A hypothetical mathematical model of a school is presented to (1) illustrate what such a model would look like, (2) determine its value to decision makers, and (3) determine its data requirements. The model relates increases in achievement to student/staff ratio, a measure of staff quality, materials used, space available, effort in community relations, and the socioeconomic background of students. The relationship is nonlinear. The model is illustrated numerically by use of hypothetical data. Use of the model in a search for optimum school resource allocation is discussed briefly, and further developments are outlined. (Author/DE)

An Hypothetical Model of a School

Roger L. Sisson*

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Abstract

An hypothetical mathematical model of a school is pre-
sented to illustrate what such a model would look like, its value
to decision-makers and data requirements. The model relates
increases in achievement, to student/staff ratio, a measure of
staff quality, materials used, space available, effort in community
relations and the socio-economic background of students. The re-
lationship is non-linear. The model is used with hypothetical data
to illustrate the model numerically; the resulting effects appear
to be reasonable. Use of the model in a search for optimum school
resource allocation is discussed briefly. Further developments are
outlined.

Operations researchers have recently become interested
in education. Attempts are being made to improve the allocation
of educational resources, the assignment of personnel and the
scheduling of classes. It is becoming clear, however, that before
there is improvement in these operational decisions, some quanti-
tative understanding of the educational process itself is required.
Until there is such understanding, the researcher is in the same
position as a designer trying to improve a railroad system in dense
fog. He has no idea of where the trains start, what route they
take or where they terminate and so he can deal with the system
only in the most gross statistical terms. Our current knowledge of
the teaching-learning process is certainly foggy.

Efforts to illuminate the process of education can proceed
on several levels. Models can be developed for entire school systems,
schools, classes, the teacher-student interaction, or for the psycho-
logical processes in the student.[1] The purpose of this note is to

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(1) Sisson, R.L., "Can We Model the Educational Process?", Journal
of Socio-Economic Planning Science. (forthcoming issue)

propose a model of a school; in particular, an elementary school. This

model needs to be validated. If valid, it would be useful in determining

optimum allocation of school budgets and in evaluating the benefits of

increases in such budgets. This note will review the reasons for

choosing the elementary school as the level of study, will state the

model, argue its reasonableness, discuss how it might be validated

and how it can be used to facilitate allocation decisions.

Choice of Objective Function

One of the major problems in applying operations research to

education is defining an objective function. First, it is difficult to

identify the decision-makers. If a decision-making group is found, such

as a Board of Education, the members are unlikely to agree on what

is to be measured to evaluate a school. In any case, the objective

function is likely to be multi-dimensional.

This problem becomes more severe as we examine higher

levels of education. As the student becomes more mature, he may

(perhaps, he should) contribute to the establishment of the objectives

of the educational activities in which he participates. Thus at the

high school and college level, the institutional objective moves to a

higher level: to permit the student-faculty groups to realize their

objectives. Because of this complexity in establishing objective

functions at high school and college levels, the model presented here

is for an elementary school. At this level the decision-making

hierarchy is reasonably well defined: classroom teachers, principal,
superintendent, board of education. Furthermore, the objectives of
the school are, at least, partly defined. The elementary schools should
teach the students certain skills basic necessary to live in the culture:
speaking, reading, writing, organizing ideas in a way which facilitates
communication, performing arithmetic and basic mathematical processes,
solving certain kinds of problems, knowing facts about the culture's
history and form of government and being able to participate in or at
least remain in a group. This list is not complete, but a finite list of
these kinds of behaviors can be drawn up which most decision-makers
and communities would agree are the behaviors which the children
leaving, say, sixth grade should be capable of performing. These
(except for social abilities) are the ones tested on the standard tests;
(e.g., Iowa, Stanford).

In this work, therefore, the objective function is defined in the
following way:

Assume a series of tests have been devised, one for each grade.
These tests measure a sample (properly drawn) of the behaviors which
children in that grade, and several grades above and below, should be
capable of exhibiting. Assume that these tests have been normalized by
giving them to a sample (also properly drawn) of children from the culture
(usually a nation) of the age level corresponding to the grade. The
question of the universe from which to draw this sample is far from

resolved but these tests are relative, not absolute measures, in any case, so the final results may be insensitive to the exact method of normalization. Furthermore, assume that the scoring of these tests is designed so that a student attaining the average for his age, a, is given a score, s, equal to the age. An average nine-year old would score s=9.

Most existing tests are designed in the fashion described above, except that they are often not good samples from the universe of desired behaviors. They do not cover the social, political and economic knowledge and skills. I believe that the standard tests could be expanded to include these classes of behaviors with a little effort. The extensions would involve, perhaps, game situation or role playing tests as well as pencil and paper tests.

Standard tests use grade rather than age score. In order to avoid confusion where a child is not promoted to the next grade, age seems more useful. It also reflects the objective of having children complete elementary school at ages close to the average (e.g., not much over 12 on leaving sixth grade). Let us assume tests such as those just described, exist. The unit of this study is any large group of E elementary students ($E > 100$); typically a single school. The standard test is given at the end of each year and the average change in "achievement" for the school is computed thus:

$$D = \frac{1}{E} \sum_{i=1}^{E} [s_i(a_i) - s_i(a_i - 1)]$$

D = a measure of the average student achievement change over a year in the school.

E = number of students. (Where there is high mobility between schools, this factor is difficult to determine, but assume test score, $s_i (a_i - 1)$, from the school just left is available for new students).

$s_i (a_i)$ = the score on the test taken when the child , i, was age a_i. It is assumed that the test given is appropriate to that age.

D will be taken as a measure of performance of the school. When a full operations research study is undertaken, one objective could be to maximize D, within resource constraints. For an average school $D = 1$.

The problem now is to develop a model of the relationship between resource expenditures and this measure of performance.

The Model

This model distinguishes two sources of influence on a student: the school, and all other influences; at the elementary level, the parents are assumed to be most influential. It is hypothesized that the school can influence the parents (for the benefit of the children), but, except, for this interaction, the nature of the other influences are assumed to be constant overtime.

The model (as developed to date) is static; it predicts the change in achievement, D, in a typical year assuming steady-state operation of the school. Extension of the model to the dynamic case will be discussed.

The factors taken into account are these:

S = the staff/student ratio in the school, where "staff" includes all professional, paraprofessional and administrative personnel.

V = a measure of staff quality. One possible measure is the average score over the staff attained on a standard verbal achievement test.[2]

M = the replacement cost per student of the educational texts, materials and equipment available to the students.

R = the area (square feet) per student of school buildings available for use by these E students.

I = the effort measured in dollars per student expended in working with parents and the community.

P = the average grade attained by parents of the children in the school, $0 \leq P \leq 16$.

S is actually computed as staff-hours/student-hours. The staff hours should exclude those staff hours which are spent in working with parents and community; the latter being included in I.

The basic phenomena of elementary education is hypothesized to be the interaction of students and adults. Thus, the staff-student ratio is a key variable which effects D. The effect is not linear, however. The difference between S = .005 and .01 (student/staff ratios of 200 and

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(2) Bowles S. and Levin, H. "Equality of Educational Opportunity = More on Multicollinearity and the Effectiveness of Schools", The Journal of Human Resources, Summer, 1968.

100) is small. On the other hand, the difference between $S = .05$ and $.1$ is usually considered to be large. Most school administrators claim "small class sizes" will improve their schools. At the other extreme, it is unlikely that $S = 2.$ (two staff/student) will be much better than $S = 1$ (direct tutoring). The logistic function models this non-linearity.

$$D = \frac{A}{1 + e^{B-CF}}$$

The terms A, B, and C, in this relationship are derived as follows.

The independent variable, F, is to be the effective staff-hours per student hour. The effectiveness of the staff is assumed to be measured by V. Thus,

$$F = k_0 SV$$

where k_0 (and all other k_i) are scaling parameters.

The variable A indicates how large D can be very large amounts of staff time are used:

$$D(F \rightarrow \infty) = A.$$

The hypothesis is that this ultimate capability of the children is determined by the early, preschool training and genetic inheritance. P will be used to estimate these influences; thus,

$$A = k_1 P.$$

Recall that we are dealing with an average for a school. This does not mean that a particular child is limited by $k_1 P$; only the average.

When $F = 0$, we have a "school" with no staff. Children will, it

is hypothesized, still learn (although there is little experimental data on this). The learning under these conditions is determined by the effective parental influence and the materials available.

Parental influence is hypothesized to be related to both P, which we are using as a measure of basic parental support and I, the effort the school makes in further motivating parents to in turn, motivate their children. N is defined as the effective parental influence resulting from these two factors. In the absence of evidence otherwise assume a linear, additive combination:

$$N = k_4 I + k_5 P.$$

Thus, even though P may be low, heavy effort on the part of the school (I), might cause indirect learning. The parameter B determines D with $F = 0$. To represent the no-staff learning we must combine the effects of N and M in defining B. Thus let:

$$B = \ln[\frac{1}{k_2 \, NM} \; -1]$$

As N and/or M increase, B decreases, and D(F=0) increases. In fact

$$D(F=0) = k_1 \, k_2 \, PNM.$$

The change in D for values $0 < F < \infty$ can be characterized by the value of D at the point of inflection.

$$D(\text{point of inflection}) = B/C.$$

This can be seen by visualizing the curves as C changes (B being already determined):
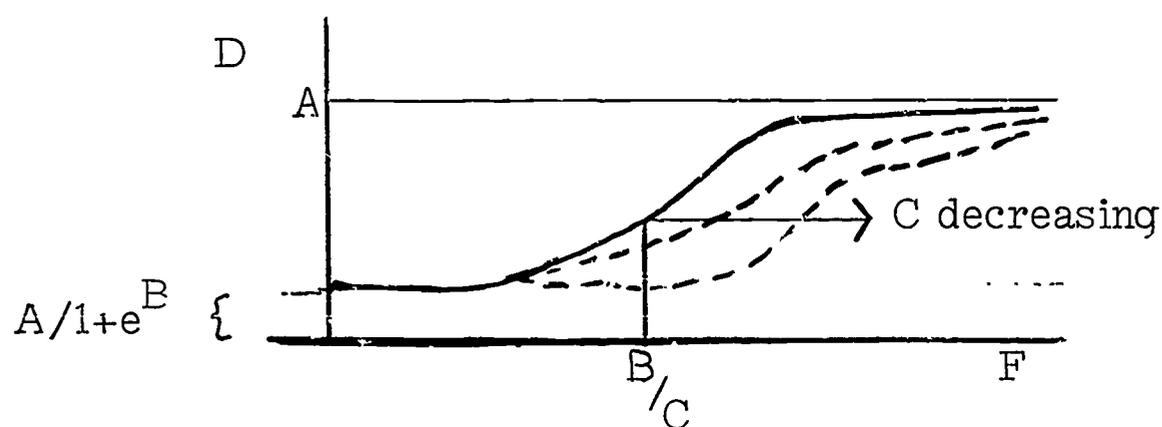
The point at which the rapid increase in achievement occurs (as staff

is increased) should be related to the materials available to the teacher-

student group, to parental support and to whether the school is crowded

or not. More parental support (high N) means the increase occurs at

lower F (higher C). Better materials and equipment (high M) have a

similar effect in that the teacher's time is more effective. Crowded

schools detract from the staff for the same increase in achievement.

These factors are combined to calculate C in that way:

$$C = \frac{NM}{k_3} (1 - e^{-R/k_6})$$

The area R is introduced in the exponential form to reflect that concept

that, on one hand, too little space is determintal(crowded classes), but,

on the other, too much does not promote learning. If R is low, $(1 - e^{-R/k_6})$

is low and the point of inflection moves to the right. But as R increases

much beyond K_6, there is little increase in $(1 - e^{-R/k_6})$ hence little

change in the D-F relationship; excess space has no effect.

The resulting model is:

$$D = \frac{k_1 P}{1 + \exp[\ln(1/k_2 NM - 1) - (FNM/k_3)(1 - \exp(-R/k_6)]}$$

Or

$$D = \frac{k_1 k_2 NMP}{k_2 NM + (1-k_2 NM) \exp[-(FNM/k_3)(1-\exp(-R/k_6))]}$$

Figure 1 shows the relationships for a number of cases.* The k's here are hypothetical; chosen to make the model behave in a reasonable fashion. The problem of how to determine the proper values of the parameters is discussed in the section after the next.

Is This a Reasonable Model?

This model is reasonable in that it produces results that conform to those found by educators:

-- The students' progress is heavi'˙ˑ ˑependent on parental and socio-economic factors, measured by P. [3]

-- Increasing staff improves achievement, but very slowly, over the range normally tested (S = .05, .07), in all but the most sophisticated communities.

-- Increasing staff does more good with students from good backgrounds.

-- Crowded schools lead to lower achievements.

-- Working with parents directly (I>0) helps considerably with low-P children, not much with high P children.

-- Better materials improve education to some extent.

(3) Coleman, et. al; Equality of Educational Opportunity, Washington D.C. G.P.O., 1966.

* Program and computations by Vinay Kumar, University of Waterloo

Curves are identified by

P,M,R,I

P = parental grade level
M = materials (\$/student)
R = area (sq, ft./student)
I = community support effort
(\$/student)

k_0V set = 1 k_4 = .01
k_1 = .083 k_5 = 1.
k_2 = .002 k_6 = 40.
k_3 = 2.5

scale change

16,10,80,0

8,20,80,0

8,10,80,0

4,10,80,0

8,10,120,0

8,10,40,0

8,10,80,100

4,10,80,100

8,10,
80,100

4,10,80,100

D   Achievement change

1.9 1.8 1.7 1.6 1.5 1.4 1.3 1.2 1.1 1.0 .9 .8 .7 .6

0 .02 .04 .06 .08 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
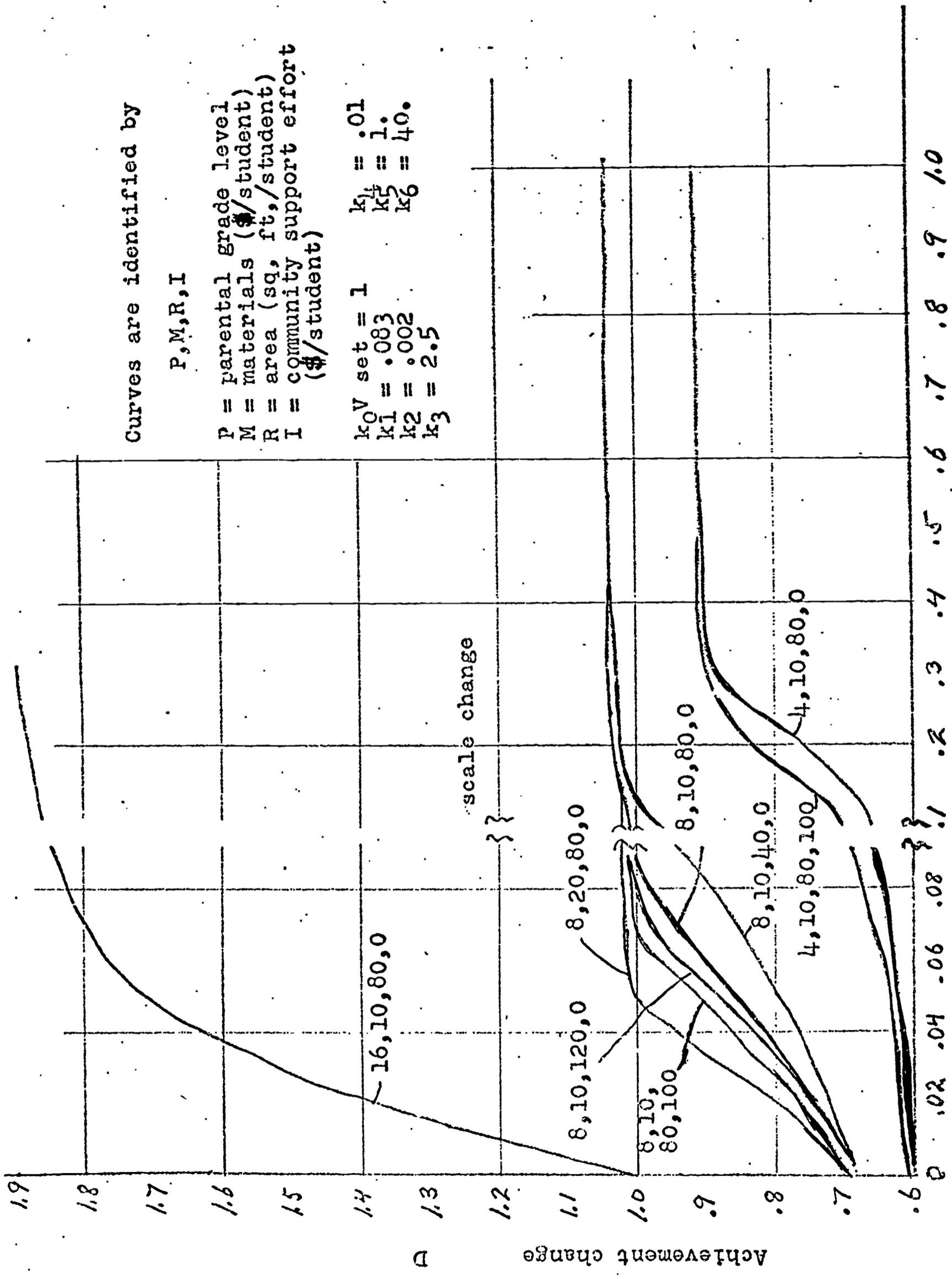
Effective staff/student ratio   F   FIGURE 1

-- Better staff improves education.

-- Students who get poor starts do worse and worse.

(Assuming the initial pre-1st grade test score is 6, then, after six years:

if $D = .6$, sixth grade score is 9.6,

if $D = 1.5$, sixth grade score is 15.0,

so that poorly educated students are at _grade_ equivalent 3.6 and very

well educated ones at 9.0).

The model, however, has some deficiencies which make it less reasonable

than desired.

-- The model is fairly complex; it is not "neat". This may simply

reflect the complexity of the educational process.

-- k_2NM is large enough, the model is invalid as it requires

taking the logarithm of a negative number.

-- the model does not account for activities by non-school

agencies designed to improve either children's or parent's motivation

for or skills in education. This activity could be included by redefining

I as activity of this sort by any agency in the community. The definition

of the kind of activity to be included in I would then have to be made

clear.

-- The model ignores the effect of curriculum content and tech-

nique changes except insofar as they involve more total staff or mater-

ials. Educational experiments seem to indicate that changes in the

details of presentation or of student-teacher interaction are not likely to

affect achievement. But major changes in technique, as from a student-paced to a structured class, may have a noticeable effect on achievement change.[4] If so, the model would have to be expanded.

-- The model contains seven parameters which represent characteristics of behavioral processes (teacher-student; staff-parent; parent-student). It is impossible at present to derive these parameters from theory. They must be obtained empirically, as discussed below.

-- The model is static; a difficulty discussed below.

Evaluating the Parameters

To use this model it is necessary to determine the values of the parameters k_i, $0 \leq i \leq 6$, that is, the parameter vector, $\overline{K}$. This will also help determine the validity of the model. The model is supposed to predict D for any school with a single set of parameters. Thus, if different samples of schools require substantially different parameters, doubt would be cast on the model structure. The most direct way to obtain parameter values would be to use a parameter identification method commonly applied in control system work.[5]

This requires a reasonably large sample of data, within which the variables cover wide ranges, if possible. Thus, for L schools we obtain: P, S, V, I, M, R and the test score difference D. To obtain

(4) Bredemeier, H.C. "Schools and Student Growth", The Urban Review, April, 1968, pp. 20-27.
(5) Zadeh, L. "On the Identification Problem", IRE Transon Circuit Theory, Vol. CT-3, No. 4, December, 1956.

D, of course, we need at least two sets of test scores, differing by a year, and the students' ages. Data should be obtained from stable schools. If, for example, the school has had an influx of teaching assistants (due, say, to Federal aid), it should not be used (see discussion below of dynamic effects).

A search criteria is formed, such as

$$\delta = \frac{1}{\iota} \sqrt{\sum_{\iota} (D_{SC} - D_{MOD})^2} \; .$$

$D_{SC} = D_{SC}$ (P, S, V, I, M, R) the change in test scores for the

real school ι and

$D_{MOD} = D_{MOD}$(P, S, V, I, M, R) the change predicted by the model

for the same independent variables.

The procedure to find $\overline{K}$, the parameter vector, is then that of Figure 2.

The search package can be one of a number of available search programs.

When δ is minimum and close to 0, the model predicts D well. If it predicts well over all schools in the sample (Var $(D_{SC} - D_{MOD})$ is small, then $\overline{K}$ (δ min.) can be accepted as a good set of parameters.

initial

$\overline{K}$
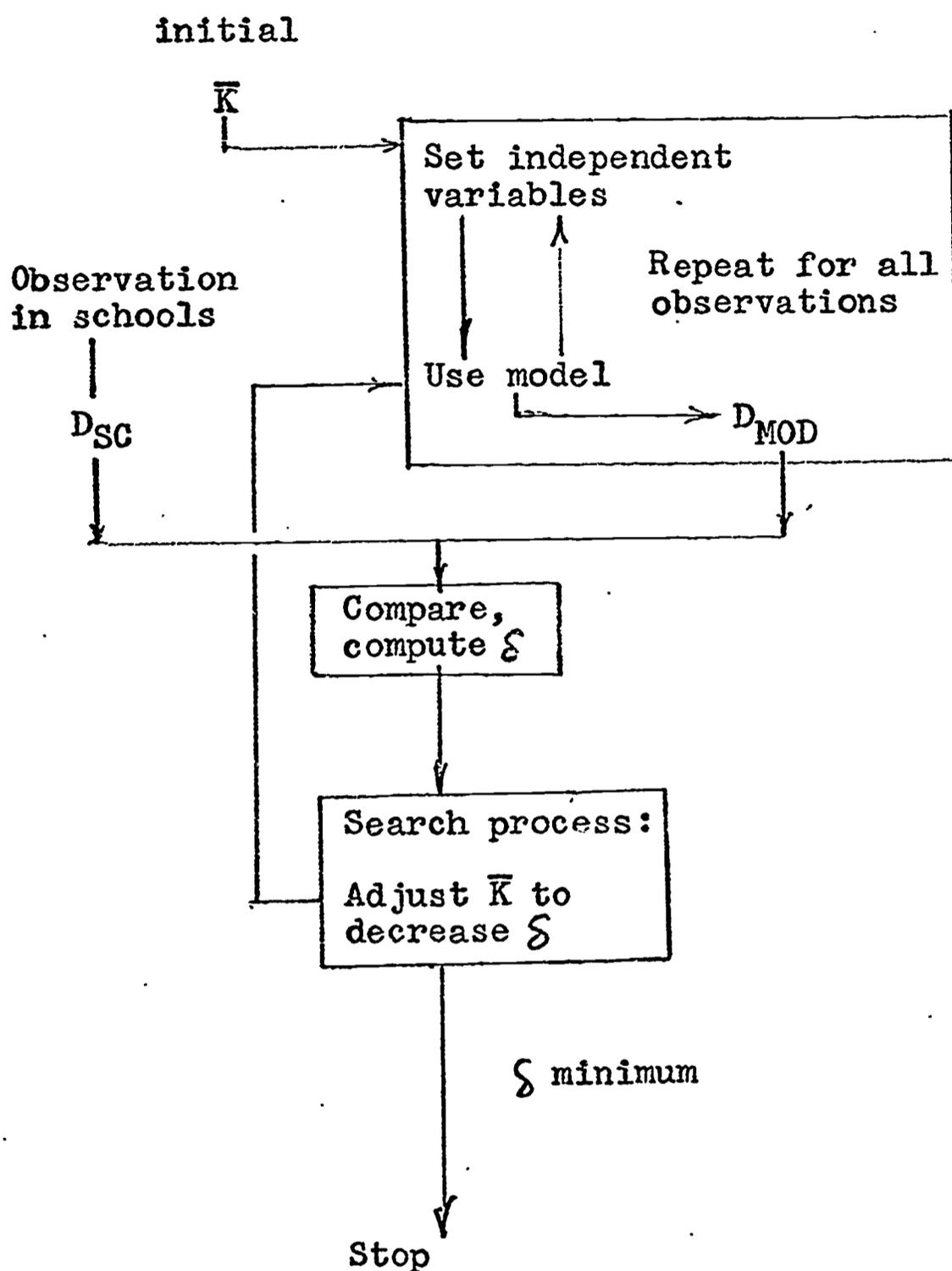
Parameter Identification Process

Figure 2

It might be possible to evaluate each parameter k_i by a series of special studies. Infact, one of the advantages of proposing a model such as the one under discussion is that it pin-points needed educational research studies. A study could focus on each of the parameters. I will describe briefly some of the experiments that are suggested by the model:

The model assumes (perhaps pessimistically) that the maximum increase in average achievement is limited to $k_1 P$. It would be most desireable to test this hypothesis and to evaluate k_1. One experiment that comes to mind (an expensive one) is this: select a sample of students stratified as to socio-economic back ground (P or equivalent). Provide these students with good tutors in all school subjects for a number of years so that F is near 1. This, according to the model, would eliminate the effects of all parameter except k_1. When the students completed the experiment and were tested in should be possible to confirm that change in achievement, D, is indeed propotional to P and to estimate k_1. Research might identify a natural experiment of this type (students who are being tutored for reasons unrelated to the model factors) which would reduce the experimental costs.

The "no staff" school is an interesting concept and worthy of study, although it is unlikely that a no staff school experiment would actually be permitted by the public. However, due to strikes, segregation politics or some other situation it might be possible to find a group

of students who did not have teachers, but did have books and parental

guidance. This would help determine k_2 and the nature of the relation-

ship between home environment, educational materials and achievement

changes.

The model makes the simple assumption that effective staff is

proportional to actual staff-hours per student-hour multiplied by some

measure of staff effectiveness, such as verbal ability. Perhaps "staff"

should be separated into teachers, other professionals, administrators

and non-professionals and a quality measure and k-factor determined

for each. But assuming the aggregate model, it would be interesting

to record carefully achievement changes in a consistent way over

several years and relate this to teacher qualitites according to the mod-

el (i.e., using the model to factor out resource levels, socio-economic

background, etc. since it is unlikely that these could be controlled).

The goal would be to evaluate the linearity of the $k_0 V$ relationship and to

estimate k_0.

k_0 might also be estimated by special studies to determine

whether a teacher quality measure, V, predicts the time required by a

teacher to successfully convey a concept to a sample group of students.

Similar studies relating space factors to achievement might help evaluate

k_6 and the proposed relationship by which area, R, effects D. There are,

unfortunately, a number of crowded schools in which data could be gathered.

k_3 determines the basic effectiveness of the school since it determines

the shape of the D-F curve. k_3 will be hard to estimate directly; but some thought should be given to possible studies.

Finally, the model assumes that the school can influence the extent to which the parents can help the education of their children (through I). Some community relations efforts are being made by some schools; for example, having special evening sessions for parents. Perhaps these efforts could be studied to obtain data to evaluate k_4, k_5 and the hypothesis that the school influence adds to the basic parental support (rather, say, then multiplying it).

k_2 and k_3 should be affected by changes in technology. Computer assisted instruction, which presumable is much more effective than texts or films, should lead to higher k_2 and lower k_3 (A "no staff" school with CAI should be more effective than one with just texts). Lower k_3 means higher C and lower point of inflection. This in turn means that a larger increase in D is obtained for a given increase in F. In other words the improved technology enhances the effectiveness of the staff. Those few school systems fortunate enough to have experimental CAI system should be running experiments to provide data to (among other things) evaluate such shifts in parameters due to technolgical advances. (Note that this does not mean that the technolgical advance is economically justified; economic allocation would be evaluated by using the new techology parameters in a resource allocation search, as discussed below).

Validation

The model must be validated by showing that it predicts changes in achievement well. For schools other than those used to determine $\overline{K}$, the model should replicate historical changes and predict future ones.

Both the problem of deriving $\overline{K}$ and the need to validate models indicate that schools should be collecting appropriate data. Unfortunately, such data is at present difficult to find.

Dynamic Considerations

The model suggested above is static. It can be applied only where the independent variables remain relatively constant overtime. The interesting cases, however, are those in which something is changing or is to be changed. If we are concerned with a "changing area", P will vary. If resources are made available then F (i.e. either or both S and V), M, or I may increase. The effect of such changes, especially in F or I, is probably not instantaneous. In the dynamic case, the effective F or I is probably related to the cumulative effect of past efforts. Some sort of exponential smoothing might approximate this, for example:

$$\text{Effective } I = \hat{I}(t) = \alpha \hat{I}(t-1) + (1-\alpha)\, I(t)$$

In the case of efforts expended toward gaining parental support, there may also be a threshold;

$$I'(t) = I(t) \qquad \text{if } I(t) > k_7$$

$$0 \text{ otherwise}$$

and $\hat{I}(t) = \alpha \hat{I}(t-1) + (1-\alpha)\, I'(t)$.

In other words, a little effort in encouraging parental support is wasted.

The addition of new space to increase R, involves the dynamics of construction lead times. In dynamic models, inflation would have to be taken into account so that factors, such as M, would be measured in constant value dollars.

The dynamic model can be studied by simulation. Such a model has several more parameters than the static version and this complicates parameter fitting. Furthermore, the time required to gather the data necessary to validate the dynamic becomes very long; many years.

This model does not account for major technological changes (e.g., CAI). Representing the dynamic charges in such parameters would be important, yet difficult.

Use of the Model

This model, if valid, provides a basis for optimizing the use of school budgets. Since the relationships are non-linear, classical programming techniques cannot be used. Search procedures should be applicable to finding the best distribution of funds between staff salaries, materials, influencing parents and costs of increasing space. Figure 3 illustrates the search procedure.

In the dynamic case, the revenues to be available would have to be estimated over several years. The objective would be to find the best allocation in each year.

There may be additional constraints such as not permitting

Initial setting
of controllables
(S,M,I,R)

```
            ┌─────────────┐
            │ Use model to│
       ┌───→│ predict D   │
       │    └─────────────┘
       │           │
       │           ↓
Revenue │    ┌─────────────────┐
limitations   │Search Process:  │
       │    │                 │
       └───→│Adjust controllables│
────────────→│to increase D    │
            └─────────────────┘
                   │
                   │   D maximum
                   ↓
                 Stop
```
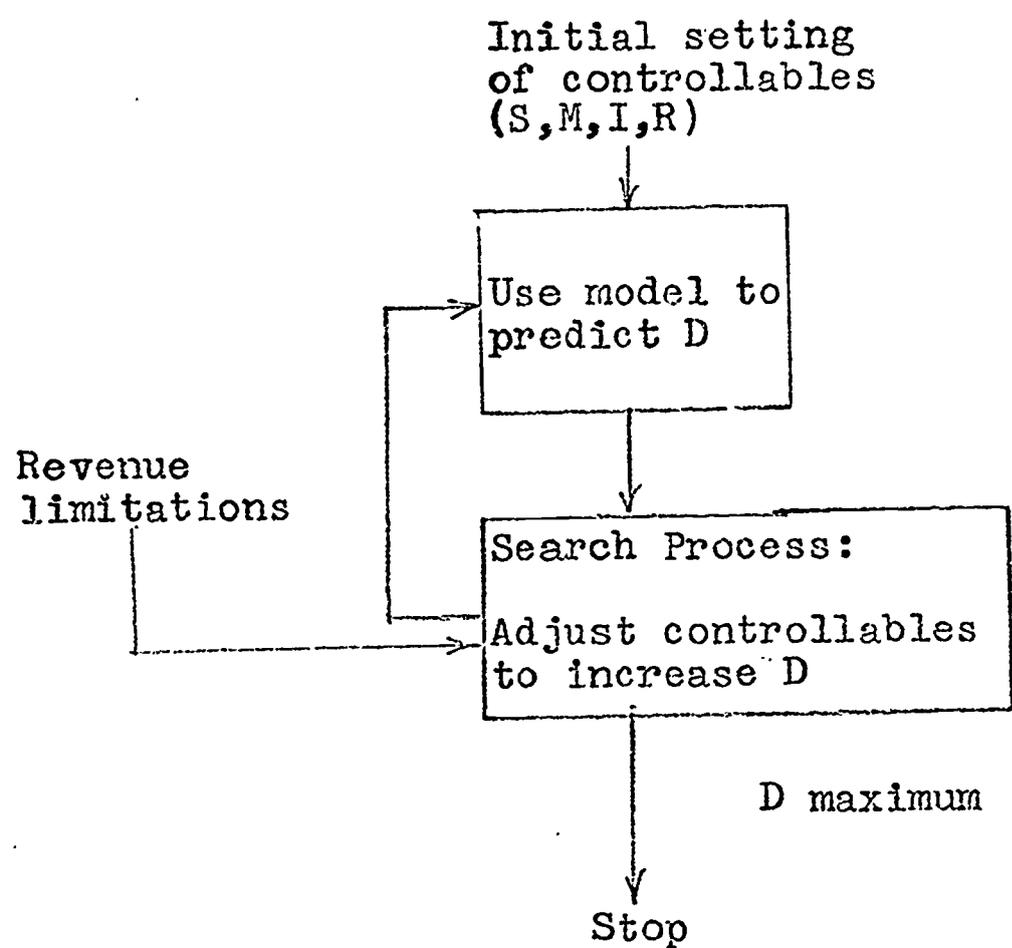
Procedure for Finding Optimum
Allocation of Resources

Figure 3

reductions in staff except through attrition.

Conclusion

A model of the elementary educational process has been proposed to initiate discussion of such quantitative models and to encourage the gathering of data that would tend to confirm the model or its sucessors. Such an effort is worthwhile, since, if a model is valid, it forms a basis for providing school administrators with quantitative guidance to the decision of allocating funds.