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Reported is the construction of a primary mathematics curriculum on two kinds of data: the structure of mathematics itself, and the developmental realities of the child. Principles which have been assumed in this development are (1) since mathematics is abstract, the process of abstracting mathematics must be well prepared, (2) since mathematics deals with generalities, the variables involved in structures to be learned must be varied as much as possible to encourage generalization, (3) the necessity to take into account the difficulties in passing from the pre-operational to the operational stage. This involves a radical re-thinking of the way of presenting many classical parts of mathematics to young children, and (4) the need to follow the logical mathematical build-up of natural number and the other number systems in designing the curriculum. In view of these principles, the author stresses the necessity to teach relations and sets, accompanied by logic. (RP)

AN ELEMENTARY MATHEMATICS PROGRAM

by
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a paper presented to

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AN ELEMENTARY MATHEMATICS PROGRAM*

PART 1: PRE-OPERATIONAL STAGE

1. The Problem:

We base the construction of our primary mathematics curriculum on two kinds of data:

1. The structure of mathematics itself
2. Developmental realities of the child.

The technical problems of constructing a viable mathematics curriculum consists in adjusting the data in these two different disciplines. The disciplines are of course the discipline of mathematics on one hand and the discipline of psychology on the other. Possibly the reasons why rationally constructed curricula do not yet exist for the most part is because mathematicians have not interested themselves in the particularly psychological problems of the details of learning mathematics and psychologists have seldom interested themselves in mathematics and even when they have, have found it extremely difficult to learn a sufficient amount of mathematics in order to be able to throw the problems around in the way in which it is necessary before suitable hypotheses may be formed which can be experimentally tested in a psychological laboratory.

For the past ten years or so, my collaborators in different parts of the world and myself acting as a team loosely connected through the International Study Group for Mathematics Learning, have been working on different aspects of this problem. Naturally, the solution of such a problem cannot be assumed to be unique. We must admit that our proposed solution is bound to be one of many. It is, however, a solution which we feel is better fitted to the requirements of the modern world than the traditional mathematics programs and methodologies or most of the so-called new mathematics programs that have flooded the markets, particularly in the United States, in the last decade.

2. Sets and Properties of Objects:

Let us look at the mathematical facts first. We shall take the view that natural number is a property of sets, consequently, is super-ordinate to sets. It is therefore a priory probable that learning about sets should precede learning about properties of sets. We can go one stage further back and ask the question of what we have to know in order to know about sets. Naturally, we need to know about the elements which make the membership of a set. Members of sets in early experience are usually objects. How can we tell whether an object does or does not belong to a set? In order to decide we must be able to distinguish one object from another object. This leads us to the idea of properties of objects, and so to classification. So it seems that one of the first things that would

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need to be thought out in order to begin the road towards natural number is the placing of objects into categories by their properties, in other words, sorting objects into different kinds. This will be one way of establishing membership or non-membership of sets. Sorting and cross-sorting will lead us to the ideas of conjunctions and disjunctions. Sorting into different kinds will give us the idea that every object is either of one or the other kind; this kind of sorting is the "exclusive disjunction" sorting. We will also do other kinds of sorting in which there is overlapping. For example, we might sort objects by colour and then we might sort them by shape or we might sort children by sex and then we might sort them by the colour of their shoes or their clothes. If we mix up the two kinds of sorting, then of course there will be overlapping. There will be overlapping if we have boys and girls on one hand and red-shoed children and black-shoed children on the other, because there will be black-shoed girls as well as black-shoed boys and there will be red-shoed girls as well as red-shoed boys. There will, of course, be children who do not have either black or red shoes and these will be a third kind.

3. Operations on Sets:

Let us sum up: one of the first items to learn is having a property, next comes belonging to a set. When we have learned about having a property we can then go on to learning about having several properties; this is when we do the cross-sorting. This leads on the logical side to the idea of using conjunctions and on the set side to using intersections. When we say that we will have objects in a set so that everyone is either one kind or the other or possibly both, then we are doing a disjunctive classification. From the set point of view we are uniting the set of objects possessing one kind of property with the set of objects possessing the other kind of property.

Conjunctions of attributes will lead us to intersections.

Disjunctions of attributes will lead us to unions of sets.

Very soon we shall have to introduce the idea of negation; this is necessary in order to develop thinking as well as for the introduction of its arithmetical counterpart, subtraction. When we split a set into two complementary subsets then every element of this set belongs to either one or the other but never to the two at the same time, so we can then say that if an element is not in one of these, it is clearly in the other and if it is in one of these then it is not in the other. Later, when we come to the idea of subtraction, we shall have to precede this by means of the operation of the removal of a sub-set; we split a set into two complementary sub-sets, remove one of these and the operation consists in finding what is left.

In order to introduce the idea of addition we shall have to fall back on the set operation of union; but not just any union. It must be a union of sets which have no common elements; in other words, the intersection set of the two sets which we unite must be empty. Clearly, here is another ingredient which must be introduced: the idea of the empty set and the corresponding arithmetical idea of zero. The empty set is the set in which there are no elements and zero is the number of elements of which there are none in the empty set.

Set operations allow us to generate sets when we have two sets before us or in some cases when we have one set before us. The former kind of set operation is known as binary and the other as unary. Examples of binary set operations are intersection and union. An example of unary set operation is the taking of the complement. We need two sets before we can find their intersections or their union but we need only one set before we can find its complement.

4. Relations:

In our mathematical work we shall not only be concerned with operations on sets but on operations on elements of sets, in other words, we shall be wanting to relate elements to elements. This is part of the story which leads to the idea of natural number. Relating something to something always involves two things: something which we relate and something to which we relate. In mathematics we introduce many different types of relations. In life itself we introduce literally thousands of different relations since every verb we utter relates the subject of the sentence to the complement of the verb if there is one. In mathematics we use a few relations but their properties must be very clearly appreciated before they can be properly used.

The relations mostly used in mathematics are equivalence relations and order relations but there are of course many others such as difference relations or geneological types of relations and so on. Equivalence relations can be defined between objects or between sets of objects. For example, if we have a set of objects, each of which is of one particular colour, then "this object has the same colour as the other" is an equivalence relation. An equivalence relation splits the sets that we are talking about into disjoint classes known as equivalence classes; any likeness relation which does that is called an equivalence relation. Of course we can speak of equivalence relations between sets. One of these is the one usually introduced in the modern mathematics books and most children grow up with the idea that this is the only kind of equivalence that exists until they come to fractions or matrices or other more sophisticated studies, when equivalences again rear their ugly heads and they have no idea in what ways these new kinds of equivalences are to do with equivalent sets.

The use of concrete materials also presents both an opportunity and a danger for introducing equivalence relations. For instance, the use of Cuisenaire rods allows the idea to be formed that there is only one equivalence relation worth talking about and that is "to occupy the same space" but of course most of us are just as much concerned with "having the same number" type of equivalence relations. "I have the same number of rods as you have" is just as much as equivalence relation as that "I have the same amount of wood as you have". Having eventually played with a number of different kinds of equivalence relations as well as with other relations, both between objects and between sets, we can single out important equivalence relations; for example, the one which is obtained between sets by taking a one to one correspondence between elements of sets as a criterion for deciding whether a set is or is not equivalent to another set.

5. Natural Number Through a Synthesis of Equivalence and Order:

The application of such an equivalence relation splits the universe of sets neatly into disjoint classes. In each class there are sets with the same number of elements and these classes or families are what are known as the natural numbers. Some prefer to say that the properties of the sets in these equivalence classes are known as the natural numbers. The property of threeness is the property of all the sets consisting of three elements and it is this property which singles out the class of sets in which there are three elements from other classes of sets in which there are either less than three or more than three elements in each set which belongs to them.

Having established these families we then have to put order into the chaos. We invent the more and less game. Given any two families we take one member-set of one family and one member-set of the other family and pair them off element by element. If in one set there are some elements that are not possible to be paired with elements in the other set, then we say that in this first set there are more than in the other and the family to which the first set belongs is said to be greater than the family to which the second set belongs. In this way we have a way of deciding which family is greater and which is by implication smaller because we say that if family A is greater than family B then the family B is smaller than the family A.

We then introduce a one more game in which, besides the more and less relation between families, we establish the one more relation between families. This will be the case if there is only one element left out of a set of one family that cannot be paired off with elements of a set from another family. In this case, the family from which the first set was taken is said to be one more than the family from which the second set is taken. This one more idea makes a chain, starting with the one family in which every set consists of one single element and this chain will reach any other family if you apply the connecting link of the chain a sufficient number of times.

This might be the first time that a relation has turned into an operator or a function, with one family being "one more" than some given family. If we have a situation in which a relation always leads to a well determined element of our universe, every time we start from any given element of the universe, then such a relation is called a function or an operator. It is rather like a machine which has an input; it does something and then produces an output. The element which we choose is the input, then the relation we apply is what we do and the element to which we are led is the output. Our machine is automatic and only has one output assuming the input or the inputs have been chosen. Many machines have of course more than one input. We might have to do more than one thing before the output is determined, but determined it is; otherwise it is not called a function or an operator.

So, we have been led through the considerations of relations in general, sets in general, to the consideration of equivalence relations and order relations in particular, and so to the equivalence classes arising out of one particular equivalence relation between sets. These are the natural numbers. They are families known as 1,2,3,4 and so on. These families are then put in order by the more-less relation and put into a chain by the one more relation. It is determined that every family has a next family which follows immediately afterwards.

6. Other Roads from Relations to Functions:

This is where we must begin to think seriously of the psychology of the matter. We have reached the idea of an operator or function along a very narrow road, namely along the arithmetical road through the consideration of sets and relations. There are many other possible relations which will lead to other possible operators and functions, for example, geometrical ones. Since the exploration of the simple properties of space can begin in the very first grade, we might as well begin to lead them through introductory exercises which will lead them to geometrical relations and subsequently to geometrical operators. A game might begin on the classroom floor by someone saying, "I am walking along the floor boards, but you are walking across the floor boards. What do I have to do to be like you?". Well, I have to turn through a right angle, but of course it does not matter whether I turn to my right or to my left, I shall still be walking across the floor boards so I shall be "like you". If "like you" means walking not just across the floor boards but in the same sense, that is, pointing towards the same wall as you are, then of course I have no choice but to turn through a right angle in a particular sense, that is, to the right or to the left as the case may be. In this case the idea of rectangularity which is a relation will give rise to the operator, "take a quarter turn towards your right" which has a uniquely determined outcome, whereas "turn through a right angle" does not have a uniquely determined outcome.

In these ways and in many others through the use of concrete materials such as the logic blocks or many other materials, we can trace the path from relations to functions. Let us now continue our discussion of operators or functions. Naturally, the "one more" operator is not going to be the only one. We shall soon introduce our "one less" operator, a "two more" operator, a "three less" operator and so on and children will soon begin to string these operators together. They will have a certain state of affairs on which they will operate yielding another state of affairs; they will begin to operate on the new state of affairs by another operator or the same one, yielding again a new state of affairs on which they will operate again and so on. In this way they will construct a chain of operators which will be interspersed with states of affairs.

7. Chains of Operators:

The states of affairs in arithmetic are expressed by natural numbers and they can be concretely represented by a set which belongs to the family about which we are speaking. For example, if we wish to represent the natural number three, all we have to do is to take a representative sample from the family three. Then we have to apply, for example, a two more operator; that is, we have to find another representative of the family to which we are led from this three family by applying the two more operator. This will, of course, be the five family. So, the next state will be represented concretely by the presence of a set of five objects and so we can go on. It will be quite difficult for children to use the one more operator in succession. If they are asked what kind of operator will take them from a particular state, not to the next state but to the next state after that, they will not necessarily realize that this is the "two more" operator. In other words, they will not necessarily know that a one more operator followed by a one more operator could have been replaced by a two more operator as far as the family membership of the set is concerned which we would have reached in one case or in the other case. In other words, even though they would know that the state one operated on by the operator one more yields the state two, they will not necessarily realize that a one more operator followed by another one more operator could be replaced by a two more operator.

But let us be careful; what do we mean by could be replaced? Here we are smuggling in another equivalence relation. Now an equivalence relation always has to operate over a universe, over a set of certain objects or entities or objects of thought. What are our objects of thought which we are relating together and making equivalence classes of? These new objects of thought are our chains of operators; the simplest chain is just one operator. There are more complicated chains which consist of operators being applied in succession in such a way that the output of each operator becomes the input of the next. What do we mean by saying that one chain is equivalent to another chain? What we mean surely is that the input-output property of one chain is just the same as the input-output property of the other. If an input of one yields an output of five in the case of one chain then an input of one will yield an output of five in the other chain, and so on. If two chains have the same input-output properties, they belong to the same family; they belong to the same equivalence class.

PART 2: OPERATIONAL STAGE

8. Operators Become States. Introduction of Integers:

We here come across a really difficult hurdle. What is the hurdle? The hurdle is to be able to speak about the operators or even chains of operators as things that we could do something to instead of them being things that we do. A one more is a thing that did something to a state and gave us another state. Now a one more becomes a chain which we connect by an equivalence relation to another chain; so what was a do before now becomes a thing. In other words, an operator has now become a state! When something that was a do becomes a thing we reach a higher level of abstraction. We are operating on an operator. This means of course in terms of the developmental theory originated by Piaget that in this particular area we have reached what is known as the operational stage. In the pre-operational stage we would be able to operate on a state and obtain another state. In the operational stage we are able to operate on another operator and obtain yet another operator. This is no mean feat and the difficulty of it should not be underestimated. For this reason states and operators are kept apart for quite a considerable time in our program in every area in which a new subject matter, a new mathematical structure is introduced. For example, when we are dealing with the properties of the geometrical bodies such as a square, an equilateral triangle or a regular tetrahedron, we take the position occupied by such a body as a state and the movements which need to be performed on the body in order for to occupy another position as the operator. In the beginning, that is, in the pre-operational stage, children merely learn how to make up chains of states and operators followed by further states and further operators. In other words, we have positions which they can change to obtain other positions which they again change and so on... The kind of problems they solve at this stage would be, for example: what state would arise as the result of a particular state operated on by a particular operator or what operator do you need to apply in order to get from one state to another given state and so on.

At the operational stage they will be looking at the movements themselves and they will try and solve problems such as: in what other way can we arrange the kind of change of states which is occasioned by a certain series of operators, by getting a certain other series of operators to operate on our initial state? It may be that the string of operators behaves in a very similar way to another string of operators. By a "similar way", we would then mean that if we feed the same state into one or into the other string of operators, we will find exactly the same state being disgorged at the other end of one string as at the other hand of the other string. This is how we define the equivalence of strings of operators. These operators can then be not just additions, subtractions but they can be rotations, reflections and also other changes whose outcomes are determinate.

Let us examine a little more closely how such strings of operators or in particular single operators to which strings might be equivalent, can themselves become states. We saw that a state described by the membership of a particular set to a family of sets can be represented by the presence of a certain concrete example of such a set. For example, if we are thinking of the state 3, we can represent this in a concrete fashion by providing an example of that state, namely a set of three objects, three elements. How can we provide an examplar of the operator two more? One way would be to give the input and then the output. For example, we might take three apples and five oranges and we could say that there are two more oranges than apples and so this represents the two more property assuming that we always go from the number of apples to the number of oranges. The number of apples is three, we apply the two more operator and we obtain the number of oranges which is five. Naturally, just as in the case of sets of three elements, many sets of three elements could represent the property of threeness. Even so now, many ways can be found of representing the property of two moreness. If we have three apples and five oranges or if we have four apples and six oranges or ten apples and twelve oranges there are always two more oranges than apples, so all these representations would do as a concrete realization of the property of their being two more. Naturally, we need not use apples and oranges, we can use boys and girls or knives and forks or any other two kinds of objects which are easily distinguishable from each other. In this way, by providing a set of inputs and to each input providing its corresponding output, we are giving a large number of representations in a concrete way of the idea of their being a certain number more of one kind of thing than of another kind of thing.

Naturally, we can also imagine that we can provide representations of lessness properties as well as moreness properties. It will then be possible to combine some of these properties by means of operations. We can combine a two more property with a three less property by taking a representation of two moreness and a representation of three lessness. For example, a two moreness could be represented by a set of ten apples and twelve oranges and three lessness by a set of ten apples and seven oranges. If we put these two sets together, we shall have twenty apples and nineteen oranges, in other words one less orange than there are apples. In other words, we shall have obtained a representative of the one lessness property. This way we can introduce the addition and the subtraction of integers.

9. Introduction of Rationals

The naturation of the operators one more, two more, one less, two less and so on to what you might call "statehood" opens the way to the study of integers. In the same way the maturation of multiplication and division operators can lead into the study of rational numbers. One easy way to introduce multiplication and division operators is by studying them as replacement operators. We have a certain set as the input for a certain operator and the operator will simply replace every member of the input set by a certain set of objects, each such set belonging to the same family. For example, every member of the input set might be replaced by a set of three elements. All these sets of three elements would then be united in a union set and this union set will be the output set. The number of members in the input set is the input number and the number of members in the output set is the output number. So, for example, if we have a certain number of plates and on each plate we put three oranges, then we have the operation which gives us three oranges for one plate. So for example, if a set of four plates is passed through this machine, a set of twelve oranges will come out as the output set. Division operators can likewise be introduced as the opposite type of operator to the multiplying operators. For example, we might say that there is one car available for taking four passengers so if the passengers are lining up then we know that, for example, for a set of 20 passengers we will need five cars, because every time four passengers line up they fill one car. Here the input set is a set of 20 passengers, the output set is a set of 5 cars. And the operator is one car for every set of four passengers.

In this way, we can soon reach the idea of a fractional operator because there is no reason why we should take one for many or many for one. We can take operators which give us many for many but of course not the same many. For example, if there are a certain number of chairs for children to sit down to watch a television show and there are too many children, we might say, "Oh well, let us put down two chairs for every three children, they can be squashed in." Say twelve children come in, then every time three children come in, we put down two chairs so there will be eight chairs for the twelve children. So, here the input set was the set of twelve children and the output set was the set of eight chairs. The operator told us to find a set of two chairs for every set of three children; in other words this is a two for three operator and we say that the number of elements in the output set is two-thirds of the number of elements in the input set.

If we wish to in some sense provide concrete representations of such fractional operators as states, Then we will need to provide an example of an input and corresponding output such as, for example, twelve children and eight chairs. But we could just as well have had six children and four chairs and so on. Here, we are again establishing equivalence classes and these are the equivalence classes to which the so-called equivalent fractions belong. Children who have studied equivalence relations and have realized how an equivalence relation invariably splits the universe into disjoint classes or families known as equivalence classes, will have no problem in realizing that certain situations are rather like each other as far as certain criteria are concerned. For example, a set of twelve children and eight chairs is similar to a set of six children and four chairs because in both cases we have two chairs for every three children. There are, of course, an infinite variety of ways in which we could represent the fraction two for three or 2/3. We can use chairs and children or, apples and oranges or pebbles and shells or any other kinds of objects which are easily distinguishable from one another and for which we can make up a plausible story for the amusement of the children.

10. Operators in Logic.

A similar situation arises in logic. In our program we introduce logic not through the propositional calculus but through what we might call the attribute calculus. We take a certain universe, usually the set of logic blocks consisting of a certain number of objects of definite colours, shapes, sizes and thicknesses. Attributes such as red or square-shaped or large or thick would be represented by the set of all those blocks possessing the particular attribute in question. Then, we use logical operators. Some of these operators are binary and there is one particular one which is unary. We have already mentioned the negation operator when we were dealing with complements of sets. The complement of a set corresponds to the negation of an attribute. The complement of the set of all the red objects is the set of all the non-red objects. Correspondingly, the negation of the attribute red is non-red or not red. So, "not" gives you an attribute as soon as an input attribute is given. For instance, blue is the input, not is the operator and not blue is the output. The other logical operators that we use have to have two inputs. These are the binary operators. For example, the logical operator of conjunction usually uses the word "and" when we use it in ordinary every day conversation. Sometimes the word "and" is even left out. When we say square and red, what we mean is that the block we are thinking of must be both square and red at the same time. So, we are thinking, in fact, of the intersection set of the set of square blocks with the set of red blocks. So the logical operator and corresponds to the intersection operator for sets.

In this way we build up certain ways in which we can operate on logical states. Naturally, we can operate on the logical state we have obtained by previous operations. For example, if we say both red and square then we can negate this, and so pile another operator on top of it. We say: not both red and square. When something is not at the same time red and square, of course it can be a not red square and not square red or a not red, not square. What it cannot be is a red square. So, if the input into our <u>not machine</u> is <u>red and square</u>, the output is <u>not (red and square)</u> not to be confused with (not-red) and (square).

The grammar and the relationships involved in proceeding from one logical state to another logical state through the use of operators is what we might call the attribute calculus. But then at one stage we begin to operate on these operators, that is when we begin to <u>reason</u>. Using logical operators on logical states may find us in a pre-operational stage of logic but operating on the operators and thus reasoning will find us in the operational stage of reasoning, normally known as the hypothetical deductive stage according to Piaget's developmental theory. For example, let us again take the attributes <u>both red and square</u>. This was obtained as an operator from <u>red</u> and from <u>square</u> by "and"-ing them. We can conclude in some vague way that such an object is red because all red squares must be red. We can also conclude that it is a square. <u>How can we conclude this?</u> One good reason is that the set of all the red squares is a sub-set of all the red blocks and we can reasonably say that if an object is in a sub-set of a set we can conclude that it is also in the set. So, we can always pass in the direction sub-set to set in reasoning. So, since the set of red squares is a sub-set of the set of red ones, we can set up a reasoning pattern which allows us to pass from <u>both red and square</u> to the attribute <u>red</u>. We are, of course, saying less when we say <u>red</u> than when we are saying <u>red and square</u> and all reasoning passes from stronger to weaker. This is how we learn to operate on what previously were operators. There are many more complex and sophisticated reasoning patterns and most of them can be based on the simple assumption that from membership of a sub-set we can conclude membership of a set. It is necessary, however, to add what we might call, the <u>intersection principle</u>. If we have more than one assumption, then our assumption from which we can reason must be based on the intersection of the set corresponding to the first assumption and the set corresponding to the second assumption. For example, if we say either red or square, then we are dealing with the union of all the red ones with all the square ones. This may be our first assumption. Then our second assumption might be that we have non-squares. So, we may only reason from the intersection of the set of all <u>non-squares</u> with the set corresponding to either <u>red or square</u>. We shall see that this interesection consists merely of the <u>red non-squares</u>. We can conclude that we are dealing with the red non-squares and consequently by the previous argument we can conclude that we are dealing with the red ones. So, from the two assumptions or premises,

(1) <u>either red or square</u>
(2) <u>not square</u>; we can first conclude by the intersection principle
<u>both red and not-square</u>
and then by the sub-set to set principle we can conclude that
<u>red</u>.

It will be noticed that all these are attributes. We are reasoning on attributes and not on statements. We are in a sense tacitly introducing existential operators, quantifiers by using the largest possible set each time whose elements possess the attribute which we want to express in concrete form by such a set. We must be careful not to assume that the negation of the attribute comes to the same thing as the negation of the quantified statement. "All the blocks are red" if negated would be: "not all the blocks are red", whereas negating the attribute red would give us "all the blocks are not red".

> All the blocks are not red
> does not by any means mean the same thing as
> not all the blocks are red.

It is to avoid this difficulty that in the beginning we do not use explicitly quantified statements but we apply the logical operators to the logical attributes themselves and not to the statements. The predicate calculus involving quantifiers comes later in our logic course and is usually reached by about sixth grade level.

11. Use of the Operational Stage in Arithmetic

Having reached the operational level, we can then safely operate by means of the operators in whose sphere such a level has been reached. It must not be assumed that the operational level is reached simultaneously in all subject matters. In logic and in geometry it might be reached much later than in arithmetic and algebra. Once the operational state is reached, then we can concentrate on the technical problems of calculations, that is of solving problems in which such arithmetical operators as addition, subtraction, multiplication and division, squaring and square-rooting and so on occur. In order to understand these computations, it is necessary to understand the way in which numerical information is passed from one person to another. In order to do this, we make use of what is known as the place value system. So, one of the tasks of the teaching of mathematics should be the exact clarification in the child's mind of what the place value system is about. The place value system is based on the properties of powers. So, the properties of powers must at least implicitly be understood by the children so that they can manipulate the successive powers of the base we use which is normally ten. In order to achieve a breadth of understanding of powers, of course more than one base is necessary. This will enable children to distinguish between generalities and particularities. For example, when they know that $11 \times 11 = 121$ is a general result and not a particular one, independently of the base used, then they are learning that $(X + 1)^2 = X^2 + 2X + 1$. There is, of course, no need to express this algebraic truth in terms of X's or frames. It is quite sufficient for the child to understand that the base number is arbitrary and the result still holds as to form in any base. It would be dangerous to assume that one example, namely the base ten, is sufficient to get a child to understand what has taken humanity thousands of years to develop. We use a large number of concrete materials, in particular the multi-base arithmetic blocks, to accustom children to manipulating these basic arithmetical ideas involved in the use of powers. Through coming across the problems themselves, children evolve their own algorithm for multiple digit multiplications and divisions, some of which are extraordinarily ingeneous. In the same way, concrete manipulations of the situations representing fractions allow them to evolve the techniques for coping with the abstractions known as fractions and finally the same applies to the development of the ideas of integers, vectors, matrices, complex numbers, groups, rings, etc., all of which form part of our elementary school curriculum.

12. Summing Up

In the foregoing I have tacitly assumed the following principles:

(1) Since mathematics is abstract, the process of abstracting mathematics must be well prepared. The path from the concrete to the abstract leads through the stages of seeing what is common to make "like" situations, and discarding the "noise", establishing the exact correspondences between such similar situations, representing such abstractions by schemata, then describing these schemata in symbolic language, and then using the symbolic language to generate further information.

(2) Since mathematics deals with generalities, the variables involved in structures to be learned must be varied as much as possible to encourage generalization. Hence the use of multibase arithmetic, for example, the use of different mathematical groups (not just the two-group) for use in multiplications, etc.

I have explicitly stressed the necessity to take into account the difficulties in passing from the pre-operational to the operational stage. This involves a radical re-thinking of the way of presenting many classical parts of mathematics to young children.

I have also tried to follow the logical mathematical build-up of natural number and the other number systems in designing the curriculum. Hence the necessity to teach relations and sets, accompanied by logic, to lead to the introduction of natural number. This is not to say that there are not other, equally valid, solutions to the problem of dove-tailing the developmental course of the child with the foundations of mathematics. In fact many such solutions, radically different from each other, are possible, and it is hoped that work will soon begin in many centres to investigate the problems involved.