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Appendix B. A Conceptual Model for the Teaching of Elementary Mathematics.

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A two-part communication model for teaching elementary mathematics is proposed. Part I delineates symbols commonly used in teaching arithmetic. Parts of mathematical language (mathematical objects, relations, operations, expressions, and sentences) are compared to analogous parts of the English language. Part 2 is a conceptualization of strategies that teachers can use in the communication process. The four distinct components of the model (manipulatory verbal stage involving the concrete, manipulatory verbal stage involving the concrete followed by symbolization, symbolization as an instruction to manipulate the concrete, and symbolization without reference to the concrete), each of which constitutes a particular sequential communication strategy, are described, and particular features of the model (individualization of instruction, identification of prerequisites, remediation, individual differences, computation, open sentences, and problem solving) are considered. Also, a problem solving model (derived from the communication model and incorporating nine heuristics of inquiry--enumeration of specific cases, deduction, inverse deduction, analogy, preservation of enabling principles, variation, continuity of form, existential counter example, and determination of limits) is presented. (This document and SP 002 155-SP 002 180 comprise the appendixes for the ComField Model Teacher Education Program Specifications in SP 002 154.) (Author/SG)

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APPENDIX B--A CONCEPTUAL MODEL FOR THE TEACHING OF ELEMENTARY MATHEMATICS

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SP002156

A CONCEPTUAL MODEL FOR THE TEACHING OF ELEMENTARY MATHEMATICS

Leon Rosseau

Introduction

The proposed model is in two parts. Part I will delineate the content variables or symbols that commonly are used in communicating arithmetic to children. Part II of the model is a conceptualization of strategies that can be used in the communicative process. The contents of the following figures are illustrative only and are not to be considered exhaustive.

PART I - CONTENT VARIABLE MODEL

As an aid to understanding the rationale behind the following classification scheme, it is useful to think of mathematics as a language in the same way that one thinks of English. The various parts of mathematical speech are therefore defined analagously to the parts of speech that the classical grammarian might use in analyzing the English language. Although reference is made explicitly in some cases to the similarity between the two languages, it should be understood that in fact there is no isomorphism between the two. The cross referencing then is made to provide an anchor by which it is hoped a clear understanding of the composition of mathematical speech will be achieved - in short, the orientation should be one of a mathematical grammarian.

Parts of Mathematical Language	Example	Analagous Parts of English Language	Example
I. Mathematical Objects		I. Name Objects	
(a) noun constants	0,2, 1/2,.6,-7	(a) nouns	boy, cat, Mary
(b) noun variables	$\square, \triangle, w, x, z$	(b) pronouns	she, he, you, me
II. Relations	equals, is less than, is greater than	II. Relations (copular verbs)	is married to, is the father of, is greener than
III. Operations	+ - x ÷	III. Active Verbs	hits, runs, swims
IV. Expressions	x + 7, 3 - y	IV. Phrases	in the house, with a smile
V. Sentences	3 + 2 = 5, 2x-y=x	V. Sentences	The boy hit the ball. He is married to her.

PART II - CONTENT VARIABLE COMMUNICATION MODEL

This part will develop:

I. A description and conceptualization of the rationale behind the proposed communication model.

II. A second rationale and its conceptualization.

III. A description of the actual communication model.

IV. A description of some derivative of the communication model. (currently being developed)

I. FIRST RATIONALE - DESCRIPTION AND CONCEPTUALIZATION

This rationale and the subsequent one are delineated in considerable detail for two reasons: First, to serve as bases for the conceptualization of the communication model and second, to make in turn the communication model plausible and reasonable as a basis for a set of teaching strategies.

To begin, a clear distinction must be made between the mathematical world of abstraction (hereafter M.W.) and the physical world of reality (hereafter P.W.). The distinction is simply that they have nothing in common or that they are completely disjoint. The model showing this distinction is given in Figure I.

Abstract M.W.

Real P.W.

Figure I

The abstract mathematical world (M.W.) deals with the manipulation of the content variables of Part I governed by the Enabling Principles. We could, in fact, define mathematizing as the maneuvering of symbols on paper or on the blackboard or in the mind. The statement "3 + 4 = 7" is one such maneuver. On the other hand, the physical world of reality (P.W.) deals with the kinds of objects perceived by the senses such as apples, oranges, buttons and the like. As an illustration, consider the distinction between the abstract noun constant 6 and six apples, applying the model of Figure I.

II. SECOND RATIONALE - DESCRIPTION AND CONCEPTUALIZATION

The description of this rationale derives from an analysis of the historical evolution of mathematics. The first kinds of problem decisions made by early man must have been of the Option I type in Rationale I. Lacking any mathematical models, he must have resorted to essentially a trial and error approach. In terms of Rationale I, Figure 4, the problem solution domains could be collapsed into one: Thus:

mw ______________________ now becomes ______________________ mw

pw [Real Problem] → [Real Solution]

Fig. 4 Rationale I

pw [Real Problem Real Solution (solved or unsolved)]

Fig. I

The model (Fig. I) indicates that the physical problems led to physical solutions or nonsolutions through the physical manipulation of the things perceived in the problem. The caveman operated in much the same way as the preschool child in that they both attacked a problematic situation by a direct trial and error assault.

The sheer repetition of the same kinds of problems leading to the same kinds of solutions may have served as an incentive for the caveman to invent "shortcuts" as labor-saving devices if nothing else. The shortcuts must necessarily have been symbolic in character. For example, the simple chore of keeping track of the numerical size of his domestic herd was solved by the recognition of each animal in the herd. The first shortcut taken was probably some sort of 1 to 1 mapping of the animals with his fingers and toes, then with tally marks in the sand and finally by a 1 to 1 mapping with some kind of "eni, mini, mini, mo," sequence--scratching the last grunt on the sand for the purpose of record keeping. By this stage the mathematizing process had begun. Its function, however, was restricted to the role of a record keeping device after the physical fact. The model (Fig. I) now becomes:

III. COMMUNICATION MODEL

The proposed model is based upon the Rationales described in Part II. It will consist of four distinct components, each of which will constitute a particular sequential communication strategy. The descriptive label for each is:

Strategy I: Manipulatory Verbal Stage Involving the Concrete

Strategy II: Manipulatory Verbal Stage Involving the Concrete Followed by Symbolization

Strategy III: Symbolization as an Instruction to Manipulate the Concrete

Strategy IV: Symbolization Without Reference to the Concrete

Preliminary Remarks. The purpose of this model is to equip the instructional manager with a set of strategies that will enable him to communicate the components of the Content Variables of Part I to elementary school children. The decisions as to which parts of mathematical speech and in what order they will be communicated are outside the scope of the model. Guidelines for these decisions are usually found in local curriculum outlines and text books. In other words, the choice of topics (addition of whole numbers, concept of fractions, etc.) and the sequencing of these topics is the content of arithmetic whereas the model deals with the process of communicating that content. The use of the model is predicated on the assumption that the instructional manager has chosen a problematic situation in the physical world as the generator of that content variable he wishes to communicate. (Rationale I Fig. 4, Rationale II Fig. I). In the primary grade especially the model would be most useful, since the child is mainly acquiring his mathematical vocabulary. As he progresses through the intermediate grades, there is a shifting from physical world generation of mathematical models to their generation from existing models. (Rationale I Figure 5.) It does not imply however that there is no need for physical world sources as initiators of mathematical models at the levels of the upper intermediate grades.

STRATEGY I: Manipulatory Verbal Stage Involving the Concrete

Suppose the instructional manager wishes to communicate the notational agreement of Place Value (Category IV of Part 1).

The translation of this agreement in physical reality is equivalent to activities of bundling objects in some arbitrary base. The variables associated with the activity would be:

(1) Content Variable - Place Value (base 10)
(2) Physical Device Variable - Coffee stirrers or soda straws, elastic bands
(3) Situation Variable - All children would have access to kits of materials
(4) Verbal Variables - Bundles (tens) leftover (units)

A brief description of the strategy would be: The instructional manager would assign the task of the children to count out say, thirteen sticks putting an elastic band around ten of them. The activity here is one of packaging thirteen into 1 bundle of ten and three leftovers. They would then be told either to unbundle what they had shown or to leave arbitrary bundles of ten and leftovers. The whole cycle is repeated again with another number. The model of Strategy I (compare with Rationale II figure 1) is:

MW

PW

count or select bundles and/or leftovers package and demonstrate

fig. 1

NOTE--As the model shows there is <u>no</u> mathematical symbolization at this stage. The instructional man would communicate bundles, tens, leftovers, units verbally (written and oral).
The symbols 1 bundle, 3 leftovers, 1 bundle + 3 leftovers 13, 15 + 3, . . . would not be used.

STRATEGY II. <u>Manipulating Verbal Stage Involving the Concrete Followed by Symbolization</u>.

The description of Strategy II is: Strategy I <u>followed</u> by an abstract recording format. The instructional manager would communicate some agreed-upon way of tallying symbolically the number of bundles and leftovers demonstrated in Strategy I. The child in effect would do, then record.

The model of Strategy II (compare with Rationale II figure 2) is:

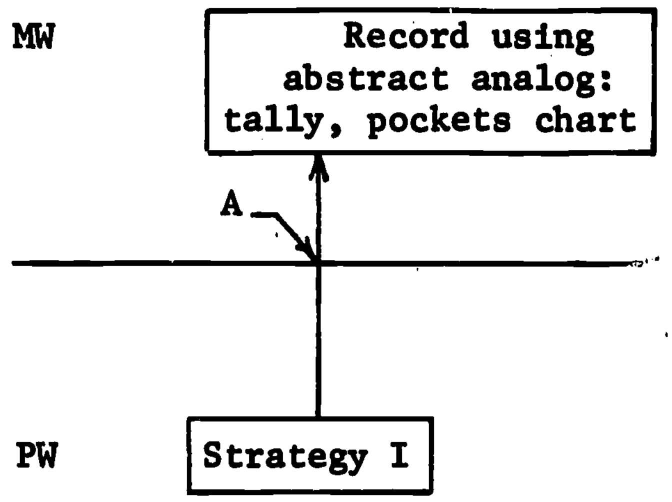


fig. 2

NOTE: The symbolization used is purely for recording purposes *after* the manipulations of Strategy I have been completed. A particular format may be

leftovers	bundles
3	

In other words the function of symbolic representation is passive in that it is descriptive of some maneuver in the Physical World.

STRATEGY III. Symbolization as an Instruction to Manipulate the Concrete

This strategy is the *precise* inverse of Strategy II in that the symbolic format constitutes an instruction to the child to perform whatever manipulation is intended (in this case bundling and demonstration). The model of Strategy III (compare with Rationale II figure 3) is:

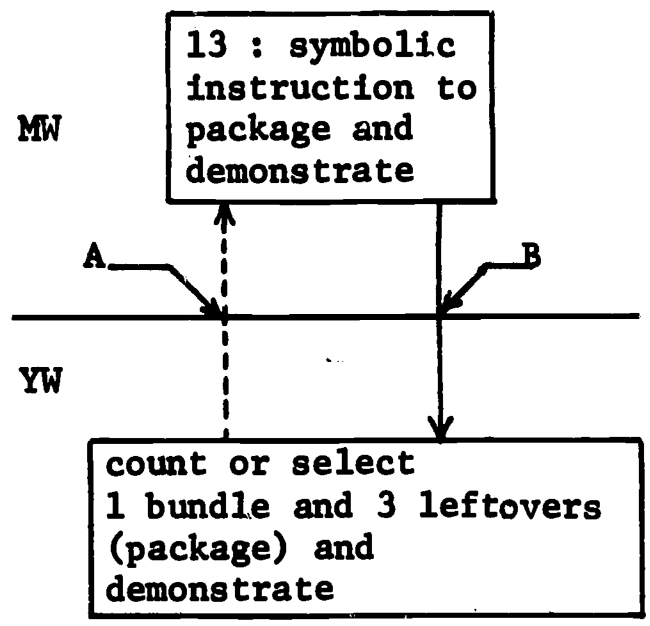


fig. 3

NOTE: This strategy completes the cycle in conceptualization of the mathematical analogs. The concrete device is in effect a mechanical computer in that the solution (in this case 1 bundle, 3 leftovers) is obtained by physical manipulations.

STRATEGY IV. Symbolization Without Reference to the Concrete

In this final strategy, the child is required to manipulate the symbolic expression without resort to physical means of verification. Using the example of Place Value, he is deliberately denied the use of the coffee stirrers or straws. This forces him to create or synthesize a mathematical model from the given one.

The model of Strategy IV (compare with Rationale II figure 4) is:

MW

13 : Symbolic Instruction to rename

→

1 bundle of 10 and 3 leftovers 10 + 3 or 13 left overs 13

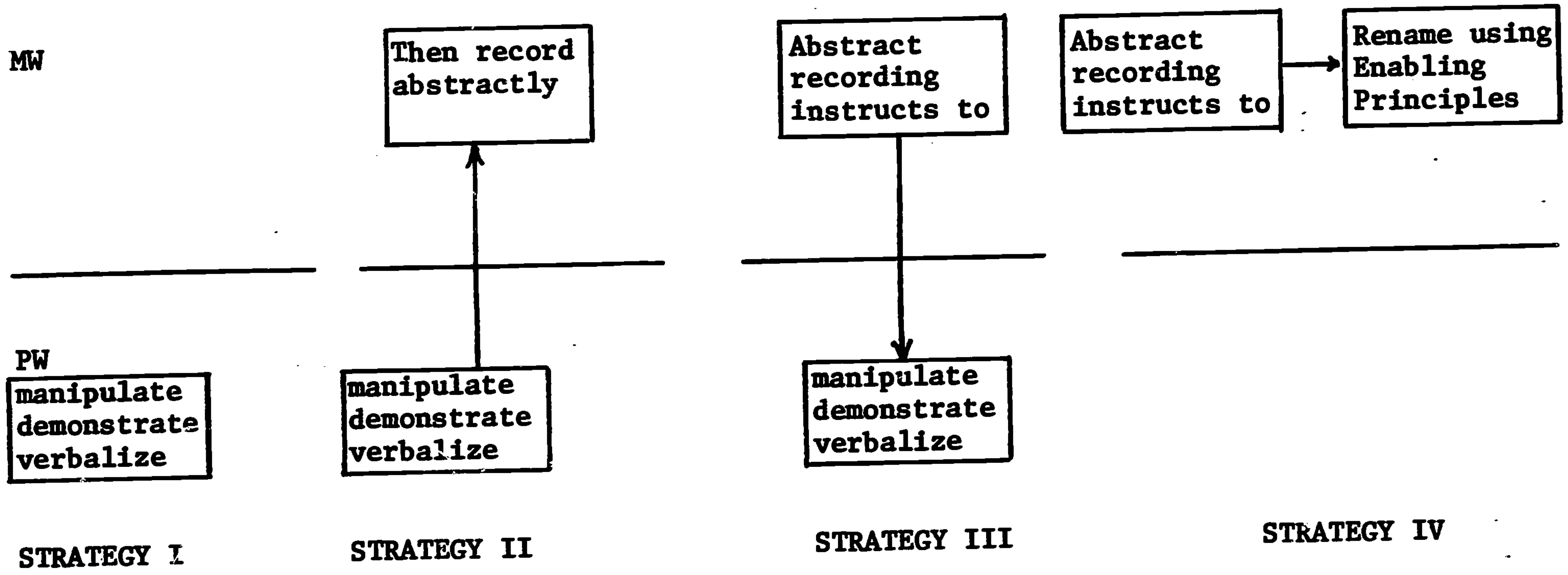
NOTE--Strategy IV actually represents in this case a primitive algorithm or computational ritual in that the child is using the Enabling Principles to rename a given expression.

PW

fig. 4

The entire Communication Model is again repeated below in abbreviated form to emphasize the 4 components and their sequencing.

COMMUNICATION PROCESS MODEL



IV. DERIVATIVES OF THE COMMUNICATION MODEL

I. Individualization of Instruction. Since the strategies comprising the Communication Model are sequential it is assumed that, in communicating a specific content variable (ex. Place Value), children would progress from one strategy to the next at different rates. For example, one child could be responding at the level of Strategy I (difficulty in counting or manipulating) while another is responding at the level of Strategy III or Strategy IV. Thus the model is capable of defining the level at which a particular child is responding.

II. Identification of Prerequisites. Suppose the instructional manager is to communicate the Enabling Principle of Place Value using the Communication Model. The first task for him would be to construct a list consisting of the components that would be required to facilitate the communication of the Principle. The explicit identification of these by the instructional manager would be necessary to assess adequacy of pupils' entering behavior. For example:

Place Value: Requisite components

1)	Rational counting	(for bundling activity)
2)	Addition	(for renaming ex. 13 = 10 + 3)
3)	Open Sentence	(13 = $\square$ tens + $\triangle$ left-overs)

Deficiencies in the understanding of any of the three Content Variables above would hinder the communication task.

III. Remediation. Assuming that one of the above three variables was not understood by way of assessment procedures, the use of the model would indicate at which strategy misunderstanding occurred (for the individual child). The remediation would consist of identifying the strategy number wherein misunderstanding occurred. This would be accomplished by starting the diagnosis at Strategy IV (most likely place of misunderstanding), then successively working through the sequence of strategies in reverse of the Communication Process to determine at which one to undertake remediation (Communication Process).

For example, suppose misunderstanding of Rational Counting. Then at Strategy IV:

M.W.	Definition of Cardinal Number	Given Cardinal Number	Record as Symbolic Shorthand	
	________ Go To →	________ Go To →	________ Go To →	________
P.W.		manipulate verbalize demonstrate	manipulate verbalize demonstrate	manipulate verbalize demonstrate
	Strategy IV	Strategy III	Strategy II	Strategy I

IV. Individual Differences. In the Communication Process the transition from Strategy III to Strategy IV is most crucial. It is here that the greatest differential is likely to occur between two children in terms of their ability to advance to the final strategy. It is within Strategy III that the child is required to mechanically compute with sticks and elastic bands, for example. The insistence of the instructional manager for this kind of solution provides the basis for his decision to move the child to the final strategy. That basis is the increased resistance offered by the child in being forced to solve by means of a concrete device. He will either demonstrate responding behavior such as "Do I have to use the sticks? I already know the answer," or he will manipulate as required after the fact that he has abstracted. Premature advancement to Strategy IV is harmful in that superficially the child appears to be mathematizing (as evidenced by his symbol pushing) but is in fact operating at Strategy III. It is true that the physical devices have been denied him (straws, stirrers, elastic bands) but he invents his own replacement (fingers). The grade 6 child who is still fingercounting in his desk is a classical illustration of this premature move.

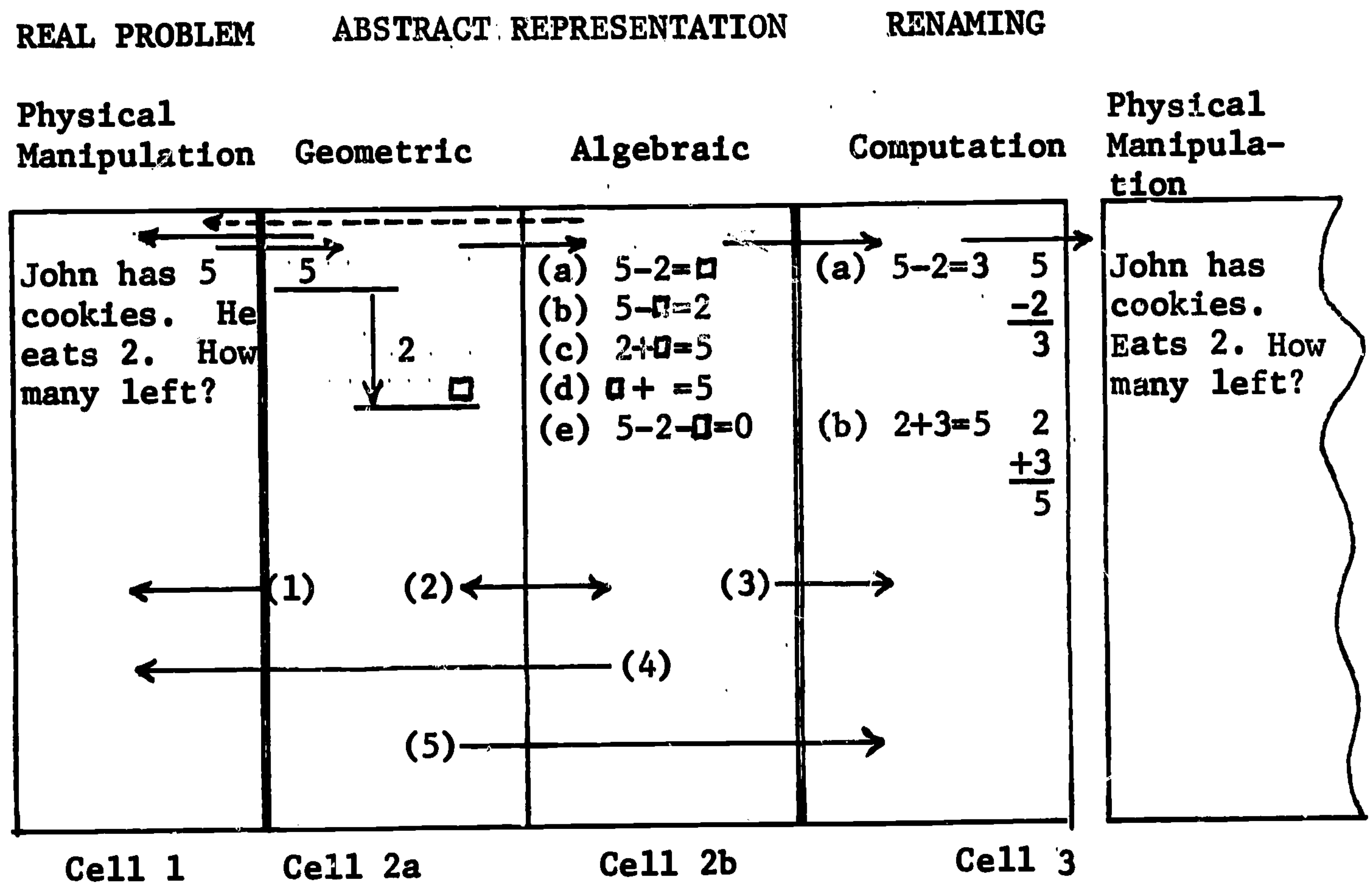
V. Computation. The model defnies two distinct computational procedures: Strategy III describes the mechanical, i.e., the physical devices are in effect primitive computers - in the example, bundles, sticks. (Rationale I, fig. 7, Rational II, fig. 7, Communication Model, fig. 3 Strategy IV describes the abstract symbolic maneuvering - i.e., the manipulation of Content Variables without any reference to the Physical World constitutes an algorithmic ritual. In the Place Value Principle, the sentence 13 = 10 + 3, is the ritual. (Rationale II, fig. 5, Communication Model, fig. 4).

VI. Open Sentences. With reference to the Communication Model, the open sentence is an extremely efficient communication device. In Strategy III, for example, it is an economical way of giving instructions to the child to go do (in the physical context). To illustrate: the sentence, 23 = $\square$ tens + $\triangle$ leftovers, explicitly requires the child to demonstrate two packages of tens and 3 leftovers. Again, the sentence, 3 tens + 2 leftovers = $\square$ $\triangle$, explicitly requires the child to program bundles and sticks into a pockets chart or some similar device. The open sentence at Strategy IV performs the same function.

VII. Problem Solving. There is no distinction made in any of the Rationales or the Communication Model between what is traditionally known as "Problem Solving" and any other part of the arithmetic curriculum. "Problems" are defined, in the context of an arithmetic program, to be any physical confrontation that is capable of generating mathematical analogs belonging to the domain of the Mathematical World (M.W.). Thus the internalization of the Cardinal Number 3 by the Grade 1 child and the solution of particular right angle triangles by the grade 7 or 8 child are both problems differing only in the facility of being conceptualized. There is a subset of physical Real World Problems corresponding to the traditional kinds of worded problems that warrant identification according to their mathematical analogs. These are singled out because they are repetitively encountered within the child's experience in the Physical World.

The Conceptualization of a Problem Solving Model.

The model is constructed from the Communication Model with a description of the corresponding parts as follows:



Ex.

Fig. 1

The analog of cell 1 is Strategy I.
The analog of cell 1 to cell 2a or 2b is Strategy II.
The analog of cell 2a or 2b to cell 1 is Strategy III.
The analog of cell 2a or 2b to cell 3 is Strategy IV.

Cell 1 represents the Real World Problem situation wherein the solution is mechanically determined. Cell 2a and 2b label 2 ways of representing the abstract analogs. One is characteristically a geometric configuration or model; the other is an algebraic model (usually an open sentence). Cell 3 represents the entire apparatus for renaming an expression. Thus it contains all the computational rituals or algorithms.

The upper set of vectors represent the direction of flow in the process of abstracting from the Physical concrete (cell 1) to the Mathematical abstract (cell 3). The projecting vector represents verification, i.e., the variable prediction process (Rationale I, figure 7). The dotted vectors are the analogs of Strategy III in the Communication Model. These would be either the geometric or the algebraic abstract representations serving the roles of instructors to go do in the physical context.

The lower set of vectors represent the heuristic strategies that may be used to facilitate transfer across the boundaries indicated.

(a) Vector (1) and Vector (4). Given abstract representations, either geometric or algebraic, the child is required to make <u>explicit</u> representation <u>implicit</u> by constructing a physical problematic situation as the referent. For example:

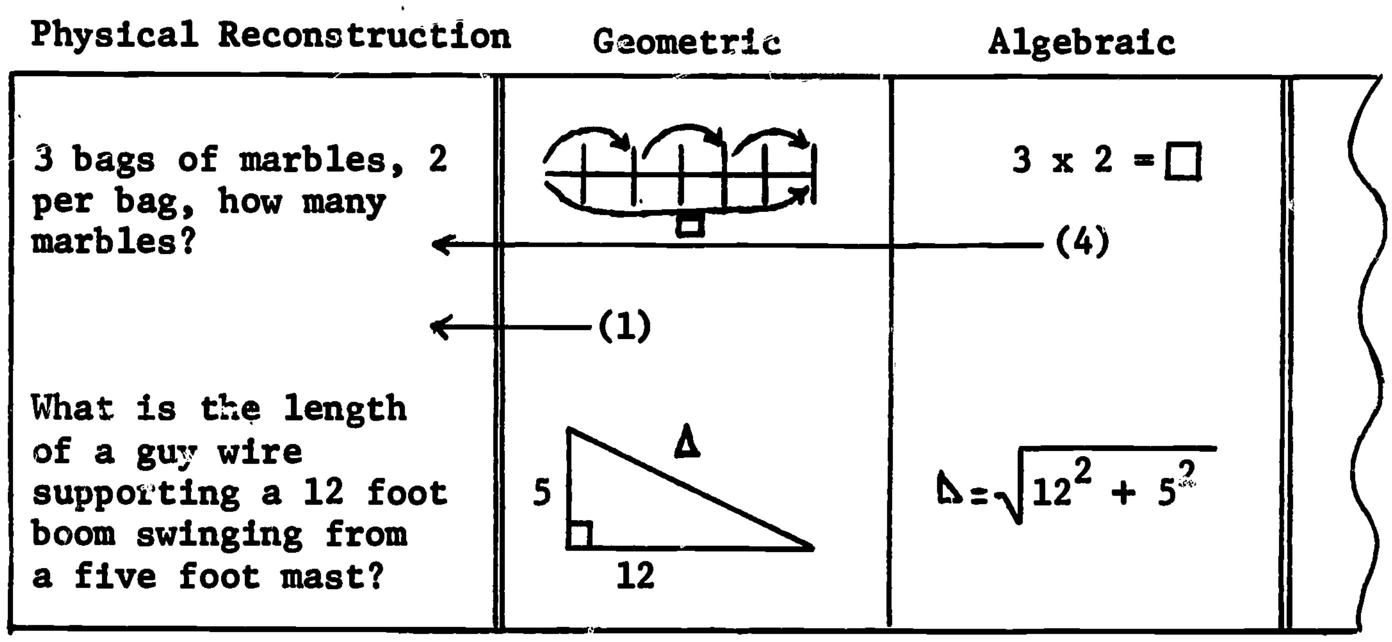


Fig. 2

(b) Vector (2). Given either geometric or algebraic representation, translate or rename the representations by the other means.

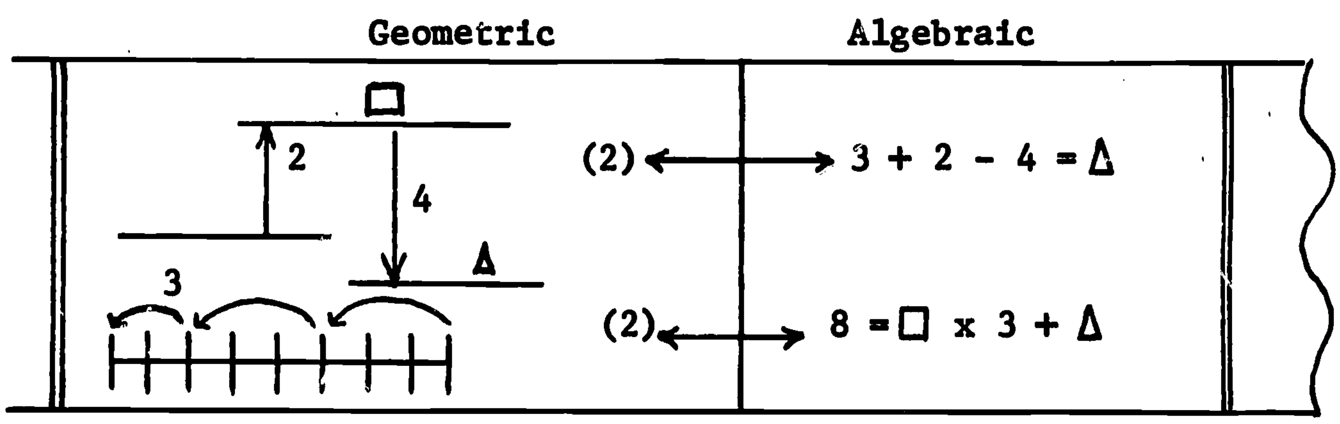


Fig. 3

(c) Vector (3) and Vector (5). Given either geometric or algebraic representations, rename using the Enabling Principles.

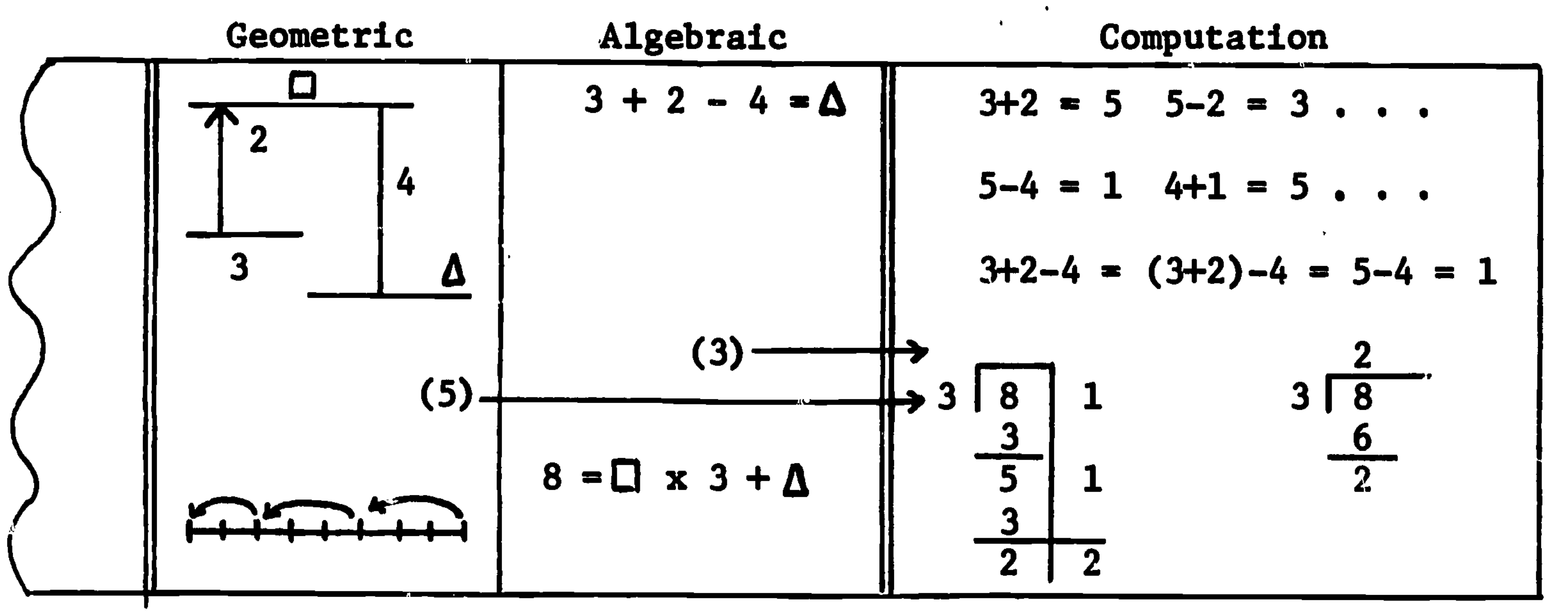


Fig. 4

(d) Vector (1) and Vector (5). Given a physical problematic situation, demonstrate the geometric and/or algebraic representation.

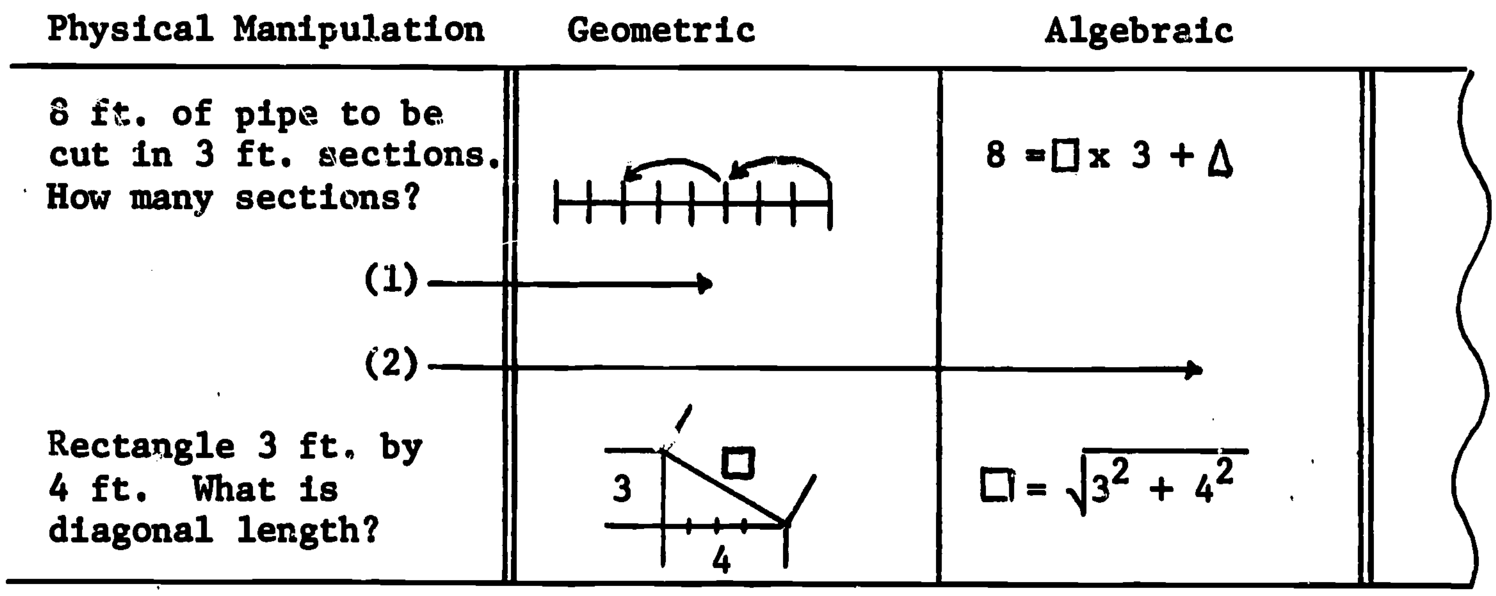


Fig. 5

In order to facilitate the process of solving Physical World Problems, pupil should be given activities involving the transfers indicated by Figures 2,3,4 and 5.

Content variables when generated as abstract analogs of physical maneuvers shall hereafter be called <u>Patterns</u>. These are the abstract geometric or algebraic representations of Real World manipulations. (Fig. 1, ABSTRACT REPRESENTATION) As an example, the commutative law of addition ($a + b = b + a$), one of the Enabling Principles, is first taught to the child as a pattern. By appropriate physical manipulations involving concrete objects (Communication Model) the abstract statement of the law is made reasonable and plausible to the child. Gradually the law is conceptualized as an unproven authority which is independent of any physical reality.

Problematic situations or physical maneuvers generally cluster into groups according to the kinds of actions perceived in the situations. Each group generates a pattern as a symbolic analog to that action. These situations are classified as follows:

	Physical Manipulation (Problematic Situation)	Pattern Name (Symbolic Representation)
I	(a) combining disjoint sets (b) separating sets into disjoint subsets (c) comparing sets	additive - subtractive (+, -)
II	(a) iterative combining of sets having same cardinality	additive - multiplicative (+, x)
III	(a) combining sets of equal cardinality (b) quotitioning sets (c) partitioning sets (d) comparing sets	multiplicative - divisive (x, ÷)
IV	(a) iterative separation of subsets of equal cardinality from a given set	subtractive - divisive (-, ÷)
V	(a) iterative separation of subsets of equal cardinality from a given set, leaving a remainder	Archimedean (-, ÷)
VI	(a) assigning distinct objects to two different groups	ratio (:)
VIII	(a) constructing sets of ordered pairs of objects	function

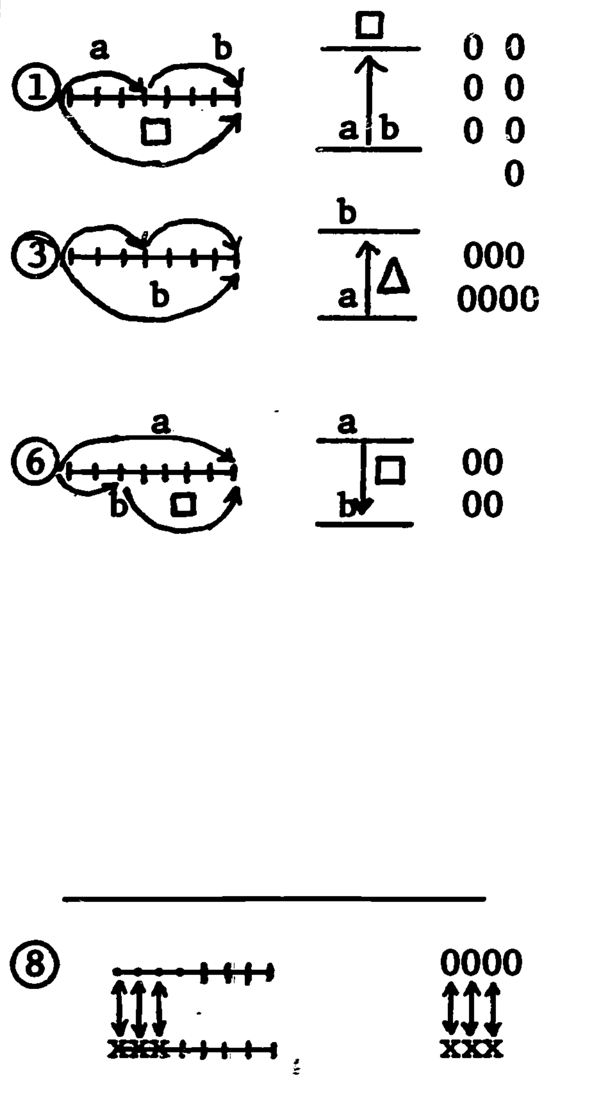
Fig. 6

Each of these generic patterns will be described by means of the Problem Solving Model (Fig. 1).

See Fig. 7 on following page.

Each of the 8 open sentences represent symbolic descriptions of different physical manipulations involving the combination, separation, or comparison of objects. For example, the open sentence a + b = describes a physical situation wherein b objects combine with a objects, resulting in a + b or objects. The sentence + a = b describes a physical situation wherein a objects are combined with an unknown number resulting in b objects. The first 7 sub-patterns would be treated as distinctly different representations of different physical situations. The + and - signs are introduced as analogs of combining (coming together) and separating (leaving) actions. Although the main objective is to communicate abstract descriptions of actions, the child may learn incidently some addition and subtraction facts, the idea of an open sentence, the concept of a sum, a difference and closure, but these are not to be stressed. Similarly the means of changing the name of an expression (computation - Fig. 7, column 4) is not stressed. Thus the child may "solve" a problematic situation by counting in either direction, adding or subtracting. Three different formats of geometric representations are shown. The number line (on paper or painted on floor) interprets the operations as vector jumps. The vertical format shows horizontal lines as beginnings or ends of transactions and the vertical lines as gains or losses. The third format shows inferred actions by graphic means -- dimming out or blocking out abstract representations of physical objects. Sub-pattern (8) derives from the comparison of two groups in terms of the notions of the fewer than concept. The abstract model describing this situation is a subtractive model but the - symbol no longer denotes a removal action. Hence the child must be led to accept this model as accurately describing the situation.

If and when the commutative law of addition and the concept of a difference as an additive inverse have been communicated, the 8 sub-patterns collapse into 4 equivalent statements, namely, a + b = c, c - b = a, c - a = b, c - b - a = 0. To illustrate, consider the problem "John had 6 marbles. After losing some he now has 4 marbles. How many did he lose?" Before the commutative law and difference concepts have been communicated, the expected sub-pattern would be 6 - □ = 4 (Fig. 7 - 6). Any other representation at that time would be incorrect, because the objective is to convince the child that the open sentence 6 - □ = 4 accurately describes that particular physical maneuver. However,

Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
As a result of the child – combining disjoint sets of objects – separating out subsets of objects – comparing sets by one-to-one mapping the Additive Subtractive Pattern (+ –) is conceptualized.		1 $a + b = \square$ direct combination 2 $b + a = \square$ direct combination 3 $a + \Delta = b$ How many conjoined? 4 $\Delta + a = b$ How many originally? 5 $a - b = \Delta$ direct removal 6 $a - \square = b$ How many separated? 7 $\Delta - a = b$ How many originally? 8 $a - b = \square$ How many fewer?	Counting Addition Subtraction

I. Additive Subtractive Pattern

Fig. 7

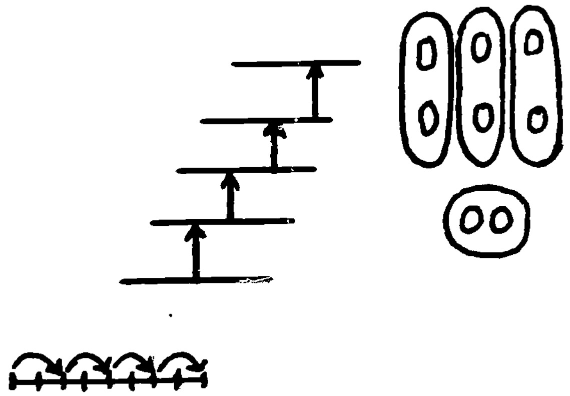
after these concepts have been taught the child would be encouraged to give many equivalent statements (open sentences) all of which are mathematically valid. These would be $6 - \square = 4$, $4 + \square = 6$, $\square + 4 = 6$, $6 - 4 = \square$, $6 - 4 - \square = 0$.

See Figure 8 on following page.

This pattern is the analog of the physical maneuver of accumulating groups of objects of the same cardinality. Its main purpose is to serve as an intuitive definition of the operation of multiplication, i.e., as a pattern the expression a x b describes the situation where the child has amassed a group of objects, b objects in each group. The x sign in effect is interpreted as an operation of setting out specific groups of objects having the same cardinality. Thus the problem, "John has 4 bags of marbles. Each bag contains 2 marbles. How many marbles does John have?" implies $4 \times 2 = \square + \square + \square + \square$ as a descriptive pattern. As before, the commutative law of multiplication, if taught explicitly, allows another representation. As an incidental learning, the child may acquire some of the simpler multiplication facts, the idea of a product and an algorithmic procedure for assessing products to two factors.

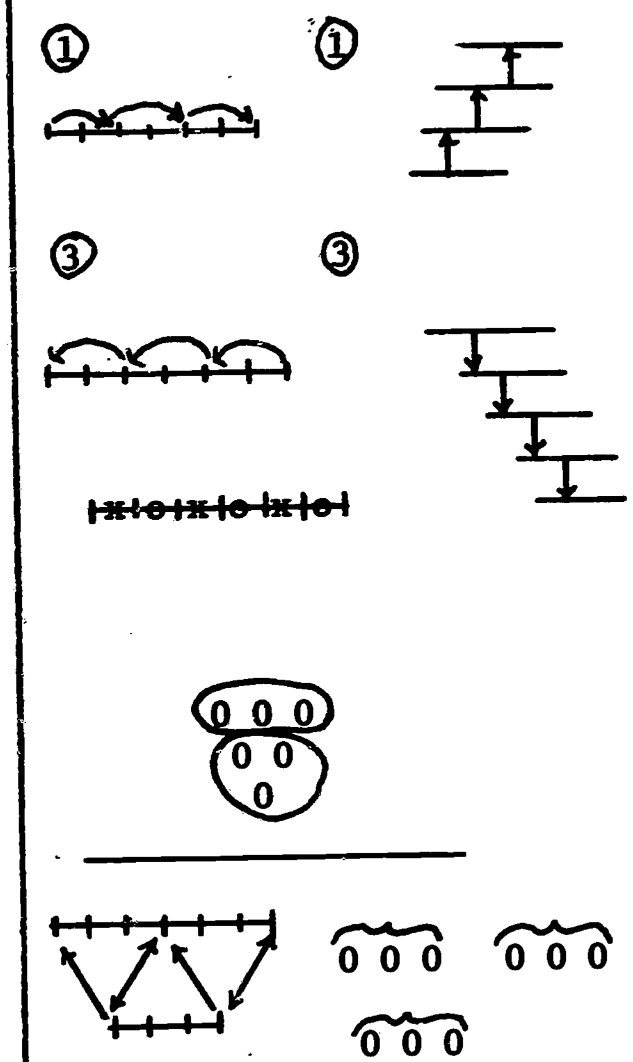
See Figure 9, following.

This pattern is the analog of Pattern I (Additive Subtractive Pattern - Figure 7). Each of the 8 open sentences represent symbolic descriptions of different physical manipulations involving: combining, quotitioning, partitioning and comparing sets. For example, the open sentence $a \times b = \Delta$ describes a physical situation wherein a groups of b objects per group are assembled resulting in a x b or Δ objects. The sentence $a \times \square = b$ describes a situation wherein b objects are to be reassembled into a groups admitting $\square$ objects per group. The first 7 sub-patterns would be treated as distinctly different representations of different physical situations. The x and ÷ signs are introduced as analogs of iterative combining and separating actions. Although the main objective is to communicate abstract descriptions of actions, the child may learn incidently some multiplication and division facts, the idea of an open sentence, the concepts of a product and a quotient, closure, but these are not stressed. Likewise the means of changing names for expressions -- computation (Figure 9, column 4) is not stressed. Thus the child may "solve" a problematic situation by adding, subtracting, or as a result of known facts. The Geometric format of representation is: the number line which interprets multiplication and division as equal internal vector jumps, the plateaus which show iterative transactions, and the lassoed outlines of set groupings. Sub-pattern (8) derives from

Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
As a result of the child combining by iteration sets of the same cardinality.		$a \times b = \underbrace{b + b + b + \ldots}_{\text{a times}}$	Addition

II. Additive Multiplicative Pattern

Fig. 8

Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
As a result of		(1) $a \cdot \times b = \Delta$ Direct multiplication	addition
(a) combining sets of equal cardinality		(2) $b \times a = \Delta$ Direct multiplication	subtraction
		(3) $a \times \square = b$ How many per group	multiplication
(b) quotitioning sets		(4) $\square \times a = b$ How many groups	
(c) partitioning sets		(5) $a \div b = \square$ Quotitioning	division
		(6) $a \div \square =$ Partitioning	
(d) comparing sets		(7) $\square \div b = a$ Direct multiplication	
		(8) $a = \Delta \times b$ How many times greater than	

III. Multiplicative Divisive

Fig. 9

the multiplicative divisive comparison of 2 groups of objects. The phrase "times as great as" describes this pattern. The model describing this situation is multiplicative or divisive but the x or ÷ symbol no longer denotes the assembly procedure. Hence the child must be led to accept this model as accurately describing the situation through example.

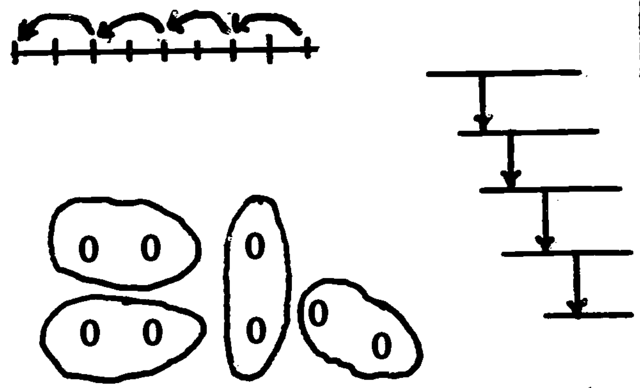
If and when the commutative law of multiplication and the concept of a quotient as a multiplicative inverse have been communicated, the 8 sub-patterns collapse into 3 equivalent statements, namely, $a \times b = c$, $c \div b = a$, $c \div a = b$. To illustrate, consider the problem, "If 8 marbles are shared equally between 4 boys, how many would each receive?" Before the commutative law and quotient concepts have been communicated, the expected sub-patterns would be either $4 \times \square = 8$ or $8 \div \square = 4$ (Figure 9 3, 6). Any other representation would be incorrect, because the objective is to convince the child that either open sentence accurately describes that particular physical maneuver. However, after these concepts have been taught, the child would be encouraged to give many equivalent statements (open sentences) all of which are mathematically valid. These would be: $4 \times \square = 8$, $\square \times 4 = 8$, $8 \div 4 = \square$, $8 \div \square = 4$.

See Figure 10 on following page.

This pattern is the complement of pattern II (Additive Multiplicative). It is the analog of the physical maneuver of iterative removal of subsets of equal cardinality from a given set. Its main purpose is to serve as an intuitive definition of the operation of division, i.e., as a pattern the expression $a \div b$ describes the situation where the child has disassembled a set of a objects into subsets, each consisting of b objects. The ÷ sign in effect is an instruction to disassemble either by quotitioning or partitioning a given set of objects. Again, as incidental learning, the child may acquire some of the simple division facts, the idea of a quotient and an algorithmic procedure for assessing quotients, given the dividend and divisor.

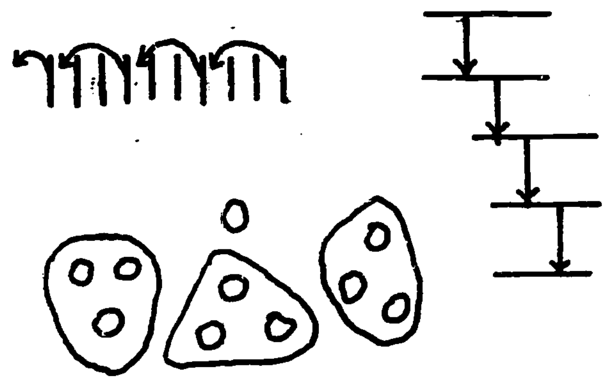
See Figure 11, following.

This pattern is an extension of the Subtractive Divising Pattern involving a remainder. The Algebraic Representation is really the Division Algorithm - a theorem of Number Theory. Translated, it states that the dividend (a) = the sum of the remainder (r) and the product of the quotient (q) and divisor (b). In practice the child is given a and b and by means of some computational ritual usually called "long division" he assesses the q and r to make the statement $a = bq + r$ and $o = r \div b$ true.

Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
As a result of the child partitioning or quotitioning out subsets of equal cardinality from a given set.		$a \div b = \underbrace{a-b-b-b- \ldots}_{c \text{ times}}$	iterative subtraction

IV. Subtractive Divisive

Fig. 10

Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
As a result of the child partitioning or quotitioning out subsets of equal cardinality with a remainder.		$a = b \times q + r \quad 0 \leq r < b$ (THE DIVISION ALGORITHM)	iterative subtraction

V. Archimedean

Fig. 11

	Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
VI. Ratio	As a result of the child assigning objects to two different groups by a specified rule.	A B a objects b objects a : b - - - - - - na : nb	a:b = c:d $\frac{a}{b} = \frac{c}{d}$	axd = bxc a:b = na:nb

Fig. 12

The pattern is descriptive of assigning objects to arbitrary groups (A,B). Ratio is defined to be a rule of assignment of objects. Thus the ratio a:b ($\frac{a}{b}$) is an instruction to assign a objects to group A and b objects to group B. Ratios are the mathematical analogs of Rate Pairs which are ordered pairs of vector quantities such as 10¢/1 lb, 40 miles/ 1 hour, 3 ft./ 1 yard. For example, the Rate Pair 10¢/1 lb. implies the ratio 10:1. This in turn is a rule which assigns 10 to the ¢ group for every 1 to the lb. group.

Symbolically, 10¢/1 lb. $\Longrightarrow$ 10:1 $\Longrightarrow$ $\underset{10}{¢}$ 10:1 $\overset{\text{lb.}}{1}$. The problematic situations having this pattern are the so-called "ratio problems." These are characterized by stating explicitly a given Rate Pair. This same Rate Pair is partially renamed. The solution consists of abstracting the ratio from the given Rate Pair, renaming the ratio (Computation, Fig. 12) then restating this ratio as a Rate Pair with the appropriate units. This pattern subsumes Pattern III (Multiplicative Divisive). For example, the statement 7 x 9 = $\square$ can be restated as 7:1 = $\square$:9 or $\frac{7}{1} = \frac{\square}{9}$. The notation a:b is to be preferred over the notation $\frac{a}{b}$ in that the latter is also used to name fractions. The concepts of fraction and ratio are quite distinct from each other; the only commonality shared by both concepts is the mechanics of name changing.

Physical Maneuver	Geometric Representation	Algebraic Representation	Computation
As a result of the child mapping sets of objects into other sets of objects under a specified rule	variable input(s) x function mapping: f variable output (s) y	$x \xrightarrow{f} y$ $[(x_1 y_1)(x_2 y_2)(x_3 y_3)\ldots]$	specific to replacement of f

VII. Function

Fig. 13

The concept of function is pervasive throughout the arithmetic curriculum. All the previous patterns (I - VI) are subsumed under the Function Pattern. The initial interpretation of a function is that of a machine (Geometric Representation). For each value of input variable (x) there corresponds an output variable (y) according to some mapping function f. This notion is suggested by various vending machines in which ordered pairs consisting of monetary values and products are formed. Three appropriate activities at this stage would be:

(1) given x and y the child is required to determine the mapping function f. <u>for example</u>, given (1,1) (2,4) (3,9) (4,16) the function f is $y = x^2$

(2) given x and f, the child is required to determine y. <u>for example</u>, given the set x: [1,2,3,4] and f:2x+1 then y: [3,5,7,9]

(3) given y and f the child is required to determine x. <u>for example</u>, given the set y: [2,4,6,8] and f:2x then x: [1,2,3,4]

The Algebraic Representation consists of sets of ordered pairs of the form (input variable, output variable). Implicit in all problem situations is the concept of function as a pattern consisting of ordered pairs of numbers. The solution of such problems consists of determining the appropriate mapping (f) and the value from that mapping. To illustrate, the problem: John has 6 marbles. AFter winning some, he now has 9 marbles. How many did he win? The Open Sentence description of this situation (Pattern I) is:

$6 + \square = 9 \quad \square + 6 = 9 \quad 9 - 6 = \square \quad 9 - \square = 6 \quad 9 - 6 - \square = 0$

As a function, the pattern is $(6, \square) \xrightarrow{f} 9 \quad (\square, 6) \xrightarrow{f} 9$

$(9,6) \xrightarrow{f} \square \quad (9, \square) \xrightarrow{f} 6$

VIII. <u>Real Problems</u>. As opposed to the worded or story book problems considered in the previous section (VII Problem Solving), there exists a class of problems which are defined to be Real -- in the sense that they are generated from experimental activities involving the Natural Sciences of the Social Studies. They are similar to those problematic situations which generated the above 7 pattern groupings in that the mathematical descriptions followed as a result of object manipulation. They differ, however, in that more than one Pattern describes the physical events -- assuming certain boundary conditions. The model of such a problem is:

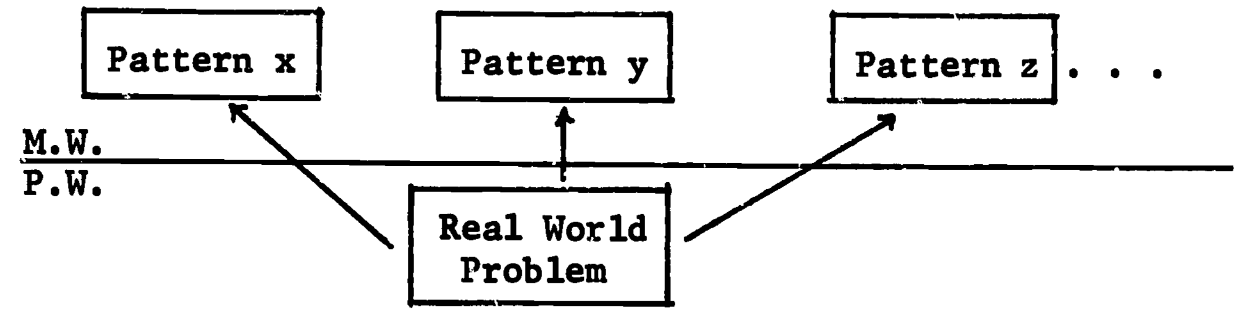


Fig. 14

As the model indicates, Pattern x may be sufficient to describe or predict the outcome of a given Real World problem -- under limiting boundary conditions. As these conditions are relaxed, the pattern is superseded by Pattern y which is capable of further generalization. This in turn is replaced by Pattern z as soon as limiting conditions are again relaxed. Two major criteria that determine whether or not a given problematic situation is in fact a Real problem are:

(1) the extent to which the situation allows the child to hypothesize a mathematical model of description from several models in his repetoire.

(2) the feasibility of the problem in terms of the laboratory kind of experimentation conducted by the child either individually or in small groups.

One illustrator of such a problem is the Law of the Lever. By imposing boundary conditions on the variables of weights and distances required to balance a lever, the child, through experimentation is able to hypothesize mathematical models which describe the conditions under which the lever balances. If, for example, both distance vectors from the fulcrum were equal then equal weights would balance the lever. This may lead one child to hypothecate that the sum, difference, product, or quotient of respective distances and weights must be equal, while another asserts that the ratio of weights to distance are equal. As restrictions are relaxed on any of the four variables, some of the existing models become invalid and must be replaced with models capable of wider generalization such as the distributive law (which can be interpreted as a pattern). Assuming the above criteria are satisfied, any formal law from the Natural or Behavior Sciences becomes a legitimate source for structuring such experiments. Some of these are Ohms Law, Pascal's Principle, Laws of Mechanics, optical laws, Bogle's and Charles' law, to name a few.

IX. Media. Any combination of physical devices which serve as generators of Patterns and mathematical concepts (Content Variable Model) is defined as Media. Two major criteria to be satisfied in the selection are:

(1) the appropriateness of the device in terms of grade level, availability
(2) the degree to which the device used implies the intended mathematical analog
(3) the substitution of an analogous device to replicate the process (This criterion if met, prevents the child from equating a particular object to its mathematical analog. For example, if fraction pie cut-outs were used solely to generate the concept of a fraction there is a possibility that the child would associate the number one-half with the physical disk in the shape of a semi-circle.)

X. Heuristics. The most generalized definition of a heuristic is any strategy used to communicate a mathematical concept in order that it may be accepted intuitively or analytically as reasonable or plausible to the child. One such heuristic is the Communication Model. Another is the representation of problematic situations according to the Pattern clusters above. In this context, the Geometric and Algebraic Representations of physical actions become heuristic devices that aid in the conceptualization of the problems inferred in those actions. In a narrower sense, a heuristic may be defined as a particular strategy of inquiry whose purpose is to reduce the number of mathematical alternatives that a child may choose from as analogs of some physical manipulation. Nine such heuristics are herein identified. These initially would be used by the Instructional Manager in communication and eventually would be self-employed by the child as a routine to structure his own inquiry. The nine heuristics are independent of the four Strategies comprising the Communication Model, i.e., they may be employed wherever appropriate to any one of the Strategies (in particular Strategy II.).

Heuristics of Inquiry

1. Enumeration of specific cases. This strategy consists of listing valid statements involving specific replacements and from these inferring that the generalized statement must be valid for all replacements. This strategy is similar to the induction step used in scientific inquiry.

Example. What is the remainder when any prime number greater than 2 is divided by 6?

2. Deduction. This strategy consists of the logical deduction (conclusion) which follows from a given premise.

Example. Given that the interior base angles of an isosceles triangle are of equal measure, what statement can be deduced about the measures of the interior angles of an equilateral triangle?

3. Inverse Deduction. This strategy consists of stating the conclusion of some mathematical argument. The child is then required to reconstruct the minimum number of antecedants comprising the premise from which the conclusion necessarily follows.

Example. Define a square.

4. Analogy. This strategy consists of establishing the validity of some mathematical argument by noting the similarities of the argument with an already known one. Isomorphisms between different systems is one such class of analogies.

Example. Given that $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ then $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$

(false analogy)

5. Preservation of the Enabling Principles. This strategy consists of defining operations in a new system so as to preserve the validity of the Enabling Principles (Postulates).

Example. Suppose the sum of two fractional numbers $\frac{a}{b}$ and $\frac{c}{d}$ were $\frac{a}{b} + \frac{c}{d}$

Which of the Postulates would no longer be valid?

6. Variation. This strategy consists of a set of antecedants making up a premise from which a conclusion is stated. The child is required to change one or more of the antecedants at a time thereby conjecturing if any part of the conclusion remains invariant in view of the restated antecedent.

Example. Given any parallelogram and the figure formed by joining adjacent mid points, what kind of figure would you get if you started with any irregular quadrilateral?

7. Continuity of form. This strategy consists of either extrapolating or interpolating from conceptualizing a mathematical pattern or trend from known cases.

Example. What is the meaning of an exponent of zero or a negative exponent?

2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	
32	16	8	4	2	$\square$	$\triangle$	$4 \times 8 \Leftrightarrow 2 + 3$
							$4 \times \square \Leftrightarrow 2 + 0 = 2$
							$4 \times \triangle \Leftrightarrow 2 + -1 = 1$

8. Existential Counter example. This strategy consists of an attempt to negate the validity of some mathematical argument by finding a specific case wherein the argument is false.

Example. There is a smallest fractional number.

9. Determination of Limits. This strategy consists of putting upper and lower or maximal and minimal limits on the range of plausible replacements needed to complete a mathematical argument. The child is allowed to estimate these limits either by intuition or previous knowledge.

Example. Determine upper and lower limits to the product of two rational numbers named by proper fractions. Since both rational numbers are less than one but greater than zero, reasonable upper and lower limits would respectively be one and zero.

Communicating by the so-called Discovery Method is equivalent to use of these strategies to advance towards some mathematical conclusion. The one property they all have in common is to force the child to exercise his intuition by guessing at probable outcomes thereby committing himself to the problem. As previously stated, the ultimate objective of using these heuristics is to provide the child with a strategy of inquiry by communicating them explicitly to him.