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These materials were written with the aim of reflecting the thinking of the Cambridge Conference on School Mathematics (CCSM) regarding the goals and objectives for school mathematics. This report deals with some seventh grade mathematical concepts taught at Cambridge Friends' School. The discovery approach was utilized by the teacher in order to involve students in the classroom discussions. The problematic areas which are dealt with in this report focus on (1) geometry as physics versus geometry as mathematics, (2) proofs and mathematical reasoning, (3) area, and (4) infinite process (approximations). Instructional procedures are described and student reactions to various procedures and activities are listed. [Not available in hard copy due to marginal legibility of original document]. (RP)

GEOMETRY REPORT

By

Gabriel Stolzenberg

This report represents much time spent thinking and planning - and much too little time spent in class - 20 meetings.

1. I met twice a week from February into May, with the 7th grade class at the Cambridge Friends' School. (There were two long breaks.) The class was small - 8 boys and 7 girls - the children were usually thoughtful and responsive, and the atmosphere for learning was good. I was helped substantially by the teacher, Mr. Thomas Waring, who has a strong feeling for the mathematical point of view - and for the child's.

2. I learned much more than I taught. Time was short. I used various approaches and did not always follow through and reinforce what was done. I tried deliberately to explore problematic, sensitive areas and, not surprisingly, there was frequent trouble and frustration.

For example:

a. They hated $\sqrt{2}$ when they found that "it doesn't come out even". So much so that they asked at one point to study only rectangles which were <u>not</u> squares in the hope of avoiding $\sqrt{2}$ (and its companions.)

b. They eventually followed the proofs of the Pythagorean Formula done on the board but they were not at all convinced it would really yield the (correct) answer if they actually measured the sides of a particular right triangle. (Often it didn't, because of their errors in measuring and inaccuracies in the construction of the triangle.) I think this is in part a reflection of their lack of confidence in the "correctness" of their reasoning. Sometimes when they seem to be following a proof they may really be saying "I can't find anything wrong with it."

Can such matters be dealt with profitably in grade 7 (or earlier)? I think so. It seemed to me that the hardest thing for the children was that these ideas were very new, unfamiliar, sometimes even startling. In that case, perhaps the earlier they're presented the better it would be, so that the long process of familiarization (and understanding) can get going.

"Familiarity breeds content" - I hope.

3. Some of my prejudices.

a. Difficult points which are natural parts of the theory and which can be understood by the student should be squarely faced. Whenever such a point is not going to be adequately treated that fact should be plainly stated. Gaps, unmotivated assumptions and devices, evasive action (even when logically legitimate) should be clearly labeled as such.

b. It is unfortunately easy to make children think that they know certain things which they plainly do not, and that certain things are obvious which obviously are not. It is notoriously difficult to undo this. This goes on all the time at all levels of mathematics instruction. In fact, I think it is "the rule". (It is attractive because it is comforting and frequently yields correct answers.) The mishmash made in treating area is a particularly scandalous situation.

c. Appeals to intuition are extremely valuable in teaching mathematics and are also extremely abused. Many different kinds of thinking are confused under this heading. For instance,

(i) Some things, especially in geometry, may be visually evident: For examples,

If a line crosses one side of a triangle it also crosses another.

Two edges of a triangle form a larger path than the third side.

The circumference of a circle is about 3 times the length of the diameter.

(ii) Sometimes one reasons by analogy.

(iii) Sometimes one generalizes after checking a few special cases.

Usually (ii) and (iii) are combined.

My feeling is that assertions gotten by (ii) or (iii) should be identified plainly as conjectures or working hypotheses. The best thing to do with them, when you can, is to verify them (or disprove them). If you can't, you frequently go ahead anyway and see what follows. But the provisional status of these assertions should be made clear.

I think visually evident things have a different character. They "feel" true - verified. In a satisfactory discussion only statements of this kind should be taken for granted and (since this is a subjective matter) even these are naturally open to challenge.

4. Here are four large problematic areas which need to be developed <u>in detail</u> and worked into the school mathematics curriculum.

a. <u>Geometry as physics versus geometry as mathematics.</u>

b. <u>Proofs, mathematical reasoning.</u>

c. <u>A satisfactory treatment of area.</u>

d. <u>Infinite processes (approximation)</u>

I will comment briefly on these on the next few pages.

a. <u>Geometry as physics versus geometry as mathematics.</u>

As a rule this relation - or contrast - is not discussed. Perhaps it is considered too sophisticated a matter or too fuzzy a problem to be amenable to teaching. Maybe. I don't know.

Neverthless, it seems that children first experience and understand geometric objects physically and so, <u>somehow</u>, have to make the transition to mathematical geometry i.e. - to abstract out the relevant formal <u>properties</u> and <u>relations</u> of the physical objects.

This is a hard series of steps. Perhaps the teacher can help. I tried. I began the first geometry class by posing finding the Pythagorean Formula as a physical problem: <u>Find a formula, in terms of the lengths of two adjacent sides of a rectangle, that will predict the result of measuring the diagonal.</u> In fact, to begin I posed this problem for three particular rectangular objects in the classroom.

I encouraged measurements and hunches.

My aim was to show them, what is to me very striking, how a line of <u>mathematical reasoning</u> can be used to solve a physical problem.

I wanted to (somehow) wean them from physical objects and measuring to imagined "idealizations" and reasoning, and to lead them through a statement and proof of $c^2 = a^2 + b^2$.

My attempt was clumsy, naive, inadequate. But an approach like this should be tried again. Perhaps starting with the 3-4-5 triangle. Perhaps working more with squares. Perhaps starting by telling what the formula is.

b. Proofs, mathematical reasoning.

I feel strongly that a proof that doesn't convince is not worth much.

Practice in proofs and mathematical ways of talking and reasoning should begin as early as possible.

It should be made clear that what's put in or left out in writing down or telling a proof is very much a matter of convention - the standards of the times or the particular classroom.

Proofmaking is a mathematical skill which should be learned along with the others. Number theory and inequalities seem natural areas to do this in at an early age.

c. A satisfactory treatment of area.

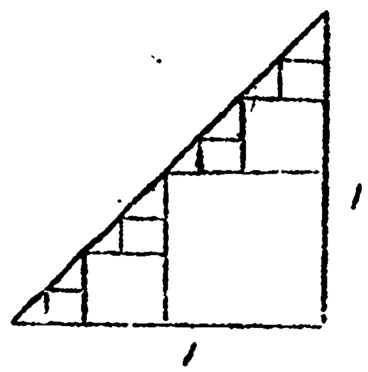
This is my main goal and I am still far from it. But I have learned some things.

I think the measure theory should be faced up to. There is much that can be done. And there are many interesting basic problems that can be tackled.

The problem, of course, is to demonstrate the existence of an area function which has all the properties it's supposed to have.

I would like to distinguish two kinds of difficulties that arise in trying to do this.

(i) There are problems of approximation. Two things have the same area if you can approximately cut one up and rearrange the pieces to form the other. Infinite processes are involved. If you try to compute the area of a triangle like this



then you have to sum $\sum \frac{1}{2^m}$.

In dealing with such questions it may be good, at first, to emphasize inequalities and bounds rather than equalities. Start with the concept - one region is smaller than another if it can be cut up and reformed to fit inside it. (After such a discussion Bolyai's Theorem on cutting up triangles is especially striking.)

(ii) The other kind of problems are like this. Take a square. Cover it with fine graph paper whose lines are parallel to the sides of the square. Count the number of boxes which hit the square. Now put the graph paper down some other way and count again. The answers will be about the same. Explain this.

Related problem. Take a 1 by 1 square and cut it up into pieces that can be rearranged to form a rectangle. Then the lenghts of the sides of that rectangle will alway satisfy the relation

$a \cdot b = 1$.

Prove it. (i.e. Show why.) (In particular, show why if the rectangle is a square, it will again be a 1 by 1 square.)

These are extremely interesting facts which ane not visually evident (though they can easily be checked empirically) and are basic to understanding area.

This is the approach to area I suggest. I think it can be taught and learned.

In the "postulational" approach to area such questions are avoided with such success that most people I've talked to find it hard (often. impossible) even to understand the questions. I had this experience last summer with a group of college graduates - mathematics majors - who were preparing to teach geometry. This surprises me even less now that I have looked through a number of the standard works on measure theory and found these matters either absent (which is fair enough), faked, or hidden in the exercises.

d. Infinite processes (approximation).

Approximation has to come up - in decimals, in area, in fact, beginning with division.

The problems on infinite processes are fascinating. Work can begin early - - How many numbers are there?

By way of illustraion here is a list of questions I once used to begin a project on infinite processes with an 11th grade class at the Commonwealth School.

1. 2,4,6,8, - what comes next?

2. $0 + 0 + 0 + \ldots = ?$

3. $1 - 1 + 1 - 1 + 1 - 1 + \ldots = ?$

4. $1 + 1 + 1 + 1 + \ldots = ?$

5. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots = ?$

6. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^m} + \ldots = 1.$ Why?

7. Is there a number N such that

   $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{N} > 3$?

   Do you think there is an N such that

   $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{N} > 1000000$?

8. About how big do the numbers

   $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2},$

   get as you take n bigger and bigger?

9. Try adding up the sums

   $1,\ 1-\frac{1}{2},\ 1-\frac{1}{2}+\frac{1}{3},\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4},\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5},$

   with more and more terms. What happens when there are many terms? Is the sum always positive? Always negative? Sometimes positive and sometimes negative? About how big does it get to be?

10. Can an infinite region have only a finite amount of area?

(I do not include a report of this project here.)

- xXXXXXXXXXX

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Before the 1st meeting, I gave out a set of problems, to be worked on at home, and then handed in. The problems were:

i) to get an idea of how they thought about geometric objects and what they knew about them (very, very roughly).

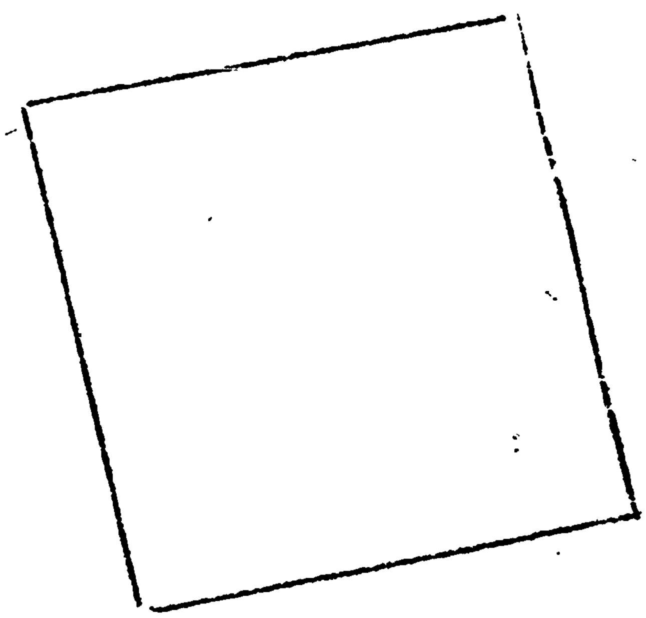
ii) to get them primed, i.e. to give them an idea of what we would be doing.

iii) (hopefully) to find some interesting leads.

Their answers were more or less what I expected and I won't comment on them in any detail here. I enclose two sample replies.

SOME PROBLEMS

Here is a square (approximately.)



Which side is shortest? Measure it.

Is there a square smaller than the tip of a needle? Could you measure it?

Is there a square bigger than the earth?

Can you make a square that is more accurate than my square?

By the way, what is a square? A circle?

Make a circle (out of something) that surrounds just as much space as my square. How long is your circle across?

How long is it around?

Finally, make a circle that is just as long around as the square is. How long around is that?

Finally, make a circle that is just as long around as the square is.

How long around is that?

How long is this circle across?

P.S. Try to finish this circle. Find the center and measure the radius.

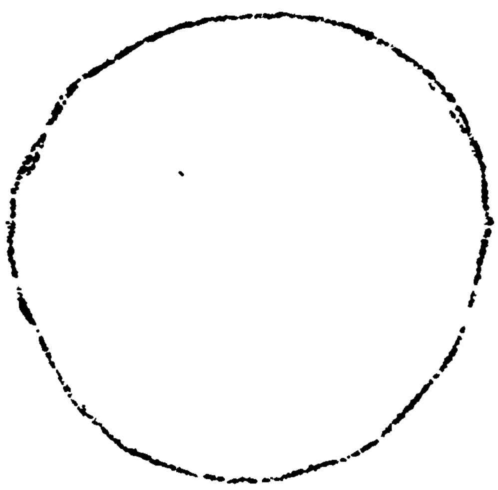
Pupil: K. Creighton

THE ANSWER TO SOME OF YOUR PROBLEMS

No sides of a square are shorter or longer. A square has equal sides.

There are squares everywhere that you can't see (like a point). You can not measure it. There is a square bigger than the earth. There are an infinite number of them and them and their made from the ∞ number of planes, and + lines. I can imagine one that is perfect square, but you can not see it.

A square has 4 line segments that endpoints only meet two other endpoints. A square is not a circle because a circle is just one perfectly round closed curve, and has no end-end.



My circle

It is two in. across.

It's circumfrence is 6"

The square isn't round

Pupil: Peter Z.

1. Yes

2. Yes

3. Yes

4. Yes

5. A square is a two-dimensional object with four equal sides

6. A circle is a two-dimensional object that is round

7. $2\frac{1}{4}$

8. 6"

9. 8"

10. $2\frac{1}{2}$"

P.S. 2"

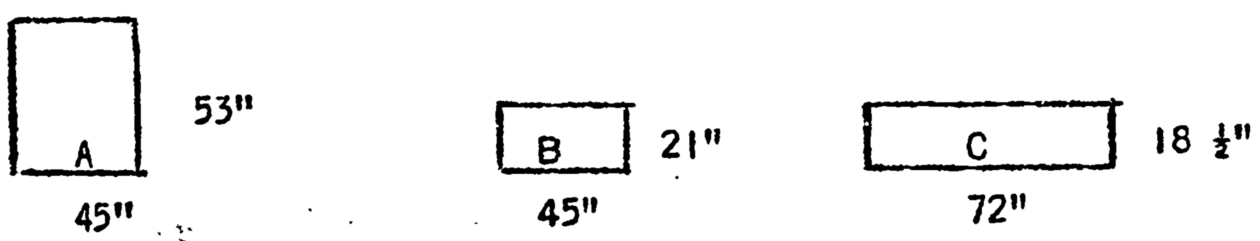
1st Meeting

To begin we picked out 3 large rectangles in the classroom: 2 large windows and a table-top.

I announced that I had a "method" whereby if I knew how long two (adjacent) sides of one of the rectangles were I could, by doing some figuring with pencil and paper (without looking at the rectangle any more), figure out how long the diagonal was.

I asked if anyone else thought they could do this. Several said yes. After a brief discussion it turned out that what they meant was that if they knew how long 2 adjacent sides were they could also tell how long the opposite sides were. I explained again what I proposed to do (determine the length of the diagonal) and this time everyone seemed to catch on. No one said he could do it.

I proposed the following "experiments". With a yardstick several children measured the sides of the 3 rectangles . I drew pictures of the 3 of them on the board, not at all precise, but at least preserving their relative shapes. I marked the appropriate lengths.



Then we did the following. Two children measured the diagonals with string and a yardstick. I worked with pencil and paper, via the Pythagorean Formula. And the other members of the class thought about the problem, looked at the objects, and tried to figure out the answer.

When we had all finished (the measurers had some trouble with the string stretching) I recorded the results on the board.

Only a few children had answers.

		A	B	C	
Student's Answers	1.	71½	55½	81¼	
	2.	98	65	90½	
	3.	98	66	90½	
	4.			76	(This was a guess)
Me	5.	69	49+	73.8	
Measured	6.	67	49	74½	

4 was a guess. 2 and 3 had added the sides. I pointed out, and everyone seemed to see immediately, that this was clearly too big.

1 was Scotty's method. She judged with her eye and estimated the length of the diagonal as the horizontal side plus half the vertical side.

We pursued her idea. Her answers were pretty close to the measured results.

First I pointed out that her answers depended on which way we looked at the objects. If we rotate them by 90° and apply her method then the answers change and in fact become quite inaccurate.

There was a lot of action at this point, various attempts to fix the method. Someone (was it Scotty?) suggested modifying it to be the long side plus half the short side. This saved B and C. The answer for A got worse (compared with the measured result and mine) but not too much worse. I thought it interesting that the answer got worse only for the more square-like figure A; several children seemed to think this curious too.

So, to fix our attention on one aspect of this problem I assigned for a homework problem to check Scotty's method for a square. That is, to find or construct several squares at home and to compare Scotty's method against the measured result.

There was surprisingly (to me!) little discussion about the fact that my answers were so close to the measured ones - (which was the main point I wanted to impress them with). I see now that this was principally due to the way I handled the situation. I treated the method as an adult secret - "not for children" my manner probably said. I openly avoided explaining what I was doing (to leave open the possibility that we would eventually "discover" it in class). This approach (which was not particularly calculated on my part) contained the seeds of subsequent failure. In effect, I challenged them. We chose sides. I had my "method". They measured.

Only one boy asked about what I had done (just after the results were tabulated). He wondered if I had done something like make a scale drawing on my paper and then measured it. I told him that I hadn't done any measuring and made a mental note to try to pursue the ideas of scaling and similarity later on. Surprisingly, it never came up again, (except briefly on the very next day). Yet this is a basic point to work on. To understand that they only have to do it for one member of any class of similar rectangles.

2nd Meeting

Four children had done the homework. They found that Scotty's method and measurement gave very close answers for the squares they tried.

(Where did they get squares? I'm really not sure. That's a problem. I think some drew them on paper and cut them out. But getting those right angles can be hard.)

Some of the results were (I've lost one)

	Side	Diagonal (Measured)	Diagonal (Scotty's Method)
1.	2"	2-7/8"	3"
2.	1"	1-3/4"	1-1/2"
3.	2"	2-5/8"	3"

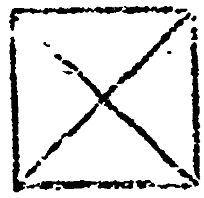
The general feeling was that the method worked pretty well.

The 2-7/8" and 2-5/8" measurements for a square 2" on a side looked curious to say the least. I asked, what about it? This caused lots of unhappiness....must have been a mistake in measuring, they said. I tried to raise the question of the accuracy of the squares themselves, of how square they really were. What about those right angles? Wouldn't inaccuracy there also throw the result off? This whole discussion didn't settle anything. I think they have great faith that any machine-cut object that looks like a right angle is one.

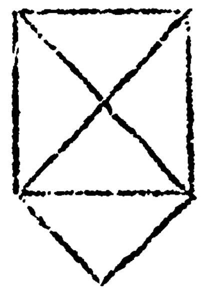
I raised the question: Could Scotty's method possibly be exactly right? (Could the discrepancy between the answers be due entirely to "experimental errors"). My intention in posing this question was to move toward abstract mathematical reasoning. I had in mind to <u>prove</u> that the method was wrong (though close). This was another blunder on my part. For the following reasons. Principally, they knew that her (Scotty's) method didn't agree with mine during the 1st meeting, so <u>I</u> must know it's wrong.

Why pretend? Why play innocent? Secondly, it has such a <u>negative</u> quality. Here am I with my "secret" adult formula proving that one of the children's formulae is wrong. No matter how positively I tried to put it, in retrospect it was a bad business.

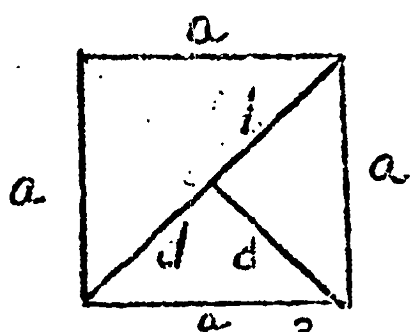
Anyway, I did it. I drew the following diagram on the board.



I waited for ideas. One boy made it into this

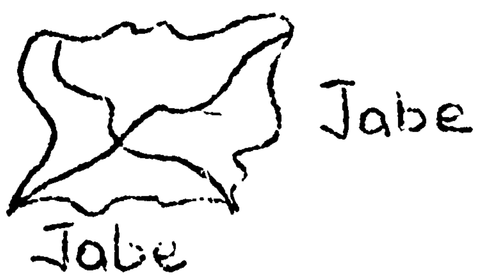


but that led nowhere. So I led them through some reasoning. First I attached labels.



I argued:---by Scotty's method $b = \frac{3}{2}a$; so $d = \frac{3}{4}a$; so, again by Scotty's method, $a = \frac{3}{2}d = \frac{9}{8}a \neq a$, a contradiction.

Some bought this; others asked "What's a ? So I did it with $a = 1$. Then they wanted to try $a = 2$. Instead, I set a = Jabe (one of the boys), drew a wavy square,



and did it again.

As I mentioned before, this didn't come off.

Also, notice how far away we <u>seem</u> to be from our original problem.

3rd Meeting

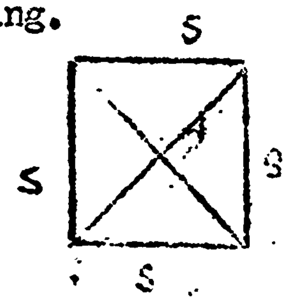
Something new has been added! Richard brought in the Pythagorean Formula, $c = \sqrt{a^2 + b^2}$. He "got" it from the sister of a friend. He tried it on several rectangles and found it worked very nicely.

What do I do? I was well aware beforehand of such an event possibly coming to pass. Still I had no plan. I'm afraid my look and manner suggested to Richard that he had done something wrong - which he hadn't.

At any rate, I asked Richard if he thought the Formula was exact (whatever that means), where the friend's sister had gotten it from, why he believed her (why not?, of course), how it might have been dicovered (trial and error, he said).

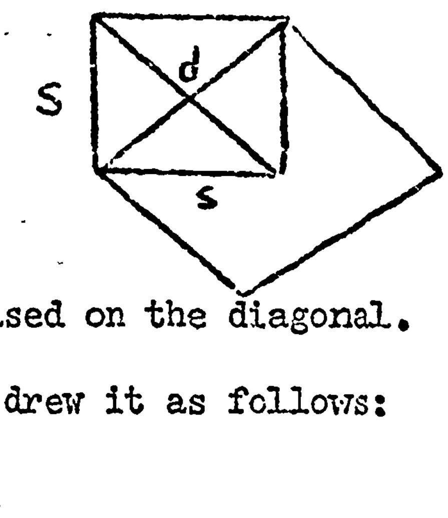
Roger and Scott also had a new formula (the same, but independently): long side plus a quarter short side. Rather than follow this up, I suggested they try on their own to apply the reasoning of the previous meeting to the new formula and see what happened. (Significantly, I realize now, I didn't suggest this to Richard).

Instead I drew the square again and tried to see if we could get any positive results by reasoning.

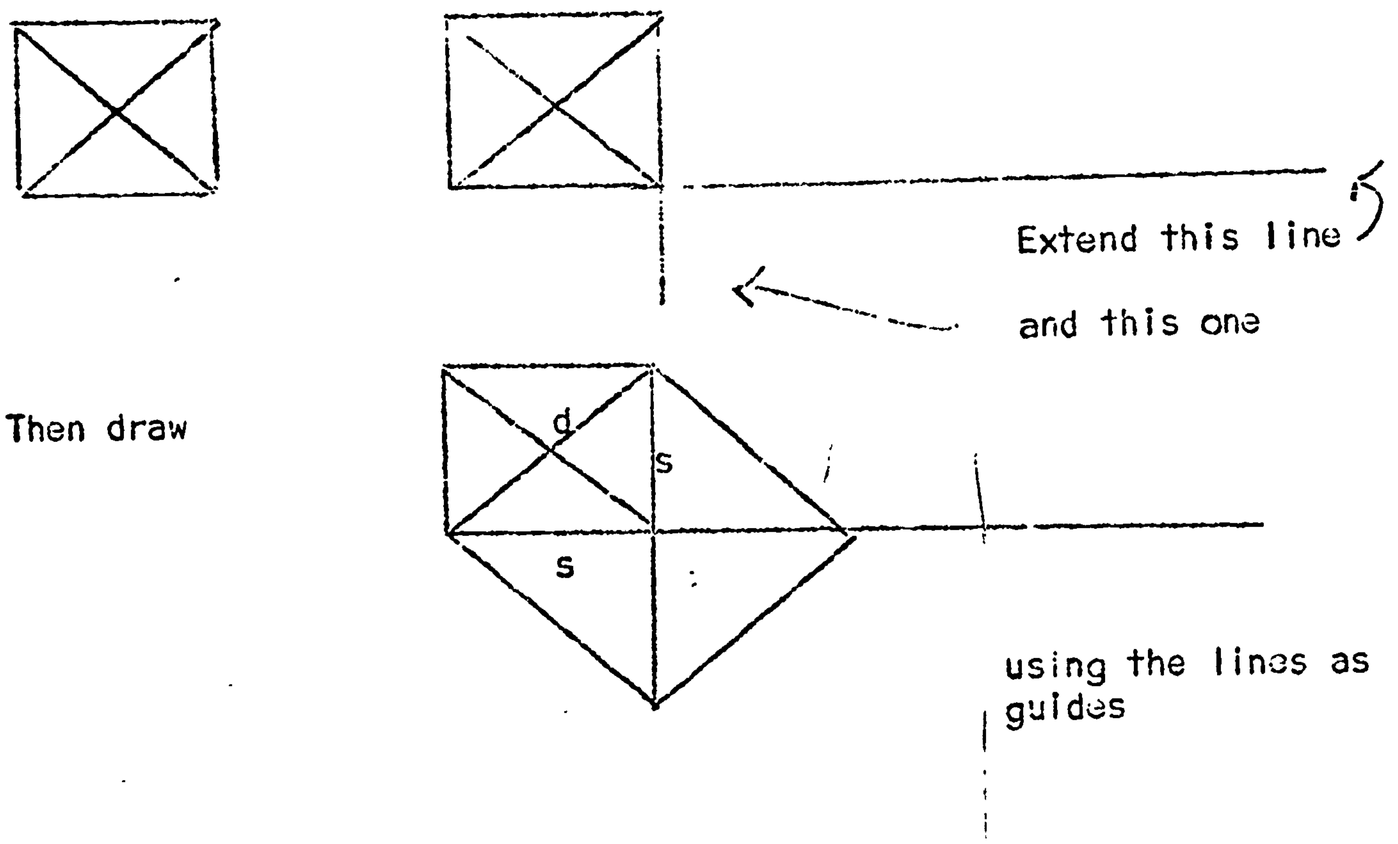


d = diagonal

Nothing happened. So I improved the picture, thus,



trying to make a square based on the diagonal. My picture looked bad. David had a good idea and drew it as follows:



After a short discussion (in which Mr. Waring, the teacher, gave the key idea) we got

Area of small square = $\frac{1}{2}$ area of big square

At this point I thought we were home; but, to my surprise, no one seemed to have learnt the formula for the area of a square of side s (or d). As it turned out, we didn't get back to this for a while. But actually they did know that, e.g. the area of a square of side 2 is 4, of side 3 is 9, of side $\frac{1}{2}$ is $\frac{1}{4}$, etc. It was the "s" that threw them.

Abstract symbols have to be introduced, but there are good and bad ways to do it.

4th Meeting.

Still responding to Richard and his Pythagorean Formula (Note: He didn't know it "by name" and I never said "Yes, that's it"). I came in with a list of possible formulae. For $b \leq a$, I listed:

1.	$a + b$	(Our first try)
2.	$a + \frac{b}{2}$	(Scotty's)
3.	$a + \frac{b}{4}$	(Roger & Scotty)
4.	$\sqrt{a^2 + b^2}$	(Richard & Pythagoras)
5.	$a + \frac{1}{2}\frac{b^2}{a} - \frac{1}{8}\frac{b^4}{a^3} + \frac{3}{48}\frac{b^6}{a^5}$	(Me, via the binomial series, first 4 terms)

I drew a big chart with these formula matched against some of our previously measured rectangles. It was unwieldly, the algebra was a little hard (and too fast) for them, and there was much too much on the board to have to look at. Richard and Pythagoras won, and no one cared.

We went back to our square. But there was time only to ask them: What is area? In particular, what is the area of a square?

Some More Problems

These were done for homework before the fifth meeting. The first 3 were exercises involving arithmetic and algebra that came up.

The last 3 were to get on into area. The responses were not particularly interesting.

SOME MORE PROBLEMS

1. Which is bigger, $\frac{7}{32}$ or $\frac{23}{100}$? $\frac{41}{83}$ or $\frac{42}{84}$?

   $\frac{1}{2}$ or $\frac{2}{3}$? $\frac{3027}{698}$ or $\frac{3028}{699}$? $\frac{a}{b}$ or $\frac{a+1}{b+1}$???

   What are your reasons?

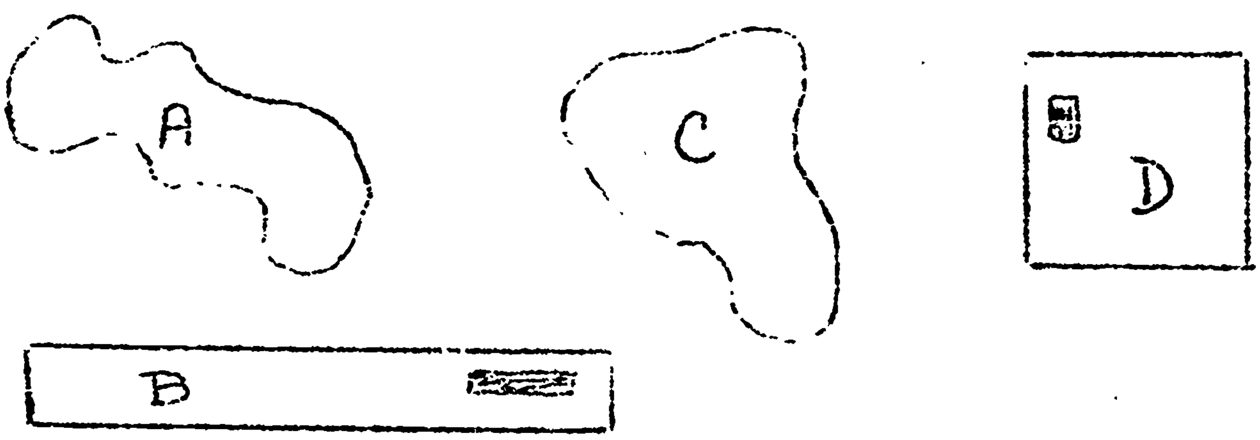
2. Find a number $\underline{a}$ such that $\underline{a}\cdot\underline{a}$ is between 5 and 6. Find another number $\underline{b}$ such that $\underline{b}\cdot\underline{b}$ is between $1\frac{1}{2}$ and $2\frac{1}{2}$.

3. What is $\frac{x}{y^2}\cdot\frac{z^3 y^2 x}{x\, z^5}$???

4. What is the area of a square whose side is $1\frac{1}{2}$" ?

5. What is the area of a square whose side is 1/3" ? 2/5" ?

6. Which one has the most area? The least?

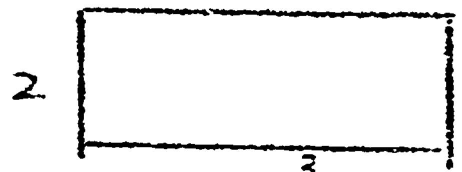


Fifth Meeting

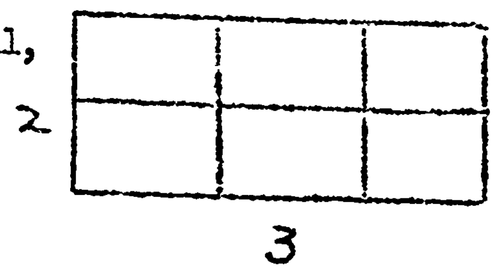
It began with a question related to a homework problem. Why was $\frac{a}{a} = 1$? That is, why doesn't it all "cancel out" (get erased from the board)? I always find it difficult to reply to a negative interrogative, (and was tempted to say "when you multiply, 1 is the zero"). Instead, I just passed.

We worked on area. It turns out they've had some classroom experience with it before(at least, some of them have) and really seem to understand well what they know.

I put a rectangle like this

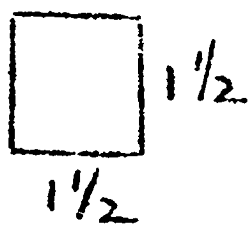


on the board. They said the area is 6. I asked why. They told me. David went to the board and made a grill,



and explained" 6 unit squares.

They could also do:



breaking up a unit square itself into 4 subsquares. I tried a square $\frac{1}{100,000}$ on a side and they did it. They understand this.

Chuck suggested using a common denominator to determine how to subdivide the general rectangle. Good idea. (But I note that quite reasonably, they think, indeed they "know" without thinking, that all numbers are rational. I plan a surprise for them - later.)

[Note: For some reason my tense changes here.]

I go to the other extreme now and ask about the area of an <u>irregular</u> figure. I draw something like this.

What is its area? How do you find it? Two methods are offered.

1. (Either Alix or Farlen). Place a string along the perimeter. Reshape it into a square. That square will have the same area.
2. (Chuck). Fill it up with squares inside and add. There's some problem with the edges.

We discuss 1 which arouses lots of interest. One girl gets up to the board with a string. I suggest that it doesn't look right for a long thin rectangle, but this idea is not picked up.

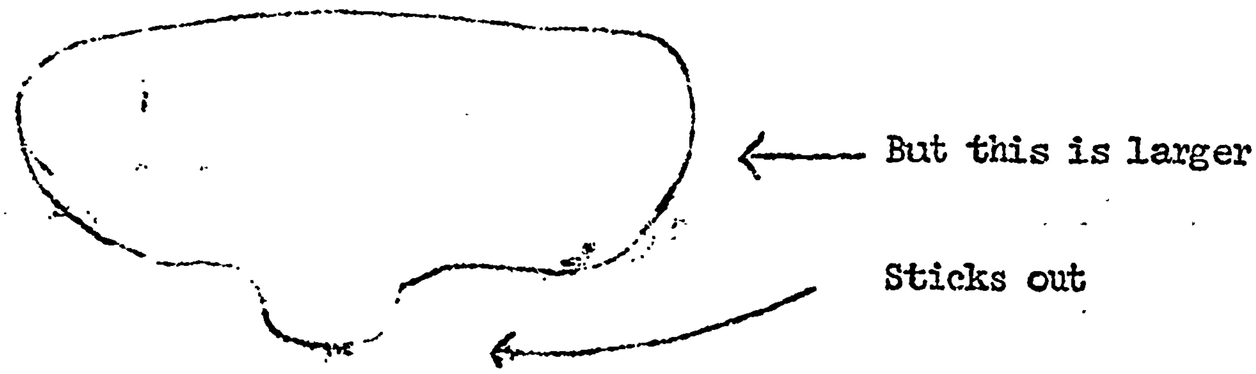
We discuss 2. What about the edges? They're curved. Someone suggests cutting up unit squares into curved pieces to make it fit exactly. His plan is to cut one (or maybe a few) unit square up into pieces like

and use them to cover the edges. I point out (and he realizes) that he must use <u>all</u> the pieces he cuts and we see that it is highly unlikely that this will work.

6th Meeting.

We begin by comparing again the two ways suggested for determining the area of a curvy region. This time I drew one like this.

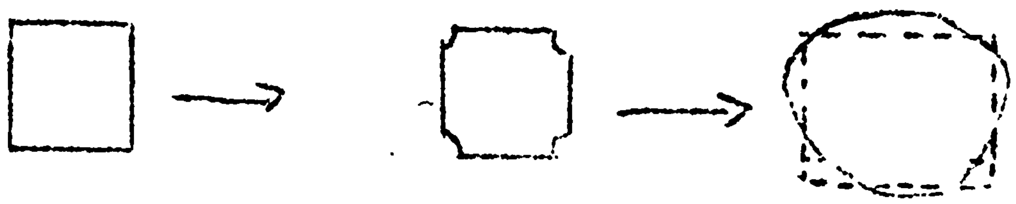
Alix sees that her method (1) won't work. She has a good idea. Namely; if (1) were right, then the figure above and the one below (gotten by a "flip") would have the same area.



Some felt this was a paradox, that the same string can determine different areas. I passed.

Then I posed the isoperimetric problem: How do you form a region with the biggest area? (Using a fixed piece of string.)

Immediately there were two answers: circle, square. We kicked this around a while. Then I simply told them that the answer was a circle, and tried to show heuristically why it couldn't be a square. I argued (with pictures) that if you push in the corners slightly, and then pull out the sides, you'll increase the area.



Then I left this and went back to the question of the formula for the area of a square. It turned out (in the 5th meeting) that they did know this after all. Namely (side) ^{2}.

From here we moved quickly and easily through our Pythagorean relation for a square. We got $d^2 = 2s^2$.

We wanted d. I said O.K. given s, what's d? We tried s=15, and got $d^2 = 450$. We tried various numbers for d. Got $21 < d < 21\frac{1}{2}$. Class unsatisfied. We tried s=3. Got $4 < d < 4\frac{1}{2}$. Class frustrated. Finally, I suggested, let's try s=1. So the problem is $d^2=2$. What is d? First response, $\frac{1}{2}$, cleared the air. $d.d = 2$. What is d? $1.1=1$. 1 is no good. Neither is 2. Neither is $1\frac{1}{2}$ (Scotty's old formula). Neither is $1\frac{1}{4}$. Class furious.

I said "There is no answer". Then I went through the usual proof that $\sqrt{2}$ is irrational. They were snowed. First by the use of abstract symbols, second by the logic of the argument (proof by contradiction), third by new ideas involving even and odd: A^2 is odd if and only if A is odd.

7th Meeting.

This was a highly unsatisfactory meeting. I tried to "patch up" the proof that $\sqrt{2}$ is irrational. First we discussed odd and even. We defined even as 2n (n whole) odd as (2m+1) (m whole). We worked on even x even, odd + odd = even, etc. There was great trouble working out $(2m + 1)(2m+1) = 4m^2 + 4m+1$.

They were clearly not ready for this.

A long "Ijust don't understand" question from Chuck provided an interlude and then we went back to an equally unhappy discussion of (again) $\sqrt{2}$ is not rational. It still doesn't go.

I am very sad (and mad).

8th Meeting.

After the last meeting, I had a good idea...a decent way to show them that $\sqrt{2}$ cannot possibly be rational. (See the section "$\sqrt{2}$ is not rational".)

They find the case 1 $\frac{\text{odd}}{\text{odd}}$ hard. It involves the fact that $(\text{even})^2$ is divisible by 4 but twice an odd is not. But they definitely seemed to believe the first two cases $\frac{\text{odd}}{\text{even}}$, $\frac{\text{even}}{\text{odd}}$ (which are easier) and there was no problem about disposing of $\frac{\text{even}}{\text{odd}}$.

At the end, I asked again: How do you explain this?

They said - you can only approximate it. One boy said - You must measure to find out exactly.

What to do????

(Notice how far away we seem to be from our original problem.)

9th Meeting.

Two boys are pretty excited and pleased. They have gotten the Pythagorean Formula from an older friend.

After a short discussion they agree that it is just the same as the one Richard has brought in earlier. (Richard meanwhile has forgotten his.)

I asked them if the formula is right. They seem to have no idea.

Then I show them, that for a square, it agrees with the formula we derived in class.

Someone suggested that since we had so much trouble with squares (i.e. Since $d = \sqrt{2s^2} = \sqrt{2}\ s$ usually didn't "come out even") we confine our attention to rectangles which aren't squares.

Or try s=3 someone else said. s = 1 is too hard. I pointed out that we had already tried s=3. We let it go at that.

The day before, at my request, Mr. Waring had reviewed the number line with the class. So I now sketched a number line on the board and marked off $\sqrt{2}$ roughly in place.

There seemed to be no question that $\sqrt{2}$ is legitimately there on the number line. That they took in stride. Also they seemed convinced (or at least accepted) that $\sqrt{2}$ is not rational. O.K.

What about representing it as a decimal? Sure. I briefly and sloppily reviewed decimals in terms of marching along the number line: unit sized steps, 1/10th sized steps, etc. Everyone seemed with it. I wrote $1.41 < \sqrt{2} < 1.42$. Still O.K.

And that's where the trouble started. I don't remember how but somehow I referred to the fact that the decimal expansion is always infinite.

That . 2734...means you've only figured it out approximately and that .2734 means .27340000...(with zeros forever).

This caused great consternation. Even 1/3 = .333...which they "know".

To them decimal notation is part of what Mr. Waring calls "Main Street mathematics". Decimals stop. And things like .33 1/3 are common.

I ended the class with a discussion about .999....(nines forever)

I tried to convince them it represents the same number as 1.000... they agree it's $\leq$ 1, and by subtracting, that the difference between 1 and it is less than .0....01, no matter how many 0's you put in. There were 3 responses to this.

a. A blank face

b. 1.00000... - .9999... = 0

but they would go no further.

c. 1.0000... - .9999... = $\underbrace{.00.........0,1}_{\text{infinitely many 0's then a 1.}}$

Peter asserted (c) saying something like, "I can't help it. That's just how it is."

10th Meeting

Mr. Waring had a memo on my desk. The day before he had asked the kids to make a square with area 2. Some tried side $1\frac{1}{2}$ or $1\frac{1}{4}$. Only Scotty thought to use what we had been doing.

To begin I wrote on the board

"If $1/3 = .3333....$and <u>if</u>

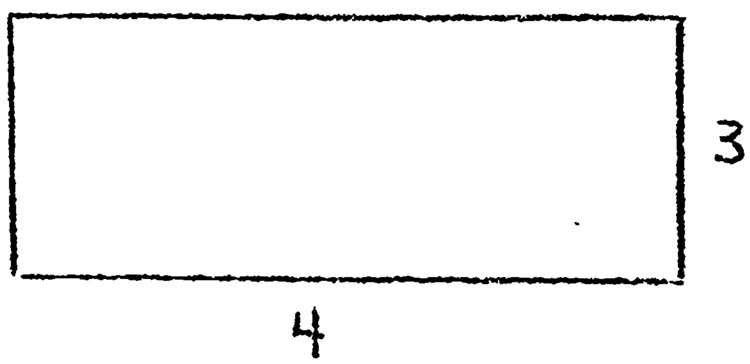
$$\begin{array}{r} .3333.... \\ \times\ 3 \\ \hline .9999...... \end{array}$$

and <u>if</u> $1/3 \cdot 3 = 1$ <u>then</u> ???"

(Complete the sentence)

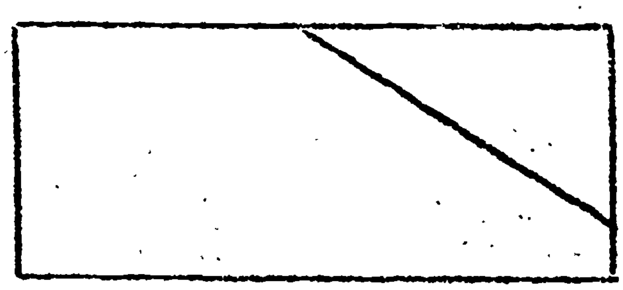
I had them copy this down. Someone murmured "Now I see why .9999.... is 1". I suggested working on this at home, not discussing it in class. (I wanted to get on with area.)

Next, I draw a rectangle on the board

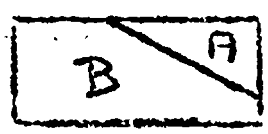


(I am never very accurate, except with circles.)

and said, "suppose it's broken into 2 pieces like this"



They said "call them A and B".



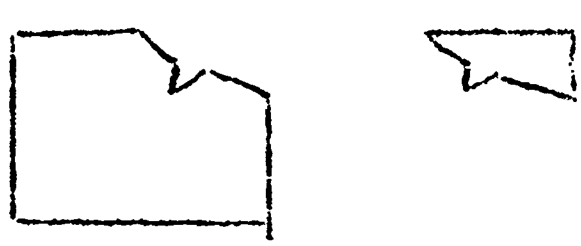
Chuck said, "Let's break them with a squiggly line". But I said let's keep it simple at first.

I asked, "suppose the 2 pieces are in 2 different countries and we want to figure out the whole area."

They: "Area A + Area B = Area R". Also someone observed that since B contains 2 adjacent sides if you have just it alone you can determine the area of R.

What about finding the area of B alone I asked, and began filling up B by squares (littler and littler). They stopped me, saying there was an easier way. David went to the board and broke B up neatly into a few rectangles and triangles. That would do it, they said.

I modified the pieces a bit, to

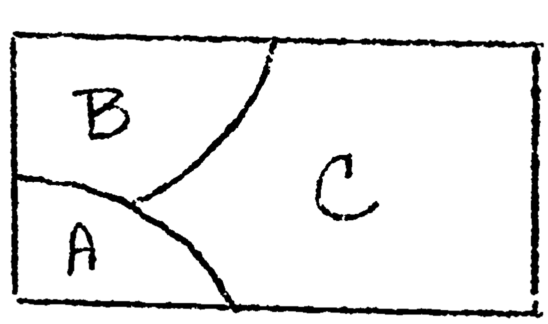


and they could still do it. (Same way.)

Jeff suggested filling it up by 1□1 's

then cutting up 1□1 's to fit into the irregular parts and "see how much was used." I didn't push him on this. (This kind of idea came up in the 5th meeting too.)

Next I broke the rectangle in 3 like this

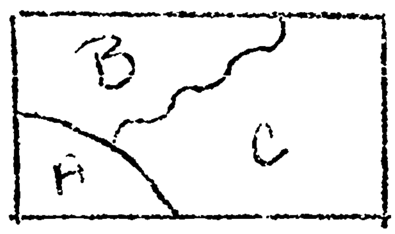


Again they said: Area A + Area B + Area C = Area R.

(They're quite at ease with symbols for "objects": <u>much less so for numbers.</u>)

They handled this setup just as before. Namely, they broke the pieces up into triangles and rectangles.

Then I took up Chuck's suggestion and made the cuts wiggly.



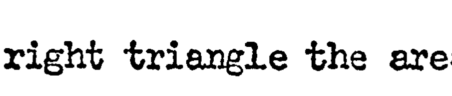
I began filling A with squares. Jeff suggested his method again to fit pieces at the edges and tried it at the board. He said (as in the 5th meeting) that you couldn't expect it to work exactly.

Chuck said "who cares about an inch?" and we kicked this around a while. I suggested that it might matter, depending on the problem. Someone said that a scientist who wasn't exact enough might wind up with the wrong result.

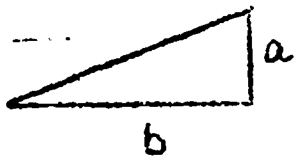
It was agreed that you could get as close as you like by Chuck's method (or Jeff's). But it will never end - they said.

I went back now to the case of a triangle,

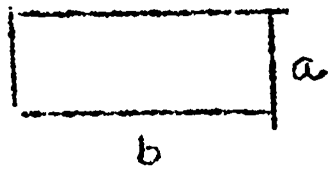
and asked how they would find its area. David got up, turned it upside down, and said - it's easier to work with right triangles. So he dropped the perpendicular



and said that for a right triangle the area is $\frac{a \cdot b}{2}$, because (he said)



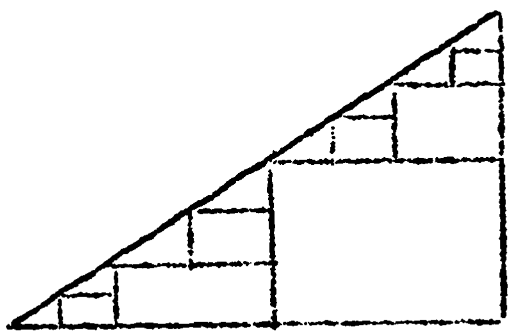
is half of



Note: Throughout the discussion there was a running argument about whether the figures were drawn accurately. Some were quite bothered. Others said, "Who cares? You're supposed to imagine it." This happened often.

Fine. Then I asked: Just suppose I didn't happen to think of the idea that a right triangle is half a rectangle. Could I still do it by filling it (approximately) with squares (or rectangles).

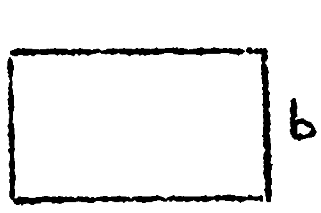
I think the question annoyed them since they had already shown me how to find the area and felt that was that. Nevertheless, I began filling in rectangles.



They agreed this was O.K. in principle, but was really a poor idea, because "it wouldn't come out exactly".

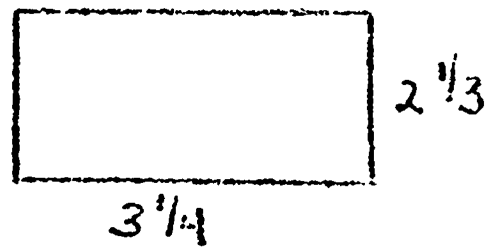
Perversely, I continued and said that by putting 2 such triangles together this would give us another way to work out the area of the rectangle. Humoring me, they agreed. But why not just do a.b?

Now I drew a rectangle on the board



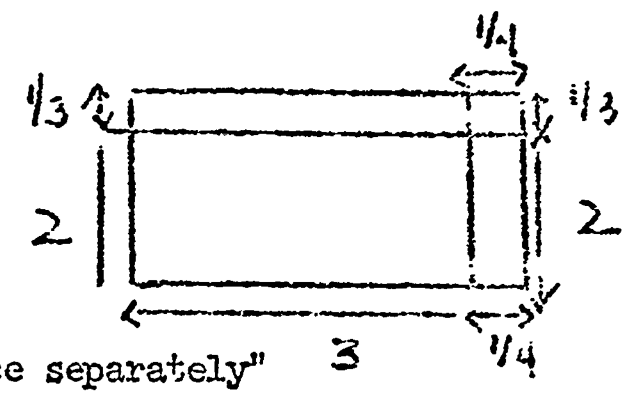
wrote Area = a.b and reminded them that the reason they had given was "you fill it up by unit squares."

Then we carefully worked out an example



After a bit of trouble a good common denominator (12) was found and they worked it out to be 28 x 39 1/12 x 1/12 squares = 1092 1/12 x 1/12 squares. After a little confusion they saw that it took 144 1/12 x 1/12 squares to cover a unit square and got $\frac{1092}{144}$ as the answer.

Chuck suggested another way. Write 2 1/3 = 2 + 1/3, $3\frac{1}{4} = 3 + \frac{1}{4}$.
"Make the picture

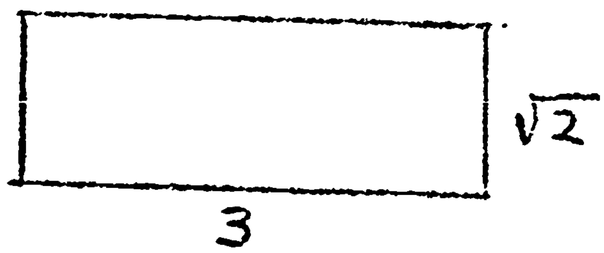


and do each piece separately"

(In writing up this report I see now that throughout the meetings Chuck has maintained a very consistent approach to area. It's always "filling in the edges".)

11th Meeting.

We reviewed the end of the last meeting and continued that discussion. I now posed - work out the area of



by the same reasoning about unit squares. I'm not sure whether they actually saw (after I tried to explain it) or just "sensed" that it wouldn't come out - since $\sqrt{2}$ was not expressable as a fraction.

Mainly, their reaction was negative.

George: "an exception....impossible to have $\sqrt{2}$ come out even"

Scott: "We don't know $\sqrt{2}$"

Jobe: "The formula is right, but it doesn't come out"

David: "If it's $\sqrt{2}$ you quit"

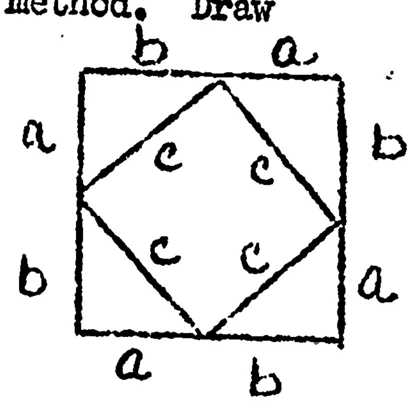
This negative mood was quite overwhelming. Still, I tried to push the point that by approximating $\sqrt{2}$ as closely as you like by fractions you could then approximate the area as closely as you like. And, that the formula was correct.

Farlan: "The area is $3\ \sqrt{2}$"

Jobe: "Ask a computer - it couldn't do it either. It would never stop".

At this point I was at a dead end. I left matters as they were and during the remaining class time I did something entirely different. Namely, I proved the Pythagorean Theorem.

I used the following method. Draw



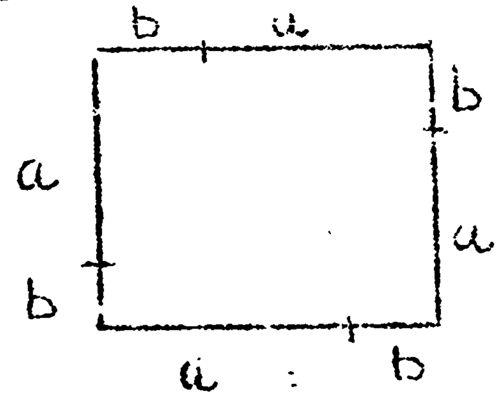
and work out the area as, on the one hand $(a + b)^2 = a^2 + 2ab + b^2$ and, on the other hand, $c^2 + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab = c^2 + 2ab$. So $c^2 = a^2 + b^2$.

Of course, the problem is "why is the area of the big square (when computed by formula) equal to the sum of the areas of its parts?" But I didn't dwell any more on it.

12th Meeting.

As for my proof of the Pythagorean Formula, last time it was as if it had never happened.

I began the proof again and drew a square a + b on a side



We spent most of the time trying lots of different ways of using this picture, none of which worked at all.

At the end I again did the proof and worked some examples (with particular numbers).

13th Meeting.

It was still as if nothing happened. So I stopped and we had a long discussion about - what would convince them that the Pythogorean Formula was correct? (They could easily follow the proof that I had given several times - but it was clearly irrelevant to them)

Various answers:

Measure.

They would believe it if the teacher told then.

Since they're tired they would say they believe it just to end matters.

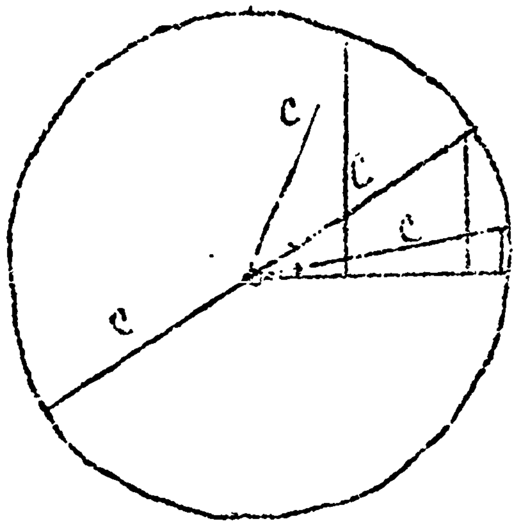
Roger said that if a friend told him he would believe it.

Peter and two girls believed the reasoning. But Peter said the crucial test for him was whether it worked when you measured. (The proof did not convince him that it must work when you measure, only that it was, in some sense, logically correct reasoning.)

We set up a test case. David measured a, b and c for some object in the room. Then we laboriously computed $a^2 + b^2$ and were about to embark on $\sqrt{a^2 + b^2}$ (to compare it with the measured c) when Jeff said (brilliantly) "Let's just square c." So we did, and compared c^2 with $a^2 + b^2$.

They were wide off. The first reaction in class was that David must have made an error in measuring. (Evidently they want the answer to come out.) He did. The string had stretched. He measured, gaining 3-3/4" on c and I assigned the comparison for homework.

The class ended with an argument between Jeff and Peter about whether c uniquely determines a and b? Jeff said yes, Peter no. Peter convinced him, by drawing various right triangles with the same hypotenuse on the board. Jeff then asked whether the area was uniquely determined. He evidently doesn't see the general principle.



14th and 15th Meetings.

We went through another proof of the Pythagorean Theorem. One with no algebra. You show explicitly how to cut up the 2 squares on the sides and rearrange them to form the square on the hypotenuse. This went somewhat better.

****

I stopped at this point. After a month I held 6 more meetings mainly devoted to lengths and areas of circles (and one class on similarity).

During one session I had them write very briefly what they recalled from the first 15 meetings. (Some responses are attached.)

I also had them, for an assignment, work out the length of the diagonal of a unit cube and this went well.

JABE

We tried to figure out square root of two. Something was cookoo first we decided that the fraction had to be odd over even. Then after we figured awhile we realized that it had to be even over even. Something was wrong. We spent about two periods trying to figure out what was going on, then we gave up and went on to the next.

PETER

I don't remember any arguements, but the measurements seemed to prove the reasoning right.

RACHEL

I don't remember very much. Scotty had a formula that didn't always work. (Long side + $\frac{1}{2}$ the short side.) Richard had a formula. Roger had something like Scotty's.

SCOTTY O'NEIL

We were trying to find what C was in terms of A and B. In otherwords we were trying to make a diagonal of a square. We had all kinds of formulae. I had one in which one side plus half the other side equaled the diagonal but that was too big. Another was that one side plus a quarter of the other equaled the diagonal but that was too small. We had several others. In the end we found that the square of the diagonal was two times the side squared. All you had to find out was what d was.

ALIX

Scotty's method long side and half of the short side. Her method did not work. Method that worked:

$$C^2 = a^2 + b^2. \quad \text{For a square } D^2 = 2S^2$$

$$D = 2.S$$

Some other ones that didn't work

$a + b$

$\sqrt{aa + b.b}$

$A + \frac{6}{4}$

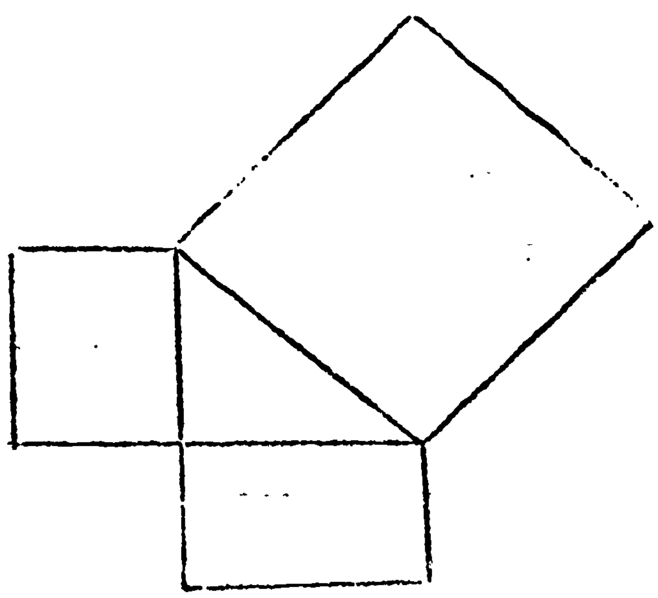
$A + \frac{1}{2}\frac{b^2}{2}$

$A + \frac{1}{2}\frac{b^2}{a}$

$A + \frac{1}{2}\frac{b^2}{a} — \frac{1}{8} \quad \frac{64}{a^3} + \frac{3}{48} \quad \frac{6^2}{a}\, 5$

ANN WISEMAN

The problems are that we tried to find how long a was compared to c in the triangle. We also tried to find the square root of two. We also tried to find two squares so that they would fit in one big square.



THE REPRESENTATION OF NUMBERS

There are several different ways of looking at numbers mathematically. There are the rationals as ratios of whole numbers, all numbers as points on the number line (or as directed lengths), and decimals. The student has to learn to be at home with all of these and to be able to go back and forth from one representation to another.

A hard problem for the student (and one which is probably rarely made explicit) is "which one is numbers?" It's even hard to say in English. I mean, decimals and points on a line certainly aren't the same thing, so "which one of them is numbers"? Several years ago I began a class at the Commonwealth School by saying something like "Let's take, for a working definition, that numbers are the points on a line." One boy objected, say that "numbers may be in one-one correspondence with the points on a line but they certainly are not the points on a line".

Mathematicians have lots to say about this, but there is no good answer. The idea which is rather sophisticated, is that lots of different things have certain analogous properties, and those properties are what we're studying. Some properties show up better from one point of view (i.e. in one representation)some in another.

I say all this because the kind of geometry I have been concerned with - really measure theory, involves different ways of looking at numbers.

As prerequisites for this kind of work I would emphasize

1. Familiarity with the number line.
2. Decimals (infinite)

But a warning about decimals. For computation a student has to become familiar with manipulating "finite decimals".

But I'm concerned, besides this, about decimals as infinite representations of numbers. This infinite aspect should be faced up to and lived with.

Rather than go further, I will only say that the development of a proper treatment of decimals is an important project in itself.*

* (It could well be part of some more general study of infinite representations and approximations of numbers.)

ALGEBRA AND ARITHMETIC

Algebra kept holding us up. Here are some things that they should be familiar with beforehand.

1. Squares and square roots (approximately).

   In particular, the formula

   $$(a + b)^2 = a^2 + 2ab + b^2$$

2. Linear operations on an equation in one unknown (i.e. multiplying by a constant or adding a constant).

3. That odd numbers are those of the form $2n + 1$ and even ones those form $2n$.

4. More generally, using letters in place of specific numbers.

Arithmetic was no problem.

(Of course, when I was in grade 7 we didn't have any algebra.)

$\sqrt{2}$ IS NOT RATIONAL

The one most successful thing I did was to make up a new presentation of the proof that $\sqrt{2}$ is not rational. It's much longer that the usual proof, much less elegant and, I hope, much easier for a child to grasp.

The usual proof goes like this:

If $\sqrt{2}$ were rational we could express it as A/B where A and B are positive integers with no common factor. In that case

$$\left(\frac{A}{B}\right)^2 = \frac{A^2}{B^2} = 2 \quad \text{so} \quad A^2 = 2B^2$$

so A^2 is even so A is even. Therefore A^2 is divisible by 4. But $A^2 = 2B^2$ so B^2 is divisible by 2 so B is also divisible by 2. Thus both A and B are divisible by 2 contrary to their having no common factor. This contradiction shows that there do not exist such A and B. This means that $\sqrt{2}$ is not rational.

Here the logic is intricate. One has to follow along line by line checking that each step does follow, without knowing where you're at. And at the end you're hit with a contradiction. The children I worked with found this very hard to follow. The symbolism, the particular facts about odd and even, and the way of reasoning were all unfamiliar. At best they agreed with each step. But no one really grasped it.

I am enclosing 2 sheets which outline my presentation. (These were given to the class after our work on $\sqrt{2}$.)

First we tried various candidates for $\sqrt{2}$. The children always expressed them in the form 1 and a fraction. To begin with they only tried "ruler numbers" 1-1/2, 1-1/4, 1-3/8,.......They say immediately that $1 < \sqrt{2} < 2$ and that squaring preserves the relevant inequality.

I encouraged them to try numbers of different types:

$1\frac{\text{odd}}{\text{even}}$, $1\frac{\text{even}}{\text{off}}$, $1\frac{\text{odd}}{\text{even}}$. It was obvious to them that $1\frac{\text{even}}{\text{even}}$ need never be tried.

We got an approximation $1.41 < \sqrt{2} < 1.42$ and then we got tired.

I proposed to eliminate all cases, categorizing them as follows.

1. $1\frac{\text{odd}}{\text{even}}$
2. $1\frac{\text{even}}{\text{odd}}$
3. $1\frac{\text{odd}}{\text{odd}}$
4. $1\frac{\text{even}}{\text{even}}$ (Immediately eliminated)

Increasing order of difficulty

1. $1\frac{\text{odd}}{\text{even}} = \frac{\text{even} + \text{odd}}{\text{even}} = \frac{\text{odd}}{\text{even}}$

$$\left(\frac{\text{odd}}{\text{even}}\right)^2 = \frac{\text{odd}^2}{\text{even}^2} = \frac{\text{odd}}{\text{even}} \neq \text{whole number}$$

(or else, odd = (whole number)$\cdot$(even) = even)

2. $1\frac{\text{even}}{\text{odd}} = \frac{\text{odd} + \text{even}}{\text{odd}} = \frac{\text{odd}}{\text{odd}}$

$$\left(\frac{\text{odd}}{\text{odd}}\right)^2 = \frac{\text{odd}^2}{\text{odd}^2} = \frac{\text{odd}}{\text{odd}} \neq 2$$ (or else odd = 2.odd = even)

3. $1\frac{\text{odd}}{\text{odd}} = \frac{\text{odd} + \text{odd}}{\text{odd}} = \frac{\text{even}}{\text{odd}}$

$\left(\frac{\text{even}}{\text{odd}}\right)^2 = \frac{\text{even}^2}{\text{odd}^2} = \frac{\text{even}}{\text{odd}}$. Now $\frac{\text{even}}{\text{odd}}$ can = 2.

For example $\frac{6}{3} = 2$. We have to do better. Going back, say instead

$$\left(\frac{\text{even}}{\text{odd}}\right)^2 = \frac{\text{even}^2}{\text{odd}^2} = \frac{\text{even}^2}{\text{odd}}$$

But even = 2$\cdot$(whole number)
so even^2 = 4$\cdot$(whole number)^2
= 4$\cdot$(whole number)

$$\text{so } \frac{\text{even}^2}{\text{odd}} = \frac{4\cdot(\text{whole number})}{\text{odd}} \neq 2$$

(or else 2.odd = 4$\cdot$(whole number) and then

odd = 2$\cdot$(whole number) = even.)

This last case is admittedly hard. I would only argue that the advantage of this approach is that the first 2 cases are easy and once the student has grasped them he really knows that at least no number of the form $1\frac{\text{odd}}{\text{even}}$ or $1\frac{\text{even}}{\text{odd}}$ can be $\sqrt{2}$.

It might also be worthwhile to run through the standard proof after this one.

D.D = 2. What is D?

$1.1 = 1 < 2$

$2.2 = 4 > 2$

$(1\frac{1}{2})(1\frac{1}{2}) = \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4} > 2$

$(1\frac{1}{4})(1\frac{1}{4}) = \frac{5}{4} \cdot \frac{5}{4} = \frac{25}{16} < 2$

$(1\frac{3}{8})(1\frac{3}{8}) = \frac{11}{8} \cdot \frac{11}{8} = \frac{121}{64} < 2$

$(1\frac{3}{7})(1\frac{3}{7}) = \frac{10}{7} \cdot \frac{10}{7} = \frac{100}{49} > 2$

$(1\frac{2}{5})(1\frac{2}{5}) = \frac{7}{5} \cdot \frac{7}{5} = \frac{49}{25} < 2$

$(1\frac{21}{50})(\frac{21}{50}) = \frac{71}{50} \cdot \frac{71}{50} = \frac{5061}{2500} > 2$

$(1\frac{41}{100})(1\frac{41}{100}) = \frac{141}{100} \cdot \frac{141}{100} = \frac{19881}{10000} < 2$

Since $1\frac{21}{50} = 1\frac{42}{100}$ it must be that

$1\frac{41}{100} < D < 1\frac{42}{100}$

In decimals $1.41 < D < 1.42$

$1\frac{odd}{even} = \frac{even}{even} + \frac{odd}{even} = \frac{even + odd}{even}$

$= \frac{odd}{even}$

$1\frac{even}{odd} = \frac{odd}{odd} + \frac{even}{odd} = \frac{odd + even}{odd}$

$= \frac{odd}{even}$

$1\frac{odd}{odd} = \frac{odd}{odd} + \frac{odd}{odd} = \frac{odd + odd}{odd}$

$= \frac{even}{odd}$

$1\frac{even}{even}$ can always be changed to one of the ones above by dividing out 2's. For example:

$1\frac{6}{16} = 1\frac{3}{8} = 1\frac{odd}{even}$

$1\frac{4}{10} = 1\frac{2}{5} = 1\frac{even}{odd}$

$1\frac{6}{14} = 1\frac{3}{7} = 1\frac{odd}{odd}$

$$\left(1\,\frac{\text{odd}}{\text{even}}\right)\left(1\,\frac{\text{odd}}{\text{even}}\right) = \left(\frac{\text{odd}}{\text{even}}\right)\left(\frac{\text{odd}}{\text{even}}\right) = \frac{\text{odd.odd}}{\text{even.even}} = \frac{\text{odd}}{\text{even}} \neq 2$$

because $\frac{\text{odd}}{\text{even}}$ can <u>never</u> be a <u>whole</u> number. (There is always something left over.)

$$\left(1\,\frac{\text{even}}{\text{odd}}\right)\left(1\,\frac{\text{even}}{\text{odd}}\right) = \frac{\text{odd}}{\text{odd}} \cdot \frac{\text{odd}}{\text{odd}} = \frac{\text{odd.odd}}{\text{odd.odd}} = \frac{\text{odd}}{\text{odd}} \neq 2,$$

<u>or else</u> the top would be <u>twice</u> as big as the bottom and so the top <u>would be even</u> (but it is odd.)

$$\left(1\,\frac{\text{odd}}{\text{odd}}\right)\left(1\,\frac{\text{odd}}{\text{odd}}\right) = \frac{\text{even}}{\text{odd}} \cdot \frac{\text{even}}{\text{odd}} = \frac{\text{even.even}}{\text{odd.odd}} = \frac{\text{2.something.2.something}}{\text{odd . odd}}$$

$$= \frac{\text{4 something.something}}{\text{odd}} \neq 2,$$ <u>or else</u> we would have

4 something.something = 2 odd, so

2 something.something = odd, but an <u>odd can't</u> be 2.<u>anything</u>.

That covers <u>all</u> cases. So what is D ???

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