecessary complication that caused you trouble in this last question I asked.

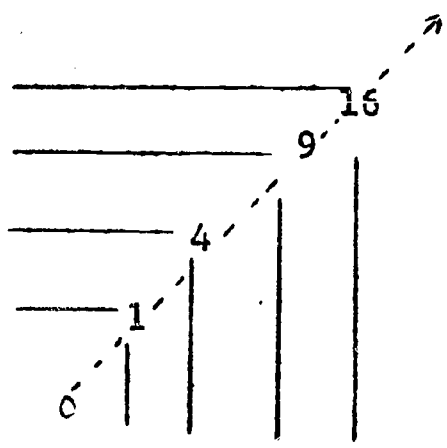
First you divided the number by 10, then you squared, and finally you multiplied by 100. $\left(\frac{50}{10}\right)^2 \times 100 = 5^2 \times 100 = 25 \times 100 = 2500$

But suppose we leave $\left(\frac{50}{10}\right)$ as the number to be squared.

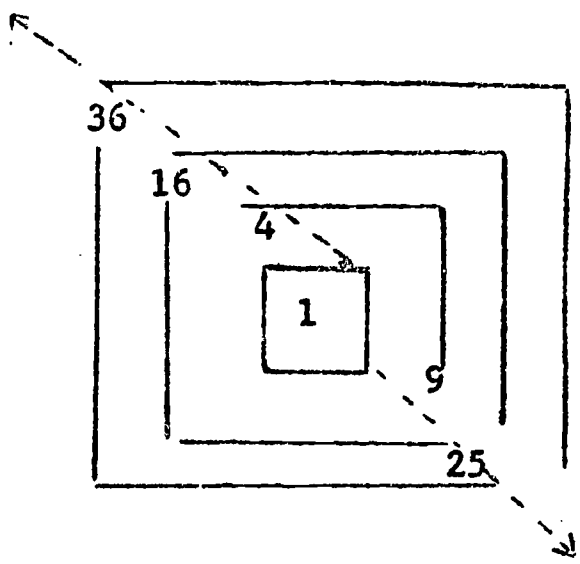
$$\left(\frac{50}{10}\right)^2 \times 100 = \left(\frac{50}{10} \times \frac{50}{10}\right) \times 100 = \left(\frac{2500}{100}\right) \times 100 = 2500$$

So you really end up dividing by 100 and multiplying by 100 which is the same thing as not doing either of these things in the first place. You could just have squared 50 right at the start.

We can explain this idea by looking at this spiral pattern of natural numbers or by thinking of where we first came across the list of perfect squares.



Some time ago, when we made the multiplication table for integers, we found the perfect squares along the main diagonal. Each was the upper-right hand corner of a square of numbers that kept getting larger.



In this spiral pattern the perfect squares again seem to line up along a diagonal, all the even squares above and all the odd squares below the center. Every time we wrap enough numbers around the unit square in the center to make a new square we have reached the next higher perfect square number. So now our original square continues to grow, but this time it grows by more blocks being wrapped around it alternately on two sides.

In both of these arrangements we start with one small block. Then we add three more, five more, seven more, etc. As each consecutive odd number of blocks is added we keep getting a square. So here are two different geometrical explanations of the fact that the sum of consecutive odd numbers is a perfect square.

$$1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$$

1st 2nd 3rd 4th 5th nth

One day last year a mathematician named Stanislaw Ulam from Los Alamos Scientific Laboratory was doodling during a boring meeting. He made this spiral array of 100 numbers and then began to cross out all the prime numbers. Since 1 is in his spiral he included it as a prime. Usually it is not listed with the prime numbers, but it is really not "non-prime" either. Mr. Ulam noticed that the prime numbers seemed to come in groups along various diagonals. Other people have continued the spiral to thousands of numbers instead of only the first 100 and these diagonal groups of primes continue throughout the pattern, although there are fewer and fewer primes as you get further away from the center of the spiral.

The Sieve of Eratosthenes

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90.
91	92	93	94	95	96
97	98	99	100	101	102
103	104	105	106	107	108
109	110	111	112	113	114
115	116	117	118	119	120
121	122	123	124	125	126
127	128	129	130	131	132
133	134	135	136	137	138
139	140	141	142	143	144
145	146	147	148	149	150
151	152	153	154	155	156
157	158	159	160	161	162

Another interesting arrangement of the natural numbers is like a six-fold lattice. (Illustration 10). If we cross out all numbers divisible by 2 on this pattern, the 2nd, 4th, and 6th columns all drop out leaving 2 as the only even prime.

Where are all the numbers which have 3 as a factor?

"In the 3 column and the 6 column (already crossed out)."

4 of course is even and all multiples of 4 are already crossed out so we now mark out all multiples of 5. Where do all these numbers come?

"In the 5 column."

Look again.

"No, they are on diagonal lines."

How about multiples of 7?

"On diagonals going the other way."

[In retrospect it would have been interesting to look at multiples of 9 and 11 on the ten-fold array to see that these diagonals are always found in one less and one more than the number of columns in the array.]

The multiples of 11 come on even steeper diagonals but all except 121 and 143 have already been crossed out.

If our picture continues into numbers greater than 162 there would be more numbers to cross out. (Multiples of 13, 17, 19, 23,,...). Suppose you shaded in the numbers which are never crossed out. Where will they be in this arrangement?

"Except for 2 and 3 they will all be in the 7 column or in the 5 column."

This means then that all prime numbers except 2 & 3 are one more or one less than a multiple of 6. We could say that all primes (except 2 and 3) can be written as $(6 \times n + 1)$ or $(6 \times n - 1)$ when n is a counting number. Not all of these numbers written to these formulas are prime. For instance if $n = 4$.

$(6 \times n + 1) = (6 \times 4) + 1 = 25$ which of course is 5×5. But all prime

numbers greater than 3 can be written by one or the other of these simple formulas.

Sometimes we want to talk about two numbers which have no common factors. (Other than 1 of course as all numbers can be divided exactly by 1) When this is true the numbers are said to be relatively prime to each other.

$9 = 3 \times 3$ $8 = 2 \times 2 \times 2$	Neither 9 nor 8 are prime numbers but they are relatively prime to each other.
$21 = 3 \times 7$ $25 = 5 \times 5$	21 and 25 are relatively prime because they have no common factor
$21 = 3 \times 7$ $24 = 3 \times 8 = 3 \times 2 \times 2 \times 2$	21 and 24 are not relatively prime because they both have 3 as a factor.

If a pair of numbers are both prime they must also be relatively prime to each other. If the first number is prime the pair is relatively prime unless the second number is a multiple of the first.

Test tomorrow!

[This test took the children much longer than was expected, but the results were otherwise encouraging. In discussing the test the next day most children admitted to either forgetting or not using short cut methods for divisibility.

John said, "I didn't forget them, but I was afraid to trust them."

It is apparent that this boy and many others developed considerable trust as the work on number theory progressed. Many of the students of this age and wide ability-span either do not read written directions or cannot distinquish the subtle distinctions between factors, proper factors, and prime factors merely by means of verbal definitions.]

TEST

Name_____

All of the questions on this test are concerned only with <u>positive whole numbers</u>. (natural, or counting numbers.)

I. In the following sentences show all possible pairs of numbers that can be put in the □ and the △ to make the sentences true. (the first problem is done as a sample. Because the pairs of numbers (1, 35) and (7, 5) are given, you do not have to show the same numbers in the opposite order (35, 1) and (5, 7).

1. □ x △ = 35
   1 x 35 = 35
   7 x 5 = 35

2. □ x △ = 36

3. □ x △ = 37

4. □ x △ = 38

5. □ x △ = 39

6. □ x △ = 40

II. In the following division problems write the answers in the form shown in the example. All remainders will be either 0 or a counting number.

1.	17 ÷ 5	=	3	with a remainder of	2
2.	49 ÷ 5	=	_____	" " " "	_____
3.	49 ÷ 7	=	_____	" " " "	_____
4.	49 ÷ 11	=	_____	" " " "	_____
5.	99 ÷ 11	=	_____	" " " "	_____
6.	999 ÷ 11	=	_____	" " " "	_____
7.	777 ÷ 7	=	_____	" " " "	_____
8.	777 ÷ 7	=	_____	" " " "	_____
9.	888 ÷ 7	=	_____	" " " "	_____
10.	1,476 ÷ 2	=	_____	" " " "	_____
11.	1,476 ÷ 9	=	_____	" " " "	_____
12.	1,476 ÷ 25	=	_____	" " " "	_____
13.	6,741 ÷ 25	=	_____	" " " "	_____
14.	6,741 ÷ 1,470	=	_____	" " " "	_____
15.	6,741 ÷ 6,742	=	_____	" " " "	_____

III. Keep in mind the distinction between these three definitions:

(a) If a first number is divided by a second number and the remainder is zero, we say that the second number is a <u>factor</u> of the first.

(b) All of the factors of a number, except itself and 1, are called <u>proper factors</u>

(c) A factor is called a <u>prime factor</u> only if it is a prime number.

1. List <u>all</u> the <u>prime factors</u> of the following numbers, showing the original number as the product of all its prime factors. (for example 12 = (2 x 2 x 3))

54 = ( )

19 = ( )

222 = ( )

264 = ( )

169 = ( )

2. List <u>all</u> the <u>factors</u> of the following numbers.

26

27

28

29

30

3. List <u>all</u> the <u>proper factors</u> of the following numbers.

51

52

53

54

55

IV. Tell which of the following numbers can be divided exactly by 2, 3, 4, 5, 6, 7, 8, 9, 10 by writing them in after the appropriate numbers. Some of the numbers will go in more than one list.

(355), (238), (97), (78), (504), (882), (3176)

Divisible by 2: (238) (78) (504) (882) (3176)

Divisible by 3:

" " 4:

" " 5:

" " 6:

" " 7:

" " 8:

" " 9:

" "10:

V. Write a paragraph about Prime Numbers. This should include a definition of what a Prime Number is and as much other information about primes as you can remember.

(continue on the back of the page if you need more space.)

V. Lowest Common Denominators, Perfect Numbers, $\phi(N)$

Some of you are confusing the idea of prime twins with pairs of number which are relatively prime. It is true that when we speak of prime twins we are talking about a pair of numbers, but these cannot be just any old numbers. First of all they must both be prime numbers and secondly the difference between them must be exactly 2. In other words, if p_1 and p_2 are any two primes, they are prime twins only if $p_1 = p_2 + 2$ or $p_2 = p_1 + 2$. When we speak of a pair of numbers which are relatively prime to each other, one or both or neither of these numbers needs to be prime by itself. A pair of numbers are relatively prime to each other only if they have no common factor other than 1.

For instance, think of the following numbers and write down all the different relatively prime pairs there are among them.

5 8 13 27 24 18 30 10

[You should find 13 different pairs.]

Try it again with these numbers.

35 17 3 16 12 23 31 28

[This time there are 23 possible pairs.]

I think you have been using the idea of relatively prime pairs of numbers for a long time without really knowing it. For instance suppose we want to add two fractions.

$\frac{2}{3} + \frac{5}{6}$

$\frac{3}{4} + \frac{5}{6}$

$\frac{2}{5} + \frac{5}{6}$

$\frac{2}{9} + \frac{5}{6}$

[The students each worked on these problems at their desks. Many of the children proceeded without hesitation to correct solutions. About 5 children had answers of this type $\frac{2}{3}+\frac{5}{6} = \frac{7}{9}$ and one girl had $\frac{2}{3} + \frac{5}{6} = 79$.]

[Mary Ann verbalized her method as follows:]

"When adding two fractions whose denominators are different, use the larger of the two denominators. If the other denominator goes into it you divide to find out how many times. Then you multiply that number times the numerator and add. If it doesn't go in exactly you keep multiplying the biggest denominator until it will."

This is a difficult thing to try to put into words. Instead let's look at the various steps you go through for these specific problems.

$$\frac{2}{3} + \frac{5}{6} = \left(\frac{2}{3} \times \frac{2}{2}\right) + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6} = 1\frac{1}{2}$$

$$\frac{3}{4} + \frac{5}{6} = \frac{3}{4} + \left(\frac{5}{6} \times \frac{2}{2}\right) = \frac{3}{4} + \frac{10}{12} = \left(\frac{3}{4} \times \frac{3}{3}\right) + \frac{10}{12} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12} = 1\frac{7}{12}$$

$$\frac{2}{5} + \frac{5}{6} = \frac{2}{5} + \left(\frac{5}{6} \times \frac{5}{5}\right) = \frac{2}{5} + \frac{25}{30} = \left(\frac{2}{5} \times \frac{6}{6}\right) + \frac{25}{30} = \frac{12}{30} + \frac{25}{30} = \frac{37}{30} = 1\frac{7}{30}$$

$$\frac{2}{9} + \frac{5}{6} = \left(\frac{2}{9} \times \frac{2}{2}\right) + \frac{5}{6} = \frac{4}{18} + \frac{5}{6} = \frac{4}{18} + \left(\frac{5}{6} \times \frac{3}{3}\right) = \frac{4}{18} + \frac{15}{18} = \frac{19}{18} = 1\frac{1}{18}$$

Of course it is possible for us to combine several of these steps, but even if we do the whole thing in our heads this is probably the path our calculations take. Let's look at the factors of the denominators, breaking each denominator of the original problem and the first solution into its prime factors.

$$\frac{2}{3} + \frac{5}{6} = \frac{2}{3} + \frac{5}{(2\text{x}3)} = \frac{9}{6} = \frac{9}{(2\text{x}3)}$$

$$\frac{3}{4} + \frac{5}{6} = \frac{3}{(2\text{x}2)} + \frac{5}{(2\text{x}3)} = \frac{19}{12} = \frac{19}{(2\text{x}2\text{x}3)}$$

$$\frac{2}{5} + \frac{5}{6} = \frac{2}{5} + \frac{5}{(2\text{x}3)} = \frac{37}{30} = \frac{37}{(2\text{x}3\text{x}5)}$$

$$\frac{2}{9} + \frac{5}{6} = \frac{2}{(3\text{x}3)} + \frac{5}{(2\text{x}3)} = \frac{19}{18} = \frac{19}{(2\text{x}3\text{x}3)}$$

We can see that the denominator of the answer must contain all the factors of the two original denominators, and so the process of finding the lowest common denominator depends upon factoring. If the two denominators are relatively prime their product is the lowest common denominator. There are several interesting things one can do when considering the counting numbers. Suppose we start a list of these numbers, list all their factors (this time we will include 1 but not the number itself) and finally list the sum of all these factors.

Number N	Factors of N (excluding N itself)	Sum of Factors
1	None	0
2	1	1
3	1	1
4	1, 2	3
5	1	1
6	1, 2, 3	6
7	1	1
8	1, 2, 4	7
9	1, 3	4
10	1, 2, 5	8
⋮	⋮	⋮

What do you notice from this list?

"For all the prime numbers the sum will always be 1 because these numbers have only 1 and themselves as factors."

"For the number 6 the sum also equals 6."

This is particularly interesting and because of this the ancient Greeks called 6 a "perfect number". By definition then a number is perfect when it equals the sum of all of its factors not counting itself.

Homework: Continue the list until you find another perfect number.

[The next morning only one student had found the next perfect number. Sheila did not usually take much part in class discussion, but she was very observant and quietly pursued ideas and patterns on her own.]

Sheila does have the right number, but before she tells what it is I'm curious what the rest of you found out.

Mark: "I tried numbers up to 20 but then I gave up."

Michael: "I went all the way to 50 and couldn't find one. But 32 almost works."

32: 1, 2, 4, 8, 16 - sum = 31

John: "I think 112 works."

112: 2 + 4 + 8 + 14 + 28 + 56 = 112

"He left out 1."

I think he must have left out some others too. Let's list all the factors of 112.

1, 2, 3, 7, 8, 14, 16, 28, 56, 112.

112 is a very nice try, John, but if we obey our original definition and use all the factors except 112 the sum is a bit bigger than 112. Michael, I think you must have made some mistake in your work because there is a number between 6 and 50 which is perfect.

Sheila: "28 works. 1 + 2 + 4 + 7 + 14 = 28".

So 6 and 28 are the first two perfect numbers. About how big do you think the next one might be?

"About 100."

"Maybe 22 more than 28 because 28-6=22."

Let's see. 28 + 22 = 50 which Michael says he's tried. I'm going to save you a lot of hard work by telling you that you would have to go all the way to 496 to find the third perfect number. For over 2000 years only twelve perfect numbers were known, but now, by using computors, several more have been found.[1] They do get very large very fast. The first five perfect numbers are:

---

[1]Note to reader: $2^{p-1}(2^p-1)$ is perfect if 2^p-1 is prime.

6 28 496 8128 33550336

Let's check 496

$$
\begin{array}{rcl}
1 & \times & (496) \text{ omit from sum} \\
2 & \times & 248 \\
4 & \times & 124 \\
8 & \times & 62 \\
16 & \times & 31 \\
\hline
31 & & 465 \\
 & & 31 \\
\hline
 & & 496
\end{array}
$$

There is quite a different way we can consider the natural numbers and their factors. Suppose this time we make a new list in which we shall think of all the numbers <u>smaller</u> than the number we are considering <u>and relatively prime</u> to it. People have counted these numbers and given them the name ϕ the greek letter Phi.

Number N	All Numbers $<N$ and relatively prime to N.	How Many Numbers in Middle Column
1	None	By definition $\phi(1) = 1$ instead of 0
2	1	$\phi(2) = 1$
3	1, 2	$\phi(3) = 2$
4	1, 3	$\phi(4) = 2$
5	1, 2, 3, 4,	$\phi(5) = 4$
6	1, 5	$\phi(6) = 2$
7	1, 2, 3, 4, 5, 6,	$\phi(7) = 6$
8	1, 3, 5, 7	$\phi(8) = 4$
9	1, 2, 4, 5, 7, 8	$\phi(9) = 6$
10	1, 3, 7, 9	$\phi(10) = 4$

"They are all even except the first one, $\phi(2)$."

"When N is a prime number all of the numbers less than N are relatively prime to it, so there will always be (N-1) of these."

Then we can say $\phi(p) = (p-1)$ when p is any prime number.

"One less than a number is always relatively prime to that number."

That's true, so after 1 and 2 we know that all the other numbers will have at least 1 and (N-1) in the second column.

$\phi(N) \geq 2$ for all N except 1 and 2.

VI. Base 7

I understand that Mr. Carroll did a little work in bases with you last fall. Of course you know that our every day number system uses 10 as a base and the real value of a digit depends on its position or place value.

Thousands $10^3=1000$	Hundreds $10^2=100$	Tens $10^1=10$	Units $10^0=1$
4	3	5	1
			8
		8	0
	8	0	0
8	0	0	0

$4351 = (1 \times 1) + (5 \times 10) + (3 \times 100) + (4 \times 1000)$

$8000 = (0 \times 1) + (0 \times 10) + (0 \times 100) + (8 \times 1000)$

There are of course more columns to the left of the thousands column (each one multiplies by 10 more) and more columns to the right of the units for showing decimal places.

As we begin now to speak about a different base I shall use two colors of chalk. Everything in white chalk will be our familiar base 10 and everything in orange chalk will be base 7. [The use of colored chalk conveniently eliminates the need of 52 subscripts, and proved an effective device especially since it avoids the misunderstanding about why there is no digit 7 in base 7.]

With base 10 all numbers, no matter how large or how small, can be written by using the same ten digits in different ways (0, 1, 2.....9). The arrangement of the digits, their place in other words, determines their actual value. In base 7 how many digits would you expect to use and what will they be?

We can make a chart and write several numbers first in base 10 and then show in orange chalk how to write the number with the same value in base 7.

Base 10 (white chalk) (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)			Base 7 (orange chalk) (0, 1, 2, 3, 4, 5, 6)	
Tens	Units		Sevens	Units
	9	→	1	2
1	2	→	1	5
1	5	→	2	1
2	2	→	3	1
1	9	→	2	5
2	1	←	3	0
3	0	→	4	2
4	2	→	6	0

In translating from base 10 to base 7, we think first how many sevens are in the number, and remainder is the units digit. Translation in the reverse direction undoes this process. You think of the place value of each digit, multiply, and then add. For instance 53_7 means $(3 \times 1) + (5 \times 7) = 3 + 35 = 38_{10}$

We can arithmetic with numbers in base 7 the same way we use base 10 numbers. In fact there's no reason why we can't combine the two different bases in the same problem.

$3_{10} + 6_{10} = \square_{10}$

$3_7 + 6_7 = \square_7$

$3_7 + 6_7 = \square_{10}$

$4_7 + 5_7 = \square_7$

$14_{10} + 3_{10} = \square_7$

$14_{10} + 14_{10} = \square_7$

$20_7 + 20_7 = \square_{10}$

$20_{10} + 20_{10} = \square_7$

$1_7 + 11_7 = \square_7$

$1_{10} + 11_{10} = \square_{10}$

$1_{10} + 11_{10} = \square_7$

$1_7 + 11_7 = \square_{10}$

$25_7 + 25_7 = \square_{10}$

$25_{10} + 25_{10} = \square_7$

$25_{10} + 25_7 = \square_{10}$

$25_{10} + 25_7 = \square_7$

One way to check arithmetic in a new base is to translate the problem into base 10, find the answer (also in base 10) and then translate this back to base 7 again. The translation should of course agree with the solution you found originally. If the two numbers do not agree then somewhere a mistake has been made.

For instance:

$$\begin{array}{rcl} 15_7 & \smile & 12_{10} \\ +\ \underline{23_7} & \smile & +\ \underline{17_{10}} \\ 41_7 & = & 29_{10} \end{array}$$

Because 29_{10} translates into 41_7 we know our addition was correct (provided we can add to base 10!!!)

Base 7		Base 10	Base 7		Base 10
6	$\smile$	6	13	$\smile$	10
15	$\smile$	12	24	$\smile$	18
4	$\smile$	4	16	$\smile$	13
+ 22	$\smile$	16	+ 5	$\smile$	5
53_7	=	38_{10}	64_7	=	46_{10}

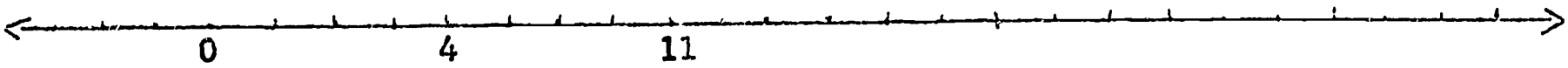
[The fact that $64_7 = 46_{10}$ was an interesting and unexpected reversal of digits that occurred in a problem chosen completely at random. This prompted a separate investigation which was not presented to the class because of its algebraic approach and because it concerned generalizations about bases that would have been premature. However, the problem of when these reversals occur for various bases would be a challenging question to pose to the more able students in junior high school at the conclusion of a study of bases. A copy of the generalization has been included at the end of this section.]

Homework: All problems in base 7

$$\begin{array}{r} 16 \\ 23 \\ 4 \\ 6 \\ +\ 13 \\ \hline \end{array} \qquad \begin{array}{r} 11 \\ 12 \\ 13 \\ +14 \\ \hline \end{array} \qquad \begin{array}{r} 25 \\ -13 \\ \hline \end{array} \qquad \begin{array}{r} 25 \\ -16 \\ \hline \end{array} \qquad \begin{array}{r} 32 \\ -24 \\ \hline \end{array}$$

$3 \times 4 = \square$ $\quad 2 \times 6 = \square$ $\quad 5 \times 5 = \square$ $\quad 6 \times 5 = \square$

Here is a number line using base 7 numbers.



Fill in the rest of the numbers.

If we made the number line long enough what will the last two digit numbers be? "66_7"

What number does that mean in base 10?

"(6 x 7) + 6 = 48"

What number comes after 66_7 then? "100_7"

If you translate 104_7 into base 10 what do you get?

"(1 x 49) + (0 x) + 4 = 53_{10}"

What is the largest three digit number in base 7?

"666"

And what is the translation of 1000_7? "7 x 49 = 343"

"Are there negative numbers in base 7?"

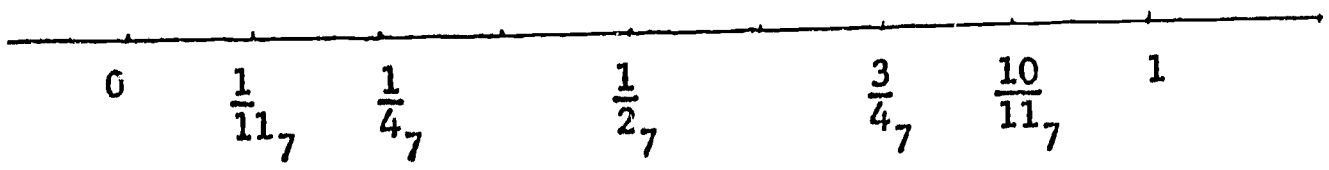
"I don't think so."

"Why not?"

[After considerable discussion it was agreed that of course there could be negative.]

How about fractions and decimals?

[By using the number line to illustrate these ideas it was shown that fractions were possible.]



This $\frac{1}{11_7}$ looks rather strange, doesn't it. But we have taken the distance

from 0 to 1 and divided it into eight equal parts. Only in base 7 we can't write 1/8 because we have no digit 8. Something just like decimals would of course be possible too, only they would not be called decimals because deca means 10. But 3.2_7 would be $3\frac{2}{7}_{10}$ and $.04_7$ would be $\frac{4}{49}_{10}$. This last one would of course be written $\frac{4}{100}_7$.

For now we won't bother about negatives, fractions, and "decimals" in other bases, but it is important to know that they are possible.

Homework: printed work sheet. (2-3-65)

There seemed to be some difficulty with changing 222_7 into base 10 and 222_{10} into base 7.

"222_7 means $(2 \times 49) + (2 \times 7) + (2 \times 1) = 114_{10}$"

Yes, that's fairly straight forward. Now, how about the other one. How do we begin?

"You see how many sevens there are in 222_{10}"

$$\begin{array}{r} 31 \text{ r } 5 \\ 7\overline{)222} \end{array} \qquad \text{so} \qquad 222_{10} = 315_7$$

Let's see if this 315_7 translates back into 222_{10}. $315_7 = (3 \times 49) + (1 \times 7) + (5 \times 1)$ which equals 159_{10} instead of the 222_{10} which we wanted. Does anyone know what was wrong with the method we used? Look carefully at the 31 we got when dividing 222 by 7.

"It's right, isn't it?"

Well $222 \div 7$ certainly does give us 31 with a remainder of 5. But what base is that 31 in? How do we write 31_{10} in base 7?

"Four sevens and three left over, 43"

So if we are trying to write the number in base 7 we'd better be sure all of our steps are in base 7.

"What do we do then, write 435_7 ?"

Let's see if it works.

$435_7 = (4 \times 49) + (3 \times 7) + (5 \times 1) = 196 + 21 + 5 = 222_{10}$.

This is exactly what we wanted.

Michael: "What about some really big numbers like 1462_7 ?"

How much is that 1 worth? "7 x 7 x 7 = 343."

So $1462_7 = (1 \times 343) + (4 \times 49) + (6 \times 7) + (2 \times 1) = 583_{10}$

How about 1462_{10} ? What would it be in base 7?

"Couldn't you think how many 343's there are in 1462_{10} ?"

Then what?

"Then you see how many 49's there are in whats left over, then how many 7's and finally how many 1's are left."

$$\begin{array}{lr} & 1462 \\ (4 \times 343) = & -\ \underline{1372} \\ & 90 \\ (1 \times 49) = & -\ \underline{49} \\ & 41 \\ (5 \times 7) = & -\ \underline{35} \\ & 6 \end{array}$$

$1462_{10} = 4156_7$

Suppose we made addition and multiplication tables for base 7. Would they look the same as the tables we made for regular numbers?

"All the numbers would use only 0-6 for digits, but the patterns of the tables would be pretty much the same."

What will the perfect squares look like?

(0 x 0)	(1 x 1)	(2 x 2)	(3 x 3)	(4 x 4)	(5 x 5)	(6 x 6)	(10 x 10)	(11 x 11)
0	1	4	12	22	34	51	100	121

Homework: Try to do these problems thinking in base 7. Then translate into base 10 to check your work.

$$\begin{array}{r} 135_7 \\ \times\ 21_7 \\ \hline \end{array} \qquad \begin{array}{r} 164_7 \\ \times\ 34_7 \\ \hline \end{array}$$

When doing arithmetic in base 7 some things look very strange to us because we are so used to thinking in base 10. But certain patterns should seem familiar. Look at the multiplication table for 6 in base 7.

0 x 6 = 0

1 x 6 = 6

2 x 6 = 15

3 x 6 = 24

4 x 6 = 33

5 x 6 = 42

6 x 6 = 51

10 x 6 = 60

John: "In the answers the digits always add up to 6."

Mark: "The units digits go 6, 5, 4, 3, 2, 1, and the sevens column goes 1, 2, 3, 4, 5, 6."

Does this remind you of anything? Look at the multiples of 9 in base 10.

0 x 9 = 0

1 x 9 = 9

2 x 9 = 18

3 x 9 = 27

4 x 9 = 36

5 x 9 = 45

6 x 9 = 54

7 x 9 = 63

8 x 9 = 72

9 x 9 = 81

10 x 9 = 90

Sheila: "Its the same pattern, only this time it always adds up to 9."

Michael: "Can you cast out sixes in base 7 the way you cast out nines in base 10?"

I would presume so. If we can, this gives us another nice way to check problems in base 7. But be careful. When casting out sixes we should do our thinking in base 7.

HOMEWORK

NAME _____

In the following problems all numbers written are in base 10 and base 7. All numbers in base 7 are indicated by an asterisk. Give the answer which is indicated by the frame.

9	=		*
19	=		*
	=	22	*
28	=		*

33	=		*
	=	50	*
46	=		*
	=	32	*

5 + 6 = [ ] *

15 + 16 = [ ] *

23 + 15* = [ ] *

23* + 15 = [ ] *

23* + 15== [ ]

4 x 9 = [ ] *

5 x [ ] = 50*

4 x 11 = [ ]

4* x 11* = [ ] *

4* x 11* = [ ]

Place Value - Base 10

		5	means (5 x 1)
	5	0	means (0 x 1) + (5 x 10)
5	0	0	means (0 x 1) + (0 x 10) + (5 x 10 x 10)
2	2	2	means (2 x 1) + (2 x 10) + (2 x 10 x 10)

Place Value - Base 7*

		5	means (5 x 1) *
	5	0	means (0 x 1) + (5 x 7) *
5	0	0	means (0 x 1) + (0 x 7) + (5 x 7 x 7) *
2	2	2	means (2 x 1) + (2 x 7) + (2 x 7 x 7) *

What is the value of 222* in Base 10_____

How would you write 222 in Base 7*_____

```
  36        36 *        61
+ 142     + 142 *     61*
-----     -------     ----
[   ]     [   ]*      [   ]
```

```
24*
30*
15*
103*
----
[   ]*
```

Check the problem on the left expressing all numbers in Base 10.

142	~	10	~	1	235	~	13	~ 4
35	~	11	~	2	x 21	~		x3
160	~	10	~	1	235			15
36	~	12	~	3	503			
436	~ 16 ~	1	~	10	5265	~	24	~ 6

You were right, Michael. Casting out sixes works in base 7 the same way casting out nines works in base 10. How about even and odd numbers? What will they look like in this base?

Even: 0, 2, 4, 6, 11, 13, 15, 20, 22, 24, 26, 31, 33,...

Odd: 1, 3, 5, 10, 12, 14, 16, 21, 23, 25, 30, 32, 34,...

Michael: "I think a number is even when the sum of its digits is even, and odd when the sum of its digits is odd."

For homework tonight why don't you test Michael's idea. Use some base 7 numbers of 3 or 4 digits and see if they always translate into an even number only if the sum of the digits is even.

[In preparation for a test several of the students were still confused about translation, particularly <u>from</u> base 10 <u>to</u> base 7.]

There is more than one way to think of this. One is by repeated subtraction and the other by repeated division. Really this all boils down to the same thing because division is a short cut for repeated subtraction in the same way that multiplication is a short cut for repeated addition.

At any rate it is important to keep in mind what the places mean in the new base.

(2401)	(343's)	(49's)	(7's)	(units)
(7x7x7x7)	(7x7x7)	(7x7)	(7)	(1)

$$244_{10} = \square_7$$

Repeated subtraction:

How many 49's in 244? $(\underline{4} \times 49) =$

$$\begin{array}{r} 244 \\ -196 \\ \hline 48 \end{array}$$

How many 7's in 48? $(\underline{6} \times 7) =$

$$\begin{array}{r} -\ 42 \\ \hline 6 \end{array}$$

How many 1's in 6? $(\underline{6} \times 1) =$

$$\begin{array}{r} -\ 6 \\ \hline 0 \end{array}$$

$$244_{10} = 466_7$$

Repeated division:

The remainders give the translation, the first remainder being the units digit so that

$244_{10} = 466_7$

$$\begin{array}{rl} 7\,/\,244 & \\ 7\,/\,34 & r\ \underline{6} \\ 7\,/\,4 & r\ \underline{6} \\ 0 & r\ \underline{4} \end{array}$$

It is important to keep dividing or keep subtracting until you get to zero. This will keep you from falling into the trap of thinking that your first step in dividing gives you the desired translation. It is very true that there are 34 sevens in 244_{10} and a remainder of 6, but this 34 is a number in base 10. By repeated divisions each step gives you a number in base 7

Let's translate several numbers.

1000's	100's	10's	1's		343's	49's	7's	1's
	2	4	4	→		4	6	6
	5	2	5	→	1	3	5	0
	4	3	6	→	1	1	6	2
1	0	0	0	→	2	6	2	6
	2	4	4	←		4	6	6
	4	5	0	←	1	2	1	2
	5	2	5	←	1	3	5	0

Let's make an addition table for base 7.

[Data was filled in by the students and the following comments were made on the resulting pattern.]

6	6	10	11	12	13	14	15	16	20
5	5	6	10	11	12	13	14	15	16
4	4	5	6	10	11	12	13	14	15
3	3	4	5	6	10	11	12	13	14
2	2	3	4	5	6	10	11	12	13
1	1	2	3	4	5	6	10	11	12
0	0	1	2	3	4	5	6	10	11
	0	1	2	3	4	5	6	10	11

Addition
Base 7

1. "The pattern is really just like it is on the base 10 addition table."
2. "Each row and each column is a piece of the number line."
3. "The main diagonal is counting by two's (even numbers)"
4. "The diagonals in the other direction are all one number."
5. "The pattern would just go on if we went into bigger numbers."
6. "If we put in negative numbers there would be all zeros on the secondary diagonal."

Test tomorrow!

[There was a wide variation in test results. Using a percentage grade the range extended from 98 to 10. Many otherwise high scores were lowered because of failure to check arithmetic in base 7 by translation into base 10 which seems to boil down to a lack of care in reading direction.]

TEST NAME: _____

I. Definition: Two numbers are said to be relatively prime to each other when they have no common factor greater than 1.

For example: take the three numbers 4 5 6
4,5 and 5,6 are each relatively prime pairs. 4,6 is not a relatively prime pair because they have 2 as a common factor.

1. Find all the possible relatively prime pairs for these six numbers. (The pair 5,6 and the pair 6,5 in this case mean the same thing and do not need to be counted twice.)
5 6 7 8 9 10

2. Find all the possible relatively prime pairs for these six numbers.

20 21 22 23 24 25

3. Below are four numbers (9, 10, 11, 12). For each of these make a list of all the smaller numbers which are relatively prime to it. 9 is completed as a sample.

Number N	All numbers less than and relatively prime to N.
9	1, 2, 4, 5, 7, 8
10	
11	
12	

4. When we want to add two fractions whose denominators are different it helps us to know whether the two denominators are a relatively prime pair of numbers. Complete these three addition problems.

a. $\frac{7}{12} + \frac{2}{3} =$

b. $\frac{7}{12} + \frac{2}{5} =$

c. $\frac{7}{12} + \frac{2}{9} =$

In which problem on the bottom of the previous page were the two denominators a relatively prime pair? _____

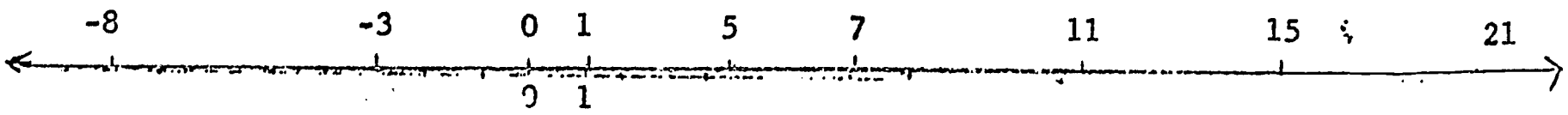
In which problem was one denominator a factor of the other? _____

II. The rest of this test has to do with two different number systems. Our common number system (base 10)* will be written in blue ink. All numbers written in the base 7** system will be shown in red ink. Remember that the only digits in this system are 0,1,2,3,4,5,6,.

Base 10 will be indicated with one asterisk.
Base 7 will be indicated with two asterisks.

1. On the number line below several points are shown above the line as numbers written in base 10.

Just below the line show what these same points would be called in base 7.



Here is another section further up the number line.
This time points are marked below the line in base 7.
Just above the number line show what these points would be called in base 10.

2. Translate the numbers shown below into what they would be if written in the other base.

Base 10		Base 7
16	=	22
29	=	
100	=	
234	=	
12	=	15
	=	66
	=	135
	=	1234

3. In the addition problems below try to do all the work and give the answer in base 7. Then translate each problem into base 10 numbers in order to check your computation.

$$\begin{array}{r} 6 \\ +\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 24 \\ +\ 31 \\ \hline \end{array}$$

$$\begin{array}{r} 41 \\ +\ 53 \\ \hline \end{array} \qquad \begin{array}{r} 35 \\ +15 \\ \hline \end{array}$$

4. Here is a partially completed multiplication table for numbers in base 7. It is arranged somewhat like the large tables you made for base 10 numbers. On this table the numbers to be multiplied together are shown in the row and column outside the double lines. All of the products (or answers) are shown in the squares above and to the right of the double lines. 4 x 2 = 11 so the number in the 4 column and the 2 row is 11. What will be the product of 2 x 4? Fill it in in the empty square that is in the 2 column and the 4 row. Complete the rest of the table by filling in all the empty squares. Remember that all numbers are in base 7. **

6	0	6		24			51
5	0	5			26		
4	0	4			22		
3	0	3		12			
2	0	2		6	11		15
1	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
x	0	1	2	3	4	5	6

5. Referring to the multiplication table you have just made, what are the first seven perfect squares in base 7? **

_____ _____ _____ _____ _____ _____ _____

What will the next perfect square be if the multiplication table is extended to numbers bigger than 6?_____ **

6. Using the base 7 multiplication table to help you, try to do these problems completely in base 7.** Translate numbers to base 10 to check your work.*

$$\begin{array}{r} 32 \\ \times\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 502 \\ \times\ 11 \\ \hline \end{array}$$

$$\begin{array}{r} 312 \\ \times\ 4 \\ \hline \end{array} \qquad \begin{array}{r} 502 \\ \times\ 26 \\ \hline \end{array}$$

VII. Russian Peasants' Multiplication, Bases 2 and 12

There is a very old method for doing multiplication in which all you need to know is how to divide by two, multiply by two and add. This is called the Russian Peasants' method and it eliminates the need for knowing the multiplication tables.

When we have two numbers to multiply, if we double one and divide the other in half the product stays the same.

(4 x 9) = (2 x 18) = (1 x 36) = 36

This same idea works for some larger numbers.

Divide by 2		Multiply by 2		Product
32	x	122	=	
16	x	244	=	
8	x	488	=	
4	x	976	=	
2	x	1952	=	
1	x	3904	=	3904

Divide by 2		Multiply by 2		Product
64	x	23	=	
32	x	46	=	
16	x	92	=	
8	x	184	=	
4	x	368	=	
2	x	736	=	
1	x	1472	=	1472

"Use some really big numbers."

Divide by 2		Multiply by 2		Product
32	x	3458	=	
16	x	6916	=	
8	x	13832	=	
4	x	27664	=	
2	x	55328	=	
1	x	110656	=	110656

Divide by 2		Multiply by 2		Product
64	x	179	=	
32	x	358	=	
16	x	716	=	
8	x	1432	=	
4	x	2864	=	
2	x	5728	=	
1	x	11456	=	11456

Michael: "I checked by casting out nines. It works!"

3458	~	2
x 32	~	5
110656	~ 19 ~	10

179	~	8
x 64	~	1
11456	~ 17 ~	8

Mark: "What if both numbers are odd?"

Well I was choosing numbers like 32 and 64 because they do divide so nicely by 2.

Michael: "When its odd, can't you divide and multiply by 3 instead?"

Let's try.

Divide by 3		Multiply by 3		Product
27	x	54	=	
9	x	162	=	
3	x	486	=	
1	x	1458	=	1458

Michael's idea works too, but if you begin to use numbers other than 2 to divide and multiply by you are well on the way to needing to use the multiplication tables.

The doubling and halving method is a very old one and there is a way to fix it up so it will work for any two numbers. When we divide any integer by 2 there is a remainder of 1 if the number was odd and a remainder of 0 if the number was even. When we kept dividing 32 or 64 by 2 the remainder was always 0 until the very end and so the product was always the same at every step. But suppose we use two odd numbers and for the moment ignore remainders.

Divide by 2		Multiply by 2		Product	
13	x	65	=	845	780 + 65 = 845
6	x	130	=	780	
3	x	260	=	780	520 + 260 = 780
1	x	520	=	520	

We know from our regular method of multiplication that 13 x 65 is 845 and not 520, so we lost something along the way. We can also see that the product remained 780 for two steps because dividing 6 in half and multiplying 130 by two doesn't make any change in the product. And so we can see that when we ignored the remainder in dividing an odd number in half we lost exactly the amount in the next column on the right. We can quickly find the true product by adding 65 + 260 + 520, in other words adding all the items in the second column except those beside an even number. Let's do another one. While I am working it out

on the board you might check it with regular multiplication and casting out nines just to prove to yourself that this method works.

Divide by 2-Multiply by 2

37	x	143		Ignore remainders when dividing but then cross out all numbers in the second column which are next to an even number in the first column.
18	x	~~286~~		
9	x	572		
4	x	~~1144~~		
2	x	~~2288~~		
1	x	4576		
Add		5291	=	(37 x 143)

With practice you might become even faster with this method than the old way. If you have trouble remembering multiplication facts this is a good method, but it does take a lot of space.

[The students were especially intrigued by Russian Peasants' multiplication and kept asking for larger and larger numbers.]

All right, for homework then you can do these.

(68 x 9876 and (58 x 305607)

Suppose we take the number 29 and divide it repeatedly by 2. Compare this to what we have just been doing.

```
     0  r 1
2 / 1  r 1
2 / 3  r 1
2 / 7  r 0
2 /14  r 1
2 /29
```

Divide by 2

1	29
0	14
1	7
1	3
1	1

Odd numbers leave remainders of 1, even numbers leave remainders of zero.

This should remind us of repeatedly dividing a number by 7 to translate it into base 7. Repeated division by 2, producing a series of remainders of either 1 or 0, translates the number into base 2. So $29_{10} = 11101_2$.

[A new color of chalk, blue, was used instead of the subscript.]

What are the possible digits in base 10? 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

What are the possible digits in base 7? 0, 1, 2, 3, 4, 5, 6

" " " " " " " 2? 0, 1,

So base 2 numbers look very strange indeed, but any number can be translated into a number using only the two digits 0 and 1.

Here is a number line. Let's try to fill in the same numbers in base 2 underneath the regular numbers.

-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
-11	-10	-1	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101

[By filling in the number line in this way the students discovered together how to write consecutive integers and of their own accord went into negative numbers in base 2.]

So you see the base 2 numbers begin to get very long, very fast and this is perhaps their main disadvantage. But arithmetic in them is very simple because there are so few basic facts to remember. If you lived in a base 2 world the only tables you would have to memorize are these:

0 + 0 = 0	0 x 0 = 0
0 + 1 = 1	0 x 1 = 0
1 + 1 = 10	1 x 1 = 1

If we think of place value for base 2 we have:

(2x2x2x2x2)	(2x2x2x2)	(2x2x2)	(2x2)	2	Units
2^5	2^4	2^3	2^2	2^1	2^0
32	16	8	4	2	1

What will the next place be after 32? "64"

And after that? "128"

So we need a number of eight places to express what in base 10 has only three digits.

64	32	16	8	4	2	1		Base 10
	1	1	1	1	1	1	=	63
		1	0	1	1	1	=	23
	1	1	0	0	0	0	=	48
1	0	0	0	1	0	0	=	68
1	0	1	0	1	0	1	=	85
1	1	1	1	1	1	1	=	255

What is a short way of doing the last one?

For homework translate several numbers from base 10 to base 2 and vice versa.

You seem to be having some trouble in translating into base 2.

Jimmy's method: $37 = 32 + 4 + 1 = 100101$

Michael's method: $37 = 7 + 30$

$$\begin{array}{rcr} 7 & = & 111 \\ 30 & = & \underline{11110} \\ 37 & = & 100101 \end{array}$$

In Michael's method we have to carry several times when we add. This is easy to do if we remember that $1 + 1 = 10$. So usually we have to carry in several columns. We can also find what 37_{10} is in base 2 by repeated division. I think perhaps if we set it up this way it will be clearer for you. We shall sort of be dividing upside down and backwards.

$$\begin{array}{cccccccl} & 1 & 0 & 0 & 1 & 0 & 1 & \text{Remainders} \\ 0 & /1 & /2 & /4 & /9 & /18 & /37 & \\ \hline & 2 & 2 & 2 & 2 & 2 & 2 & \end{array}$$

So the row of remainders at the top gives us the translation. Be sure to keep on dividing until you get to zero. How about some larger numbers.

$$\begin{array}{ccccccc} & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & /1 & /3 & /6 & /12 & /25 & /50 & /100 \\ \hline & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

Try some arithmetic with base 2 numbers. Check your work by translating the problem into base 10.

```
  10011        10110
+ 1101       x  101
```

For homework you can make up some of your own problems.

[The students brought in the following examples which they put on the board.]

```
  11110        10111        100110
x   111        x1010       x 11010
```

```
  10101010101010101
   1010101010101010
  10101010101010101
           101110
+            1111
  11010101011100111
```

[Michael's addition problem was a source of utter amazement from the other and it contained an unexpected lesson in carrying.]

When we add the ones in the units column of Michael's enormous problem we write down 1 and carry 1. Now there are five ones in the 2 column and in base 2 five is written 101 so we have to write one 1 and carry 10 into the next <u>two</u> columns. This idea is continued until we have added up all the columns, carrying into how ever many are necessary. Would someone like to find out the place value of all the columns in Michael's answer.

"<u>The last column on the left</u> is worth 65536."

Shall we make a multiplication table for base 2.

1000	0	1000	10000	11000	100000	101000	110000
111	0	111	1110	10101	11100	100011	101010
110	0	110	1100	10010	11000	11110	100100
101	0	101	1010	1111	10100	11001	11110
100	0	100	1000	1100	10000	10100	11000
11	0	11	110	1001	1100	1111	10010
10	0	10	100	110	1000	1010	1100
1	0	1	10	11	100	101	110
0	0	0	0	0	0	0	0
x	0	1	10	11	100	101	110

Perfect squares:

Base 10;	0	1	4	9	16	25	36	49
Base 2;	0	1	100	1001	10000	11001	100100	110001

Test tomorrow!

[The test included a section on base 12 which was not obligatory. About half of the students tried it, a few of these doing very well although base 12 had never been worked with or even mentioned in class. The concept of evenness and oddness had also deliberately not been included in class discussion. It was hoped that the students would be led to discover for themselves the simple test for parity in base 2. Several students did but Michael jumped to the erroneous conclusion that it had to do with the sum of the digits, instead of only the units digit.]

Some of you did very well indeed on the test although it included several new ideas. If we start counting in base 2, sorting the members into two piles, what do you notice?

TEST

Name_____

For this test all numbers written will be base 10 unless indicated by an asterisk which will indicate base 2. Remember that all numbers in base 2 have as digits only 0 and 1.

I. Use repeated division and multiplication by 2 to do these multiplication problems. (This way of multiplying is often called the Russian Peasants' Method.)

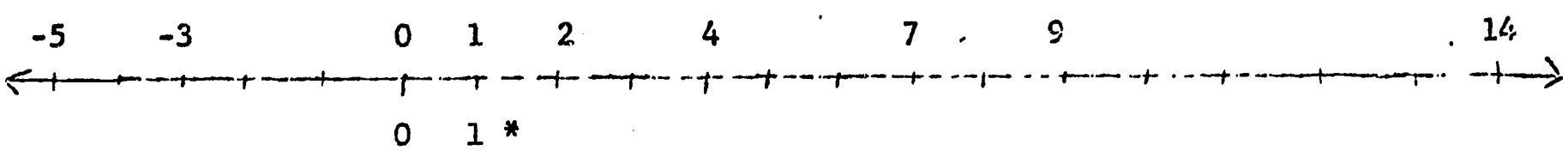
Divide by 2		Multiply by 2	Divide by 2		Multiply by 2
32	x	53	64	x	105

32 x 53 = [ ] 64 x 105 = [ ]

Divide by 2		Multiply by 2	Divide by 2		Multiply by 2
17	x	39	44	x	215

17 x 39 = [ ] 44 x 215 = [ ]

II. Here is a number line showing some points marked in base 10 numbers just above the line. Show what these same points would be called in base 2 just below the line.*



On the second number line several points are shown in base 2 numbers.* Just above the line show what these same points would be called in base 10.