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This programed booklet is designed for the engineering student who is familiar with integral calculus and electrical networks. The first portion of this booklet is concerned with sinusoids, their properties, and their mathematical and graphical representations. The second portion is concerned with phasors and the mathematical relationship between phasors and sinusoids. Opportunity is provided for the student (1) to see how the use of phasors can simplify the work of adding or subtracting sinusoids of the same frequency, and (2) to see how different forms of the same expression emphasize different characteristics of the function. (RP)

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SINUSOIDS AND PHASORS

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Section 1, Sinusoids

The first portion of this booklet will be concerned with sinusoids, their properties and their representation.

To start, let's define a periodic function as one whose form repeats itself over and over in intervals of time called the period. In Fig. 1 the function in (a) is periodic with a period of 6 seconds. Starting at the point marked x, the function does not start repeating its shape until 6 seconds have passed.

In the space below, state whether each of the other functions plotted in Fig. 1 is periodic or non-periodic. For each function which is periodic, state its period.

a. Periodic; 6-second period.

b.

c.

d.

2

Answer:

b. Periodic; 4-second period.

c. Non-periodic.

d. Periodic; 10-second period.

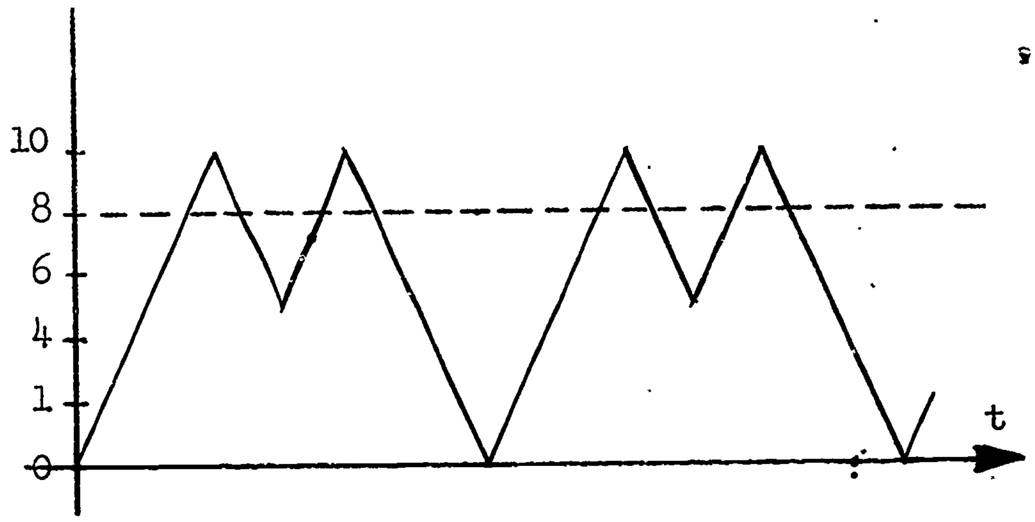

Fig. 2

The variation of the seasons through the year is periodic, or cyclic; one complete traverse of spring, summer, autumn and winter being a cycle. In the same way, the sequence of values of a periodic function over the interval of one period is called a cycle.

In Fig. 3, how many times per cycle does the function take on the value 8?

4

<u>Answer:</u>

4 times.

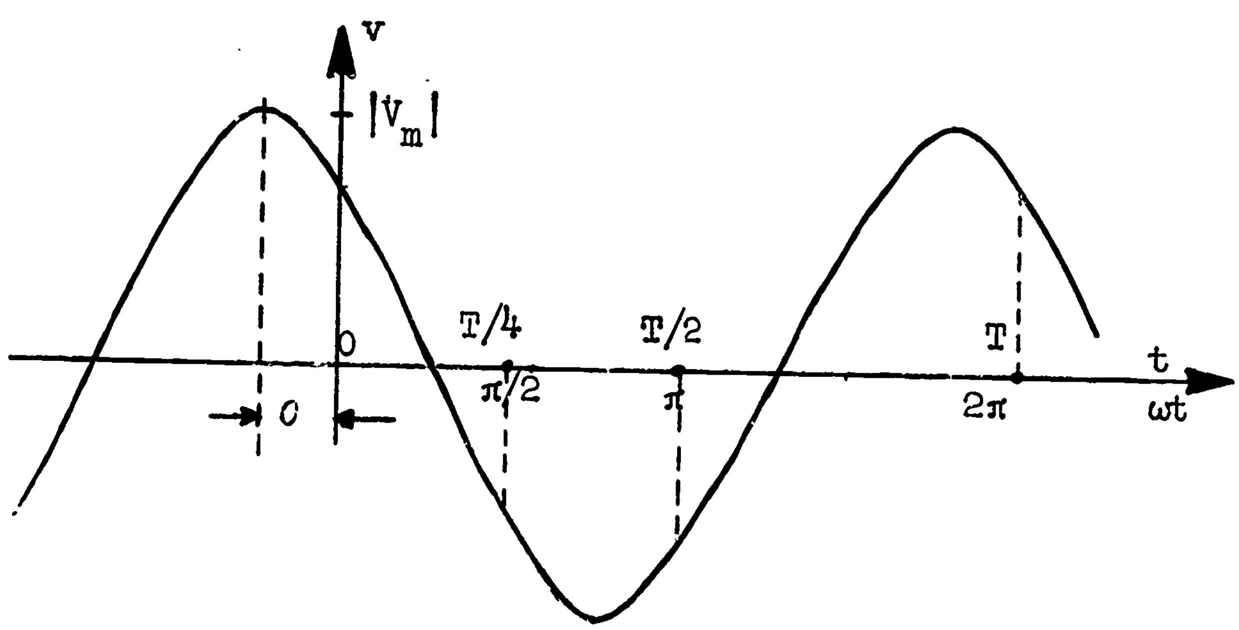

Fig. 3

A very important periodic function in all of engineering is the sinusoid. By a sinusoid we mean a periodic function like that shown in Fig. 3 whose equation is:

$$v(t) = V_m \cos(\omega t+\theta)$$

(The function is taken to be a voltage only for purposes of discussion. Also, the reasons for using a cosine function as the general expression for a sinusoid will be explained later.)

Two scales are shown for the abscissa, one for time t and one for ωt. From the diagram, the period is seen to equal __________.

As time goes on, the function alternately takes on positive and negative values. Hence, it is sometimes referred to as a (an) __________.

Answer: The period equals T, an alternating function. (You may have said oscillating, which indicates you have the right idea.)

Note: A lower case t is used as the variable representing time. The constant period is represented by a capital T. This use of lower-case letters for variables and upper-case letters for constants is common, although not universal, in engineering.

Note from Fig. 3 that the change in $\underline{\omega t}$ over one period equals ____________.

Hence, the time elapsed during one cycle of the waveform -- which is simply the period T -- can be expressed in terms of ω and equals ____________.

8

<u>Answer</u>:

Change in ωt over one period equals 2π.

$T = \underline{2\pi/\omega}$

The <u>frequency f</u> of the sinusoid is the number of cycles completed per unit of time. If time is measured in seconds, the unit of frequency is <u>cycles per second</u> (cps). Since the <u>period</u> is the time required to complete one cycle, give an expression relating f and T; then give one relating ω and f.

$f =$ ________

$\omega =$ ________

10

Answer:

$$f = \underline{1/T}$$

$$\omega = \underline{2\pi f}$$

ω (which is the Greek letter "omega") is related to the frequency by $\omega = 2\pi f$; it is called angular frequency and is measured in radians/sec.

For the following sinusoids state the values and units of the frequency and the angular frequency.

a) $10 \cos (377t - \theta_1)$

$f =$

$\omega =$

b) $5 \cos (2\pi t + \theta_2)$

$f =$

$\omega =$

swer:

a) $f = 60$ cps.

$\omega = 377$ rad/sec.

b) $f = 1$ cps.

$\omega = 2\pi$ rad/sec.

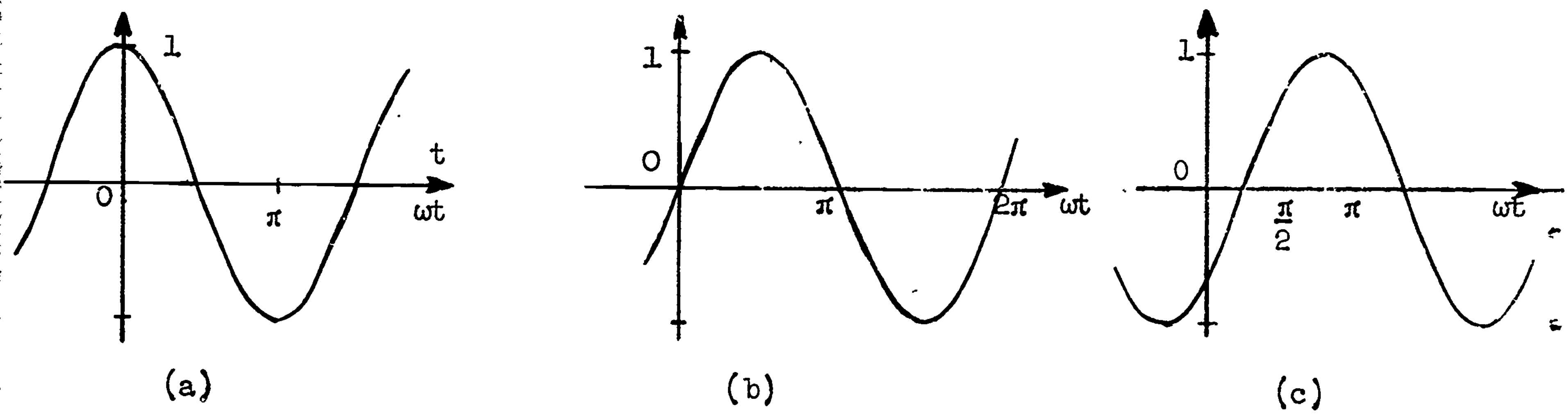

Fig. 4

The maximum, or peak, value of the sinusoid is called its amplitude. Thus, in the function $8 \cos (30t + \pi)$ the amplitude equals ___________.

In Fig. 3, V_m is the _________.

In Fig. 3, the quantity θ is called the initial angle since $(\omega t + \theta)$ reduces to θ when $t = 0$. It is measured from the positive peak of the sinusoid to the origin. In Fig. 3, θ is a positive quantity. State the values of θ for the sinusoids in Fig. 4.

a) $\theta =$ ____________

b) $\theta =$ ____________

c) $\theta =$ ____________

<u>Answer</u>:

a) $\theta = 0$ radians

b) $\theta = \pi/2$ radians

c) $\theta = 3\pi/4$ radians

(If you got any signs wrong, note that the origin is to the left of the first peak in a) and c), and so the direction <u>from</u> the peak <u>to</u> the origin is opposite to the reference direction of the ωt axis.)

The equation representing Fig. 4(a) is cos ωt, which results by setting $\underline{\theta}$ = 0 in the general expression for a sinusoid.

For Fig. 4(b), if θ is set equal to $-\pi/2$, the general expression becomes $\cos(\omega t-\pi/2)$. Using a trigonometric identity for the cosine of the difference of two angles, put this expression in a different form.

$$\cos(\omega t-\pi/2) = \underline{\hspace{4cm}}$$

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<u>Answer</u>:

$$\cos(\omega t - \pi/2) = \underline{\sin\ \omega t}$$

In fact, because of the trigonometric identities

$$\cos x = \sin(x + \pi/2)$$
$$\sin x = \cos(x - \pi/2)$$

a sinusoid can be written either as a cosine function or as a sine function, but with angles that differ by $\pi/2$ radians.

Write the following sinusoids as sine functions:

a) $\cos(\omega t + \pi/6) = \sin$ (____________)

b) $\cos(10t - \pi/4) = \sin$ (____________)

Write the following sinusoids as cosine functions:

c) $\sin(8\pi t - \pi/3) = \cos$ (____________)

d) $\sin(\omega t + 5\pi/6) = \cos$ (____________(

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<u>Answer</u>:

a) $\cos(\omega t + \pi/6) = \sin(\underline{\omega t + 2\pi/3})$

b) $\cos(10t - \pi/4) = \sin(\underline{10t + \pi/4})$

c) $\sin(8\pi t - \pi/3) = \cos(\underline{8\pi t - 5\pi/6})$

d) $\sin(\omega t + 5\pi/6) = \cos(\underline{\omega t + \pi/3})$

But when a sinusoid is written as a sine function, the value of $(\omega t+\theta)$ when $t = 0$ is different from what it is when it is written as a cosine function; it is, in fact, $\pi/2$ radians less. So, in order to have a unique meaning for the term initial angle, we shall henceforth agree to write sinusoids as cosine functions.

The initial angle is sometimes referred to as the phase of the sinusoid. Since this word, phase, has another meaning in a slightly different context, we will restrict ourselves here to the term "initial angle".

Sketch a voltage sinusoid having an initial angle of $\pi/4$ radians, a frequency of 100 cps., and an amplitude of 10 volts. Label the axes and all pertinent properties.

Answer:

Your sketch should have its positive peak 1/8 of a period to the left of the time origin. Its maximum value should be labeled 10 volts and its period 0.01 sec. or 10 milliseconds. Its angular frequency is 200π rad/sec. If any of these items are missing from your sketch, add them now.

We have seen that the general form of a sinusoid can be written as $V_m \cos(\omega t+\theta)$. A sinusoid which is written as a sine function can always be converted to the cosine form to obtain its general form.

Another form in which a sinusoid can be written is obtained by using a trigonometric identity for the cosine of the sum of two angles. Use that identity to rewrite the following sinusoid:

$$10 \cos(\omega t+\pi/6) = \underline{\hspace{6cm}}$$

Answer:

$$10 \cos(\omega t + \pi/6) = (10 \cos \pi/6) \cos \omega t - (10 \sin \pi/6) \sin \omega t$$

$$= 8.66 \cos \omega t - 5 \sin \omega t.$$

It is not hard to see that no matter what the amplitude or initial angle of the sinusoid, the general form of a sinusoid can be expanded as follows:

$$V_m \cos(\omega t + \theta) = A \cos \omega t - B \sin \omega t$$

where A and B may be either positive or negative quantities. Thus, any sinusoid can be represented as the sum of two other, related functions.

As shown on the preceding page, for the sinusoid $10 \cos(\omega t+\pi/6)$; $\theta = \pi/6$, $V_m = 10$, $A = 10 \cos \pi/6 = 8.66$, and $B = 10 \sin \pi/6 = 5$. An interesting relationship exists between $\underline{A}$ and $\underline{B}$, and $\underline{V_m}$ and $\underline{\theta}$ which we shall use at a later time. Note that

$$A^2 + B^2 = (8.66)^2+(5)^2 = 75+25 = 100 = 10^2 = (V_m)^2$$

and

$$\frac{B}{A} = \frac{10 \cos \pi/6}{10 \sin \pi/6} = \tan \pi/6 = \tan \theta .$$

In each of the following expanded forms, state the values of A, B, and V_m.

1) $V_{m_1} \cos(\omega t+\theta_1) = 6\cos\omega t - 3\sin\omega t$; A = ____, B = ____, V_{m_1} = ____.

2) $V_{m_2} \cos(\omega t+\theta_2) = -8\cos\omega t - 2\sin\omega t$; A = ____, B = ____, V_{m_2} = ____.

3) $V_{m_3} \cos(\omega t+\theta_3) = 4\cos\omega t + 5\sin\omega t$; A = ____, B = ____, V_{m_3} = ____.

4) $V_{m_4} \cos(\omega t+\theta_4) = -5\cos\omega t + 4\sin\omega t$; A = ____, B = ____, V_{m_4} = ____.

<u>Answer</u>: 1) $A = 6$, $B = 3$, $V_{m_1} = 3\sqrt{5}$

2) $A = -8$, $B = 2$, $V_{m_2} = 2\sqrt{17}$

3) $A = 4$, $B = -5$, $V_{m_3} = \sqrt{41}$

4) $\underbrace{A = -5,\ B}_{} = -4$, $V_{m_4} = \sqrt{41}$

If you got any <u>signs</u> wrong, examine the reasons for your own errors and then correct them.

Two constants, or parameters, are sufficient to describe the characteristics of a sinusoid. In the general expression $V_m \cos(\omega t+\theta)$, the parameters are the amplitude (V_m) and the initial angle (θ). When this general form is given, it is simple to obtain the expression ($A \cos \omega t - B \sin \omega t$).

Write an expression for A, and another for B, in terms of the parameters, V_m and θ.

$A =$

$B =$

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<u>Answer:</u>

$$A = V_m \cos \theta.$$

$$B = V_m \sin \theta.$$

But suppose the expanded form is given and we want to write it as a general cosine function. That is, A and B are known and we wish to get V_m and θ.

Write expressions for V_m and θ in terms of A and B.

$V_m =$

$\theta =$

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Answer:

$$V_m = \sqrt{A^2 + B^2}$$

$$\theta = \tan^{-1} \frac{B}{A}$$

These are obtained (1) by squaring and adding the answers of the previous frame, and (2) by taking the ratio of those answers. If you did not obtain the correct answer, carry through these steps.

Write the value of the amplitude and initial angle of each of the following sinusoids:

a) $v_1 = 8.66 \cos \omega t + 5 \sin \omega t$; $V_m =$ _____, $\theta =$ _____

b) $v_2 = 5 \cos \omega t - 8.66 \sin \omega t$; $V_m =$ _____, $\theta =$ _____

c) $v_3 = 43.3 \cos \omega t - 25 \sin \omega t$; $V_m =$ _____, $\theta =$ _____

Answer:

a) $V_m = 10,\ \theta = -\pi/6$

b) $V_m = 10,\ \theta = \pi/3$

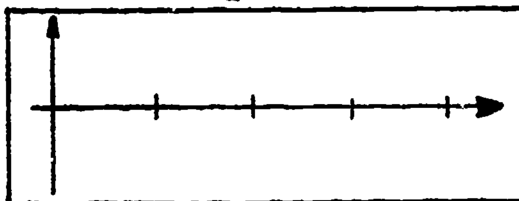
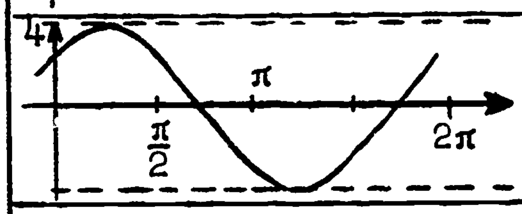
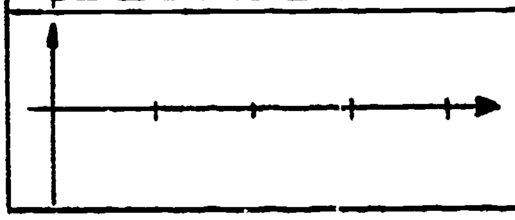
c) $V_m = 50,\ \theta = 5\pi/6$

Remark

Although it is necessary to express angles in radians when manipulating trigonometric functions, it is convenient to deal with degrees rather than radians in making numerical calculations. Henceforth, when convenient, we shall work with degrees, always remembering that in carrying out mathematical operations we assume that angles will be converted to radians.

You should now be able to express a sinusoid in any one of three forms: (1) a time-plot of the function, (2) the general form, and (3) the expanded form of a sinusoid. You should also be able to translate any one of these expressions into each of the other two.

For each of the expressions below, complete the other two forms.

Time-plot	General Form	Expanded Form
	$8 \cos\left(2\pi t + \frac{\pi}{2}\right)$	
		
		$3 \cos \omega t - 4 \sin \omega t$

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No answers are given here.
Review the last few pages
if you need to be sure your
answers are correct.

Section 2, The "rms" Value of a Sinusoid

The next few pages will discuss a characteristic of sinusoids which make it possible to compare the effect of a sinusoidal function to the effect of an equivalent constant function. This characteristic will be the rms value of the sinusoid. For example, if the rms value of the sinusoidal current through a resistor is 5 amps., this varying current will produce the same heating as a constant current (DC) of 5 amps.

You will learn to determine this rms value of a sinusoid both mathematically and graphically.

In many applications it is convenient to use another quantitative measure of a sinusoid, rather than its amplitude. This measure is called its rms value. (These are the initials of root-mean square.) The rms value of a sinusoid is defined as the square Root of the average or "Mean" value of the Square of the sinusoid, the averaging being done over one period.

To determine the rms value of a sinusoid quantitatively, let the initial angle be chosen to be zero.

Thus $$v = V_m \cos\omega t$$

and $$v^2 = V_m^2 \cos^2 \omega t$$

To find the rms value, we integrate v^2 over one period, divide by T, and then take the square root.

$$(v^2)_{ave} = \frac{1}{T}\int_0^T V_m^2 \cos^2 \omega t \; dt$$

$$= \frac{V_m^2}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)dt$$

$$= \frac{V_m^2}{2T}\Big[\text{\_\_\_\_\_\_\_\_}\Big]_0^T$$

$$(v^2)_{ave} = \text{\_\_\_\_\_\_\_\_}$$

Therefore $$V_{rms} = \text{\_\_\_\_\_\_\_\_}$$

Answer:

$$(v^2)_{ave} = \frac{V_m^2}{2T}\left(t + \frac{\sin 2\omega t}{2\omega}\right)\Bigg|_0^T = \frac{V_m^2}{2}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}} = .707\ V_m$$

(If you want to have a little more discussion of rms values in general, go to the next page. If not, go on to page 45.)

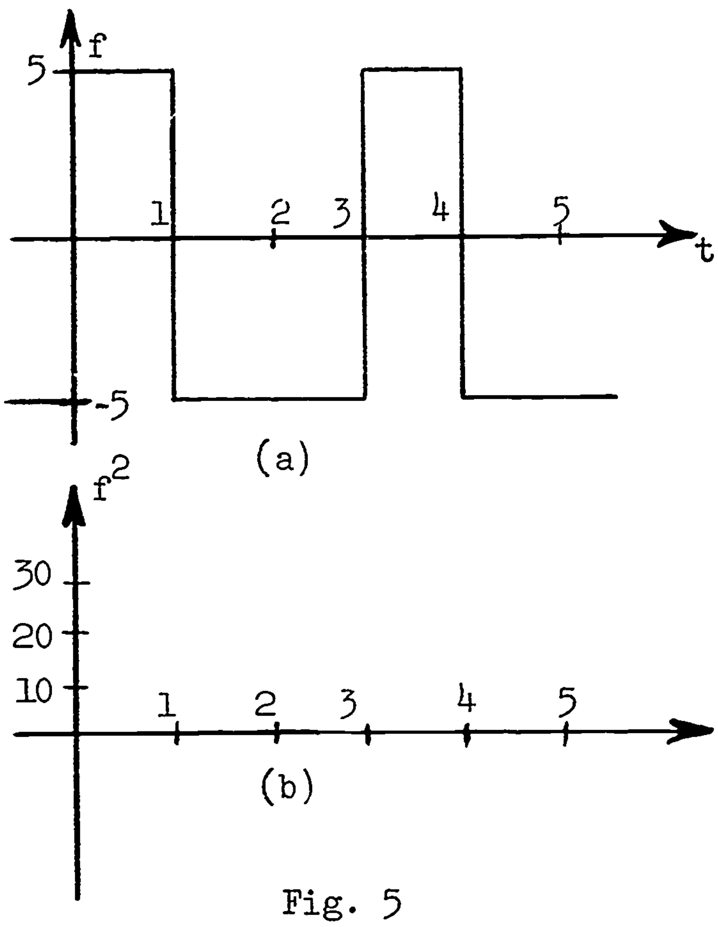

Fig. 5

Neither the process of finding a root-mean-square, or rms value, nor the usefulness of its interpretation, is limited to sinusoids. The rms value has significance for any periodic functions. (It is also applicable to random functions, which we shall not consider here.)

For any periodic function, f(t), the rms value is defined as:

$$F_{rms} = \sqrt{\frac{1}{T} \int_{t_1}^{t_1+T} f^2(t)dt}$$

where T is the period and t_1 is any arbitrary time. That is, the integration need not be started at t = 0, as long as it is carried out over one period.

Figure 5 shows a simple periodic function. Find its rms value by applying the above formula and by graphical means.

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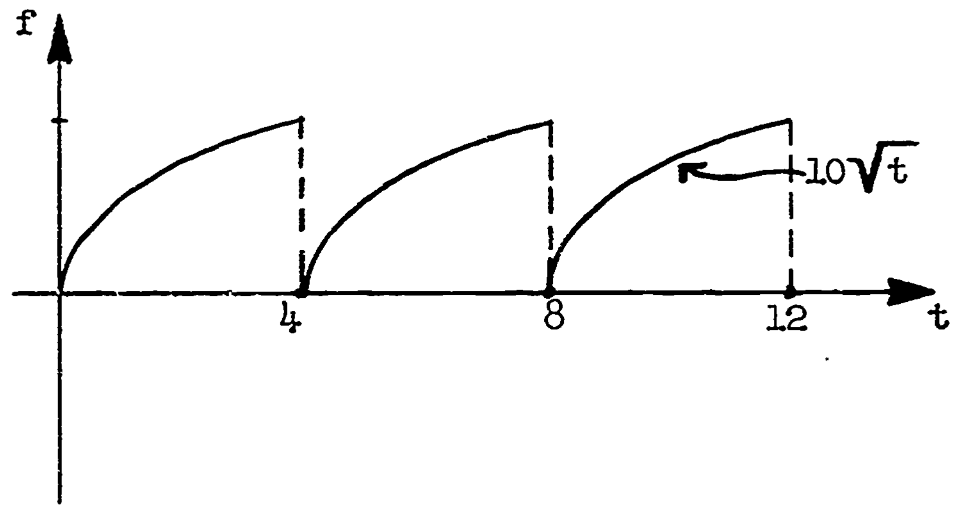
<u>Answer:</u>

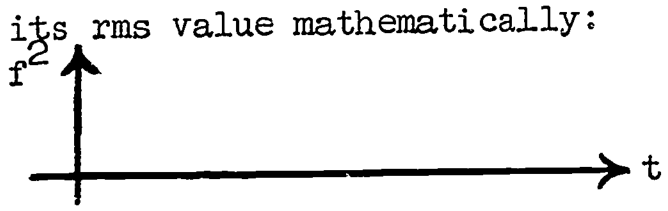
$F_{rms} = 5$



The same mathematical and graphical procedures can be applied to more complex functions.

Figure 6 shows another periodic function.

Find its rms value mathematically:



Sketch the function $f^2(t)$, obtain the rms value by observation, and check your above answer.

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The correct answer is not needed as you can now check your own results.

An interpretation of the rms value in electrical terms is extremely useful. For purposes of discussion, let's talk about a current $i(t)$ which is a periodic function of time. If this current flows through a resistor R for an interval of time, some energy will be dissipated as heat. A quantitative measure of $i(t)$ is obtained if this heat is compared with the heat dissipated when a standard current goes through the same resistor for the same length of time. Let's choose as a standard a constant current I. Recalling the relationship of the <u>power</u> dissipated in a resistor to the current, the <u>energy</u> dissipated in the resistor over one period of the function starting at a time t_1, when the periodic function $i(t)$ is flowing there, is

$$W_R = \int_{t_1}^{t_1+T} R\, i^2(t)dt$$

Compute the energy dissipated in the same resistor, over the same time interval, when the constant current I is flowing.

W_r = ____________________

If the two values of energy computed above are to be equal [when I = rms value of $i(t)$] write an expression for the required value of I (upper-case letter) in terms of $i(t)$, (lower-case letter).

I = ____________________

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Answer:

$$I = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} i^2(t)dt}$$

The constant current I gives the same heating effect as the periodic current i(t). It is thus called the effective value of the current. However, a look at the expression just found shows that this effective value, computed from the physical consideration of heat dissipation, is the same function that has already been defined mathematically and called the _____ ______. We thus have a meaning for the rms value of a periodic function -- at least when the function is a current. Since current and voltage in a resistor are proportional, the same physical interpretation applies to a voltage.

In a network, an average power of 20 watts is dissipated in a resistor R by a periodic current i(t). When a constant current of I_1 amps. flows in the same resistor, it dissipated 5 watts. State the rms value of the periodic current in terms of I_1.

I_{rms} = ____________________

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Answer:

rms value

$I_{rms} = 2I_1$

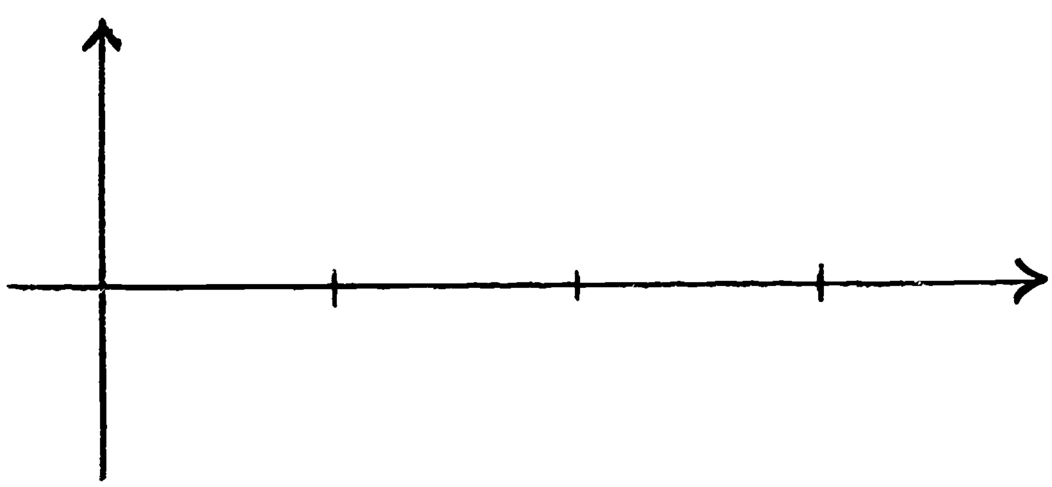
(The power dissipated by $i(t)$ is $RI_{rms}^2 = 20$; the power dissipated by I_1 is $RI_1^2 = 5$ watts. The result follows.)

Since the use of rms as a subscript in V_{rms} or I_{rms} is inconvenient, we will avoid it, and simply use a capital letter to designate the rms value. Thus, the rms value of a voltage $v_2(t)$ and a current $i_1(t)$ will be written (capital) V_2 and I_1, respectively, except when there is the possibility of confusion, in which case the more descriptive subscript (rms) will be used.

Write an expression for a sinusoidal voltage whose frequency is 60 cps., whose initial angle is $\pi/6$ radians and whose rms value is 110 volts.

$v(t) =$ ____________

Make an approximate sketch of this expression and identify each portion of your function on the sketch.

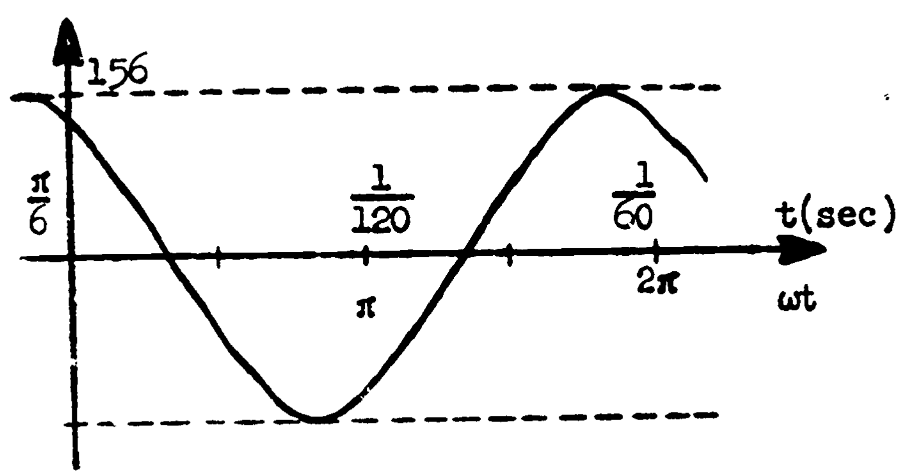


Answer:

$$v(t) = \underline{110\sqrt{2}\cos(377t + \pi/6)} \text{ volts}$$

or

$$v(t) = \underline{156 \cos(2\pi \times 60t + \pi/6)} \text{ volts.}$$



This completes this introduction to sinusoids.

Section 3, Phasors

Very often it is desired to find the sum of a number of sinusoids all having the same frequency. For example, if the currents in all branches connected at a node but one are sinusoidal, by Kirchhoff's current law, this one current will equal a sum of sinusoids. We shall now demonstrate that the sum of two sinusoids of the same frequency is a sinusoid of the same frequency.

Let $v_1 = 8\cos(\omega t+\pi/6)$

$v_2 = 10\sin(\omega t+40^\circ)$

Both of these can be expanded by an appropriate trigonometric identity, and then added. Do this, and collect terms so that the result is in the form $A\cos\omega t - B\sin\omega t$.

$v_1 =$

$v_2 =$

$v_1+v_2 =$

Answer:

$$v_1 = 6.93 \cos \omega t - 4 \sin \omega t$$

$$v_2 = 6.42 \cos \omega t + 7.65 \sin \omega t$$

$$v_1 + v_2 = 13.4 \cos \omega t + 3.65 \sin \omega t$$

The expression (Acosωt - Bsinωt) is one form of a sinusoid. But if we wish to know the amplitude (or rms value) and initial angle, it must still be transformed back to the general form.

Write the rms value and initial angle of (v_1+v_2), as calculated on the preceding page.

V = ______________________

θ = ______________________

Answer:

$$V = \underline{9.82} \quad \left(V = \frac{V_m}{\sqrt{2}} = \frac{\sqrt{13.4^2 + 3.65^2}}{\sqrt{2}} = \frac{13.9}{\sqrt{2}}\right)$$

$$\theta = \underline{-15.2^\circ} \quad \left(\theta = \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{-3.65}{13.4}\right)$$

(Remember that "V" stands for rms value, while "V_m" represents the amplitude, or maximum value.)

Although we have carried out this demonstration by a numerical example, you can certainly appreciate the general nature of the result; that is, that the sum of two sinusoids is a sinusoid (of the same frequency). But look how much computational work is involved:

Given two sinusoids whose amplitudes (or rms values) and initial angles are known, to find the rms value and initial angle of their sum we must first _______ ____;

then collect terms to put the result in the form ______________________________;

and finally get the rms value and initial angle from the formulas:

V = ______________ and θ = _________________.

<u>Answer</u>:

we must first <u>expand each sinusoid by means of a trigonometric identity</u>; then put the result in the form <u>A cosωt - B sinωt</u>; and finally,

$$V = \underline{\frac{\sqrt{A^2 + B^2}}{\sqrt{2}}}$$ (Did you forget the $\sqrt{2}$, for the <u>rms</u> value?)

$$\theta = \underline{\tan^{-1} \frac{B}{A}}$$

This is quite a bit of numerical work. To avoid this work (and to simplify other computations involving sinusoids (which we have not yet discussed), another representation of sinusoids has been developed. This representation is extremely important and has been of inestimable value in the development of electrical engineering. We shall spend the rest of this unit discussing this representation.

The starting point is <u>Euler's formula</u>

$$e^{jx} = \cos x + j \sin x$$

where $j^2 = -1$. (You have probably been used to the symbol i for $\sqrt{-1}$; but in electrical engineering we use i to represent current, so we need another symbol.)

For each value of x, the right-hand side of Euler's formula is a <u>complex number</u>; for $x = \pi/4$, for example, the right-hand side equals the complex number $.707+j.707$. (It is necessary for you to know well the algebra of complex numbers in order to go on. If you feel it necessary, go to Program Text No. 8, Complex Numbers, before you proceed.)

It is possible to prove Euler's formula in a number of ways.

However, the proof requires some more advanced knowledge than is assumed here. For our purposes, it is enough to accept Euler's formula as an unproved (but provable) formula. It essentially defines the exponential e^{jx}, which has the same properties as ordinary, real exponentials. For example, the law of exponents applies: $(e^{jx_1})(e^{jx_2}) = e^{j(x_1+x_2)}$; the formula for differentiation applies: $d(e^{jx})/dx = je^{jx}$, and so on. We shall not prove these properties here but will use them as if they had been proved.

Write the following as single exponentials.

a) $e^{j2}e^{-j5t}$ = ____________________

b) $e^{-t}e^{j3t}$ = ____________________

c) $e^{j\omega t}e^{j\theta}$ = ____________________

Express the following as products of two exponentials.

d) $e^{-j(4t-\pi/4)}$

e) $e^{j(\omega t-\alpha)}$

Answer:

a) $e^{j2}e^{-j5t} = \underline{e^{j(2-5t)}}$ or $\underline{e^{-j(5t-2)}}$.

b) $e^{-t}e^{j3t} = \underline{e^{(-1+j3)t}}$

c) $e^{j\omega t}e^{j\theta} = \underline{e^{j(\omega t+\theta)}}$.

d) $e^{-j(4t-\pi/4)} = \underline{e^{-j4t}e^{j\pi/4}}$

e) $e^{j(\omega t-\alpha)} = \underline{e^{j\omega t}e^{-j\alpha}}$

We see from Euler's formula, $e^{jx} = \cos x + j \sin x$, that the cosine and sine functions can be written in terms of the imaginary exponential as:

$$\cos x = \mathrm{Re}(e^{jx})$$

$$\sin x = \mathrm{Im}(e^{jx})$$

where Re is read "real part of" and Im is read "imaginary part of". Thus, the cosine and sine functions can be obtained from the imaginary exponential by the operation of taking the real part and the imaginary part, where Re is read "real part of" and Im is read "imaginary part of". However, another, and perhaps more clear-cut, expression for the cosine and sine functions can be obtained as follows. Suppose x in Euler's formula is replaced by -x. Remembering the values of the cosine and sine of a negative angle, write the resulting expression.

58

<u>Answer</u>:

$$e^{-jx} = \cos x - j \sin x$$

(This follows because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.)

Next, add the two expressions (Euler's formula and the result in the last frame) and solve for cos x.

cos x = ____________________

60

<u>Answer:</u>

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

We see that a cosine function is given as the sum of two imaginary exponentials, one with a positive exponent and the other with a negative exponent.

Write the following cosines as a sum of exponentials.

a) $\cos \omega t =$ ________________

b) $\cos \pi/3 =$ ________________

c) $\cos(\omega t + \pi/3) =$ ________________

Answer:

a) $\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$

b) $\cos \pi/3 = \frac{e^{j\pi/3} + e^{-j\pi/3}}{2}$

c) $\cos(\omega t+\pi/3) = \frac{e^{j(\omega t+\pi/3)} + e^{-j(\omega t+\pi/3)}}{2}$

A similar expression can be found for the sine function. Thus, write Euler's formula; write it again with x replaced by -x; then subtract the two. The result will be

sin x = ______________________

Answer:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

We again find that a real function, sin x, is expressible as the difference between two imaginary exponentials. But this time, there is an additional factor j in the denominator.

Let us now turn to sinusoidal voltages and currents, which are of interest to us, and see how we may express them as exponentials. Let's consider a current

$$i(t) = \sqrt{2}\, I \cos(\omega t+\theta)$$

where capital I is the _______ value. We can insert into this expression the representation of the cosine in terms of exponentials. The only thing new here is the appearance of a multiplier of the cosine function. The result will be:

$$i(t) = \rule{6cm}{0.4pt}$$

66

Answer:

$$i(t) = \frac{\sqrt{2}}{2}\left[Ie^{j(\omega t+\theta)} + Ie^{-j(\omega t+\theta)}\right]$$

(You may have factored out the I and written it outside the brackets, which is OK. But for subsequent use we want I to remain as a factor with each term.)

Next, use the law of exponents to rewrite each exponential as a product of exponentials.

$$i(t) = \underline{\hspace{6cm}}$$

Answer:

$$i(t) = \frac{\sqrt{2}}{2}\left(Ie^{j\theta}e^{j\omega t} + Ie^{-j\theta}e^{-j\omega t}\right)$$

(You probably did not put the factors in this order, but that is all right.)

Notice the grouping of factors $Ie^{j\theta}$ and $Ie^{-j\theta}$. You will recognize these as complex numbers written in terms of their magnitudes and angles. Since we will be dealing extensively with complex numbers, we will want to give them a special designation in order to distinguish them from real numbers. We shall use letters with a bar above them as symbols for complex numbers. Thus, $\bar{I}$, $\bar{V}$ and $\bar{A}$ are complex numbers; when reading them, we say "I bar", "V bar" and "A bar". The magnitude of a complex number will have the same letter symbol, but without the bar.

The complex quantity $Ie^{j\theta}$ can, therefore, be represented simply by

$$\bar{I} = Ie^{j\theta}$$

Note that the magnitude and angle of the complex number are, respectively, equal to the ______________ and ______________ of the sinusoid with which we started.

70

Answer:

rms value and **initial angle**.

Now look at the number $Ie^{-j\theta}$. How does this differ from $\bar{I} = Ie^{j\theta}$ in terms of magnitude and angle? ______________________________

From your knowledge of complex numbers you should know that $Ie^{-j\theta}$ is called the ______________ of $Ie^{j\theta}$.

Answer:

$Ie^{-j\theta}$ has the same magnitude as $Ie^{j\theta}$ but its angle has the opposite sign.

$Ie^{-j\theta}$ is called the complex conjugate of $Ie^{j\theta}$.

To designate a complex conjugate, a superscript asterisk is used. Thus, the conjugate of $\bar{I}$ is written $\bar{I}^*$. (The conjugate of a complex number $\bar{V} = Ve^{j\alpha}$ is designated ____________.)

In rectangular form, the complex conjugate of a number has the same <u>real part</u> as the number but the negative <u>imaginary part</u>. Thus the complex conjugate of 3 -j7 is ____________. The conjugate of a + jb is ____________.

74

<u>**Answer:**</u>

$\bar{V}*$

$3 + j7$

$a - jb$

Sinusoids and Phasors

In this unit, we will be concerned with quantities which vary in a consistent and repeated manner as time passes. Many natural phenomena act in this way.

First, we will develop the trigonometric functions which describe a large class of these periodically-varying quantities. You will have practice in interpreting the terms in an expression such as:

$$V_m \cos(10\pi t - \frac{5\pi}{6})$$

You will also practice manipulating this expression mathematically, and will see how different forms of the same expression emphasize different chara teristics of the function.

Finally, we will introduce still another set of mathematical terms to represent the same time-varying functions. When you become accustomed to working with expressions such as:

$$\bar{V}_m = 10e^{j\pi/3} \quad \text{and} \quad v = \mathrm{Re}(\bar{V}_m e^{j\omega t})$$

you will find you can do some important manipulations that would be intolerably tedious without the use of this type of mathematics.

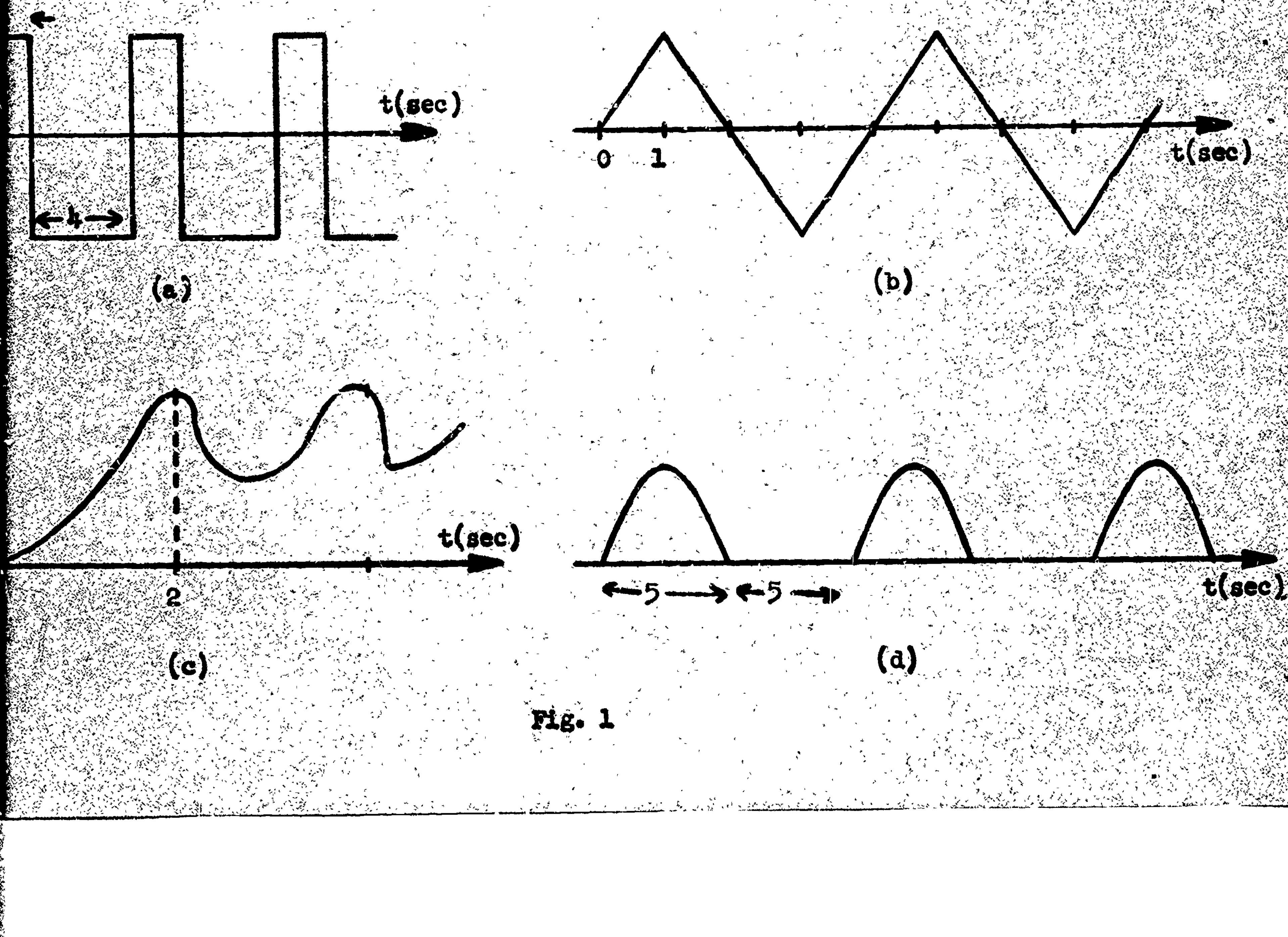


Fig. 1

With the introduction of these complex numbers, we can return to the representation of the sinusoid previously obtained:

$$i(t) = \sqrt{2}\, I\cos(\omega t+\theta) = \frac{\sqrt{2}}{2}(Ie^{j\theta}e^{j\omega t} + Ie^{-j\theta}e^{-j\omega t}).$$

With the agreed-upon designation of complex numbers and conjugates inserted here, this can be rewritten as

$$\sqrt{2}\, I\cos(\omega t+\theta) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

<u>Answer:</u>

$$\sqrt{2}\,I\cos(\omega t+\theta) = \frac{\sqrt{2}}{2}\left(\bar{I}e^{j\omega t}+\bar{I}^{*}e^{-j\omega t}\right)$$

The complex quantity $\bar{I}$ (remember that you read this "eye bar") is specified in terms of its magnitude I and its angle θ. But these quantities are also the only ones necessary to describe a sinusoid, for a fixed frequency. Thus, knowing the value of the complex quantity $\bar{I}$ leads immediately to a knowledge of the sinusoid. We shall give the complex quantity $\bar{I}$, which is the coefficient of the positive exponential in the exponential representation of a sinusoid, the general name <u>phasor</u>. Thus, once a phasor is known, the <u>corresponding</u> sinusoid can be written down immediately.

The following is a <u>phasor current</u>; write the corresponding sinusoid:

a) $\bar{I} = 8e^{j\pi/6}$

$i(t) =$ ____________

Now, here are two <u>phasor voltages</u>; write the corresponding sinusoids.

b) $\bar{V} = 30 + j40$

$v(t) =$ ____________

c) $\bar{V} = -2$

$v(t) =$ ____________

Answer:

a) $i(t) = \underline{8\sqrt{2}\cos(\omega t+\pi/6)}$

b) $v(t) = \underline{50\sqrt{2}\cos(\omega t+53.2^{o})}$

c) $v(t) = \underline{2\sqrt{2}\cos(\omega t+\pi)} = \underline{-2\sqrt{2}\cos\omega t}$
(Did you forget to multiply by $\sqrt{2}$?)

In b), the voltage phasor is given in the rectangular form of a complex number; to be able to write the sinusoid, it is necessary to convert it first into the polar form that puts its magnitude and angle in evidence.

Let's now try the converse. Write the phasors corresponding to the following sinusoids.

a) $v(t) = 141 \cos(\omega t - 60^\circ)$; $\bar{V} =$ __________

b) $i(t) = 30 \sin(\omega t + 125^\circ)$; $\bar{I} =$ __________

<u>Answer:</u>

a) $\bar{V} = 100e^{-j\pi/3}$

b) $\bar{I} = 21.2e^{j35^{\circ}}$

(I hope you do not still forget to use $\sqrt{2}$ properly!)

In b) the sinusoid is expressed as a sine function. It should first be rewritten as a cosine, which can be done simply by subtracting 90° from its initial angle. Thus, $\sin(\omega t + 125^{\circ}) = \cos(\omega t + 35^{\circ})$.

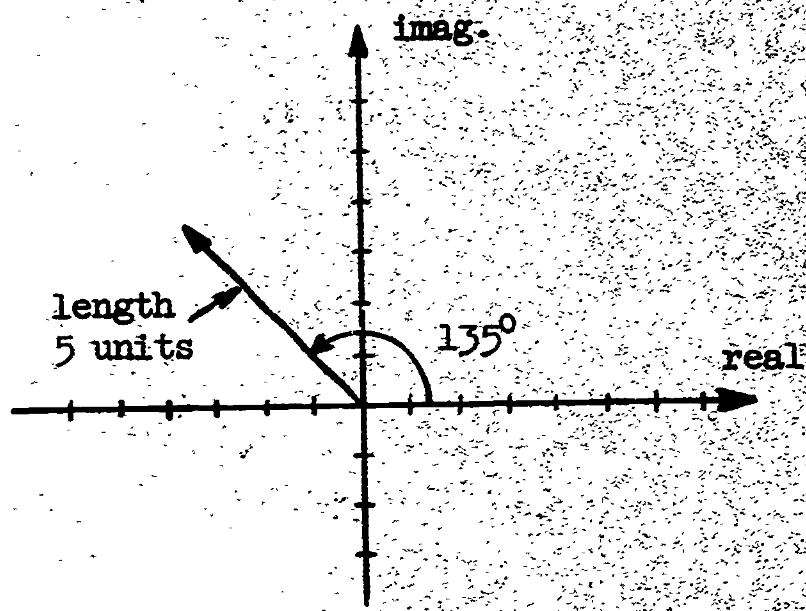


Fig. 7

Since phasors are complex numbers, they can be represented in a plane like complex numbers. Thus, the complex number $5e^{j3\pi/4}$ can be drawn on a set of axes in a plane as a line whose length is 5 units, making an angle of 135° ($3\pi/4$ radians) with the positive horizontal axis, as in Fig. 7.

Sketch the phasors from the preceding page approximately to scale on a set of axes.

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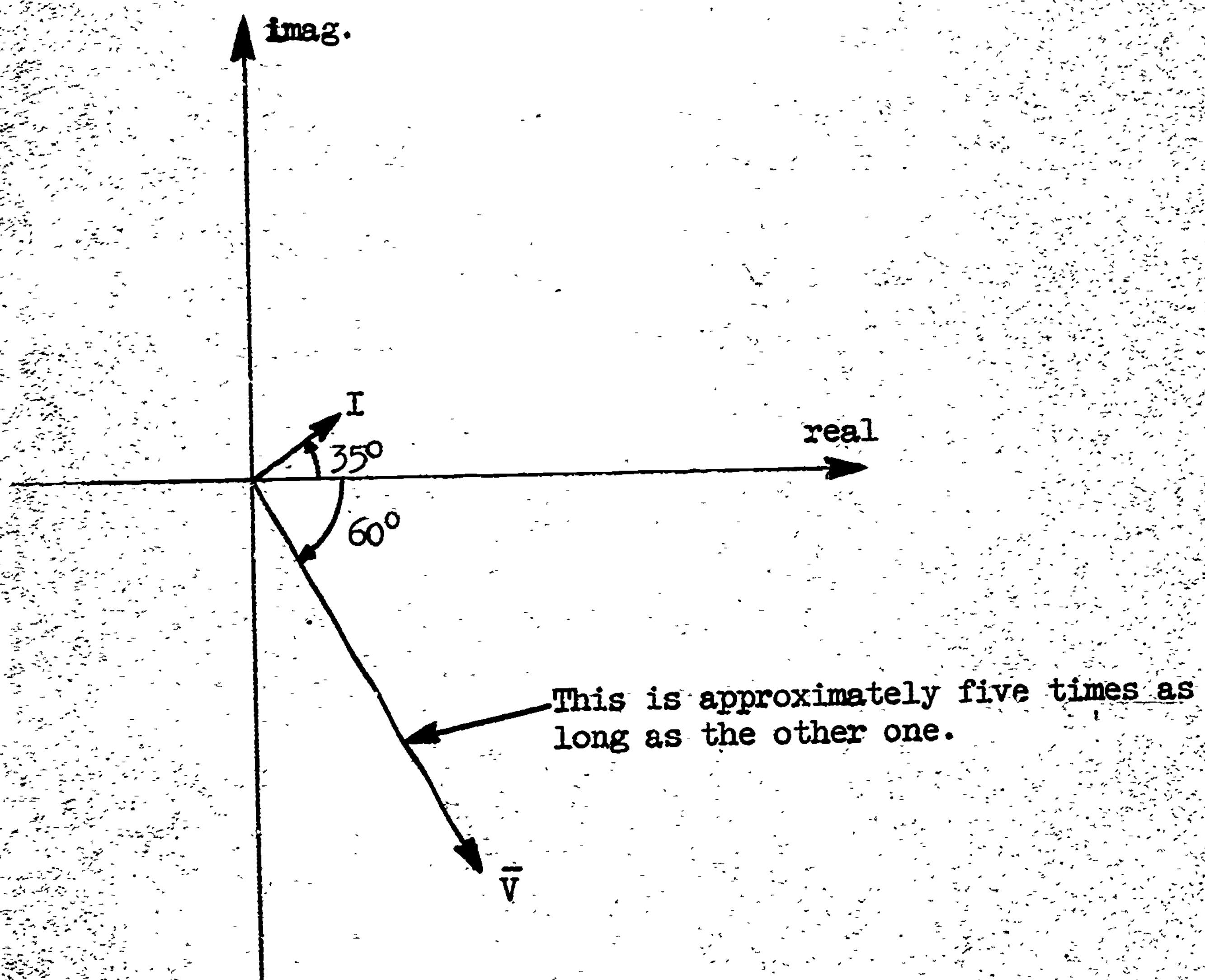
<u>Answer:</u>


imag.
real
I
35°
60°
$\bar{V}$
This is approximately five times as long as the other one.

Now let's see how the use of phasors can simplify the work of adding (or subtracting) sinusoids of the same frequency. Suppose we have two sinusoidal currents of the same frequency.

$$i_1(t) = \sqrt{2}\, I_1 \cos(\omega t + \theta_1)$$
$$i_2(t) = \sqrt{2}\, I_2 \cos(\omega t + \theta_2)$$

First we express them in terms of exponentials, which involves writing the phasors (and their conjugates) corresponding to each sinusoid.

$$i_1(t) = \frac{\sqrt{2}}{2}(\bar{I}_1 e^{j\omega t} + \bar{I}_1{}^* e^{-j\omega t})$$
$$i_2(t) = \frac{\sqrt{2}}{2}(I_2 e^{j\omega t} + \bar{I}_2{}^* e^{-j\omega t})$$

Then we add these two expressions, collecting the positive and negative exponential terms. Do this.

84

<u>Answer</u>:

$$i_1 + i_2 = \frac{\sqrt{2}}{2}\left[(\bar{I}_1 + \bar{I}_2)e^{j\omega t} + (\bar{I}_1{}^* + \bar{I}_2{}^*)e^{-j\omega t}\right]$$

The coefficient of the first exponential is a sum of two phasors. The coefficient of the negative exponential is the sum of the conjugates of these two phasors. But in the expression for a sinusoid in terms of exponentials, the coefficient of the negative exponential should be the <u>conjugate</u> of the coefficient of the positive exponential. $\left(\bar{I}\,e^{j\omega t} + \bar{I}^*\,e^{-j\omega t}\right)$

this is the conjugate of this.

In the preceding frame, then, we would want $\bar{I}_1^* + \bar{I}_2^*$ to be the conjugate of $\bar{I}_1 + \bar{I}_2$; i.e., $I_1 + I_2 = (\bar{I}_1 + \bar{I}_2)^*$.

From your knowledge of complex numbers you should know that this is, in fact, the case. (<u>The conjugate of the sum of two complex numbers is the sum of the conjugates of each number.</u>) Demonstrate this general theorem for the following numerical values.

$$\bar{I}_1 = 2 + j6$$
$$\bar{I}_2 = 13 - j20$$
$$\bar{I}_1^* + \bar{I}_2^* = \underline{\qquad\qquad\qquad}$$
$$(\bar{I}_1 + \bar{I}_2)^* = \underline{\qquad\qquad\qquad}$$

86

Answer:

$$\bar{I}_1^* + \bar{I}_2^* = (2-j6) + (13+j20) = 15 + j14$$

$$(\bar{I}_1 + \bar{I}_2)^* = (2+j6+13-j20)^* = (15-j14)^* = 15+j14.$$

Returning now to the development, we found the sum of the two sinusoids to be

$$i_1 + i_2 = \frac{\sqrt{2}}{2}\left[(\bar{I}_1+\bar{I}_2)e^{j\omega t} + (\bar{I}_1{}^*+\bar{I}_2{}^*)e^{-j\omega t}\right]$$

Using what has just been demonstrated, and letting $\bar{I} = \bar{I}_1 + \bar{I}_2$, the sum of the two sinusoids can be written

$$i = i_1 + i_2 = \frac{\sqrt{2}}{2}(\bar{I}e^{j\omega t} + \bar{I}^*e^{-j\omega t})$$

Once the sum of the two phasors $\bar{I} = \bar{I}_1 + \bar{I}_2$ is obtained the corresponding sinusoid can be easily written.

Although it may look like we have performed an awful lot of complicated manipulations, the actual process is very simple. To review, to find the sum of two sinusoids of the same frequency, we first find the ________ corresponding to each one. We then ________ the phasors to get a single phasor. The sinusoid corresponding to this phasor, which can be written immediately, is the sum of the two sinusoids.

88

Answer:

You probably do not need confirmation of what you wrote.

Let's carry out this proces with a few numerical examples. Find as a sinusoid the sum of

$$i_1 = 4\sqrt{2}\cos(\omega t - \pi/3)$$

$$i_2 = 6\sqrt{2}\cos(\omega t + \pi/4)$$

phasors

$\bar{I}_1$ = ______________

$\bar{I}_2$ = ______________

$\bar{I}_1 + \bar{I}_2$ = ______________

$i = i_1 + i_2$ = ______________

Answer:

$\overline{I}_1 = \underline{4e^{-j\pi/3}}$

$\overline{I}_2 = \underline{6e^{j\pi/4}}$

$$\overline{I}_1+\overline{I}_2 = (4e^{-j\pi/3}+6e^{j\pi/4})$$
$$= 4(\cos\frac{\pi}{3}-j\sin\frac{\pi}{3}) + 6(\cos\frac{\pi}{4}+j\sin\frac{\pi}{4})$$
$$= \underline{6.24 + j.78} = \underline{6.28e^{j7.1^\circ}}$$

$$i = i_1 + i_2 = 6.28\sqrt{2}\cos(\omega t+7.1^\circ)$$

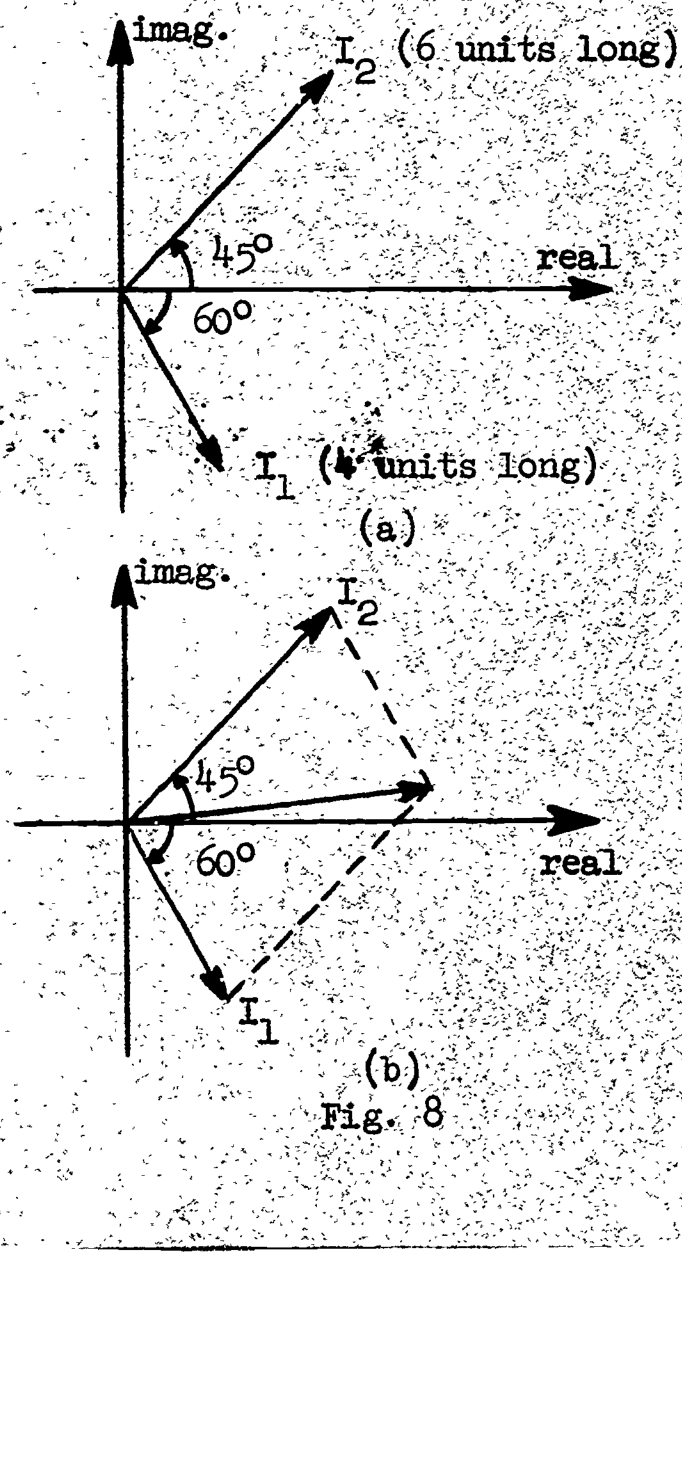


Fig. 8

Since phasors are complex numbers, all means by which complex numbers can be added and subtracted can be used to add and subtract phasors. One of these methods is a graphical one. As we have observed, a phasor can be sketched on a set of axes as a line of appropriate length and making an appropriate angle with the horizontal axis.

In the example under discussion, each of the phasors can be drawn as shown in Fig. 8a. Their graphical sum can then found by the usual parallelogram rule, as shown in Fig. 8b.

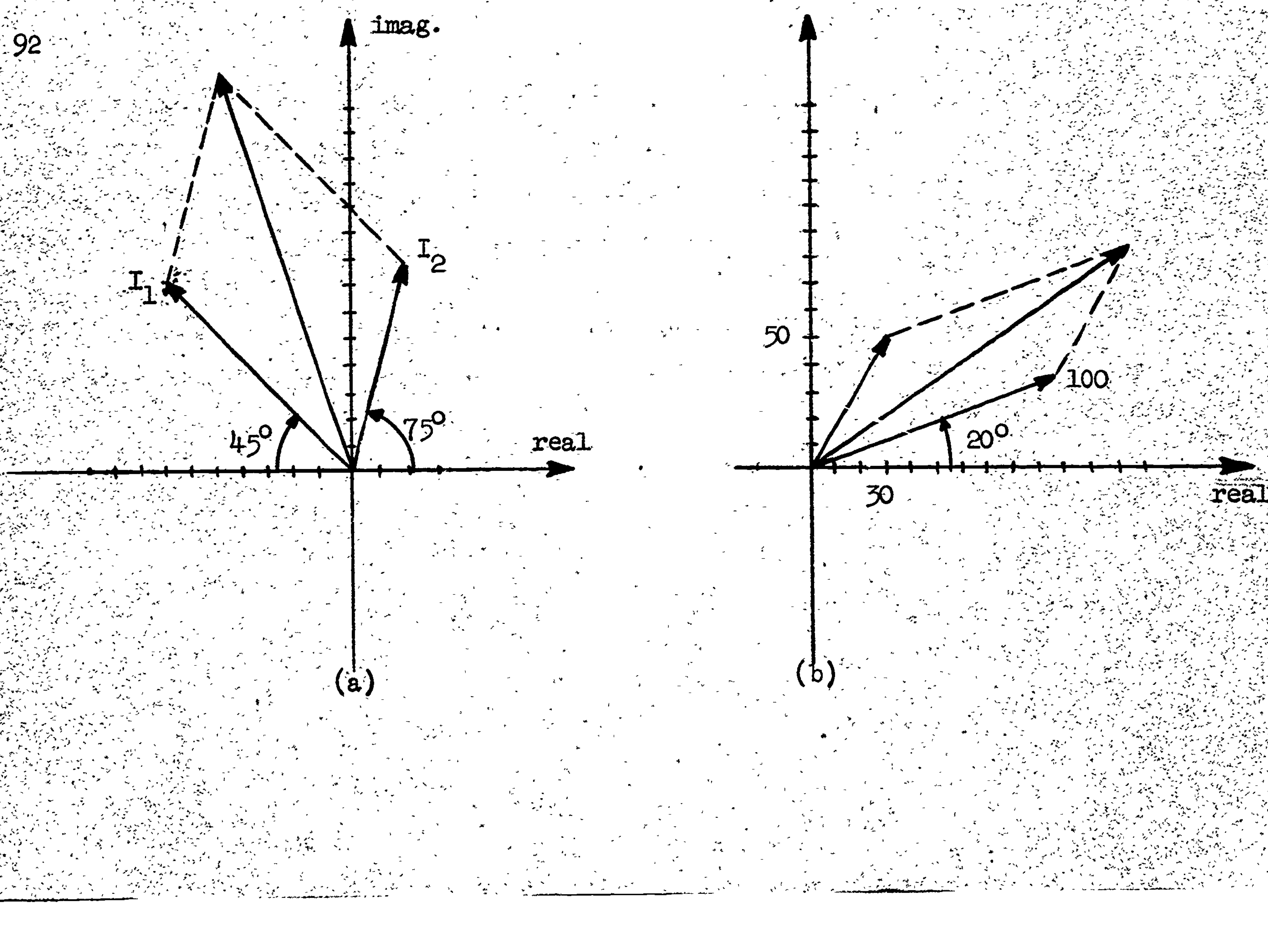
Find the sum of the following pairs of phasors graphically. Draw the phasor to scale.

a) $\bar{I}_1 = 10e^{j135^\circ}$

$\bar{I}_2 = 8e^{j75^\circ}$

b) $\bar{V}_1 = 100e^{j20}$

$\bar{V}_2 = 30 + j50$


imag.
I_1
I_2
45°
75°
real
(a)
50
100
20°
30
real
(b)

As another example, find as a sinusoid the sum of

$$v_1 = 8\cos(\omega t+\pi/6)$$

$$v_2 = 10\sin(\omega t+40^\circ)$$

Also sketch approximately to scale the phasors and their sum. (This was done by direct addition starting on page 47.)

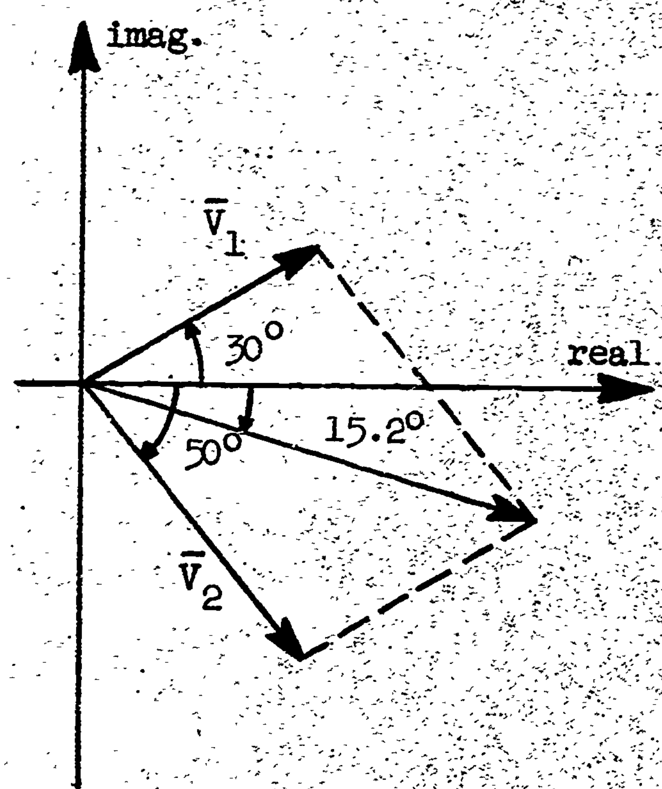
Answer:

$$\bar{V}_1 = \frac{8}{\sqrt{2}}e^{j\pi/6} = \frac{1}{\sqrt{2}}(6.93+j4)$$

$$\bar{V}_2 = \frac{10}{\sqrt{2}}e^{j(40^\circ-90^\circ)} = \frac{10}{\sqrt{2}}e^{-j50^\circ} = \frac{1}{\sqrt{2}}(6.42-j7.65)$$

$$\bar{V}_1 + \bar{V}_2 = \frac{1}{\sqrt{2}}(13.4-j3.65) = \frac{13.9}{\sqrt{2}}e^{-j15.2^\circ}$$

$$v = v_1 + v_2 = 13.9\cos(\omega t-15.2^\circ)$$



(Compare with the previous result on page 50.)

This completes the discussion of phasors.

Summary:

In this section you learned how to express a sinusoidal function in terms of exponentials with imaginary exponents. Thus, $v(t) = 20\sqrt{2}\cos(\omega t - 40^\circ)$ can be written as

$$v(t) = \frac{\sqrt{2}}{2}(\bar{V} e^{j\omega t} + \bar{V}^* e^{-j\omega t})$$

where $\bar{V}$ is called a ______________. It is a complex number whose numerical value, in polar and rectangular form, for the given sinusoid is:

$\bar{V}$ = ______________________ (polar form)

= ______________________ (rectangular form)

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Answer:

phasor

$\overline{V} = \underline{20e^{-j40^\circ}}$

$= \underline{20\cos 40^\circ - j20\sin 40^\circ}$ or $\underline{15.3 - j12.8}$

Conversely, if the numerical value of a phasor is known, the corresponding sinusoidal function can be immediately written. Thus, if

$$\bar{I} = 5e^{j55^\circ}$$

the corresponding **sinusoid** is

$$i(t) = \underline{\hspace{4cm}}$$

The **magnitude** of the **phasor equals** the ________________ of the sinusoid; the angle of the phasor equals the ________________ of the sinusoid.

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Answer:

$$i(t) = \underline{5\sqrt{2}\cos(\omega t+55^{\circ})}$$

rms value

initial angle

Describe a procedure using phasors which can be used for finding the sum of two sinusoids having the same frequency.

Given two sinusoids $v_1(t)$ and $v_2(t)$ having the same frequency, ________________
____
____
____
____

<u>Answer</u>:

Your answer should contain each of the steps below. If it does not, turn back and complete your answer.

(1) <u>first find the phasor corresponding to each sinusoid</u>;

(2) <u>then add the phasors to get a single phasor</u>;

(3) <u>then write the sinusoid corresponding to the sum phasor</u>.

(<u>The addition of the phasors can be performed graphically</u>.)