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AUTHOR	Davis, R.
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Abstract

These materials were written with the aim of reflecting the thinking of the Cambridge Conference on School Mathematics (CCSM) regarding the goals and objectives for school mathematics. They represent a practical response to a proposal by CCSM that some elements of probability be introduced in the elementary grades. These materials provide children with a variety of activities involving probability and statistics in a laboratory setting. Opportunities are provided for children to gain experiences in various types of situations - performing experiments, recording data, graphing experimental data, determining mathematical models for chance events, and computing. The experiences described in this report are intended to give students the opportunity to become familiar, by direct experiment, with important probability concepts before they are to be studied at a more sophisticated level. [Not available in hardcopy due to marginal legibility of original document.] (RP)

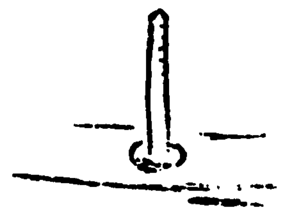
CCSM
Feasibility Study No. 7
R. Davis

PROBABILITY

Section I

Most people find it difficult to think about chance events. Let's see if we can agree on answers to the following questions:

Question 1: If we drop a nail from a height of about 4 ft., will it land in a "point-up" position,



or in a "point-down" position?

Question 2: If we drop a thumbtack from a height of about 4 ft, will it land in a "point-up" position,

or in a "point-down" position?

Question 3: Alan says that if we drop a thumbtack 20 times, from a height of about 4 ft., it will land "point-up" 7 times, and "point-down" 13 times. Do you agree?

Performing an Experiment and Recording Data

In dropping a thumbtack 20 times, and recording the outcomes, we are performing an experiment. It is usually desirable to drop the thumbtack in nearly the same way each time. We can do this by setting the tack "point-up" on a desk, and slowly pushing it off the desk by means of the edge of a book. You can think of many other ways to achieve uniformity: for example, you can rest your forearm on a desk and hold the tack in a paper cup, then turn the cup quickly upside down, so that the tack falls out.

The way that you record your data is also important. You want to preserve as much of the data as possible, so that you can use it to answer new questions that may arise in the future. One way to do this, in the thumbtack experiment, is to use the letter "U" to mean "point-up" and to use the letter "D" to mean "point-down".

Record each outcome in the order of occurrence, grouping the symbols in groups of 5, so that your record for 20 drops might look like this:

U U U D U
D U U D D
D U U D D
D U U D U

Question 4: Jerry says that you can't be sure of the outcome when you drop a tack 20 times, because 20 is too small a number. He says that Alan could guess the outcome more closely if we used 40 drops, instead of 20. What do you think?

A Big Experiment

In order to answer Jerry's question, Alex suggested a co-operative experiment by the entire class.

Each person dropped a tack 20 times, and recorded the "U's" and "D's" in the order in which they occurred, grouping them into groups of 5.

After this data had been recorded, the class tried to decide whether it was easier to guess the outcome for 20 drops or for 40.

They decided that part of the problem was the question of "consistency" or "stability". Here is what Alex did with the data recorded by Marilyn, Jerry, Harold and Ellen. Their original data looked like this:

Marilyn:	D D D U U	Jerry:	D U D U U
	D D D U D		D U U D U
	U D D U D		U U U U D
	U D U U D		U U U U U
Harold:	U U D U D	Ellen:	U D U U D
	U U D U D		U U U U U
	D D U D U		D D U U U
	D U D D D		D D D U D

Alex made 4 groups of 20 drops, as follows:

Marilyn 12 "downs" and 8 "ups"
Jerry 5 "downs" and 15 "ups"
Harold 11 "downs" and 9 "ups"
Ellen 8 "downs" and 12 "ups":

Alex was trying to see how much variation there was in the number of times the tack landed "point-up" in 20 drops.

Marilyn	8	"ups"
Jerry	15	"ups"
Harold	9	"ups"
Ellen	12	"ups"

Question 5: Do you think that these numbers vary so much that it is hopeless to try to guess the number of U's that will appear in 20 drops?

Alex combined Marilyn's and Jerry's data, to get a group of 40 drops:

Marilyn and Jerry: 17 "downs" and 23 "ups"

Combining Harold and Ellen's data, he made another group of 40:

Harold and Ellen: 19 "downs" and 21 "ups"

In order to get 2 more groups of 40 drops each, Alex used the data recorded by 4 other members of the class:

Tony:	U U U U D	Richard:	D U D D D
	U D D U U		D U D D D
	U U U U D		D U U D U
	U D D U U		D U D U D
Nancy:	D D U D U	Susan:	U U U D D
	D U U U U		D U D U U
	U U U D D		U U D D U
	U D U U U		U U D D D

Tony and Richard: 6 + 13 = 19 "downs"

14 + 7 = 21 "ups"

Nancy and Susan: 7 + 9 = 16 "downs"

13 + 11 = 24 "ups"

Question 6: For 20-drop groups, the number of "ups" in each of 4 groups were: 8, 15, 9, 12

For 40-drop groups, the number of "ups" in each of 4 groups were: 23, 21, 21, 24.

Which would you rather try to predict, the outcome for a group of 20 drops, or the outcome for a group of 40 drops?

Question 7: Jerry says there is not enough data here to be convincing. Can you suggest a way to get more data?

Here is the data taken by other members of the class:

Joan:	U U D D U	Jim:	D U U D D
	D U U D U		D D U U D
	U D U D U		U D D U U
	U U D D U		U D D U U
Francis:	D D U D U	Tony:	D D D U D
	D U U D D		U D U D D
	D U D D U		U D U D D
	D U D D U		U D D D U
Marge:	D U D U D	Rene:	U U U U U
	U D D D U		D D D D D
	D D U U U		D D U U U
	D U D D U		U U D U D
George:	D D D U D	Steve:	U U U D U
	U D U U U		U U D U D
	D D D U D		D U U D U
	U U U D U		U D D D U
Jeff:	U D U D U	Mary:	U D D D D
	D U D D D		D U U U D
	U D D U D		D U D U D
	D D U D D		D U U U D
Ann:	U D U U D	Jake:	D D D D U
	U U D U D		U U U U U
	D D D U U		D D D U U
	D U U U U		D U U U U

Jerry now made up 10 groups of 5 drops each, using the first five drops from the first 10 students:

<u>Number of "ups" in each group of 5 drops:</u>

2, 3, 3, 3, 4, 1, 2, 3, 3, 2

Similarly, he made up 10 groups of 10 drops each, by using the last 10 drops of each of the last 10 students:

Number of "ups" in each group of 10 drops

6, 6, 5, 3, 5, 5, 6, 5, 4, 6

He made up 10 groups of 20 drops each, by using the last 10 drops from every student:

Number of "ups" in each group of 20 drops

14, 7, 12, 12, 10, 10, 9, 10, 11, 11

He made up 10 groups of 40 drops each, by combining the work of pairs of students:

Number of "ups" in each group of 40 drops

23, 21, 21, 24, 20, 17, 20, 22, 16, 24

Question 8: Is it easier to predict the outcome for 5 drops, or for 10 drops, or for 20 drops, or for 40 drops.

Section II

Permanent Experiment # 1

Why don't you perform a big thumb-tack experiment with your class? If you have 20 or more people in class, have each person drop a tack 40 times. Have him record each "Up" or "Down" as it occurs, and separate his answers into groups of 5 each. Then, by combining groups, you will be able to get 10 groups of 5, or 10 groups of 10, or 10 groups of 20, or 10 groups of 40, or 10 groups of 80.

Keep all your data! We will be able to make use of it again in the future.

Is the number of U's more predictable in a large sample, or in a small sample?

A. Record the number of U's in each of 10 groups of 5:

___, ___, ___, ___, ___, ___, ___, ___, ___, ___.

B. Record the number of U's in each of 10 groups of 10:

___, ___, ___, ___, ___, ___, ___, ___, ___, ___.

C. Record the number of U's in each of 10 groups of 20:

___, ___, ___, ___, ___, ___, ___, ___, ___, ___.

D. Record the number of U's in each of 10 groups of 40:

___, ___, ___, ___, ___, ___, ___, ___, ___, ___.

E. Record the number of U's in each of 10 groups of 80:

___, ___, ___, ___, ___, ___, ___, ___, ___, ___.

Question 1: Looking at your data above, where is it easier to predict the number of U's, in groups of 5 or in groups of 20, or in groups of 80?

Question 2: Can you describe what we mean by the "variability" in a set of numbers? Which set of numbers shows the greatest variability, the set recorded under A, or the set recorded under C, or; the set recorded under E?

We need some good methods for studying how much "variation" there is in a set of numbers. Here are 5 methods:[1]

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1. We shall take our data from Section I. Why don't you use data from the experiment that your class did.

I. The Method of "Just Looking".

For groups of 5, we got these numbers: (counting "Ups")

2, 3, 3, 3, 4, 1, 2, 3, 3, 2.

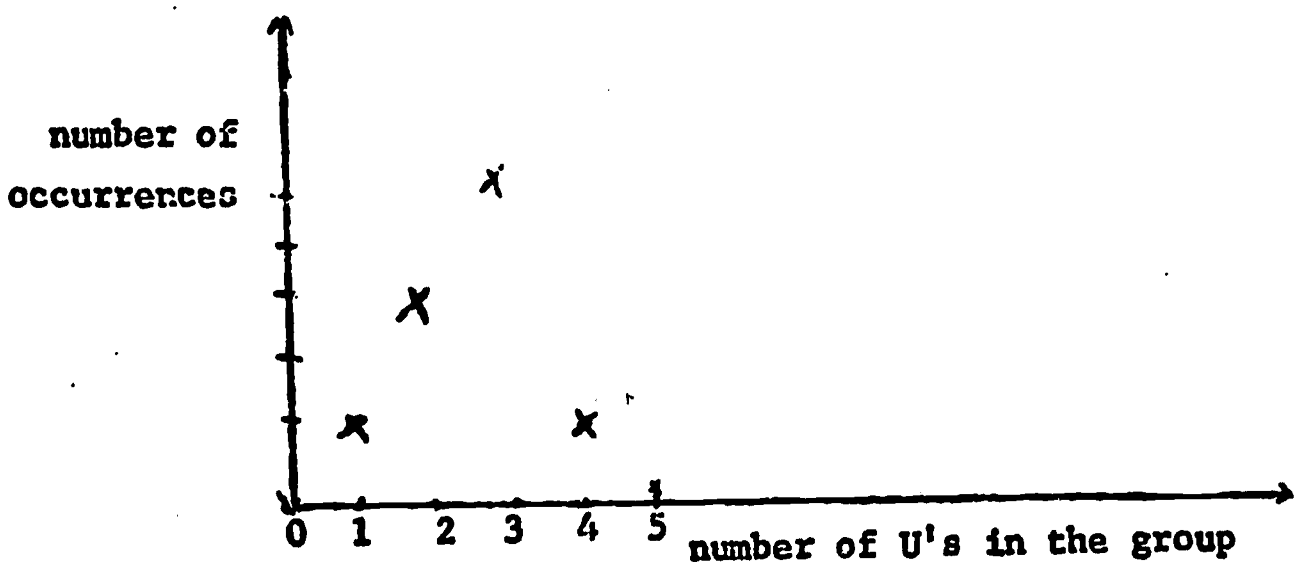
For groups of 20, we got these numbers:

14, 7, 12, 12, 10, 10, 9, 10, 11, 11.

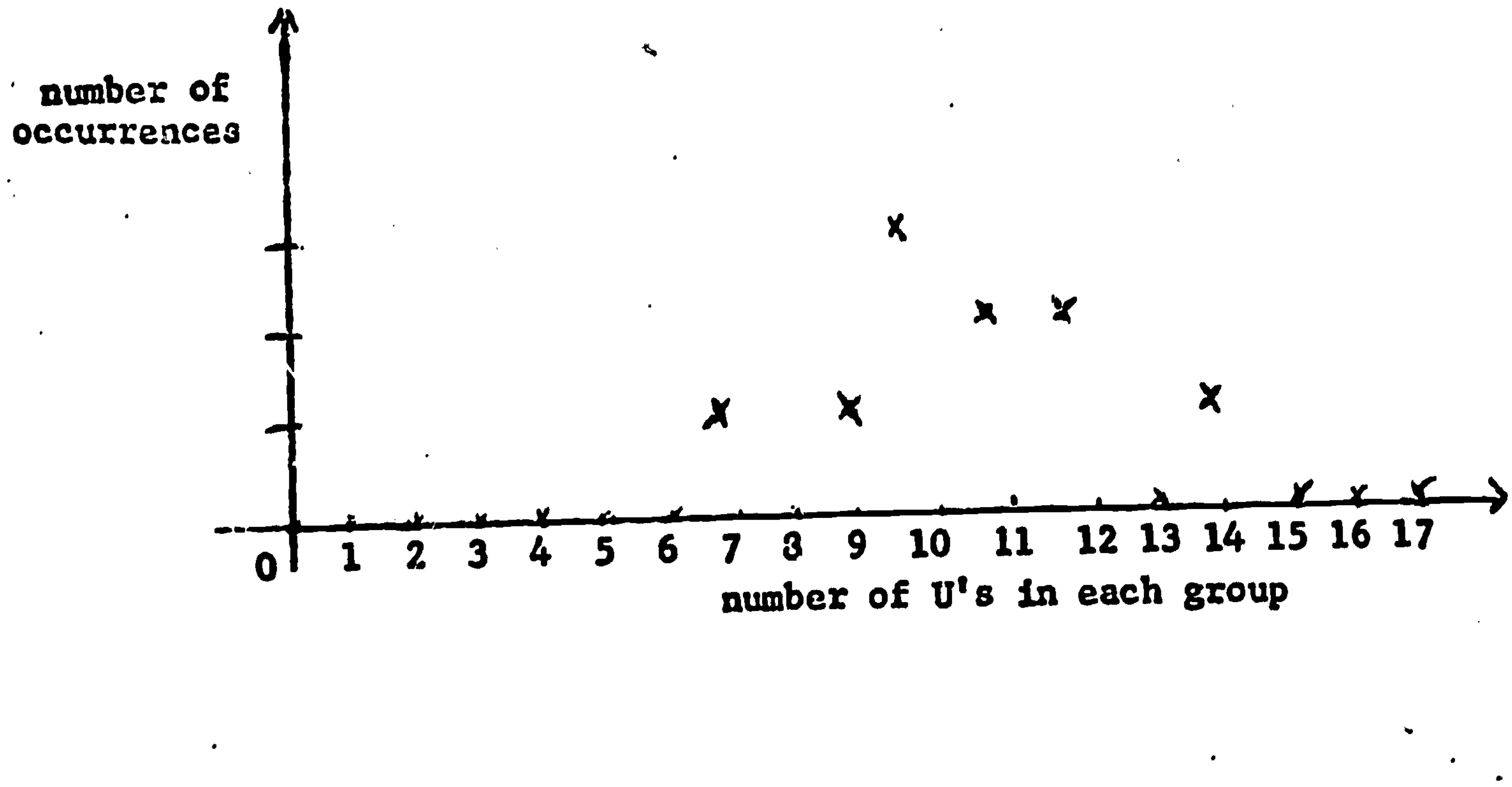
By just looking at these numbers, which set of numbers seems to show greater "variation"?

II. The Method of Graphs.

We can show the first set of numbers of a graph like this:



Number of "Ups" in 10 groups of 5 drops



Number of "Ups" in 10 groups of 20 drops.

From looking at these two graphs, which set of numbers seems to show greater variability?

III. The Method of Mean Absolute Deviation from the Mean.

One good method requires that we compute the "average" or "mean" for each set of numbers:

$$2 + 3 + 3 + 3 + 4 + 1 + 2 + 3 + 3 + 2 = 26$$

$$\frac{26}{10} = 2.6$$

$$14 + 7 + 12 + 12 + 10 + 10 + 9 + 10 + 11 + 11 = 106$$

$$\frac{106}{10} = 10.6$$

We then compute the distance (on the number line) between each number and the mean:

$$|2 - 2.6| = 0.6$$

$$|3 - 2.6| = 0.4$$

$$|3 - 2.6| = 0.4$$

$$|3 - 2.6| = 0.4$$

$$|4 - 2.6| = 1.4$$

$$|1 - 2.6| = 1.6$$

$$|2 - 2.6| = 0.6$$

$$|3 - 2.6| = 0.4$$

$$|3 - 2.6| = 0.4$$

$$|2 - 2.6| = 0.6$$

We have now computed the "deviations from the mean" for our first set of numbers. We now proceed to compute the <u>average deviation</u> by averaging these new numbers:

$$0.6 + 0.4 + 0.4 + 0.4 + 1.4 + 1.6 + 0.6 + 0.4 + +0.4 + 0.6 = 6.8$$

$$\frac{6.8}{10} = 0.68$$

This is the mean absolute deviation from the first set of numbers (groups of 5 drops).

Now, let's do the same thing with our second set of numbers (groups of 20 drops):

$|14 - 10.6| = 3.4$

$|7 - 10.6| = 3.6$

$|12 - 10.6| = 1.4$

$|12 - 10.6| = 1.4$

$|10 - 10.6| = 0.6$

$|10 - 10.6| = 0.6$

$|9 - 10.6| = 1.6$

$|10 - 10.6| = 0.6$

$|11 - 10.6| = 0.4$

$|11 - 10.6| = 0.4$

$3.4 + 3.6 + 1.4 + 1.4 + 0.6 + 0.6 + 1.6 + 0.6 + 0.4 + 0.4 = 14$

$\frac{14}{10} = 1.4$

This is the mean absolute deviation for the second set of numbers (groups of 20 drops).

From this method of comparison, which set of numbers seems to vary more?

IV. The Method of Comparing Ranges.

For the first set of numbers

2, 3, 3, 3, 4, 1, 2, 3, 3, 2

the smallest is 1 and the largest is 4. The <u>range</u>, therefore, is

$4 - 1 = 3$

For the second set of numbers

14, 7, 12, 12, 10, 10, 9, 10, 11, 11,

the smallest is 7 and the largest is 14. The <u>range</u>, therefore is: $14 - 7 = 7$.

From comparing the <u>ranges</u>, which set of numbers seems to show the greater variability?

V. The Method of Comparing Trimmed Ranges.

To use this method, we arrange the numbers in order of size:

1, 2, 2, 2, 3, 3, 3, 3, 3, 4

7, 9, 10, 10, 10, 11, 11, 12, 12, 14.

We then "trim" each set by discarding (say) the two "largest" and the "smallest" numbers in each:

2, 2, 3, 3, 3, 3

10, 10, 10, 11, 11, 12

For these "trimmed" sets of numbers, we compute the ranges:

3 - 2 = 1 trimmed range for first set of numbers (groups of 5)

12 - 10 = 2 trimmed range for second set of numbers (groups of 20).

By using this method of comparison, which set of numbers seems to show greater variability?

<u>Question 3</u>: Which set of numbers, in <u>your</u> data, shows greater variability, the set recorded under C, or the set recorded under E?

<u>Question 4:</u> Can you predict the total number of "Ups" more accurately in small numbers of tosses, or in large numbers of tosses?

<u>Question 5</u>: If we want to get a set of numbers showing twice as much variability, should we use sample sizes twice as large? One-half as large? four times as large? One-fourth as large? Or what?

<u>Question 6</u>: Do you know how mathematicians express the answer to Question 5?

<u>Question 7</u>: What advantages and disadvantages can you find to help choose between the 5 different methods for comparing variability?

Question 8: Jerry says that, even though the second set of numbers seemed to show more variability, there is some sense in which it really shows less variability. What do you think? How would you suggest we deal with these two sets of numbers?

2925-66

Section III

Proportional Occurrence of U's

Question 1: Ellen says that even though the total number of U's is harder to predict for larger samples, the proportional occurrence of U's is easier to predict for larger samples. What do you think? What does your data suggest?

Let's test Ellen's suggestion by each of our 5 methods for comparing variability. In Section II we compared the variability of the total number of U's. We now compare the variability of the proportional or fractional number of U's.

Question 2: How do you expect the variability of the fractional number of U's in the 5 drop case will compare with the fractional number of U's in the 20 drop case?

Method I:

The fractional number of U's in the 5 drop case can be found by taking the total number of U's:

2, 3, 3, 3, 4, 1, 2, 3, 3, 2

and dividing by the total number of drops (in this case, 5):

$$\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}$$

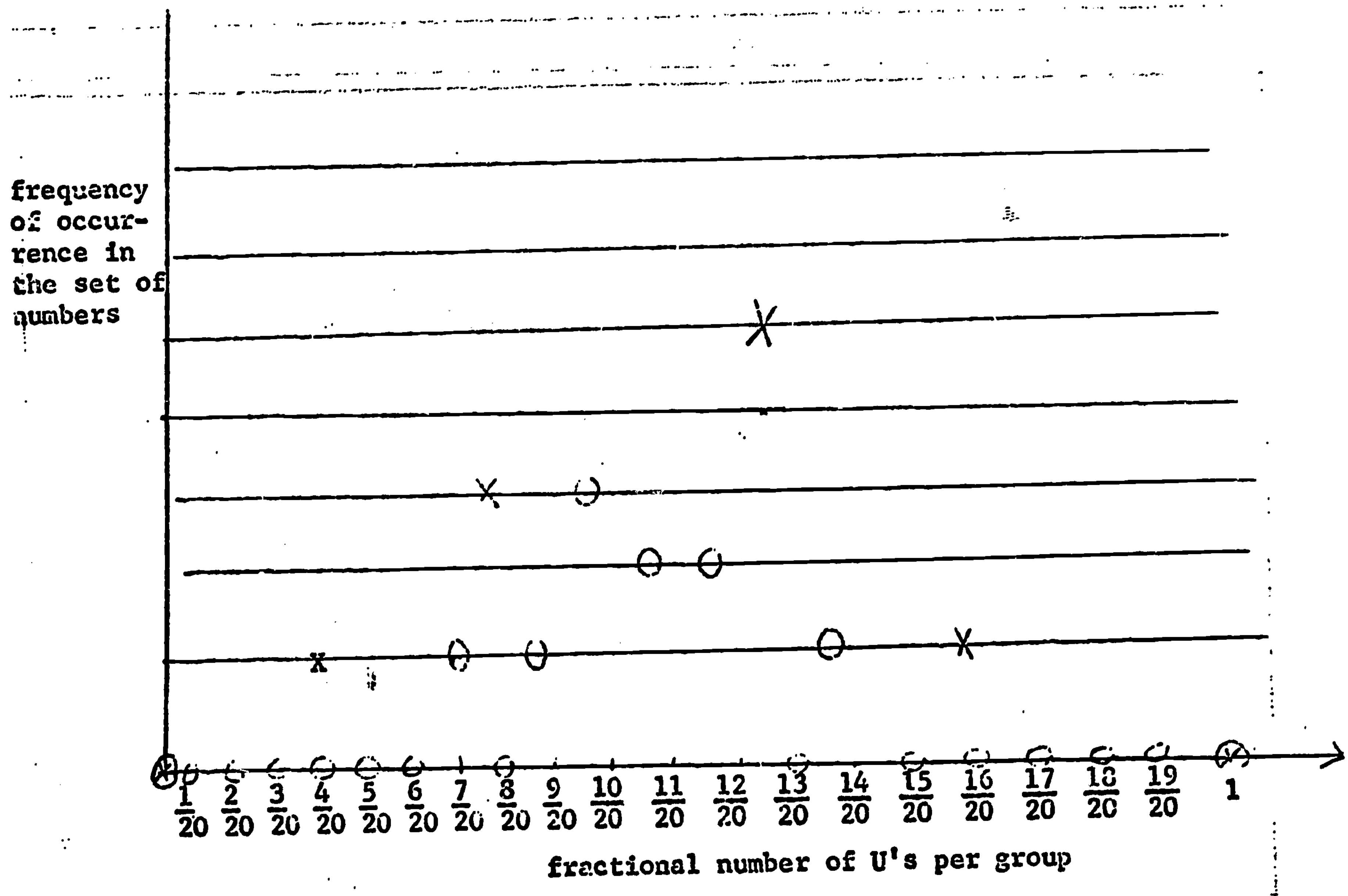
For the groups of 20 drops we get

$$\frac{14}{20}, \frac{7}{20}, \frac{12}{20}, \frac{12}{20}, \frac{10}{20}, \frac{10}{20}, \frac{9}{20}, \frac{10}{20}, \frac{11}{20}, \frac{11}{20}$$

Can you tell, by just looking, which set of numbers varies more?

Method II: Comparison by Graphs.

We shall mark both sets of numbers on the same graph, using x's to indicate the 1st set, and O's to indicate the 2nd set:



Which set of numbers shows greater consistency? Wich shows greater variability? Did it work out the way you expected?

Method III: Comparison of Mean Absolute Deviation from the Sample Mean.

Evidently, the mean for the 1st set of numbers must be

$$\frac{2.6}{5} = 0.52$$

The absolute deviations from 0.52 are

$|0.4 - 0.52| = 0.12$

$|0.6 - 0.52| = 0.08$

$|0.6 - 0.52| = 0.08$

$|0.6 - 0.52| = 0.08$

$|0.8 - 0.52| = 0.28$

$|0.2 - 0.52| = 0.32$

$|0.4 - 0.52| = 0.12$

$|0.6 - 0.52| = 0.08$

$|0.6 - 0.52| = 0.08$

$|0.4 - 0.52| = 0.12$

Averaging these deviations, we get

$0.12 + 0.08 + 0.08 + 0.08 + 0.28 + 0.32 + 0.12 + 0.08 + 0.08 + 0.12 = 1.36$

$\frac{1.36}{10} = 0.136 \leftarrow$ This is the mean absolute deviation for the 1st set of numbers (group of 5 drops).

We can now do the same thing for the 2nd set of numbers (groups of 20 drops):

The mean is

$$\frac{10.6}{20} = 0.53.$$

The absolute deviations from this mean are:

$|0.7 - 0.53| = 0.17$

$|0.35 - 0.53| = 0.18$

$|0.6 - 0.53| = 0.07$

$|0.6 - 0.53| = 0.07$

$|0.5 - 0.53| = 0.03$

$|0.5 - 0.53| = 0.03$

$|0.45 - 0.53| = 0.08$

$|0.5 - 0.53| = 0.03$

$|0.55 - 0.53| = 0.02$

$|0.55 - 0.53| = 0.02$

0.17 + 0.18 + 0.07 + 0.07 + 0.03 + 0.03 + 0.08 + 0.03 + 0.02 + 0.02 = 0.7

$\frac{0.7}{10}$ = 0.07 ← This is the mean absolute deviation for the 2nd set of numbers (group of 20 drops, using proportion of U's rather than total number of U's).

Which set of numbers seems to vary more? How much more?

IV. The Method of Comparing Ranges

The 1st set of numbers (proportion of U's in group of 5 drops) is

0.2, 0.4, 0.4, 0.4, 0.6, 0.6, 0.6, 0.6, 0.6, 0.8

The smallest number is 0.2, and the largest is 0.8. Consequently, the range is 0.8 - 0.2 = 0.6.

The 2nd set of numbers (proportion of U's in groups of 20 drops) is

0.35, 0.45, 0.50, 0.50, 0.50, 0.55, 0.55, 0.60, 0.60, 0.70,

The largest is 0.70, and the smallest is 0.35. Consequently the range is 0.70 - 0.35 = 0.35.

V. The Method of "Trimmed" Ranges

For the 1st set of numbers, we delete the two largest and the two smallest, to get a "trimmed" set of numbers:

0.4, 0.4, 0.6, 0.6, 0.6, 0.6.

The range is now 0.6 - 0.4 = 0.2.

For the 2nd set of numbers (groups of 20 drops), if we omit the 2 largest and 2 smallest we get the "trimmed" set of numbers:

0.50, 0.50, 0.50, 0.55, 0.55, 0.60

The range of this trimmed set is 0.60 - 0.50 = 0.10.

Question #3: Does the total number of U's vary more in large samples, or in small samples?

Question #4: Does the proportion of U's vary more in large samples, or in small samples?

Question: #5: Can you summarize what we have learned? What does your data seem to indicate?

2926-66

Section IV

Variability of Total Number of U's, and of Proportion of U's, in Large Samples and in Small Samples.

(Summary of Sections I-III)

Alex says that mathematicians talk about our thumb-tack experiment this way:

When we were using 10 groups of 5 drops each, they would say we had a sample size n, equal to 5.

When we were using 10 groups of 20 drops each, they would say we had a sample size n, equal to 20.

In general, when we increase the sample size by making it 4 times as big, the variability of the total number of U's would be expected to increase by a factor of 2. Consequently, mathematicians say that the variability of the total number of U's increases as $\sqrt{n}$.

In the fractional proportion of U's, however, the situation is quite different. Here, if we multiply the sample size by 4, the variability of the fractional proportion of U's decreases by a factor of 2. Consequently, mathematicians say that the fractional proportion of U's decreases like $\frac{1}{\sqrt{n}}$

Question 1: Is this what your data seemed to indicate?

Question 2: Could you come closer, in predicting the number of U's, from a small number of drops, or from a large number of drops?

Question 3: Does your data become more variable or less variable, as you go to larger-sized samples?

Question 4: Can you summarize what we have learned?

Question 5: Why do you think we use fractions so much in the theory of probability?

2955-66

Section V

A Telephone Book Experiment

Experiment II. Look at some "randomly chosen" page well into the phone book. Make a record of the last digit of the 1st 40 numbers of the page, grouping by fives as usual. Each student should collect this data independently, so that we can combine into a "big experiment." From this record, determine the frequency of occurrence of each digit, and the relative frequency of each. Study the variability of these frequencies as a function of sample size, as was done in Experiment I.

Here is some typical data (although you will undoubtedly want to work with data collected by your own class):

Harold's data:

4, 9, 5, 6, 3

5, 6, 5, 4, 4

2, 9, 4, 2, 2,

8, 0, 8, 4, 3

4, 4, 0, 5, 6

8, 9, 9, 0, 8

0, 2, 0, 7, 3

0, 1, 8, 4, 6

Judy's data:

3, 3, 4, 7, 8

5, 3, 0, 9, 6

2, 8, 7, 9, 4

8, 5, 9, 6, 4

9, 5, 9, 3, 9

8, 9, 9, 6, 7

1, 7, 7, 9, 7

1, 6, 6, 8, 8

Using only Harold's and Judy's data we find:

digit	total number of occurrences	relative proportion of occurrences
0	0, 0, 0, 1, 1, 1, 2, 1, 0, 1	$0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{5}$
1	0, 0, 0, 0, 0, 0, 0, 1, 0, 0	$0, 0, 0, 0, 0, 0, 0, \frac{1}{5}, 0, 0$
2	0, 0, 3, 0, 0, 1, 2, 1, 0, 1	$0, 0, \frac{3}{5}, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{5}$
3	1, 0, 0, 1, 0, 0, 1, 0, 2, 1	$\frac{1}{5}, 0, 0, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, \frac{2}{5}, \frac{1}{5}$
4	1, 2, 1, 1, 2, 0, 0, 1, 1, 0	$\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, 0, 0, \frac{1}{5}, \frac{1}{5}, 0$
5	1, 2, 0, 0, 1, 0, 0, 0, 0, 1	$\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{1}{5}, 0, 0, 0, 0, \frac{1}{5}$
6	1, 1, 0, 0, 1, 0, 0, 1, 0, 1	$\frac{1}{5}, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, \frac{1}{5}$
7	0, 0, 0, 0, 0, 0, 1, 0, 1, 0	$0, 0, 0, 0, 0, 0, \frac{1}{5}, 0, \frac{1}{5}, 0$
8	0, 0, 0, 2, 0, 2, 0, 1, 1, 0	$0, 0, 0, \frac{2}{5}, 0, \frac{2}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0$
9	1, 0, 1, 0, 0, 2, 0, 0, 0, 1	$\frac{1}{5}, 0, \frac{1}{5}, 0, 0, \frac{2}{5}, 0, 0, 0, \frac{1}{5}$

In order to study 10 groups of 10, we need more raw data. Here is Marilyn's data:

9, 2, 0, 4, 3

8, 7, 7, 7, 4

2, 1, 6, 1, 4

3, 3, 9, 7, 9

7, 0, 1, 8, 0

7, 9, 4, 1, 6

4, 1, 4, 9, 8

4, 4, 1, 3, 7

Occurrences of Digits in Each of 10 Groups of 10

digit	total number of occurrences	relative proportion of occurrences
0	0, 1, 2, 3, 1, 0, 0, 0, 1, 0	0, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\frac{1}{10}$, 0, 0, 0, $\frac{1}{10}$, 0
1	0, 0, 0, 1, 0, 0, 0, 2, 1, 0	0, 0, 0, $\frac{1}{10}$, 0, 0, 0, $\frac{2}{10}$, $\frac{1}{10}$, 0
2	0, 3, 0, 1, 0, 1, 0, 0, 1, 1	0, $\frac{3}{10}$, 0, $\frac{1}{10}$, 0, $\frac{1}{10}$, 0, 0, $\frac{1}{10}$, $\frac{1}{10}$
3	1, 1, 0, 1, 3, 0, 1, 0, 1, 2	$\frac{1}{10}$, $\frac{1}{10}$, 0, $\frac{1}{10}$, $\frac{3}{10}$, 0, $\frac{1}{10}$, 0, $\frac{1}{10}$, $\frac{2}{10}$
4	3, 2, 2, 1, 1, 2, 0, 0, 2, 1	$\frac{3}{10}$, $\frac{2}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, 0, 0, $\frac{2}{10}$, $\frac{1}{10}$
5	3, 0, 1, 0, 1, 1, 1, 0, 0, 0	$\frac{3}{10}$, 0, $\frac{1}{10}$, 0, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, 0, 0, 0
6	2, 0, 1, 1, 1, 1, 1, 2, 0, 1	$\frac{2}{10}$, 0, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, 0, $\frac{1}{10}$
7	0, 0, 0, 1, 1, 1, 1, 3, 3, 1	0, 0, 0, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{3}{10}$, $\frac{1}{10}$
8	0, 2, 2, 1, 1, 2, 1, 2, 1, 0	0, $\frac{2}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, 0
9	1, 1, 2, 0, 1, 2, 4, 1, 1, 2	$\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, 0, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{4}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$

In order to consider 20 groups of 20 numbers each ("samples with n equal to 20"), we need more data:

Tom's data:

5, 8, 4, 5, 0

7, 5, 3, 3, 0

0, 8, 0, 9, 1

8, 2, 4, 0, 8

2, 5, 4, 2, 5

1, 1, 9, 7, 4

1, 9, 6, 6, 7

8, 9, 7, 6, 7

Bills data:

0, 4, 6, 2, 3

0, 2, 5, 7, 2

0, 0, 5, 5, 1

0, 7, 5, 8, 9

9, 0, 2, 4, 7

7, 0, 8, 4, 1

4, 2, 1, 4, 9

0, 3, 1, 4, 4

<u>Occurrences of Digits in Each of 10 Groups of 20</u>

Digit	total number of occurrences	relative proportion of occurrences
0	1, 5, 1, 0, 1, 2, 5, 0, 5, 3	$\frac{1}{20}, \frac{5}{20}, \frac{1}{20}, 0, \frac{1}{20}, \frac{2}{20}, \frac{5}{20}, 0, \frac{5}{20}, \frac{3}{20}$
1	0, 1, 0, 2, 2, 4, 1, 3, 1, 3	$0, \frac{1}{20}, 0, \frac{2}{20}, \frac{2}{20}, \frac{4}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{3}{20}$
2	3, 1, 1, 0, 2, 0, 1, 2, 2, 2	$\frac{3}{20}, \frac{1}{20}, \frac{1}{20}, 0, \frac{2}{20}, 0, \frac{1}{20}, \frac{2}{20}, \frac{2}{20}, \frac{2}{20}$
3	2, 1, 3, 1, 3, 1, 2, 0, 1, 1	$\frac{2}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{2}{20}, 0, \frac{1}{20}, \frac{1}{20}$
4	5, 3, 3, 0, 3, 4, 2, 2, 1, 6	$\frac{5}{20}, \frac{3}{20}, \frac{3}{20}, 0, \frac{3}{20}, \frac{4}{20}, \frac{2}{20}, \frac{2}{20}, \frac{1}{20}, \frac{6}{20}$
5	3, 1, 2, 1, 0, 0, 3, 2, 4, 0	$\frac{3}{20}, \frac{1}{20}, \frac{2}{20}, \frac{1}{20}, 0, 0, \frac{3}{20}, \frac{2}{20}, \frac{4}{20}, 0$
6	2, 2, 2, 2, 1, 1, 0, 3, 1, 0	$\frac{2}{20}, \frac{2}{20}, \frac{2}{20}, \frac{2}{20}, \frac{1}{20}, \frac{1}{20}, 0, \frac{3}{20}, \frac{1}{20}, 0$
7	0, 1, 2, 4, 4, 3, 1, 3, 2, 2	$0, \frac{1}{20}, \frac{2}{20}, \frac{4}{20}, \frac{4}{20}, \frac{3}{20}, \frac{1}{20}, \frac{3}{20}, \frac{2}{20}, \frac{2}{20}$
8	2, 3, 3, 3, 1, 2, 4, 1, 1, 1	$\frac{2}{20}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{1}{20}, \frac{2}{20}, \frac{4}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}$
9	2, 2, 3, 6, 3, 2, 1, 3, 1, 2	$\frac{2}{20}, \frac{2}{20}, \frac{3}{20}, \frac{6}{20}, \frac{3}{20}, \frac{2}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{2}{20}$

We can now test the suggestion that the variability of totals increases like $\sqrt{n}$, and the variability of fractional occurrences decreases like $\frac{1}{\sqrt{n}}$, where n is the so-called "sample size."

We shall use three methods: the method of ranges, the method of trimmed ranges, and the method of average ranges. The first two of these methods we used in Experiment I; the method of average ranges will, however, be new.

Method of Ranges: For the total number of occurrences of the digit 0, in sample sizes of 5 (n=5), we have:

0, 0, 0, 0, 1, 1, 1, 1, 1, 2

Evidently, the range is 2-0=2.

For the total number of occurrences of the digit 0, m' sample sizes of 20 (n=20), we have:

0, 0, 1, 1, 1, 2, 3, 5, 5, 5

Evidently the range is 5-0=5. It is reasonably close to our generalization that, if we multiply the sample size by 4, we double the variability (in this case, we double the range).

Here are some further comparisons:

	Total number of occurrences		Fractional proportion of occurrences	
digit	range for n = 5	range for n = 20	range for n = 5	range for n = 20
0	2	5	0.4	0.25
1	1	4	0.2	0.2
2	3	3	0.6	0.15
3	2	3	0.4	0.15
4	2	6	0.4	0.3
5	2	3	0.4	0.15
6	1	3	0.2	0.15
7	1	4	0.2	0.2
8	2	3	0.4	0.15
9	2	5	0.4	0.25

This table does not seem to show very decisive agreement without generalization about variability. What do your data show?

Method of Trimmed Ranges

If we delete the two largest and two smallest members from each set, we get

	trimmed set for total no. of occurrences	
digit	group with n=5	group with n=20
0	0, 0, 1, 1, 1, 1	1, 1, 1, 2, 3, 5
1	0, 0, 0, 0, 0, 0	1, 1, 1, 2, 2, 3
2	0, 0, 0, 1, 1, 1	1, 1, 1, 2, 2, 2
3	0, 0, 0, 1, 1, 1	1, 1, 1, 1, 2, 2
4	0, 1, 1, 1, 1, 1	2, 2, 3, 3, 3, 4
5	0, 0, 0, 0, 1, 1,	0, 1, 1, 2, 2, 3
6	0, 0, 0, 1, 1, 1	1, 1, 1, 2, 2, 2
7	0, 0, 0, 0, 0, 0	1, 2, 2, 2, 3, 3
8	0, 0, 0, 0, 1, 1	1, 1, 2, 2, 3, 3
9	0, 0, 0, 0, 1, 1	2, 2, 2, 2, 3, 3

For the trimmed ranges we get:

digit	trimmed range for total no. of occurrences		trimmed range for fractional proportion of occurrences	
	n=5	n=20	n=5	n=20
0	1	4	0.2	0.2
1	0	2	0	0.1
2	1	1	0.2	0.05
3	1	1	0.2	0.05
4	1	2	0.2	0.1
5	1	3	0.2	0.15
6	1	1	0.2	0.05
7	0	2	0	0.1
8	1	2	0.2	0.1
9	1	1	0.2	0.05

Method of Average Range

Combining our data for all digits, we can compute the average range and average trimmed range as follows:

Average range				Average Trimmed range			
Total no. of occurrences		Proportional fraction of occurrences		Total no. of occurrences		Proportional fraction of occurrences	
n=5	n=20	n=5	n=20	n=5	n=20	n=5	n=20
1.8	3.9	0.36	.19	0.8	1.9	1.6	.09

This table appears to fit in quite nicely with our generalization that the variability of total number of occurrences increases like $\sqrt{n}$, while the variability of the fractional proportion of occurrences decreases like $\frac{1}{\sqrt{n}}$, as the sample size n increases.

What do your data show?

2927-66

Section VI

An Experiment with a Coin

Experiment III: Each member of your class can toss a coin 40 times[1], recording each occurrence of heads and tails in order. Keep these records in groups of 5 tosses each. Keep this data permanently -- we can use it repeatedly in the future! You can study the variability of total number of heads, and fractional proportion of heads, as functions of the sample size n.

What do you expect to find? Here is the record of 2,000 tosses of U.S. coins:

H H H T H	T H T T T
T T T T H	T H T H H
H H H T T	T T T H H
H H H T T	H T H H T
H T T T T	T H T T H ___ 100
H T H T H	H H T T T
H H T T T	T T T T T
H T T H T	H T T T T
H T T T H	T T H H T
H T T H T ___ 50	T T H T T
T H H H H	H T T H T
H H H H H	T T H T T
H H T H T	H T H T T
T T H H T	T H H T H
T H T T T	H T T T T ___ 150
H T T H T	T T T T H
H T T H H	T H T T H
H H H H T	T T T H H
H T T T H	T T T H H
H T H H H	H T T T T
T H H T T	H T T T T
T H T T H	H T T T T
H H T H H	H T T H T
H H H T H	H T T T H
H T T T T ___ 200	T H H T H ___ 350

1. You may want to get records of even more tosses; perhaps a total of at least 2,000.

H H H T H	T H H T T
T H T T H	T H T H T
T T T H H	H H T H H
H T H T T	T T T H T
H H T H T	T T H T H
T T T H H	H H H H H
H T H H H	H H H T H
H T H H H	T T H H T
T H T H H	T H T T H
H H T T H ___ 250	H H H H T ___ 400
T H T H H	H T H H T
T H H T H	T H H H T
T T T T T	T H T T T
H H T T T	T T T H T
T T T H H	H H H H H
T H T T T	H H H H H
H T T H T	H H H T H
T T T H T	T T T H T
T H T T T	H T H H H
T H H H H ___ 300	H T T H H ___ 450
T T H T H	H H H H H
T H H T T	H H T H H
T T T H H	H T T H H
T H H H H	T H T H H
H T T H H	T H H T T
H H T H T	H H T H H
H H T T H	H H H T T
T H T H H	
H H H T T	
T H H T T ___ 500	

(Section VI is temporarily left incomplete. In the completed version, one would treat this data as in the preceding sections, studying empirically the variability of totals and ratios as a function of sample size.)

(This coin data would also be used later for an empirical comparison of the "compensation" vs. "swamping" theories of the law of large numbers.)

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Section VII

An Abstract Model for Chance Events

In the preceding sections, we have made empirical studies of variability, using thumbtacks, telephone directories, and coins. We have seen that as we make our samples larger, the variability of the total number of occurrences of (say) an "Up", or of a "head", becomes larger. However, the fractional proportion of "Ups" or "heads" varies less for larger samples.

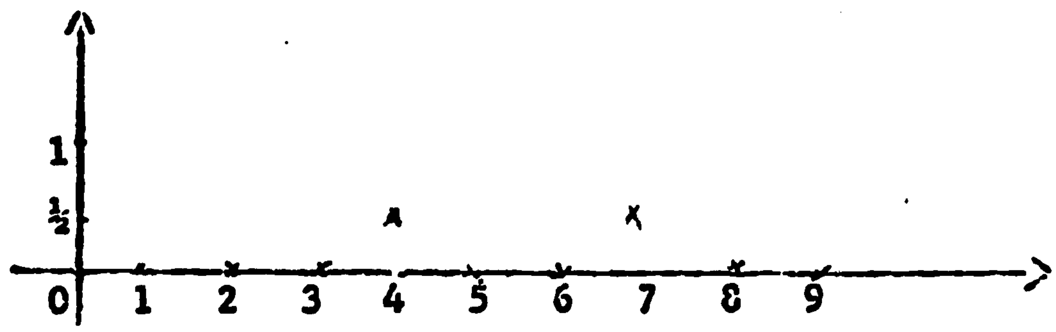
Can we use this apparent stability of the fractional proportion of heads as the foundation for a mathematical model? We would like our model to help describe "chance" events. Let's see if we can make one that will have some usefulness.

To begin with, let us think of the example of the last digit of a telephone number. We can make a 2-dimensional graph by representing the possible outcomes along the horizontal axis

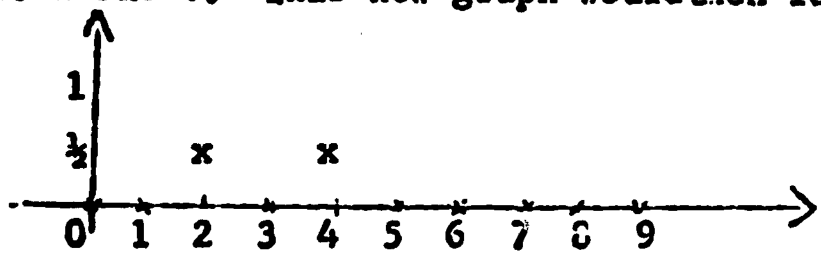
and representing the fractional proportion of occurrences along the vertical axis. Suppose, for 2 numbers, the last digit of one was 7, and of the other was 4. Then the fractional proportion of occurrences would be

Digit	0	1	2	3	4	5	6	7	8	9
Fractional Proportion of Occurrence	0	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0

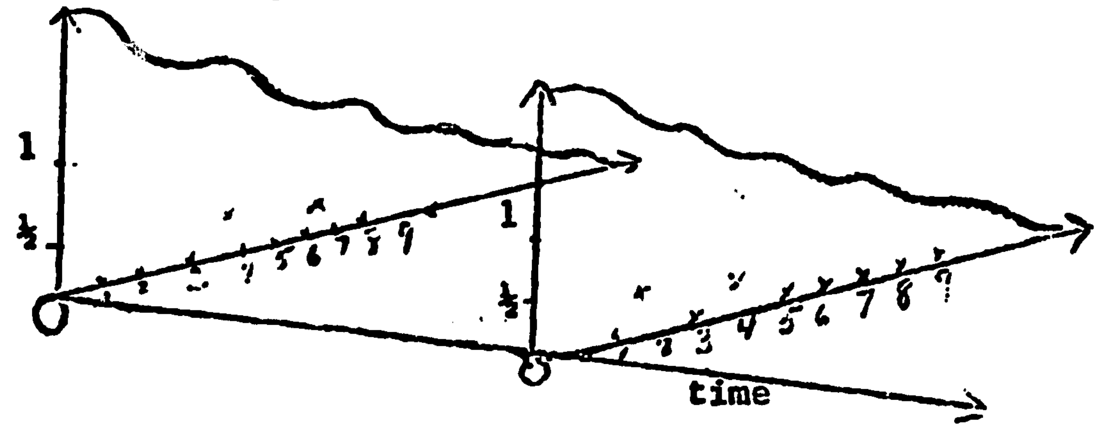
and the corresponding graphical representation would be



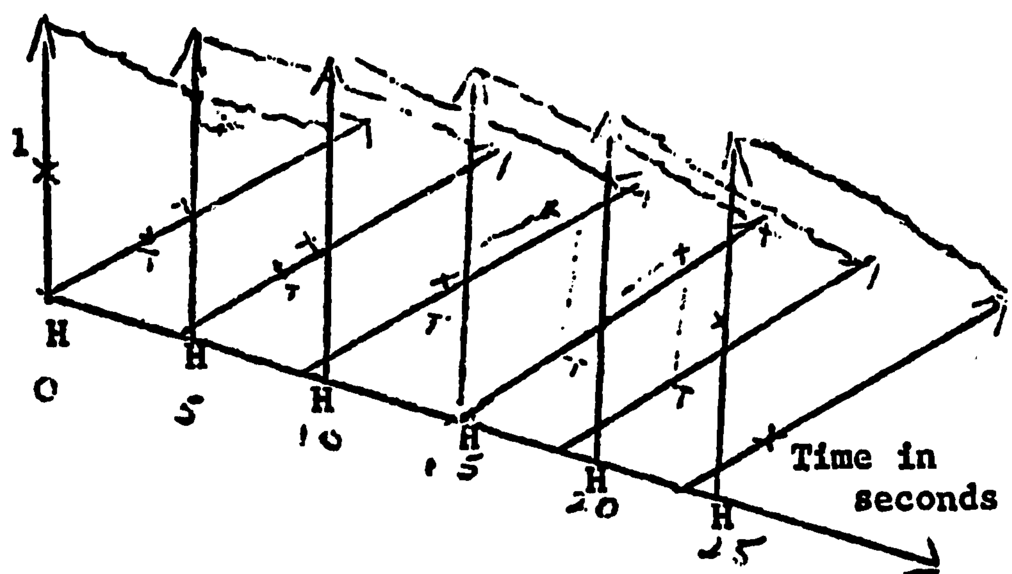
Suppose the experiment of selecting 2 numbers were repeated, and the final digits were 2 and 4. This new graph would then look like this?



We can make a 3-dimensional picture by arranging these two planes parallel at two different points on a "time" axis:

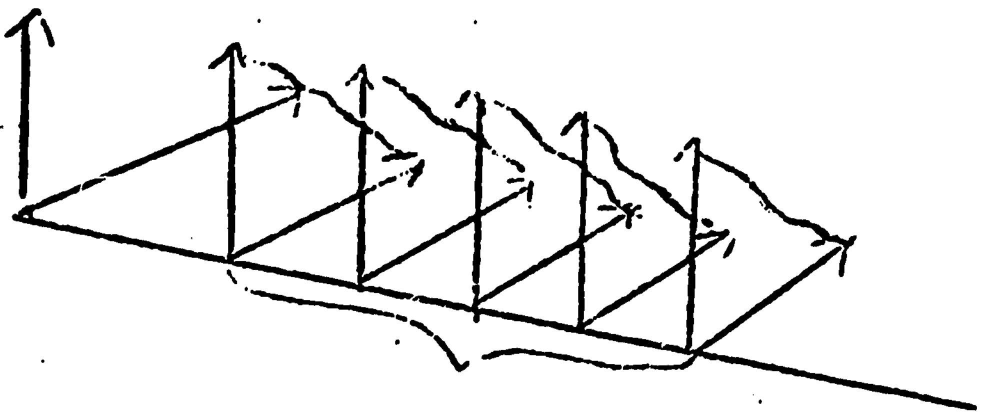


Suppose we toss a coin 6 times, at 5 second intervals. We could represent the outcome by a 3-dimensional picture as follows:

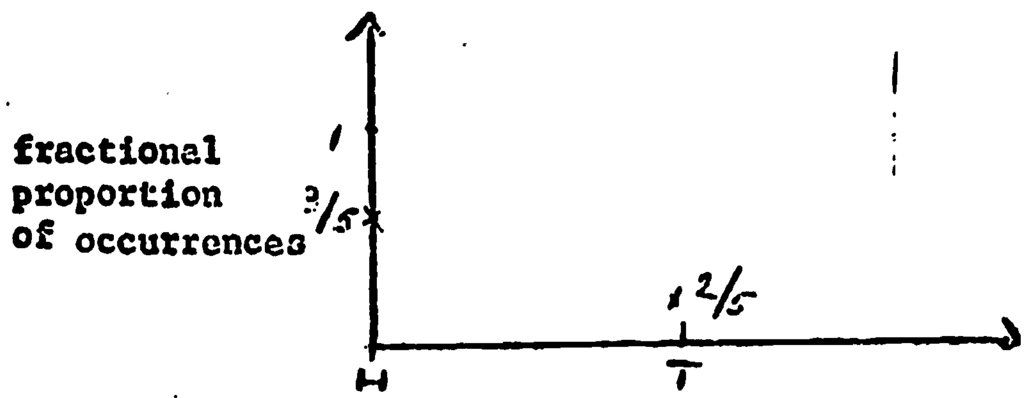


From this picture, we can see that the outcome of the 1st toss was "Heads", the outcome of the 2nd toss was "Heads", of the 3rd also "Heads", the outcome of the 4th toss was "Tails", and so on.

Now what did we seem to be observing in our empirical studies of probability? For one thing, we computed the fractional average, not of a single toss, but cumulatively over many tosses. We took a fairly long section along the time axis

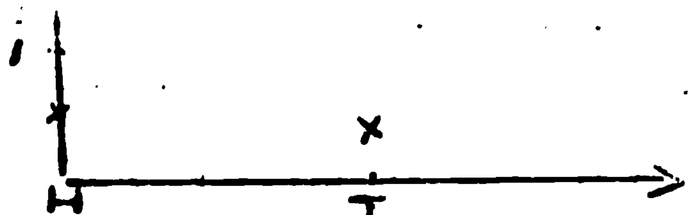


and computed an average for all of the tosses included within this time interval. The resulting 2-dimensional graph might look like this:

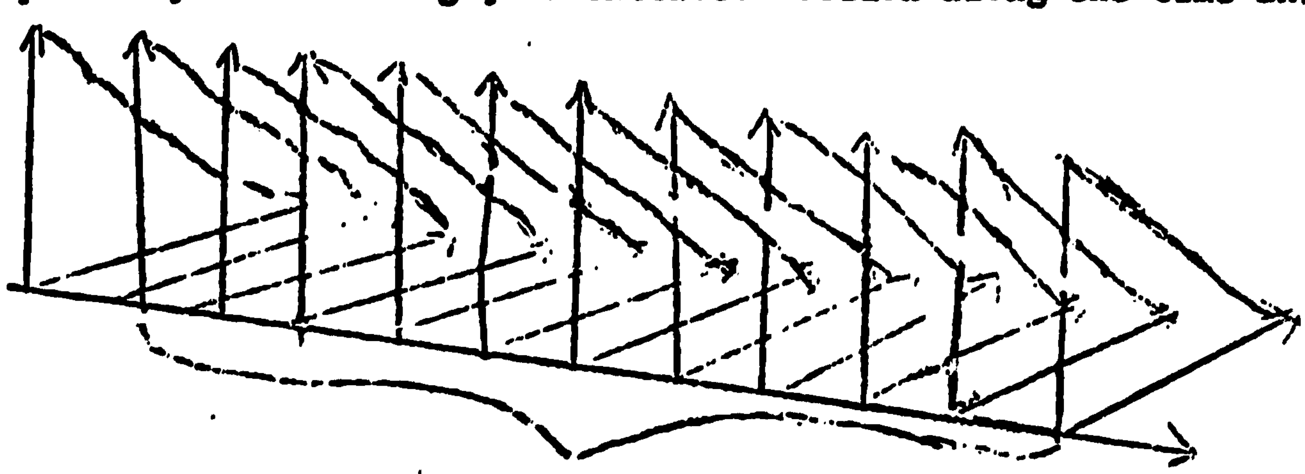


If we take longer and longer sections along the time axis, the variability of the fractional proportion of occurences will become smaller and smaller. The fractions appear to be "homing in" on some constant values, from which they do not deviate very much in large samples.

We might, then base our model upon the idea of a long-range average



which can represent, as an average, an extended section along the time axis:



Question I. What do you think a 2-dimensional "long-range average" would look like for:

a) the thumbtack experiment

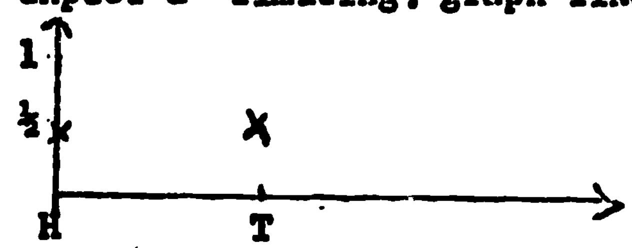
b) the last digit of telephone numbers

c) the coin-tossing experiment.

Question II. If you computed a 2-dimensional graph of fractional occurrences from a very long average along the time axis, would your 2-dimensional graph be relevant to some other long average along the time axis?

We evidently can get slightly different, but quite similar, graphs by averaging over different long sections of the time axis. It is convenient to assume a "limiting" graph towards which our long-range average graphs are tending.

We can frequently use logical analysis to determine what this "limiting" graph should be. In the case of the coin-tossing experiment, we can argue that the coin is reasonably symmetric, and so each side should be as likely as the other. Consequently, we can expect a "limiting: graph like this:



Such logic, unfortunately, fails us in the case of the thumbtack, and we are forced to rely upon our long-range averages computed from empirical data.

For the coin we have a good theory; for the thumbtacks we have none at all. The case of the last digit of the telephone numbers lies somewhere in between: we might believe that all digits are equally likely, on the grounds that the telephone company uses essentially consecutive numbers without gaps. On the other hand, it is harder to be sure just how telephone numbers are assigned, and so we are less confident that all digits really are equally likely. It is, however, possible to compare our "equally-likely" theoretical limit graph against graphs obtained empirically from long averages along the time axis. This comparison might be quite interesting.

We shall make one further modification of our 2-dimensional "limits" graph. The various outcomes of an experiment are usually things like "heads", "tails", "point-up", "point-down:, and so on. These outcomes do not naturally arrange themselves along a number line. We shall consequently dispense with the graphical arrangement, and shall concern ourselves only with the <u>set</u> of possible outcomes, which we shall call a <u>sample space</u>.

Examples:

1) If we toss a coin once, the set of possible outcomes (or "sample space") might evidently be written $\{H,T\}$.

2) If we toss a single die, it can come to rest showing, 1, 2, 3, 4, 5, or 6 on its uppermost face. We can represent this set of possible outcomes as $\{1, 2, 3, 4, 5, 6\}$.

3) If we toss one dime and one quarter, we can list the outcome in a definite order, giving the outcome for the dime first, then the outcome for the quarter. Thus, HT would mean the dime showed heads, the quarter showed tails. Using this convention, the sample space might be written

$$\{HH, HT, TH, TT\}.$$

4) If we throw two dice simultaneously, and care only about the total obtained by adding the two numbers on the uppermost faces, we might write the sample space this way:

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

5) For our thumbtack, the sample space might be written U, D, where "U" means the tack came to rest point-up, and "D" means that the tack came to rest point-down.

We have replaced our horizontal axis by a simple listing of the possible outcomes of an experiment. We must, however, retain the numberal values which our 2-dimensional limit graph exhibited along the vertical axis. We shall do this by means of a function f whose range is a subset of the set of real numbers.

Examples:

1) For our single coin experiment, the sample space is

$$\{H, T\}$$

and the function f is defined as

$$f(H) = \tfrac{1}{2}$$
$$f(T) = \tfrac{1}{2}$$

2) For the thumbtack experiment, use your own data to determine f(U) and f(d). Depending upon the kind of thumb-tack that you used, the surface onto which it fell, and the method of dropping it, you may get different ratios of U's and D's. If, in <u>a</u> drops you got <u>b</u> U's and <u>a - b</u> D's, then your estimated limit graph might result in this function:

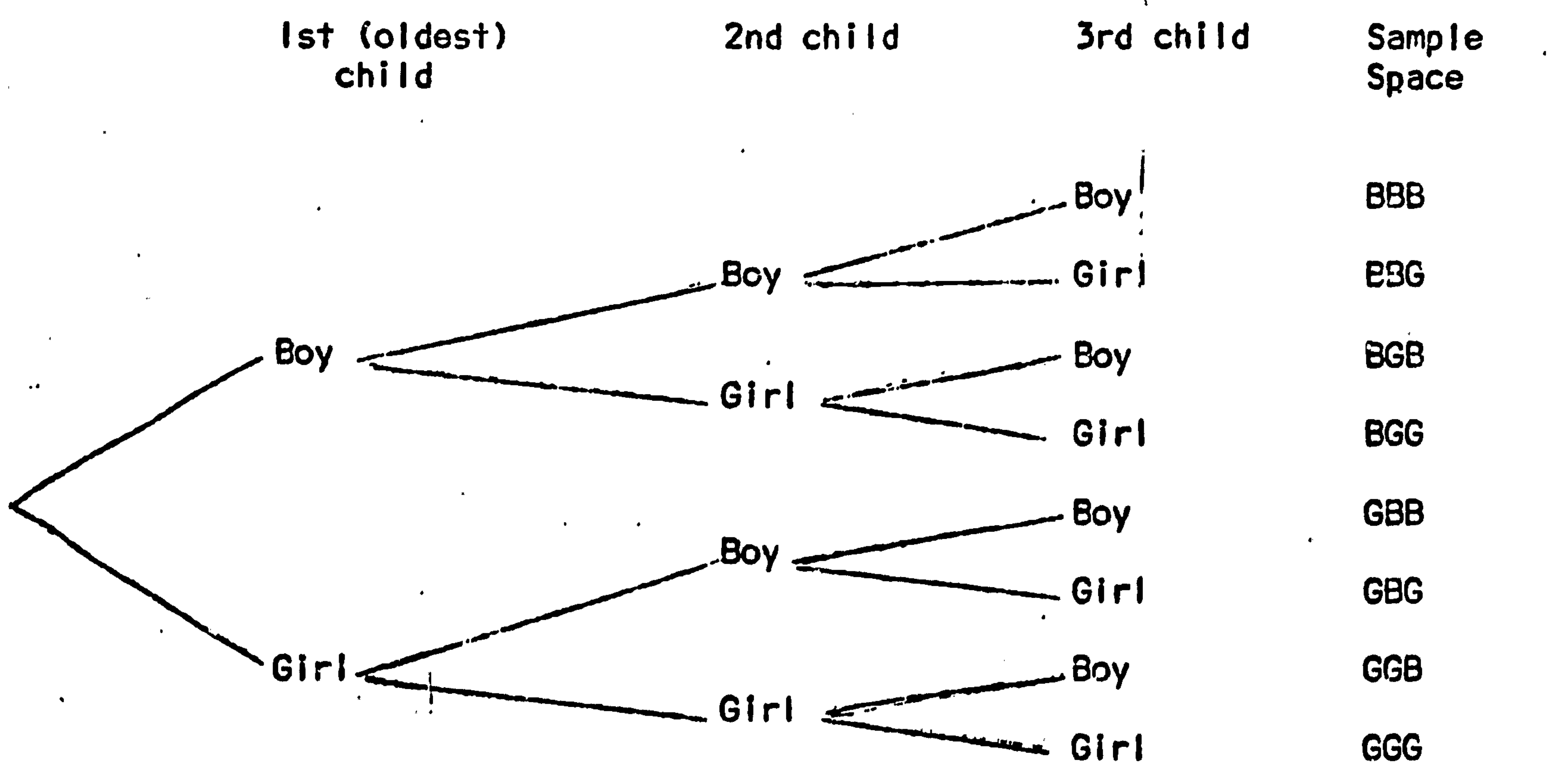
$$f(U) = \frac{b}{a}$$
$$f(D) = \frac{a-b}{a}$$

<u>Question III.</u> Even without knowing the actual experiment and the actual sample space that someone has in mind, can you describe certain limitations on the function f which he must use?

The Use of Tree Graphs

The task of deciding upon a sample space is sometimes simplified by using a "tree graph". We can illustrate this method by an example:[1]

<u>Three-child families</u>. To study the distribution of boys and girls in families having three children, a survey of such families is made. What is a sample space for the experiment of drawing one family from a population of three-child families? We can construct a "tree graph" like this:



In the usual set notation, we could write the sample space as

$\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

Suggested continuation of Section VII

1) Discuss "events" as subsets of the sample space.
2) Describe the function f, extending its domain to the set of subsets of the sample space. Include additive property.

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1. This example is quoted from <u>Probability: A First Course</u>, by Mosteller, Rourke and Thomas (Addison-Wesley, 1961), pp. 64, 65.

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Section II

Permanent Experiment # 1

Why don't you perform a big thumb-tack experiment with your class? If you have 20 or more people in class, have each person drop a tack 40 times. Have him record each "Up" or "Down" as it occurs, and separate his answers into groups of 5 each. Then, by combining groups, you will be able to get 10 groups of 5, or 10 groups of 10, or 10 groups of 20, or 10 groups of 40, or 10 groups of 80.

Keep all your data! We will be able to make use of it again in the future.

Is the number of U's more predictable in a large sample, or in a small sample?

A. Record the number of U's in each of 10 groups of 5:

___. ___. ___. ___. ___. ___. ___. ___. ___. ___.

B. Record the number of U's in each of 10 groups of 10:

___. ___. ___. ___. ___. ___. ___. ___. ___. ___.

C. Record the number of U's in each of 10 groups of 20:

___. ___. ___. ___. ___. ___. ___. ___. ___. ___.

D. Record the number of U's in each of 10 groups of 40:

___. ___. ___. ___. ___. ___. ___. ___. ___. ___.

E. Record the number of U's in each of 10 groups of 80:

___. ___. ___. ___. ___. ___. ___. ___. ___. ___.

Question 1: Looking at your data above, where is it easier to predict the number of U's, in groups of 5 or in groups of 20, or in groups of 80?

Question 2: Can you describe what we mean by the "variability" in a set of numbers? Which set of numbers shows the greatest variability, the set recorded under A, or the set recorded under C, or; the set recorded under E?

We need some good methods for studying how much "variation" there is in a set of numbers. Here are 5 methods:[1]

---

1. We shall take our data from Section I. Why don't you use data from the experiment that your class did.

I. The Method of "Just Looking".

For groups of 5, we got these numbers: (counting "Ups")

2, 3, 3, 3, 4, 1, 2, 3, 3, 2.

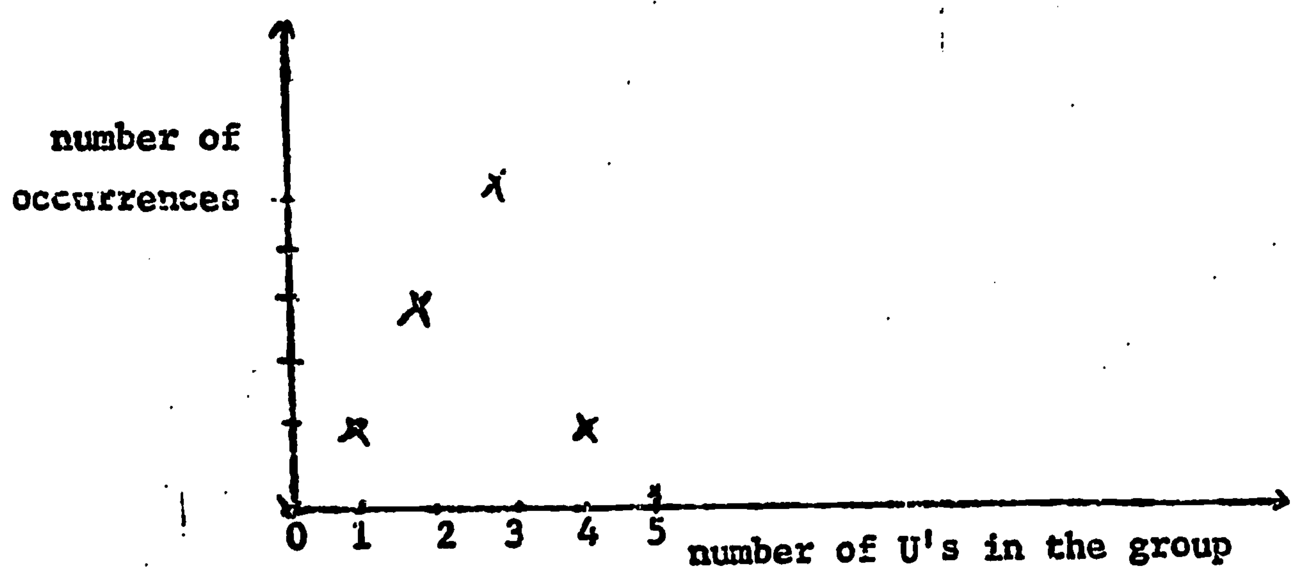
For groups of 20, we got these numbers:

14, 7, 12, 12, 10, 10, 9, 10, 11, 11.

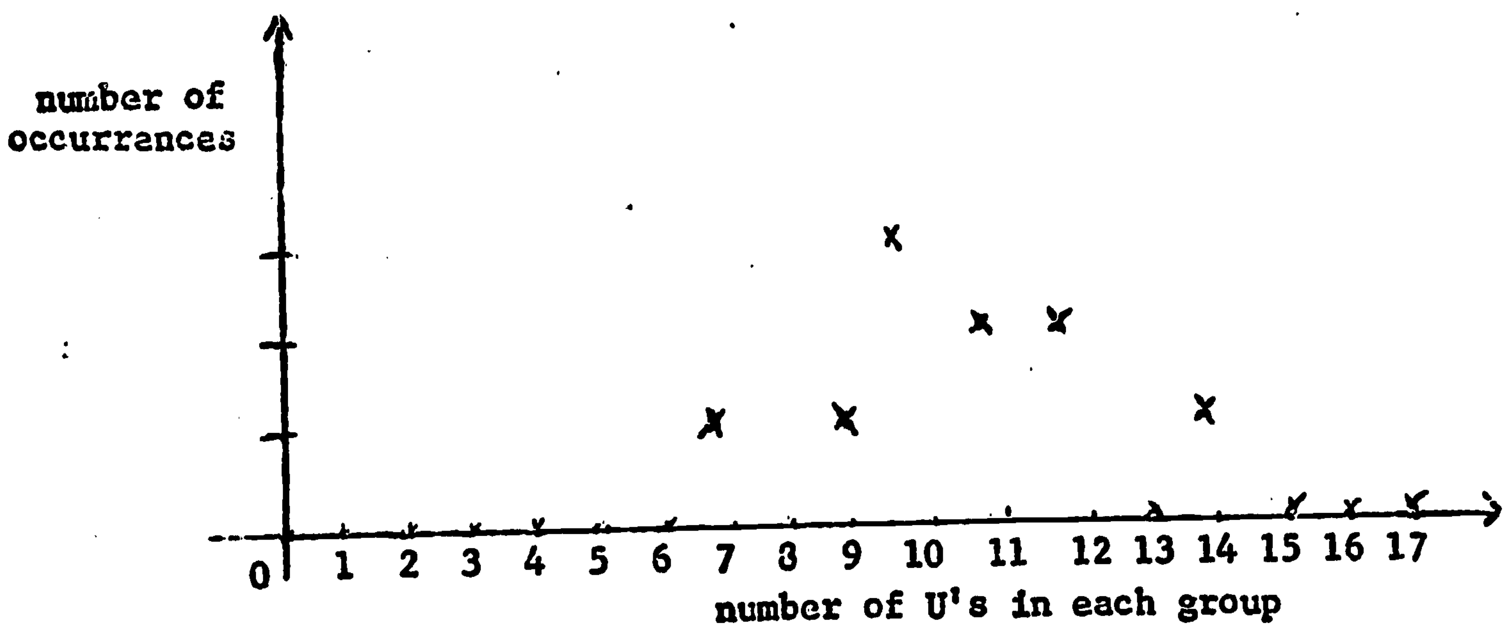
By just looking at these numbers, which set of numbers seems to show greater "variation"?

II. The Method of Graphs.

We can show the first set of numbers of a graph like this:



Number of "Ups" in 10 groups of 5 drops



<u>Number of "Ups" in 10 groups of 20 drops.</u>

From looking at these two graphs, which set of numbers seems to show greater variability?

III. The Method of Mean Absolute Deviation from the Mean.

One good method requires that we compute the "average" or "mean" for each set of numbers:

$$2 + 3 + 3 + 3 + 4 + 1 + 2 + 3 + 3 + 2 = 26$$

$$\frac{26}{10} = 2.6$$

$$14 + 7 + 12 + 12 + 10 + 10 + 9 + 10 + 11 + 11 = 106$$

$$\frac{106}{10} = 10.6$$

We then compute the distance (on the number line) between each number and the mean:

$$|2 - 2.6| = 0.6$$
$$|3 - 2.6| = 0.4$$
$$|3 - 2.6| = 0.4$$
$$|3 - 2.6| = 0.4$$
$$|4 - 2.6| = 1.4$$
$$|1 - 2.6| = 1.6$$
$$|2 - 2.6| = 0.6$$
$$|3 - 2.6| = 0.4$$
$$|3 - 2.6| = 0.4$$
$$|2 - 2.6| = 0.6$$

We have now computed the "deviations from the mean" for our first set of numbers. We now proceed to compute the <u>average deviation</u> by averaging these new numbers:

$$0.6 + 0.4 + 0.4 + 0.4 + 1.4 + 1.6 + 0.6 + 0.4 + +0.4 + 0.6 = 6.8$$

$$\frac{6.8}{10} = 0.68$$

This is the mean absolute deviation from the first set of numbers (groups of 5 drops).

Now, let's do the same thing with our second set of numbers (groups of 20 drops):

$|14 - 10.6| = 3.4$

$|7 - 10.6| = 3.6$

$|12 - 10.6| = 1.4$

$|12 - 10.6| = 1.4$

$|10 - 10.6| = 0.6$

$|10 - 10.6| = 0.6$

$|9 - 10.6| = 1.6$

$|10 - 10.6| = 0.6$

$|11 - 10.6| = 0.4$

$|11 - 10.6| = 0.4$

$3.4 + 3.6 + 1.4 + 1.4 + 0.6 + 0.6 + 1.6 + 0.6 + 0.4 + 0.4 = 14$

$$\frac{14}{10} = 1.4$$

This is the mean absolute deviation for the second set of numbers (groups of 20 drops).

From this method of comparison, which set of numbers seems to vary more?

IV. The Method of Comparing Ranges.

For the first set of numbers

2, 3, 3, 3, 4, 1, 2, 3, 3, 2

the smallest is 1 and the largest is 4. The range, therefore, is

$4 - 1 = 3$

For the second set of numbers

14, 7, 12, 12, 10, 10, 9, 10, 11, 11,

the smallest is 7 and the largest is 14. The range, therefore is: $14 - 7 = 7$.

From comparing the ranges, which set of numbers seems to show the greater variability?

V. The Method of Comparing Trimmed Ranges.

To use this method, we arrange the numbers in order of size:

1, 2, 2, 2, 3, 3, 3, 3, 3, 4

7, 9, 10, 10, 10, 11, 11, 12, 12, 14.

We then "trim" each set by discarding (say) the two "largest" and the "smallest" numbers in each:

2, 2, 3, 3, 3, 3

10, 10, 10, 11, 11, 12

For these "trimmed" sets of numbers, we compute the ranges:

3 - 2 = 1 trimmed range for first set of numbers (groups of 5)

12 - 10 = 2 trimmed range for second set of numbers (groups of 20).

By using this method of comparison, which set of numbers seems to show greater variability?

Question 3: Which set of numbers, in your data, shows greater variability, the set recorded under C, or the set recorded under E?

Question 4: Can you predict the total number of "Ups" more accurately in small numbers of tosses, or in large numbers of tosses?

Question 5: If we want to get a set of numbers showing twice as much variability, should we use sample sizes twice as large? One-half as large? four times as large? One-fourth as large? Or what?

Question 6: Do you know how mathematicians express the answer to Question 5?

Question 7: What advantages and disadvantages can you find to help choose between the 5 different methods for comparing variability?

Question 8: Jerry says that, even though the second set of numbers seemed to show more variability, there is some sense in which it really shows less variability. What do you think? How would you suggest we deal with these two sets of numbers?

2925-66

Section III

Proportional Occurrence of U's

Question 1: Ellen says that even though the total number of U's is harder to predict for larger samples, the proportional occurrence of U's is easier to predict for larger samples. What do you think? What does your data suggest?

Let's test Ellen's suggestion by each of our 5 methods for comparing variability. In Section II we compared the variability of the total number of U's. We now compare the variability of the proportional or fractional number of U's.

Question 2: How do you expect the variability of the fractional number of U's in the 5 drop case will compare with the fractional number of U's in the 20 drop case?

Method I:

The fractional number of U's in the 5 drop case can be found by taking the total number of U's:

2, 3, 3, 3, 4, 1, 2, 3, 3, 2

and dividing by the total number of drops (in this case, 5):

$$\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}$$

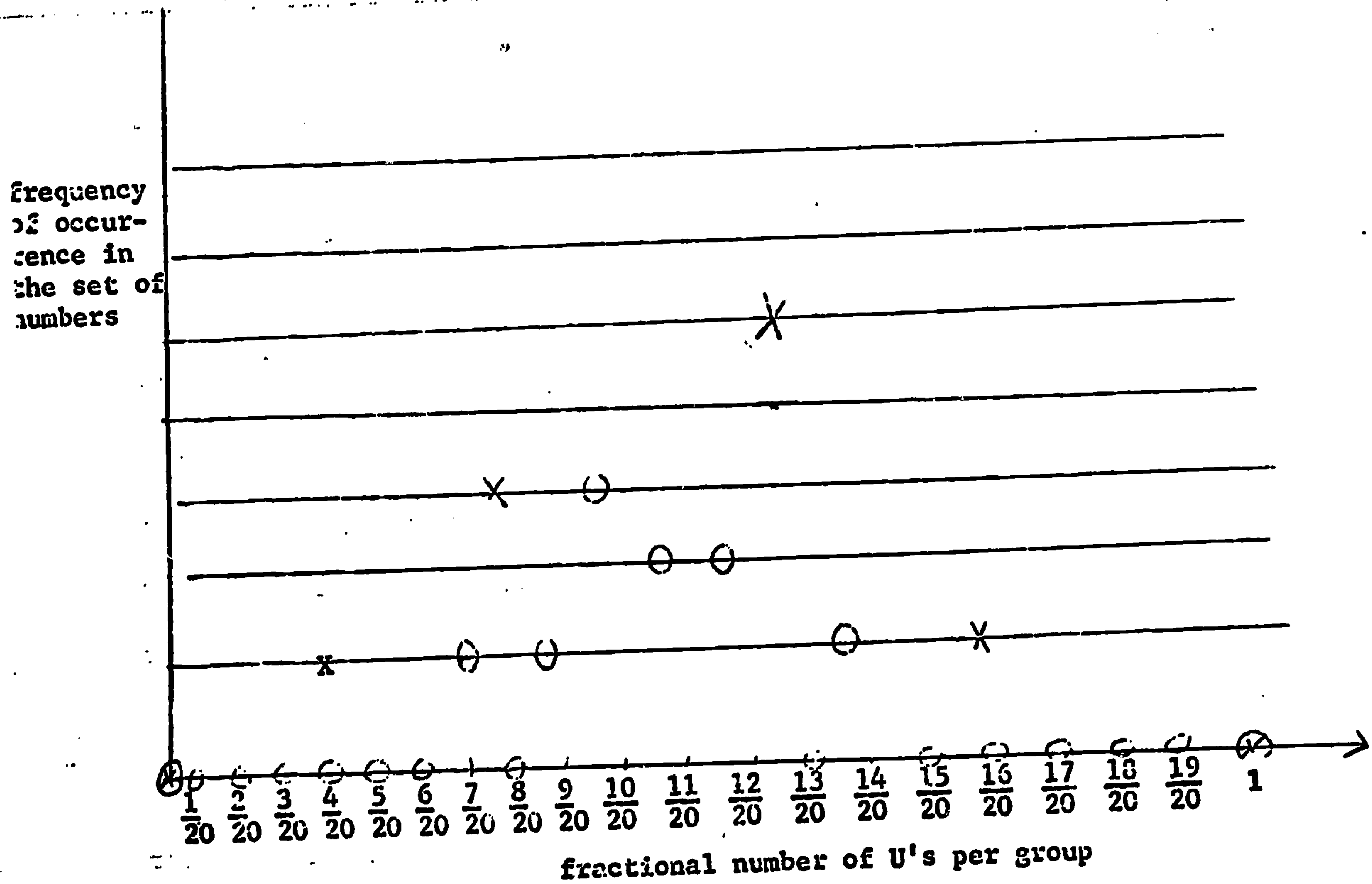
For the groups of 20 drops we get

$$\frac{14}{20}, \frac{7}{20}, \frac{12}{20}, \frac{12}{20}, \frac{10}{20}, \frac{10}{20}, \frac{9}{20}, \frac{10}{20}, \frac{11}{20}, \frac{11}{20}$$

Can you tell, by just looking, which set of numbers varies more?

Method II: Comparison by Graphs.

We shall mark both sets of numbers on the same graph, using x's to indicate the 1st set, and O's to indicate the 2nd set:



Which set of numbers shows greater consistency? Wich shows greater variability? Did it work out the way you expected?

Method III: Comparison of Mean Absolute Deviation from the Sample Mean.

Evidently, the mean for the 1st set of numbers must be

$$\frac{2.6}{5} = 0.52$$

The absolute deviations from 0.52 are

$|0.4 - 0.52| = 0.12$

$|0.6 - 0.52| = 0.08$

$|0.6 - 0.52| = 0.08$

$|0.6 - 0.52| = 0.08$

$|0.8 - 0.52| = 0.28$

$|0.2 - 0.52| = 0.32$

$|0.4 - 0.52| = 0.12$

$|0.6 - 0.52| = 0.08$

$|0.6 - 0.52| = 0.08$

$|0.4 - 0.52| = 0.12$

Averaging these deviations, we get

$0.12 + 0.08 + 0.08 + 0.08 + 0.28 + 0.32 + 0.12 + 0.08 + 0.08 + 0.12 = 1.36$

$\frac{1.36}{10} = 0.136$ ← This is the mean absolute deviation for the 1st set of numbers (group of 5 drops).

We can now do the same thing for the 2nd set of numbers (groups of 20 drops):

The mean is

$$\frac{10.6}{20} = 0.53.$$

The absolute deviations from this mean are:

$|0.7 - 0.53| = 0.17$

$|0.35 - 0.53| = 0.18$

$|0.6 - 0.53| = 0.07$

$|0.6 - 0.53| = 0.07$

$|0.5 - 0.53| = 0.03$

$|0.5 - 0.53| = 0.03$

$|0.45 - 0.53| = 0.08$

$|0.5 - 0.53| = 0.03$

$|0.55 - 0.53| = 0.02$

$|0.55 - 0.53| = 0.02$

$0.17 + 0.18 + 0.07 + 0.07 + 0.03 + 0.03 + 0.08 + 0.03 + 0.02 + 0.02 = 0.7$

$\frac{0.7}{10} = 0.07 \leftarrow$ This is the mean absolute deviation for the 2nd set of numbers (group of 20 drops, using proportion of U's rather than total number of U's).

Which set of numbers seems to vary more? How much more?

IV. The Method of Comparing Ranges

The 1st set of numbers (proportion of U's in group of 5 drops) is

0.2, 0.4, 0.4, 0.4, 0.6, 0.6, 0.6, 0.6, 0.6, 0.8

The smallest number is 0.2, and the largest is 0.8. Consequently, the range is $0.8 - 0.2 = 0.6$.

The 2nd set of numbers (proportion of U's in groups of 20 drops) is

0.35, 0.45, 0.50, 0.50, 0.50, 0.55, 0.55, 0.60, 0.60, 0.70,

The largest is 0.70, and the smallest is 0.35. Consequently the range is $0.70 - 0.35 = 0.35$,

V. The Method of "Trimmed" Ranges

For the 1st set of numbers, we delete the two largest and the two smallest, to get a "trimmed" set of numbers:

0.4, 0.4, 0.6, 0.6, 0.6, 0.6.

The range is now $0.6 - 0.4 = 0.2$.

For the 2nd set of numbers (groups of 20 drops), if we omit the 2 largest and 2 smallest we get the "trimmed" set of numbers:

0.50, 0.50, 0.50, 0.55, 0.55, 0.60

The range of this trimmed set is $0.60 - 0.50 = 0.10$.

Question #3: Does the total number of U's vary more in large samples, or in small samples?

Question #4: Does the proportion of U's vary more in large samples, or in small samples?

Question: #5: Can you summarize what we have learned? What does your data seem to indicate?

2926-66

Section IV

Variability of Total Number of U's, and of Proportion of U's, in Large Samples and in Small Samples.

(Summary of Sections I-III)

Alex says that mathematicians talk about our thumb-tack experiment this way:

When we were using 10 groups of 5 drops each, they would say we had a sample size n, equal to 5.

When we were using 10 groups of 20 drops each, they would say we had a sample size n, equal to 20.

In general, when we increase the sample size by making it 4 times as big, the variability of the total number of U's would be expected to increase by a factor of 2. Consequently, mathematicians say that the variability of the total number of U's increases as $\sqrt{n}$.

In the fractional proportion of U's, however, the situation is quite different. Here, if we multiply the sample size by 4, the variability of the fractional proportion of U's decreases by a factor of 2. Consequently, mathematicians say that the fractional proportion of U's decreases like $\frac{1}{\sqrt{n}}$

Question 1: Is this what your data seemed to indicate?

Question 2: Could you come closer, in predicting the number of U's, from a small number of drops, or from a large number of drops?

Question 3: Does your data become more variable or less variable, as you go to larger-sized samples?

Question 4: Can you summarize what we have learned?

Question 5: Why do you think we use fractions so much in the theory of probability?

2955-66

Section V

A Telephone Book Experiment

Experiment II. Look at some "randomly chosen" page well into the phone book. Make a record of the last digit of the 1st 40 numbers of the page, grouping by fives as usual. Each student should collect this data independently, so that we can combine into a "big experiment." From this record, determine the frequency of occurrence of each digit, and the relative frequency of each. Study the variability of these frequencies as a function of sample size, as was done in Experiment I.

Here is some typical data (although you will undoubtedly want to work with data collected by your own class):

Harold's data:

4, 9, 5, 6, 3

5, 6, 5, 4, 4

2, 9, 4, 2, 2,

8, 0, 8, 4, 3

4, 4, 0, 5, 6

8, 9, 9, 0, 8

0, 2, 0, 7, 3

0, 1, 8, 4, 6

Judy's data:

3, 3, 4, 7, 8

5, 3, 0, 9, 6

2, 8, 7, 9, 4

8, 5, 9, 6, 4

9, 5, 9, 3, 9

8, 9, 9, 6, 7

1, 7, 7, 9, 7

1, 6, 6, 8, 8

Using only Harold's and Judy's data we find:

digit	total number of occurrences	relative proportion of occurrences
0	0, 0, 0, 1, 1, 1, 2, 1, 0, 1	$0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{5}$
1	0, 0, 0, 0, 0, 0, 0, 1, 0, 0	$0, 0, 0, 0, 0, 0, 0, \frac{1}{5}, 0, 0$
2	0, 0, 3, 0, 0, 1, 2, 1, 0, 1	$0, 0, \frac{3}{5}, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{5}$
3	1, 0, 0, 1, 0, 0, 1, 0, 2, 1	$\frac{1}{5}, 0, 0, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, \frac{2}{5}, \frac{1}{5}$
4	1, 2, 1, 1, 2, 0, 0, 1, 1, 0	$\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, 0, 0, \frac{1}{5}, \frac{1}{5}, 0$
5	1, 2, 0, 0, 1, 0, 0, 0, 0, 1	$\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{1}{5}, 0, 0, 0, 0, \frac{1}{5}$
6	1, 1, 0, 0, 1, 0, 0, 1, 0, 1	$\frac{1}{5}, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, 0, \frac{1}{5}, 0, \frac{1}{5}$
7	0, 0, 0, 0, 0, 0, 1, 0, 1, 0	$0, 0, 0, 0, 0, 0, \frac{1}{5}, 0, \frac{1}{5}, 0$
8	0, 0, 0, 2, 0, 2, 0, 1, 1, 0	$0, 0, 0, \frac{2}{5}, 0, \frac{2}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0$
9	1, 0, 1, 0, 0, 2, 0, 0, 0, 1	$\frac{1}{5}, 0, \frac{1}{5}, 0, 0, \frac{2}{5}, 0, 0, 0, \frac{1}{5}$

In order to study 10 groups of 10, we need more raw data. Here is Marilyn's data:

9, 2, 0, 4, 3

8, 7, 7, 7, 4

2, 1, 6, 1, 4

3, 3, 9, 7, 9

7, 0, 1, 8, 0

7, 9, 4, 1, 6

4, 1, 4, 9, 8

4, 4, 1, 3, 7

Occurrences of Digits in Each of 10 Groups of 10

digit	total number of occurrences	relative proportion of occurrences
0	0, 1, 2, 3, 1, 0, 0, 0, 1, 0	0, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\frac{1}{10}$, 0, 0, 0, $\frac{1}{10}$, 0
1	0, 0, 0, 1, 0, 0, 0, 2, 1, 0	0, 0, 0, $\frac{1}{10}$, 0, 0, 0, $\frac{2}{10}$, $\frac{1}{10}$, 0
2	0, 3, 0, 1, 0, 1, 0, 0, 1, 1	0, $\frac{3}{10}$, 0, $\frac{1}{10}$, 0, $\frac{1}{10}$, 0, 0, $\frac{1}{10}$, $\frac{1}{10}$
3	1, 1, 0, 1, 3, 0, 1, 0, 1, 2	$\frac{1}{10}$, $\frac{1}{10}$, 0, $\frac{1}{10}$, $\frac{3}{10}$, 0, $\frac{1}{10}$, 0, $\frac{1}{10}$, $\frac{2}{10}$
4	3, 2, 2, 1, 1, 2, 0, 0, 2, 1	$\frac{3}{10}$, $\frac{2}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, 0, 0, $\frac{2}{10}$, $\frac{1}{10}$
5	3, 0, 1, 0, 1, 1, 1, 0, 0, 0	$\frac{3}{10}$, 0, $\frac{1}{10}$, 0, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, 0, 0, 0
6	2, 0, 1, 1, 1, 1, 1, 2, 0, 1	$\frac{2}{10}$, 0, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, 0, $\frac{1}{10}$
7	0, 0, 0, 1, 1, 1, 1, 3, 3, 1	0, 0, 0, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{3}{10}$, $\frac{1}{10}$
8	0, 2, 2, 1, 1, 2, 1, 2, 1, 0	0, $\frac{2}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{1}{10}$, 0
9	1, 1, 2, 0, 1, 2, 4, 1, 1, 2	$\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, 0, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{4}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{2}{10}$

In order to consider 20 groups of 20 numbers each ("samples with n equal to 20"), we need more data:

Tom's data:

5, 8, 4, 5, 0

7, 5, 3, 3, 0

0, 8, 0, 9, 1

8, 2, 4, 0, 8

2, 5, 4, 2, 5

1, 1, 9, 7, 4

1, 9, 6, 6, 7

8, 9, 7, 6, 7

Bills data:

0, 4, 6, 2, 3

0, 2, 5, 7, 2

0, 0, 5, 5, 1

0, 7, 5, 8, 9

9, 0, 2, 4, 7

7, 0, 8, 4, 1

4, 2, 1, 4, 9

0, 3, 1, 4, 4

Occurrences of Digits in Each of 10 Groups of 20

Digit	total number of occurrences	relative proportion of occurrences
0	1, 5, 1, 0, 1, 2, 5, 0, 5, 3	$\frac{1}{20}, \frac{5}{20}, \frac{1}{20}, 0, \frac{1}{20}, \frac{2}{20}, \frac{5}{20}, 0, \frac{5}{20}, \frac{3}{20}$
1	0, 1, 0, 2, 2, 4, 1, 3, 1, 3	$0, \frac{1}{20}, 0, \frac{2}{20}, \frac{2}{20}, \frac{4}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{3}{20}$
2	3, 1, 1, 0, 2, 0, 1, 2, 2, 2	$\frac{3}{20}, \frac{1}{20}, \frac{1}{20}, 0, \frac{2}{20}, 0, \frac{1}{20}, \frac{2}{20}, \frac{2}{20}, \frac{2}{20}$
3	2, 1, 3, 1, 3, 1, 2, 0, 1, 1	$\frac{2}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{2}{20}, 0, \frac{1}{20}, \frac{1}{20}$
4	5, 3, 3, 0, 3, 4, 2, 2, 1, 6	$\frac{5}{20}, \frac{3}{20}, \frac{3}{20}, 0, \frac{3}{20}, \frac{4}{20}, \frac{2}{20}, \frac{2}{20}, \frac{1}{20}, \frac{6}{20}$
5	3, 1, 2, 1, 0, 0, 3, 2, 4, 0	$\frac{3}{20}, \frac{1}{20}, \frac{2}{20}, \frac{1}{20}, 0, 0, \frac{3}{20}, \frac{2}{20}, \frac{4}{20}, 0$
6	2, 2, 2, 2, 1, 1, 0, 3, 1, 0	$\frac{2}{20}, \frac{2}{20}, \frac{2}{20}, \frac{2}{20}, \frac{1}{20}, \frac{1}{20}, 0, \frac{3}{20}, \frac{1}{20}, 0$
7	0, 1, 2, 4, 4, 3, 1, 3, 2, 2	$0, \frac{1}{20}, \frac{2}{20}, \frac{4}{20}, \frac{4}{20}, \frac{3}{20}, \frac{1}{20}, \frac{3}{20}, \frac{2}{20}, \frac{2}{20}$
8	2, 3, 3, 3, 1, 2, 4, 1, 1, 1	$\frac{2}{20}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{1}{20}, \frac{2}{20}, \frac{4}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}$
9	2, 2, 3, 6, 3, 2, 1, 3, 1, 2	$\frac{2}{20}, \frac{2}{20}, \frac{3}{20}, \frac{6}{20}, \frac{3}{20}, \frac{2}{20}, \frac{1}{20}, \frac{3}{20}, \frac{1}{20}, \frac{2}{20}$

We can now test the suggestion that the variability of totals increases like $\sqrt{n}$, and the variability of fractional occurrences decreases like $\frac{1}{\sqrt{n}}$, where n is the so-called "sample size."

We shall use three methods: the method of ranges, the method of trimmed ranges, and the method of average ranges. The first two of these methods we used in Experiment I; the method of average ranges will, however, be new.

Method of Ranges: For the total number of occurrences of the digit 0, in sample sizes of 5 (n=5), we have:

0, 0, 0, 0, 1, 1, 1, 1, 1, 2

Evidently, the range is 2-0=2.

For the total number of occurrences of the digit 0, m' sample sizes of 20 (n=20), we have:

0, 0, 1, 1, 1, 2, 3, 5, 5, 5

Evidently the range is 5-0=5. It is reasonably close to our generalization that, if we multiply the sample size by 4, we double the variability (in this case, we double the range).

Here are some further comparisons:

	Total number of occurrences		Fractional proportion of occurrences	
digit	range for n = 5	range for n = 20	range for n = 5	range for n = 20
0	2	5	0.4	0.25
1	1	4	0.2	0.2
2	3	3	0.6	0.15
3	2	3	0.4	0.15
4	2	6	0.4	0.3
5	2	3	0.4	0.15
6	1	3	0.2	0.15
7	1	4	0.2	0.2
8	2	3	0.4	0.15
9	2	5	0.4	0.25

This table does not seem to show very decisive agreement without generalization about variability. What do your data show?

Method of Trimmed Ranges

If we delete the two largest and two smallest members from each set, we get

	trimmed set for total no. of occurrences	
digit	group with n=5	group with n=20
0	0, 0, 1, 1, 1, 1	1, 1, 1, 2, 3, 5
1	0, 0, 0, 0, 0, 0	1, 1, 1, 2, 2, 3
2	0, 0, 0, 1, 1, 1	1, 1, 1, 2, 2, 2
3	0, 0, 0, 1, 1, 1	1, 1, 1, 1, 2, 2
4	0, 1, 1, 1, 1, 1	2, 2, 3, 3, 3, 4
5	0, 0, 0, 0, 1, 1,	0, 1, 1, 2, 2, 3
6	0, 0, 0, 1, 1, 1	1, 1, 1, 2, 2, 2
7	0, 0, 0, 0, 0, 0	1, 2, 2, 2, 3, 3
8	0, 0, 0, 0, 1, 1	1, 1, 2, 2, 3, 3
9	0, 0, 0, 0, 1, 1	2, 2, 2, 2, 3, 3

For the trimmed ranges we get:

digit	trimmed range for total no. of occurrences		trimmed range for fractional proportion of occurrences	
	n=5	n=20	n=5	n=20
0	1	4	0.2.	0.2
1	0	2	0	0.1
2	1	1	0.2	0.05
3	1	1	0.2	0.05
4	1	2	0.2	0.1
5	1	3	0.2	0.15
6	1	1	0.2	0.05
7	0	2	0	0.1
8	1	2	0.2	0.1
9	1	1	0.2	0.05

Method of Average Range

Combining our data for all digits, we can compute the average range and average trimmed range as follows:

Average range				Average Trimmed range			
Total no. of occurrences		Proportional fraction of occurrences		Total no. of occurrences		Proportional fraction of occurrences	
n=5	n=20	n=5	n=20	n=5	n=20	n=5	n=20
1.8	3.9	0.36	.19	0.8	1.9	1.6	.09

This table appears to fit in quite nicely with our generalization that the variability of total number of occurrences increases like $\sqrt{n}$, while the variability of the fractional proportion of occurrences decreases like $\frac{1}{\sqrt{n}}$, as the sample size n increases.

What do *your* data show?

Section VI

An Experiment with a Coin

Experiment III: Each member of your class can toss a coin 40 times[1], recording each occurrence of heads and tails in order. Keep these records in groups of 5 tosses each. Keep this data permanently -- we can use it repeatedly in the future! You can study the variability of total number of heads, and fractional proportion of heads, as functions of the sample size n.

What do you expect to find? Here is the record of 2,000 tosses of U.S. coins:

H H H T H	T H T T T
T T T T H	T H T H H
H H H T T	T T T H H
H H H T T	H T H H T
H T T T T	T H T T H___100
H T H T H	H H T T T
H H T T T	T T T T T
H T T H T	H T T T T
H T T T H	T T H H T
H T T H T ___50	T T H T T
T H H H H	H T T H T
H H H H H	T T H T T
H H T H T	H T H T T
T T H H T	T H H T H
T H T T T	H T T T T___150
H T T H T	T T T T H
H T T H H	T H T T H
H H H H T	T T T H H
H T T T H	T T T H H
H T H H H	H T T T T
T H H T T	H T T T T
T H T T H	H T T T T
H H T H H	H T T H T
H H H T H	H T T T H
H T T T T___200	T H H T H___350

1. You may want to get records of even more tosses; perhaps a total of at least 2,000.

H H H T H	T H H T T
T H T T H	T H T H T
T T T H H	H H T H H
H T H T T	T T T H T
H H T H T	T T H T H
T T T H H	H H H H H
H T H H H	H H H T H
H T H H H	T T H H T
T H T H H	T H T T H
H H T T H ___250	H H H H T ___400
T H T H H	H T H H T
T H H T H	T H H H T
T T T T T	T H T T T
H H T T T	T T T H T
T T T H H	H H H H H
T H T T T	H H H H H
H T T H T	H H H T H
T T T H T	T T T H T
T H T T T	H T H H H
T H H H H ___300	H T T H H ___450
T T H T H	H H H H H
T H H T T	H H T H H
T T T H H	H T T H H
T H H H H	T H T H H
H T T H H	T H H T T
H H T H T	H H T H H
H H T T H	H H H T T
T H T H H	
H H H T T	
T H H T T ___500	

(Section VI is temporarily left incomplete. In the completed version, one would treat this data as in the preceding sections, studying empirically the variability of totals and ratios as a function of sample size.)

(This coin data would also be used later for an empirical comparison of the "compensation" vs. "swamping" theories of the law of large numbers.)

2928-66

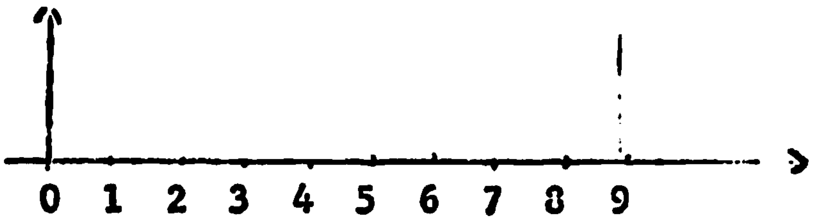
Section VII

An Abstract Model for Chance Events

In the preceding sections, we have made empirical studies of variability, using thumbtacks, telephone directories, and coins. We have seen that as we make our samples larger, the variability of the total number of occurrences of (say) an "Up", or of a "head", becomes larger. However, the fractional proportion of "Ups" or "heads" varies less for larger samples.

Can we use this apparent stability of the fractional proportion of heads as the foundation for a mathematical model? We would like our model to help describe "chance" events. Let's see if we can make one that will have some usefulness.

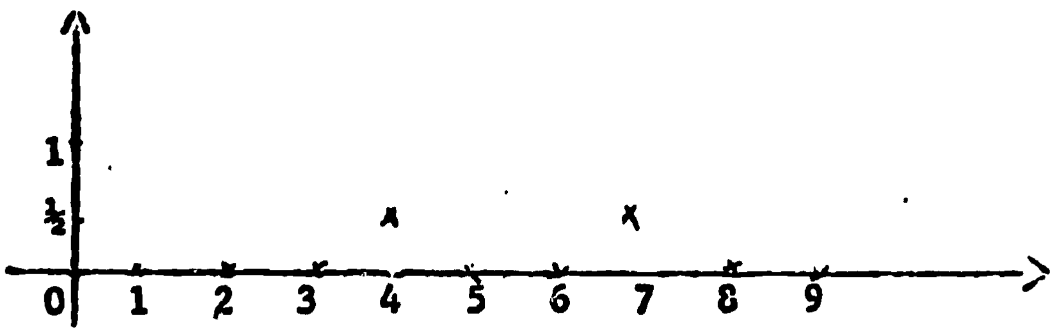
To begin with, let us think of the example of the last digit of a telephone number. We can make a 2-dimensional graph by representing the possible outcomes along the horizontal axis



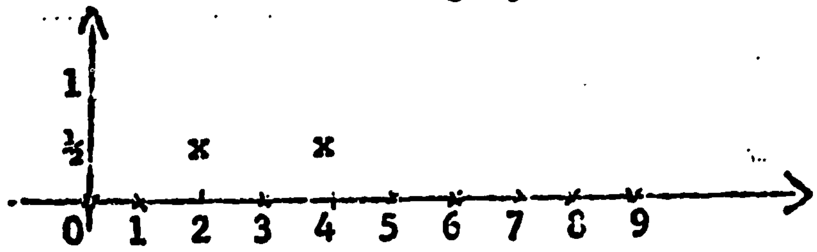
and representing the fractional proportion of occurrences along the vertical axis. Suppose, for 2 numbers, the last digit of one was 7, and of the other was 4. Then the fractional proportion of occurrences would be

Digit	0	1	2	3	4	5	6	7	8	9
Fractional Proportion of Occurrence	0	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0

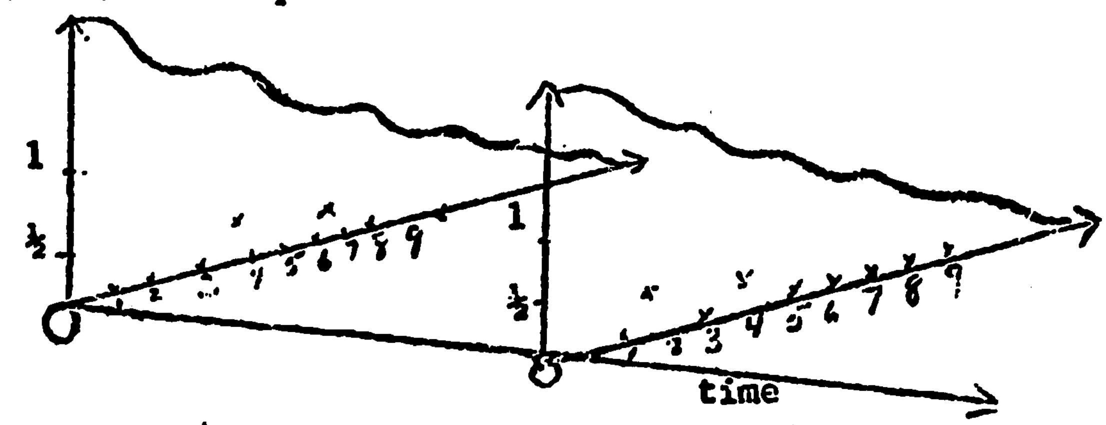
and the corresponding graphical representation would be



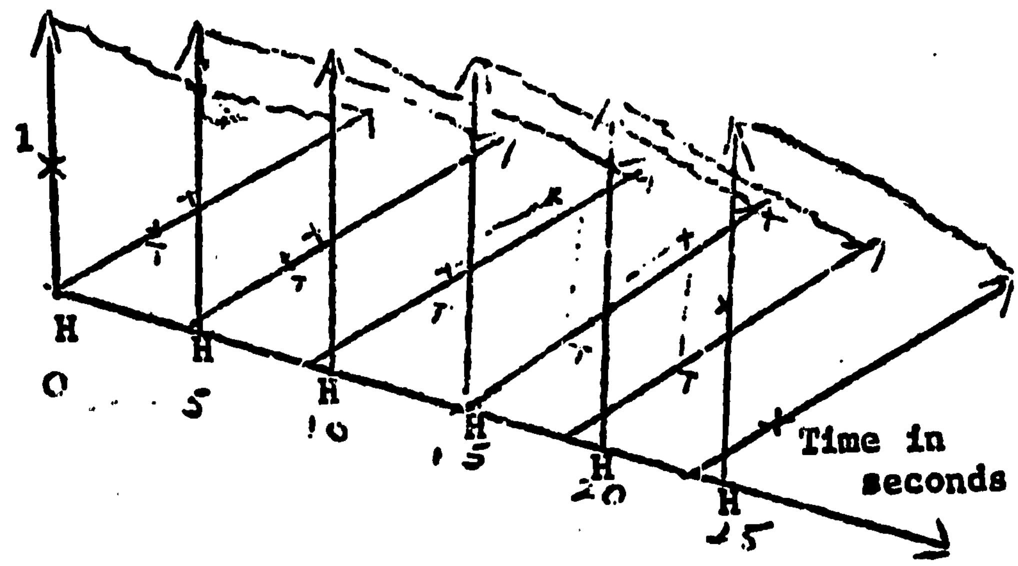
Suppose the experiment of selecting 2 numbers were repeated, and the final digits were 2 and 4. This new graph wouldthen look like this?



We can make a 3-dimensional picture by arranging these two planes parallel at two different points on a "time" axis:

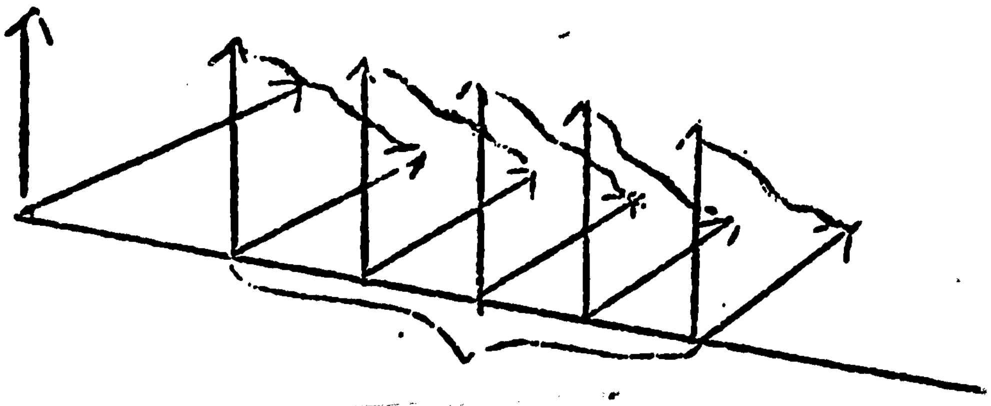


Suppose we toss a coin 6 times, at 5 second intervals. We could represent the outcome by a 3-dimensional picture as follows:

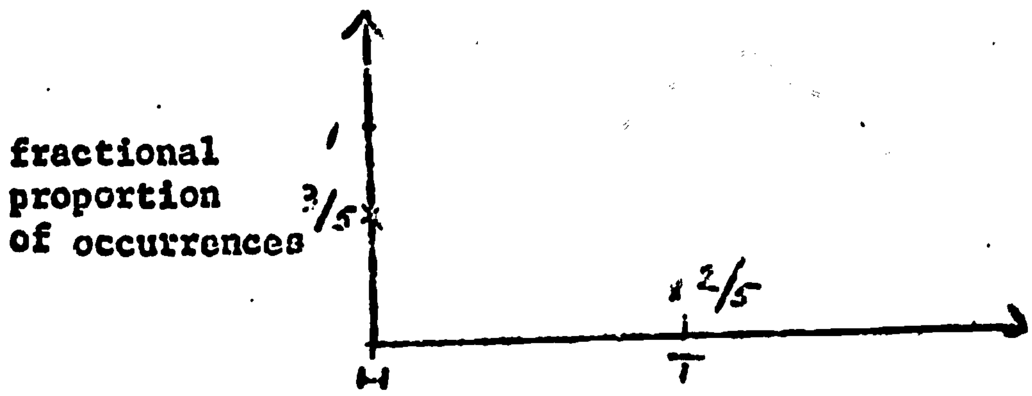


From this picture, we can see that the outcome of the 1st toss was "Heads", the outcome of the 2nd toss was "Heads", of the 3rd also "Heads", the outcome of the 4th toss was "Tails", and so on.

Now what did we seem to be observing in our empirical studies of probability? For one thing, we computed the fractional average, not of a single toss, but cumulatively over many tosses. We took a fairly long section along the time axis

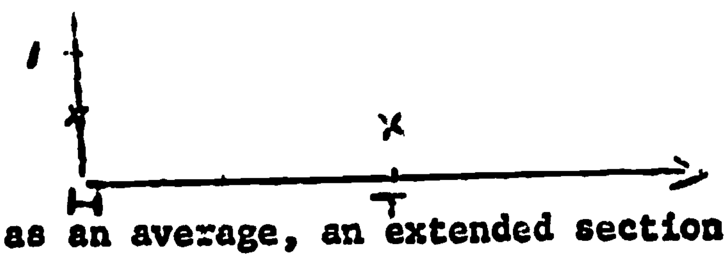


and computed an average for all of the tosses included within this time interval. The resulting 2-dimensional graph might look like this:

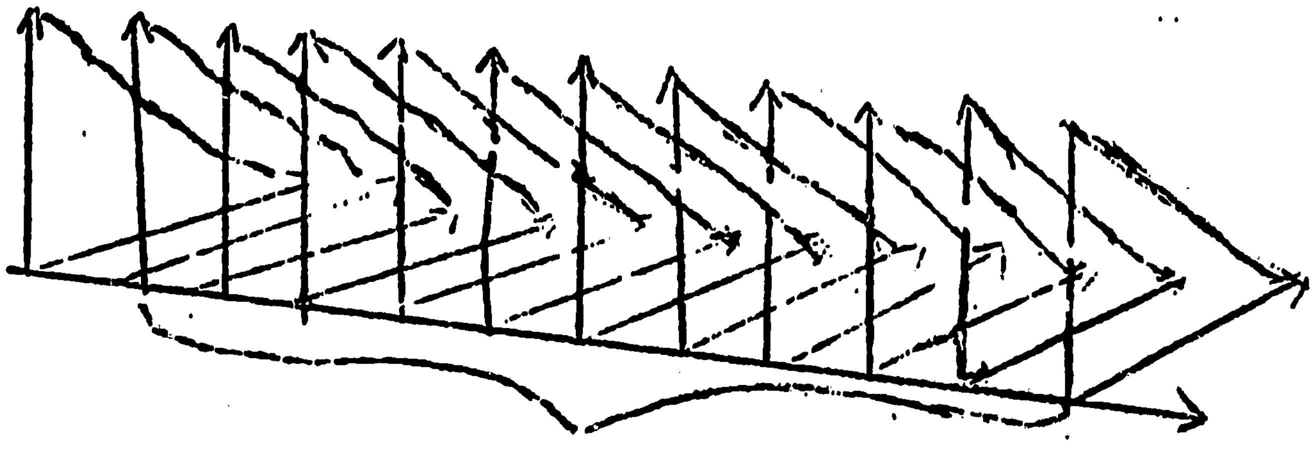


If we take longer and longer sections along the time axis, the variability of the fractional proportion of occurences will become smaller and smaller. The fractions appear to be "homing in" on some constant values, from which they do not deviate very much in large samples.

We might, then base our model upon the idea of a long-range average



which can represent, as an average, an extended section along the time axis:



<u>Question I</u>. What do you think a 2-dimensional "long-range average" would look like for:

a) the thumbtack experiment

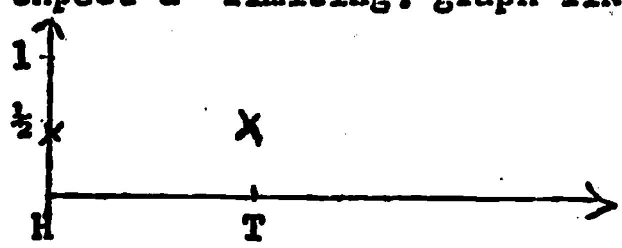
b) the last digit of telephone numbers

c) the coin-tossing experiment.

<u>Question II</u>. If you computed a 2-dimensional graph of fractional occurrences from a very long average along the time axis, would your 2-dimensional graph be relevant to some <u>other</u> long average along the time axis?

We evidently can get slightly different, but quite similar, graphs by averaging over different long sections of the time axis. It is convenient to assume a "limiting' graph towards which our long-range average graphs are tending.

We can frequently use logical analysis to determine what this "limiting" graph should be. In the case of the coin-tossing experiment, we can argue that the coin is reasonably symmetric, and so each side should be as likely as the other. Consequently, we can expect a "limiting: graph like this:



Such logic, unfortunately, fails us in the case of the thumbtack, and we are forced to rely upon our long-range averages computed from empirical data.

For the coin we have a good theory; for the thumbtacks we have none at all. The case of the last digit of the telephone numbers lies somewhere in between: we might believe that all digits are equally likely, on the grounds that the telephone company uses essentially consecutive numbers without gaps. On the other hand, it is harder to be sure just how telephone numbers are assigned, and so we are less confident that all digits really are equally likely. It is, however, possible to compare our "equally-likely" theoretical limit graph against graphs obtained empirically from long averages along the time axis. This comparison might be quite interesting.

We shall make one further modification of our 2-dimensional "limits" graph. The various outcomes of an experiment are usually things like "heads", "tails", "point-up", "point-down:, and so on. These outcomes do not naturally arrange themselves along a number line. We shall consequently dispense with the graphical arrangement, and shall concern ourselves only with the set of possible outcomes, which we shall call a sample space.

Examples:

1) If we toss a coin once, the set of possible outcomes (or "sample space") might evidently be written $\{H,T\}$.

2) If we toss a single die, it can come to rest showing, 1, 2, 3, 4, 5, or 6 on its uppermost face. We can represent this set of possible outcomes as $\{1, 2, 3, 4, 5, 6\}$.

3) If we toss one dime and one quarter, we can list the outcome in a definite order, giving the outcome for the dime first, then the outcome for the quarter. Thus, HT would mean the dime showed heads, the quarter showed tails. Using this convention, the sample space might be written

$$\{HH, HT, TH, TT\}.$$

4) If we throw two dice simultaneously, and care only about the total obtained by adding the two numbers on the uppermost faces, we might write the sample space this way:

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

5) For our thumbtack, the sample space might be written U, D, where "U" means the tack came to rest point-up, and "D" means that the tack came rest point-down.

We have replaced our horizontal axis by a simple listing of the possible outcomes of an experiment. We must, however, retain the numberal values which our 2-dimensional limit graph exhibited along the vertical axis. We shall do this by means of a function f whose range is a subset of the set of real numbers.

Examples:

1) For our single coin experiment, the sample space is

$$\{H, T\}$$

and the function f is defined as

$$f\ (H) = \tfrac{1}{2}$$
$$f\ (T) = \tfrac{1}{2}$$

2) For the thumbtack experiment, use your own data to determine f(U). and f(d). Depending upon the kind of thumb-tack that you used, the surface onto which it fell, and the method of dropping it, you may get different ratios of U's and D's. If, in <u>a</u> drops you got <u>b</u> U's and <u>a - b</u> D's, then your estimated limit graph might result in this function:

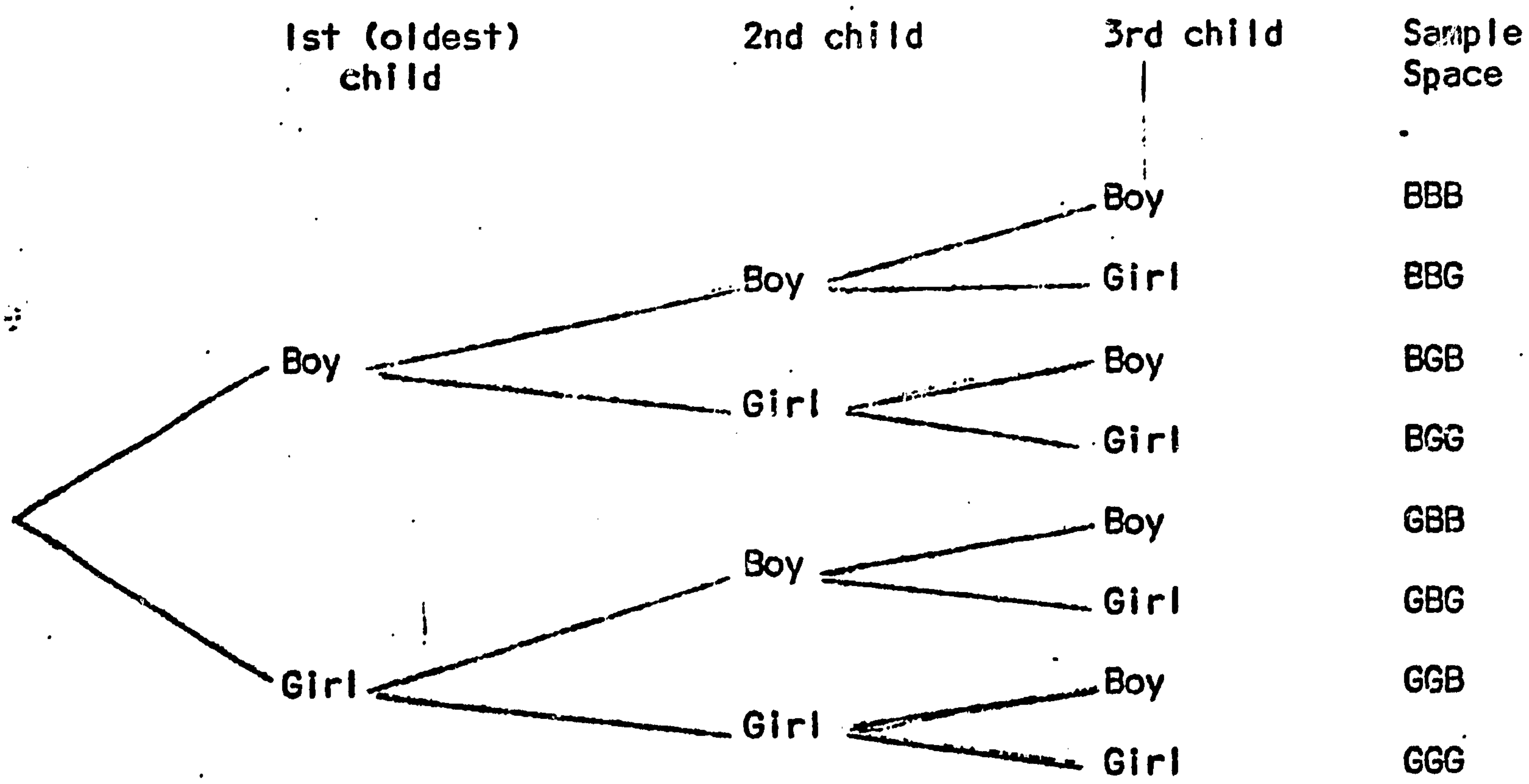
$$f(U) = \frac{b}{a}$$
$$f(D) = \frac{a-b}{a}$$

<u>Question III.</u> Even without knowing the actual experiment and the actual sample space that someone has in mind, can you describe certain limitations on the function f which he must use?

The Use of Tree Graphs

The task of deciding upon a sample space is sometimes simplified by using a "tree graph". We can illustrate this method by an example:[1]

<u>Three-child families</u>. To study the distribution of boys and girls in families having three children, a survey of such families is made. What is a sample space for the experiment of drawing one family from a population of three-child families? We can construct a "tree graph" like this:



In the usual set notation, we could write the sample space as

$\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

Suggested continuation of Section VII

1) Discuss "events" as subsets of the sample space.

2) Describe the function f, extending its domain to the set of subsets of the sample space. Include additive property.

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1. This example is quoted from <u>Probability: A First Course</u>, by Mosteller, Rourke and Thomas (Addison-Wesley, 1961), pp. 64, 65.