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THE EFFECTS OF SELECTED EXPERIENCES ON THE ABILITY OF KINDERGARTEN AND FIRST-GRADE CHILDREN TO CONSERVE NUMEROUSNESS.

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THE EFFECTS OF SELECTED EXPERIENCES ON THE ABILITY OF KINDERGARTEN AND FIRST-GRADE CHILDREN TO CONSERVE NUMEROUSNESS

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Report from the Mathematics Concept Learning Project

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PREFACE

Contributing to an understanding of cognitive learning by children and youth—and improving related educational practices—is the goal of the Wisconsin R & D Center. Activities of the Center stem from three research and development programs, one of which, Processes and Programs of Instruction, is directed toward the development of instructional programs based on research on teaching and learning and on the evaluation of concepts in subject fields. Since the inception of the Center in 1964, Professor Van Engen and his staff have been concurrently developing and testing "Patterns in Arithmetic," a televised instructional program for Grades 1—6, and conducting related research in children's learning of mathematical concepts.

In this Technical Report is described a study of instruction designed to enhance children's ability to conserve numerousness and the validation of a test of numerousness. Though neither of the authors are currently affiliated with the Center, during 1966-1967 Professor Harper held a USOE Postdoctoral Fellowship here and Professor Steffe was a research associate.

Herbert J. Klausmeier
Director

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ABSTRACT

This study was designed to test the effects of a sequence of twelve lessons on the ability of kindergarten and first-grade children to recognize and conserve numerousness. Two pretests were administered to the children in each grade level, the Lorge-Thorndike Intelligence Test (nonverbal) and a test of numerousness. One posttest was administered to each grade level, the test of numerousness. A separate analysis of covariance was performed at each grade level where the covariates were the scores from the two pretests and the dependent measure was the score obtained from the posttest of numerousness. Significant differences were observed between the adjusted means of the experimental and control groups at the kindergarten level in favor of the experimental group, even though both groups had gained. High mean scores on the test of numerousness were present for both the experimental and control groups at the first-grade level. At this time it is not possible to determine whether the test is not appropriate for first graders or whether the sample involved was unusual.

The sequence of lessons involved the following concepts: 1) one-to-one correspondence, 2) perceptual rearrangement, 3) as many as, 4) more than, 5) fewer than, 6) additions, and 7) subtractions. It was concluded that, for the kindergartners, the lessons were successful in enhancing the children's ability to conserve numerousness.

I
INTRODUCTION

The writings of Jean Piaget have variously been attacked and eulogized in recent years. His concern for conservation as it relates to mathematical experience has been the subject of many research projects. The present study is another in the broad area of children's ability to recognize numerical properties of sets of objects.

With the advent of "modern mathematics" programs for elementary school children have come many new experimental arithmetic series and curriculums. Parents and teachers have expressed concern for the mounting load of subject matter elementary children are responsible for mastering. This study is not another attempt to further burden the children, but rather an attempt to relieve this burden as it relates to their experiences in mathematics. The investigators are not trying to present something "new" (for as one looks at the history of teaching elementary mathematics this would be most presumptuous) [15] but rather to test an idea in a carefully controlled experiment in a fairly "normal" Midwest community.

The first portion of this paper will consist of three parts: Mathematical Background, Psychological Background, and Related Studies. A fairly extensive treatment of the cardinal and ordinal numbers will be given in the section on mathematical background to provide a sound theoretical backdrop for the psychological theory of Jean Piaget. His theory is discussed in part in the section on psychological background. Some of the validation studies of Piaget's theory will be reviewed in the section on related studies. Particular reference will be given to testing instruments. Also, studies involving attempts to accelerate the acquisition of concrete operational thinking in young children will be discussed as they relate to the lessons employed in this study.

MATHEMATICAL BACKGROUND

Cardinal and Ordinal Numbers

Two sets, A and B, are called equivalent if there exists a one-to-one correspondence between their elements. The symbol "$\sim$" will be used to denote that A is equivalent to B. An object $\underline{a}$, called a cardinal number, may be associated with each set A such that if $A \sim B$, then $\underline{a} = \underline{b}$. If $A \sim B_1$ where $B_1 \subset B$, then $\underline{a} \leq \underline{b}$. Strict inequality occurs in the case where there does not exist an A_1 such that $A_1 \subset A$ and $A_1 \sim B$.

A binary order relation "α" may be defined on a set A such that the relation is (1) nonreflexive, (2) asymmetric, and (3) transitive. (The symbol α may be read "is followed by.") That is, (1) for all $a \in A$, it is not true that $a \alpha a$; (2) for all $a, b \in A$, if $a \alpha b$, then it is not true that $b \alpha a$; and (3) if $a \alpha b$ and $b \alpha c$, then $a \alpha c$ [19, p. 49]. Any set A on which such a relation is defined is called an ordered set. Two ordered sets are called similar (1) if they are equivalent and (2) if the one-to-one correspondence preserves order [19, p. 51]. That is, if $a_1, a_2 \in A$ correspond to $b_1, b_2 \in B$, and if $a_1 \alpha a_2$, then $b_1 \alpha' b_2$, where α and α' are the orderings on A and B respectively.

If α is an order relation defined on a set A, then α is said to well order A if every subset A_1 of A possesses the property that there exists an element $a_1 \in A_1$ such that $a_1 \alpha x$ for all $x \in A_1$ (a_1 is called the first element of A_1). If two well-ordered sets are similar, then they possess the same ordinal number. Two sets with the same ordinal number, then, necessarily possess the same cardinal number.

Every element $\underline{a}$ of a well-ordered set A determines a segment P, where $P = \{x \in A: x \alpha \underline{a}\}$. If $Q = \{y \in A: y \notin P\}$, then $A = P \cup Q$.

In this case, $\underline{a}$ is the first element of Q [19, p. 68]. A result of this definition is that a well-ordered set is never similar to one of its segments [19, p. 69].

For two ordinal numbers γ and β, $\gamma < \beta$ means that A is similar to a segment of B, where γ corresponds to A and β to B. If $O(\gamma) = \{\text{ordinal numbers } \beta: \beta < \gamma\}$, then it is true that $O(\gamma) = \{0, 1, 2, \ldots, \sigma, \ldots\}$ where $\sigma < \gamma$ [19, p. 70]. Moreover, if A is any well-ordered set, then $A = \{a_0, a_1, a_2, \ldots, a_\sigma, \ldots\}$ where $\sigma < \gamma$, γ being the ordinality of A, and where the index of every element is just the ordinal number of the segment belonging to it.

In summary, the following points have been covered:

1. Two sets with the same ordinal number have the same cardinal number.
2. A well-ordered set is never similar to one of its segments.
3. If γ is an ordinal number, then $O(\gamma) = \{0, 1, 2, \ldots, \sigma, \ldots\}$ where $\sigma < \gamma$.
4. If A is a well-ordered set, then $A = \{a_0, a_1, \ldots, a_\sigma, \ldots\}$, where $\sigma < \gamma$ and γ is the ordinality of A.

Finite Sets

Set A is a finite set if it is not equivalent to one of its proper subsets. Since any set can be well-ordered [19, p. 65], any finite set can be well-ordered. It will now be shown that if α and β are ordinal numbers which correspond to a finite set A, then $\alpha = \beta$. Exactly one of the following is true: (1) $\alpha = \beta$, (2) $\alpha < \beta$, or (3) $\beta < \alpha$. If $\alpha < \beta$, then A is similar to a segment of B, where B is nothing more than A ordered in a different way. But then A would be equivalent to one of its proper subsets, which contradicts the definition of a finite set. If $\beta < \alpha$, the same result holds. It must be true, then, that $\alpha = \beta$.

Let α be the ordinal number corresponding to the well-ordered finite set A. Since $O(\alpha) = \{0, 1, 2, \ldots, \sigma, \ldots\}$ where $\sigma < \alpha$ and where each σ corresponds to a segment of A, the elements of A can be indexed by the elements of $O(\alpha)$ in such a way that $A = \{a_0, a_1, a_2, \ldots, a_\sigma, \ldots\}$, where the index of each element is the ordinal number of the segment belonging to it. If $O(\alpha)$ does not contain a terminating element, then neither does A. But then A would be equivalent to a proper subset of itself. Hence, $O(\alpha)$ has a last element and so does A. Since each number of $O(\alpha)$ has an immediate predecessor, $O(\alpha) = \{0, 1, 2, \ldots, \alpha-1\}$, and $A = \{a_0, a_1, a_2, \ldots, a_{\alpha-2}, a_{\alpha-1}\}$, which is the usual notation for finite sets. Any ordering of a finite set, then, is a well-ordering. It is impossible to distinguish the orderings with reference to the ordinal number of the set, since all well-orderings give the same ordinal number. Since the ordinal and cardinal numbers correspond, it is possible to find the cardinal number of a finite set by a process of counting, that is, by indexing the elements of the set A by the set of ordinal numbers $\{0, 1, 2, \ldots, \alpha - 1\}$ by virtue of successive selection of single elements. That is, select some A_0, then some A_1, etc., until the last one, $A_{\alpha-1}$, is selected. Then α will be the cardinal number of the set. This process is often referred to as rational counting.

The notion of equivalence classes of finite sets is implicit in the above discussion, since "$\sim$" is an equivalence relation. The set $\{0, 1, 2, \ldots, \alpha - 1\}$ of cardinality α can be considered as the standard set of an equivalence class of sets each of cardinality α [32, p. 28].

PSYCHOLOGICAL BACKGROUND

When one is asked the question "How many objects are there in this set?" the only way of establishing the number, assuming it is large and finite, is by a counting process. Sophisticated counting processes are available, but they all reduce to the counting process just discussed. It is known that young children have difficulty with this counting process. They may attach the verbal labels "one," "two," and so on to objects of the set with no awareness of the segments of the set; this is evidenced by the fact that children often count in a circular fashion. Many children may obtain the correct cardinal number of a set by the counting process but still have little notion of what they have done [23, p. 64]. In fact, many educators have found that preschool children have more difficulty counting sets of concrete objects than they do just repeating the number names in a rote fashion. Indeed, one could take the position that in order for a child to master the counting process, he must first master the notion of the cardinality of a set and the notion of one-to-one correspondence. This position is not without support, for many contemporary arithmetic curriculums do proceed on these assumptions. Moreover, when one looks at the counting process from a mathematical standpoint, he also must come to these conclusions; for the notion of a similarity mapping is based in part on the notion of equivalence and hence cardinality.

When asked to make a comparison between two sets, a child may do so in one of several or only the two mentioned ways: e.g., (1) he may set up a one-to-one correspondence between the objects of the sets and base his comparison on this correspondence, or (2) he may count the objects in each set and base his comparison on the counting process. If the two sets are A and B, he indexes set A with the counting numbers 1 through the cardinality of A, then indexes set B with 1 through the cardinality of B. It is then necessary for him to make a similarity mapping between the two counting sets. This may be accomplished, in the case that $|A| > |B|$,[1] by realizing that to proceed to the ordinal number of A, he had to pass the ordinal number of B, so that $|A| > |B|$. It is important to observe that the notion of equivalence is implicit in this process.

In Case 1 above, the child must realize that no matter how the correspondence is established, his conclusion will not change as different comparisons are made. There are $\frac{a!}{(a-b)!}$ different ways of setting up correspondences between A and B, where $|B| < |A|$.

In Case 2 above, the child must realize that no matter how he counts a set, the cardinal number of that set will always be the same. For a set A, there are $a!$ different ways of permuting the order of the elements, each of which leads to the same cardinal number.

Jean Piaget has shown that young children may have great difficulty in maintaining the equality of the cardinal numbers of two sets when the correspondence has been changed for the child by an alteration of the elements of one of the two sets or an alteration of the order. A discussion of Piaget's developmental theory and results of experimentation follows.

Concrete Operations

In The Child's Conception of Number, Piaget has two main goals: 1) to demonstrate stages in the development of particular concepts and 2) to demonstrate the development of a conceptualizing ability that underlies the formation of any particular concept. Piaget lists four main stages in the development of this conceptualizing ability: 1) Sensory-motor, preverbal stage; 2) preoperational representation; 3) concrete operations; 4) formal operations [26, pp. 9-10]. With reference to the first goal, there are three main stages: 1) Absence of conservation; 2) intermediary reactions; 3) necessary conservation [23, pp. 5-13]. These three stages are spanned by the two stages of preoperational representation and concrete operations.

Concrete operations are a part of the cognitive structure of children from about 7-8 years of age to 11-12 years of age [25, p. 123]. Piaget postulates that this cognitive structure has the form of what he calls groupings. He cites the following five properties of his groupings [25, pp. 40-42].

1. Combinativity: $x + x' = y$; $y + y' = z$
2. Reversibility: $y - x = x'$ or $y - x' = x$
3. Associativity: $(x + x') + y' = x + (x' + y')$
4. General Operation of Identity: $X - X = 0$
5. Special Identities: $X + X = X$

Eight major groupings are identified, each of which is supposed to satisfy the above five properties [25, pp. 42-47]. In the following discussion only three of these groupings will be considered. The idea of an operation is central to these groupings. According to Piaget,

> ...an operation is an interiorized action ...in addition, it is a reversible action; that is, it can take place in both directions....
>
> Above all, an operation is never isolated. It is always linked to other operations, and as a result, it is always a part of a total structure ....
>
> To understand the development of knowledge we must start with an idea which seems central to me ... the idea of an operation. Knowledge is not a copy of reality. To know an object, to know an event, is not simply to look at it and make a mental copy or image of it. To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed. An operation is thus the essence of knowledge; it is an interiorized action which modifies the object of knowledge. For instance, an operation would consist of joining objects in a class .... In other words, it is a set of actions modifying the object, and enabling the knower to get at the structure of the transformation [26, pp. 9f].

According to this quotation, the groupings are the structures of which the operations are a part. As stated by Flavell, "Intellectual development is an organization process, and what are organized are active, intellectual operations; their organization into systems with definable structure is the sine qua non for 'good' cognition ... [17, p. 168]." The differences in the groupings reside in the various operations which are to be organized. The elements of the groupings are what Piaget refers to as "classes" and "asymmetrical

[1] $|A|$ means the cardinality of A.

relations." These correspond to the cognitive operations of combining individuals in classes and assembling the asymmetrical relations which express differences in the individuals [23, pp. 43f].

In The Psychology of Intelligence, Piaget apparently selects special classes for part of his elements in the first grouping [25, p. 43]. These classes must satisfy the following pattern: $\phi \subset A_1 \subset A_2 \subset \ldots \subset \bigcup_{\sigma \varepsilon A} A_\sigma$, where $\sigma \varepsilon A$ and A is the index set. If "$\subset$" is interpreted to mean "$\subseteq$", then the above sets constitute a lattice, which is a partially ordered system in which any two elements have a greatest lower bound and a least upper bound [7, p. 352]. Clearly, "$\subset$" is a partial ordering of the sets in question since it is (1) reflexive, (2) anti-symmetric, and (3) transitive. Moreover, for any two elements A_γ and A_β, $A_\gamma \cap A_\beta$ is the greatest lower bound and $A_\gamma \cup A_\beta$ is the least upper bound. In the above lattice, the following laws of classes hold [7, p. 352]. If X, Y, and Z are any elements of the lattice, then:

(1) $X \cup X = X$: Idempotent Law
(2) $X \cup Y = Y \cup X$: Commutative Law
(3) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$: Associative Law
(4) If $X \subset Y$, then $X \cup Y = Y$: Resorption Law

This lattice structure is not all that Piaget wants for his first grouping. He apparently also wants classes of the form $A_\sigma' = A_\gamma - A_\sigma$ where $A_\sigma \subset A_\gamma$. The notation A_i' is used in the case of $A_{i+1} - A_i$. The classes A_σ' included along with the elements of the lattice apparently are the elements of this first grouping. If one interprets Piaget's "+" to be "$\cup$", then he gives (embedded in a zoological classification) statements analogous to the following [25, p. 43]:

(1) Combinativity: $A_\sigma \cup A_\sigma' = A_\gamma$
(2) Reversibility: If $A_\sigma \cup A_\sigma' = A_\gamma$, then $A_\sigma = A_\gamma - A_\sigma'$.
(3) Associativity: $(A_\sigma \cup A_\sigma') \cup A_\gamma' =$. $A_\sigma \cup (A_\sigma' \cup A_\gamma')$.
(4) General Operation of Identity: $A_\sigma \cup \phi = A_\sigma$.
(5) Special Identities: (a) $A_\sigma \cup A_\sigma = A_\sigma$, (b) $A_\sigma \cup A_\gamma = A_\gamma$ where $A_\sigma \subset A_\gamma$.

If all the sets are included that Piaget would want included, then Property 1 does not hold for any two sets. Take some A_σ, and consider some A_γ' such that $A_\sigma \subset A_\gamma$ but $A_\gamma' = A_\beta - A_\gamma$. There then is no guarantee that $A_\sigma \cup A_\gamma'$ is in the system. One could, of course, include enough sets to remove these restrictions. It is not clear, however, that Piaget would want these sets to be included. This is really of no special concern, since it is just a matter of correct interpretation. The interpretation given above for this grouping is not the only one Piaget himself gives. Flavell states,

> Although Piaget does clearly state that the mathematical elements of the groupings are class and relation equations [$A_i + A_i' = A_{i+1}$] rather than related classes and relations ..., it is often possible (and usually more convenient) to treat classes and relations as the grouping elements, provided the special rules just alluded to are invoked. Piaget himself oscillates between the equation level and the single-term level in his descriptions of grouping properties [17, p. 176].

Even if the equation form is taken, however, difficulties of a "good" mathematical interpretation still exist.

Though the two remaining groupings only receive cursory discussion here, interested readers should refer to a detailed account in Flavell [17, pp. 164-187].

The second grouping to be discussed is commonly known as the addition of asymmetrical relations. The asymmetrical relations referred to will be interpreted to mean a strict partial ordering. That is, an ordering that is (1) transitive, (2) asymmetric, and (3) non-reflexive. Moreover, it will be assumed that the relation is linear. That is, if $\underline{x}$ and $\underline{y}$ are elements of the set A on which the relation is defined, then either $\underline{x}\,\alpha\,\underline{y}$ or $\underline{y}\,\alpha\,\underline{x}$. Employing this order relation, A satisfies the necessary condition for a chain [22, p. 29], which implies that the ordered set A is a lattice.

Piaget apparently uses the transitive property to define some of the compositions of "two" relations. That is, if $\underline{a}\,\alpha\,\underline{b}$ and $\underline{b}\,\alpha\,\underline{c}$, then $\underline{a}\,\alpha\,\underline{c}$. This composition is associative and has special identities by virtue of the transitive law.

The resorption property needs special attention as do the reversibility and general identity properties. The resorption property takes the form of the following: If $\underline{a}\,\alpha\,\underline{b}$ and $\underline{a}\,\alpha\,\underline{c}$, then $\underline{a}\,\alpha\,\underline{c}$. Logically this is certainly a valid conclusion. The general identity and reversibility properties are contained in the statement, If $\underline{a}\,\alpha\,\underline{b}$ and $\underline{b}\,\not\alpha\,\underline{a}$, then $\underline{a} = \underline{a}$. $\underline{b}\,\not\alpha\,\underline{a}$ is called the reciprocal (inverse) of $\underline{a}\,\alpha\,\underline{b}$. They are defined to mean the same thing. In a sense, a quasi-transitive law has been identified along with an equivalence relation "=" which is distinct from "α."

Piaget calls the third grouping the bi-univocal multiplication of relations. Apparently, one has to have two or more orderings of the type discussed above defined on a set A. For simplicity, these orderings will be taken to well-order the set. In the matrix table below, $\underline{a}_i\,\alpha_1\,\underline{a}_{i+1}$ and $\underline{a}_i'\,\alpha_2\,\underline{a}_{i+1}'$, and all $\underline{a}_i \in A$ as well as all $\underline{a}_i'$.

	$\underline{a}_1$	$\underline{a}_2$	$\underline{a}_3$	...
$\underline{a}_1'$	$\underline{a}_1\underline{a}_1'$	$\underline{a}_2\underline{a}_1'$	$\underline{a}_3\underline{a}_1'$	...
$\underline{a}_2'$	$\underline{a}_1\underline{a}_2'$	$\underline{a}_2\underline{a}_2'$	$\underline{a}_3\underline{a}_2'$	...
$\underline{a}_3'$	$\underline{a}_1\underline{a}_3'$	$\underline{a}_2\underline{a}_3'$	$\underline{a}_3\underline{a}_3'$	...
⋮	⋮	⋮	⋮	

This table serves as a "composition" table for the two relations. By horizontal movements along the rows and vertical movements along the columns, one can move around in the matrix. To give an interpretation, consider a cylindrical container in which the dimensions may vary. Imagine this container partially filled with beads. Let α_1 be the relation between the height of the beads at two different observations of the container and let α_2 be the relation between the cross sections of the beads at the same observations. Then the pair $\underline{a}_i\underline{a}_j$ represents the dimensional state in which the beads are in at any instant. At this instant, the beads may have the quality of being both higher and having a smaller cross section than the beads at a later instant represented by the pair $\underline{a}_{i+\delta}\,\underline{a}_{j+\gamma}$, where $\delta \geq 0$ and $\gamma \geq 0$. This is an interpretation of what Piaget means by the multiplication of relations when he states, "By multiplication of these relations; we mean their seriation from two or more points of view simultaneously [23, p. 11]." Schematically, we could write $(\underline{a}_i\underline{a}_j'\alpha_1\,\underline{a}_{i+\gamma}\,\underline{a}_j')$ and $(\underline{a}_{i+\delta}\,\underline{a}_j\,\alpha_2\,\underline{a}_{i+\delta}\,\underline{a}'_{j+\gamma})$ which implies $(\underline{a}_i\underline{a}_j')[\alpha_1,\alpha_2]\underline{a}_{i+\delta}\,\underline{a}'_{j+\gamma})$, where $[\alpha_1,\alpha_2]$ is to be interpreted to mean $a_i a_j$ is in both relations to $\underline{a}_{i+\delta}\,\underline{a}_{i+\gamma}$.

In the foregoing discussions of groupings, it is important to be aware of the fact that if they are to be models of cognitive operations, then necessarily a grouping's operation must correspond to some cognitive operation. Piaget calls the cognitive operations discussed concrete operations because "they operate on objects, and not yet on verbally expressed hypotheses [6, p. 9]." These groupings do not exhaust all of those which Piaget postulates. Of these groupings, he states,

> Psychologically, a "grouping" consists of a certain form of equilibrium of operations, i.e., of actions which are internalized and organized in complex structures [25, p. 36].
>
> The psychological existence of a grouping can in fact be easily recognized from the overt operations of which a subject is capable. But that is not all; without the grouping, there could be no conservation of ... wholes, whereas the appearance of a grouping is attested by the appearance of a principle of conservation. For example, the subject who is capable of reasoning operationally in accordance with the structure of groupings will know in advance that a whole will be conserved independently of the arrangements of its parts ... [25, p. 42].

He also makes explicit the relationship, as he sees it, of the first two groupings discussed in this paper and the additive group of integers.

> Further, we should note that the best proof of the natural character of the totalities constituted by these groupings of operations is that it is only necessary to fuse together the groupings formed by simple combinations of classes and those formed by seriations in order to obtain what is no longer a qualitative grouping but the "group" constituted by the series of positive and negative whole numbers. In fact, to combine individuals in classes means considering them as equivalent, while serializing them according to an asymmetrical relation expresses their differences. Now, when we consider the qualities of objects we cannot simultaneously group them as both equivalent and different. But, if we abstract qualities, by the very fact we render them equivalent to each other and capable of being serialized according to some form of enumeration, we thus transform them into ordered "units," and the additive operation which constitutes a whole number is just that [25, p. 46].

It may be helpful to refer to the section on mathematical background for clarification of this quotation. There, it was observed that the ordinality and cardinality of any finite set are indistinguishable and the ordinal numbers implicitly involve a similarity mapping. This may be what Piaget is referring to when he states that "number is neither merely a uniting class nor merely a seriating relation, but both a hierarchical class and a series [23, p. 156]." However, whether these two groupings give rise to the positive and negative whole numbers is questionable, since this group requires inverse elements. With reference to this point Flavell reminds us, "If Piaget has something valuable to say which we somehow fail to hear, it is our loss, and the loss is absolute [17, p. 406]."

As has been pointed out, the notion of cardinal number is based on the notion of one-to-one correspondence and the notion of ordinal number is based on the notion of a similarity mapping. Piaget discusses quite extensively the relationship between his groupings and these two basic concepts. Moreover, he emphasizes the types of quantitative comparisons of which children are capable and the relationship of these to the groupings and to one-to-one correspondence. The remainder of this section will summarize these discussions.

Quantitative Comparisons

Piaget identifies three types of quantitative comparisons which are observable in children. He defines these in terms of quantity itself. First, gross quantity is

> no more than the asymmetrical relations between qualities, i.e., comparisons of the type "more" or "less" contained in judgments such as "it's higher," "not so wide," etc. These relations depend on perception, and are not as yet relations in the true sense, since they cannot be co-ordinated one with another [23, p. 5].

This would seem to indicate to Piaget the absence (for children) of the third grouping of concrete operations discussed above. The second type, intensive quantification, may be described in the following manner: given a set A such that $A_1 \cup A_2 = A$, it is known that $A_1 \subset A$, $A_2 \subset A$ but no information is available about A_1 relative to A_2 in terms of set inclusion. In terms of asymmetrical relations, one may know $\underline{a}_1 \alpha \underline{a}_3$ and $\underline{a}_2 \alpha \underline{a}_3$ without knowing anything at all about $\underline{a}_1$ relative to $\underline{a}_2$ [17, p. 171]. The third type, extensive quantification, entails precise comparisons among component parts or subclasses [17, pp. 171f].

With regard to the conservation of a liquid, Piaget states,

> The main characteristic of the perceptual relationships of gross quantity used by the child at this first level is that they cannot be composed one with another either additively or multiplicatively. When the child thinks that the quantity increases because the level rises, he is disregarding the cross section, and when he takes the cross section into account he disregards the level, and so on [23, p. 11].

Piaget goes on to describe a more advanced state of reasoning,

> At this stage, the child is attempting to co-ordinate the perceptual relations involved .... the child at the second stage tries to take the two relations into account simultaneously, but without success .... it is only when the levels are equal that he attempts a logical multiplication of the relations of height and width [23, pp. 15-16].

However, the logical multiplication of relations is not enough to insure the conservation of the liquid as "intensive graduation must be completed by an extensive quantification, i.e., it must be possible to establish a true proportion ... between the gain in height and the loss in width [23, p. 16]." Multiplication of relations, then, is an intermediary between gross quantification and extensive quantification, the extensive quantification being made possible by intensive quantification.

One-to-One Correspondence

Piaget also lists the types of correspondence children are capable of making between two different sets. Qualitative correspondence is "correspondence that is based only on the qualities of corresponding elements [23, p. 70]." Numerical correspondence is correspondence "in which each element is considered as a unit, irrespective of the qualities ... [23, p. 70]." Intuitive correspondence is "correspondence that is entirely based on perception and is consequently not preserved outside the actual field of perception (or clear recollection) [23, p. 70]." Operational correspondence "is based on relationships of an intellectual nature, its distinctive characteristics therefore being the fact that it is preserved independently of actual perception, and its 'reversibility' [23, p. 70]."

Qualitative correspondence may be either intuitive or operational. A child making a correspondence between two sets based on the qualities of the elements may not be able to preserve the correspondence if the configuration of the elements is altered; in this case, the qualitative correspondence is intuitive and not operational. If the child is able to preserve the correspondence, it is an operational correspondence (the elements that were altered always have the possibility of being placed back in the original position). A numerical correspondence is essentially operational.

Piaget spells out the relationships between different types of quantitative comparisons and the different types of correspondences, i.e., "Global evaluation corresponds to 'gross quantity,' qualitative correspondence to 'intensive quantity,' and numerical correspondence to 'extensive quantity' [23, p. 90]." Operational or numerical correspondence thus results from the type of mental operations related to relations a child is capable of utilizing.

If two sets of objects are placed in rows in front of a child capable of qualitative correspondence (and hence of intensive quantification) and one of two sets is altered, then a proper judgment could arise in the case of

(1) equal length and equal density of the two sets;
(2) greater length and greater density of one of the sets;
(3) equal length and greater or less density, or greater or less length and equal density, of one of the sets;

but not in the case of:

(4) greater length and smaller density, or greater density and smaller length, since he must be able to deduce the proportionality of differences [23, p. 91].

The relationship between cardinal and ordinal numbers and between similarity and one-to-one correspondence is made clear by Piaget's statement that "... the three stages in coordination of cardinal and ordinal numbers correspond to the three stages in seriation, which in turn correspond to the three stages in cardination and cardinal correspondence ... [23, p. 153.]." [2]

RELATED STUDIES

Conservation

In his first study replicating Piaget's experiments, Elkind gives the following summary:

> Eighty ... children were divided into three age groups (4, 5, 6–7) and tested on the three Types of Material for three Types of Quantity in a systematic replication of Piaget's investigation of the development of quantitative thinking. Analysis of variance showed that success in comparing quantities varied significantly with Age, Type of Quantity, Type of Material and two of the interactions ....
>
> The results were in close agreement with Piaget's finding that success in comparing quantity developed in three, age related, hierarchically ordered stages ... [14, pp. 37–46].

The types of material Elkind used were (1) wooden sticks 1/4" square by 1 1/4"; (2) orange colored water, a tall narrow glass, a 16-ounce drinking glass, and an 8-ounce drinking glass; and (3) large wooden beads that would just fit into the tall narrow glass. The types of quantity he compared were (1) gross quantity, (2) intensive quantity, and (3) extensive quantity. Of these three types of quantity he stated,

> It is the development of <u>logical multiplication</u> (the operation of coordinating two judgments of perceived relations) which makes possible the step from gross to intensive quantification and it is logical multiplication plus the development of equation of differences (the operation of coordinating judgments of the magnitude of differences between objects) which makes possible the step from intensive to extensive quantification [14, p. 45].

Gross quantities seemed easiest to compare, intensive quantities were a little more difficult, and extensive quantities were most difficult to compare, with no difference between sticks and beads. There was a significant interaction of age groups and the quantity compared. Comparisons involving intensive quantities were quite difficult for the four-year-old group and became increasingly easier for the two older groups. The same was true for comparisons involving extensive quantities, but these comparisons remained more difficult than the comparisons involving intensive quantities.

Since Piaget defined his stages for particular concepts in terms of the type of quantitative comparisons children are capable of making, it is clear from Elkind's study that a child may be able to make extensive quantitative comparisons using materials of one kind and thereby be classified at Stage 3, but changing the type of material could affect the type of quantitative comparison the child is capable of making and thereby alter the stage classification. However, there is a definite

[2] For a full discussion of the stages referred to in this quotation, see reference 23.

statistical relationship between age group and stages as exemplified by the interaction of age groups and quantity compared and by the high and significant correlations between types of material.

Dodwell also observed variability of stages in a study involving 250 children in an age range of about five to eight years. He studied, among other things, (1) relation of perceived size to number using beakers and beads, (2) provoked correspondence using eggs and egg-cups, (3) unprovoked correspondence using red and blue poker chips [11, pp. 191-195]. In the first category above, about 25% of the children at six years two months of age showed Stage 3 responses; in the second category, about 60% of the children at six years two months showed Stage 3 responses; and in the third category, about 20% of the children showed Stage 3 responses. In this same study Dodwell observed a low but significant correlation (-.24) between IQ and Stage 3 responses, indicating that intelligence is a factor in conservation problems. His findings seem to corroborate Piaget's stages of performance.

With regard to the relation of intelligence to conservation Van Engen and Steffe (in a study involving 100 first-grade children) observed a low (.24) but significant correlation between IQ (as measured by the Kuhlmann-Anderson Intelligence Test) and the success of the children in four tasks involving concepts of conservation of number applied to problems involving addition [33].

Dodwell and Elkind have also performed replications of Piaget's experiments on the ability of children to include partial classes within a total class; i.e., if $A \cup B = C$ $(A \cap B = \phi)$, then $A \subset C$ or $B \subset C$. For his subjects, Elkind selected 25 children from each of the grades kindergarten to third. The question asked of each child was, "Are there more boys (or girls depending upon the sex of the child being questioned) or more children in your class?" Other questions were also asked to gain assurance the children understood the above question. On the basis of the responses, the children were placed in one of these three stages: Stage 1 if either $C \subset A$ or $C \subset B$ (A = boys, B = girls, and C = children), Stage 2 if $C = A$ or $C = B$, or Stage 3 if either $A \subset C$ or $B \subset C$. A χ^2 calculated on age groups by stages was significant. Fifty percent of the five-year olds, 32% of the six-year olds, 12% of the seven-year olds, and 8% of the eight-year olds were in Stage 1. Correspondingly 48, 56, 76, and 92% were in Stage 3 [13, pp. 152-159].

Dodwell investigated the responses to class-inclusion questions and responses made on the tests of provoked and unprovoked correspondence discussed earlier. His results indicated that the

> ability to answer correctly questions which involve simultaneous consideration of the whole class and its (two) component sub-classes, appears to develop to a large extent independently of understanding of the concept of cardinal numbers (as measured by the tests for provoked and unprovoked correspondence) ... [12, pp. 152-159].

With reference to the above studies, a question of immediate concern is, if a child is on a given stage at a given point in time with reference to a particular situation and particular materials, will he be on the same stage at a different point in time all other things remaining constant? Dodwell, using the tests devised in an earlier study, made a test-retest reliability study with intervals of one week and three months. He commented,

> The short term reliability of the test is highly satisfactory, and compares well with the reliabilities of commercially available cognitive tests. The long term reliability indicates considerable stability in the development of number concepts ... [11, p. 30].

Almy, Chittenden and Miller have conducted a longitudinal study concerned with the progress of 65 children through certain conservation tasks in the kindergarten, first, and second grades. They utilized two basic types of conservation: (1) conservation of number and (2) conservation of liquid. In conservation of number, two sub-types were identified, i.e., (a) conservation of number without counting and (b) conservation of number with counting. The data indicated that many children at the kindergarten and first-grade levels give evidence of conservation in Case 1 above but not in Case 2. For example, at the beginning of the first grade, 37 children showed some evidence of being able to conserve number but not liquid; only 18 children conserved liquid and number. In the spring of the first grade, the numbers were 32 and 27 respectively [3, pp. 85-110]. These results are consistent with the observations made by Elkind that conservation of liquid seems to be more difficult than the conservation of number.

A study by Kenneth Feigenbaum suggests that the number of objects in the collection may affect the child's ability to ignore his perception. Feigenbaum's population consisted of 90 children whose ages ranged from four to seven years. The children were placed in three experimental groups that received the same treatments with different materials. Treatment I involved (a) a correspondence test using 28 beads and two glasses of the same size, (b) a conservation test using the same beads and two glasses of different sizes, (c) a test for the understanding of "more" and "bigger," and (d) a correspondence and conservation test not related to our discussion here. Treatment II was the same as Treatment I except half as many beads were used. Treatment III was the same as Treatment II except the size differential of the glasses used in the conservation test was less. Feigenbaum cited evidence that the complexity of the stimuli (number of beads) affected the subjects' frequency of success in cases of incomplete assimilation of the principle of one-to-one correspondence [16, pp. 423-432].

The complexity of the stimuli was not significant in a study conducted by Van Engen and Steffe. Children received four tasks which involved five, nine, twenty-five and fifty candies. For each task, the candies were first separated into two groups of about equal number, then were pushed together to form one group. The subjects were asked, "If I let you take the candies for your friends, would you take the two piles of candy or the one pile of candy after I put them together, or would it make any difference?...Why [33, pp. 3-5]?" The tasks were constructed to model addition situations.

Uzgiris [31] replicated some of Piaget's work to verify the following stages of conservation:

1. That a child's reasoning becomes operational in mathematical and logical operations at about the age of seven;
2. That under the age of seven a child is not convinced of the constancy of substance, i.e., weight, measure, sets of objects, volume, etc., and
3. That conservation of substance is acquired at about seven years, conservation of weight at about nine years, and conservation of volume at about twelve years.

Piaget contends that "the conservation of weight always implies the conservation of substance and the conservation of volume always implies the conservation of both weight and substance [31, p. 832]." Uzgiris' findings corroborated Piaget's levels and sequences.

Carpenter [8] attempted to verify Piaget's stages of development in a study involving groups of four girls at each age level, five years old to nine years old. Each group contained one child of below average intelligence, two of average intelligence, and one of above average intelligence. The study involved five tasks which are explained below.

1. The children observed colored beads being placed in a tube and then they were asked questions about the order in which the beads would emerge (considering different rotations of the tube). The same procedure was used with a wooden tunnel and a model railroad train.
2. The children were asked questions concerning a family of brothers and sisters to determine ability to keep the numbers straight as different questions were posed.
3. Tests were devised to determine the children's ability to relate the concept of "left" and "right" to situations not involving concrete objects.
4. Tests similar to those in 3 above were used with a pencil, a rubber band, and a coin. Various arrangements and questions concerning these arrangements were utilized.
5. The final test was a replication of Piaget's conservation-of-liquid experiments involving colored water and containers of differing size.

Carpenter [8] substantiated Piaget's stages of development. She noticed, however, that achievement on the various tests occurred at an earlier age with the subjects in this study than those in Piaget's studies. The investigator also discovered that there was a higher correlation between mental age and total score (.86) than between chronological age and total score (.68) on the tests employed.

Estes [15] conducted a study designed to replicate some of Piaget's tests of conservation. She claims to have results which do not support his findings. Dodwell [11], however, conducted similar studies and found that development of conservation proceeded in much the same fashion as that indicated by Piaget, namely, from global to intuitive to concrete-operational levels.

Learning Experiences

Not many studies have been conducted to see whether conservation of numerousness can be enhanced through teaching. Piaget has not been altogether clear on his stand on this issue. At one time he indicated that teaching could, no doubt, have an important effect on a child's ability to conserve number. Piaget has stated [26, p. 1] that the development of the intellectual capacity of children depends on at least the following factors: (1) maturation, (2) encounters with experience, (3) social transmission (teaching, etc.), and (4) equilibrium [which has been described as

"...the mental activity of the subject when confronted with cognitive conflict ..." [30, p. 325].

Duckworth interprets Piaget's stand on teaching as follows:

> Good pedagogy must involve presenting the child with situations in which he himself experiments, in the broadest sense of that term—trying things out to see what happens, manipulating things, manipulating symbols, posing questions and seeking his own answers, reconciling what he finds one time with what he finds at another, comparing his findings with those of other children [26, p. 2].

Adler also emphasizes the importance of environmental stimulation:

> Piaget's critics have often complained that his emphasis on inward maturation and growth leaves no room for the effects of a stimulating environment. This view involves a partial misunderstanding of his theory, and the difficulty could be resolved easily by the realization that Piaget assumes continuous <u>interaction</u> between the child and his environment [2, p. 300].
>
> Every normal pupil is capable of sound mathematical reasoning if his own initiative is brought into play.
>
> The real cause of the failure of formal education must be sought primarily in the fact that it begins with language (accompanied by illustrations and fictitious or <u>narrated action</u>) instead of beginning with real practical action. The preparation for subsequent mathematical teaching should begin in the home by a series of manipulations involving logical and numerical relationships, the idea of length, area, etc., and this kind of practical activity should be developed and amplified in a systematic fashion throughout the whole course of primary education ... [2, p. 301].

The order in which a child progresses through the four major stages of mental growth is fixed, but his rate of progress is not fixed. The transition from one stage to the next can be hastened by enriched experience and good teaching. Carpenter quotes Piaget's expression of this idea from his book <u>The Growth of Logical Thinking from Childhood to Adolescence:</u>

> The maturation of the nervous system can do no more than determine the totality of possibilities and impossibilities at a given stage. A particular social environment remains indispensable for the realization of these possibilities. It follows that their realization can be accelerated or retarded as a function of cultural and educational conditions [24, p. 74].

There are other occasions, however, where his statements make one wonder whether he really believes that the acquisition of conservation can be effected at all through teaching.

> It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching. On the contrary, to a remarkable degree he develops them himself, independently and spontaneously. When adults try to impose mathematical concepts on a child prematurely, his learning is merely verbal; true understanding of them comes only with his mental growth [24, p. 74].

"Children go through certain stages of intellectual development from birth through adolescence. These stages materialize, fully constructed, when their time has come, and there is little we can do to advance them." [26, p. 1] The above quotation is Eleanor Duckworth's interpretation of Piaget.

Piaget is very ellusive with regard to his personal convictions on the matter of enhancing learning of conservation at early ages. However, there is no question concerning his beliefs about the importance of children being able to conserve both number and substance. He states, "...children must grasp the principle of conservation of quantity before they can develop the concept of number."

Many educators have expressed concern for the amount of pressure that is exerted on elementary children today in terms of curricular load. Some feel we should not have formal lessons in any subject at the kindergarten level; while others feel we are wasting time by not exposing even the preschool child to formal learning experiences. The investigators involved in this study are of the opinion that there may be some advantage to exposing children to conservation activities at a very early period in their school life. A study conducted during 1965 in Wisconsin indicated that first-grade children who possess conservation of numerousness perform at a significantly higher level than those who do not conserve [29, pp. 46f], from which comes the impetus to develop effective lessons which might help children acquire this skill at an early age.

Several investigators have attempted the teaching of conservation but with little success. According to Stendler, most of these experiments have failed "because experimenters have not paid attention to the processes of assimilation and accomodation in equilibrium theory [30, p. 334]."

> Piaget sees the process of equilibration as a process of balance between assimilation and accommodation in a biological sense. An individual assimilates the world—which comes down to saying he sees it in his own way. But sometimes something presents itself in such a way that he cannot assimilate it into his view of things, so he must change his view—he must accommodate if he wants to incorporate this new item [26, p. 4].

The studies which have shown any degree of success have utilized some of the following ideas which seem to be useful in attempts at acceleration:

> 1. It has been possible to accelerate the development of logical intelligence by inducing cognitive conflict in subjects ....
>
> 2. Training children to recognize that an object can belong to several different classes at once aids in the development of logical classification ....
>
> 3. There is a tendency for conservation of number to be accelerated in children trained to see that addition and subtraction of elements change numerical value ....
>
> 4. To help children move from the preoperational stage to the stage of concrete operations, it is helpful to make gradual transformations in the stimulus and to call the child's attention to the effects of a change in one dimension to a change in another [30, p. 334].

One of the studies which has produced significant positive results was conducted by Churchill [9]. The subjects were 16 five-year-old children who were matched on understanding of number concepts. The children were taught in two groups of eight. The experimental group received two sessions a week of guided play in seriation, matching, ordering, sharing, comparing, grouping, etc. They were urged to discover numerical invariance on their own. The results of the posttest indicated that the experimental group was able to perform significantly better on number questions at the "operational level" than the control group. A retest conducted three months later indicated that the experimental group was still significantly superior to the control group [9, p. 45]. Churchill's major conclusion was that children's conservation abilities may be enhanced by means of much use of concrete experiences. It is interesting to note that good teachers have long advocated the use of such materials to enhance learning. Christian Trapp promoted the use of objects in teaching arithmetic as early as the 1780's [18, p. 61].

Wallach and Sprott [34] also produced significant results in teaching conservation of number to first- and third-grade children. It is not entirely clear whether group instruction or individual instruction was utilized. The training period employed only the concept of reversals. Children matched checkers on cards or dolls in beds. The sets were then rearranged so that one row was shorter than the other. The children were questioned and then the sets were reassembled as in the first instance. The technique employed was to impress children with the idea that a set can always be returned to its original state as a check on numerousness. The pretest, posttest and post-posttest consisted of a very limited number of items. However, the results indicated that conservation might be taught if an emphasis is placed on reversals.

Wohlwill and Lowe [36] conducted a study with 72 kindergarten children divided into four subgroups of 18 each. One group of children was exposed primarily to counting activities involving equivalent sets where spatial rearrangement was effected. A second group was exposed to a similar treatment only repeated addition and subtraction of elements in the two sets was added. The third group was given much practice in noticing that the cardinal value of a set is unchanged regardless of the length of rows into which objects are placed. The fourth group was a control group receiving no treatment. The results of the study indicated no significant differences between the various experimental subgroups and the control group. The study seems to be more of a number of readiness study than a conservation study. The children were repeatedly asked to make judgments concerning the number of objects pasted on a card and the numerical symbol representing that set. Whether this is a true conservation situation is open to question since it involves the use of number names.

The other studies which have been conducted to investigate the feasibility of teaching conservation have dealt not with the conservation of number but rather with the conservation of substance and geometric figures. (See Lovell [20], Lovell and Ogilvie [21], and Smedslund [27].) Almy [3] pointed out in her report that there is little evidence in any of these studies to indicate that the transition from nonconservation to conservation can be effected by any short term manipulation.

STATEMENT OF THE PROBLEM

This study is designed to test the feasibility of teaching conservation of numerousness to kindergarten and first-grade children through a set of twelve thirty-minute lessons taught one each week.

It is the hope of the investigators that this study will provide two things: a test for locating basic strengths and weaknesses in mathematical maturity in kindergarten and first-grade children and a set of lessons which are designed to enhance a child's chances of acquiring conservation of number at an earlier age and thus improve his chances of success in mathematics.

II

DEVELOPMENT OF THE TESTING INSTRUMENT AND LESSONS

CONSERVATION TESTS

A test of conservation of numerousness was essential to the study. Such a test was to be based on the types of quantitative comparisons that children are known to make—gross quantitative comparisons, intensive quantitative comparisons, and extensive quantitative comparisons. The tests Dodwell constructed [11], which have already been discussed, are direct replicas of those used by Piaget to assess number concepts in young children. There are five different situations on which the children are tested and on which they perform quite differently. Dodwell reported that the probability of a child's making extensive quantitative comparisons in the case of unprovoked correspondence was .8, given that he had correctly answered questions relative to cardinal and ordinal numbers. If a child did not respond correctly to questions involving cardinal and ordinal numbers (was not operational), then the probability of his making extensive quantitative comparisons in the case of unprovoked correspondence was .06. Because of these probabilities, situations that can be classified as unprovoked correspondences were selected for use in this study. Dodwell's test could not be directly utilized since it involved subtests which were not concerned with the conservation of numerousness. When Dodwell reported the results of a group paper and pencil test based on his original test, he concluded,

> ... although the group test measures understanding of number and related concepts in situations apparently similar to those used in the individual test, it is arguable that it in fact measures a different aspect of the child's cognitive abilities. In the individual test one is measuring ability or understanding when the child actually perceives the transformation on the test materials; in the group test, on the other hand, the child is faced with fixed alternatives between which he has to choose, and therefore has to imagine the transformations... [10, p. 35].

A definition of the conservation of numerousness is appropriate at this point since that is what this test is designed to measure. Conservation of numerousness means that 1) irrespective of how a set of objects is rearranged, the number of objects remains the same and 2) if two sets are in one-to-one correspondence, then the number of objects in each is the same, regardless of the arrangement of the objects [29, p. 3].

The investigators' first concern was the construction of a test which could be administered to groups of children. Because of Dodwell's comment that an individual test may measure different abilities than a group test because the objects in the tests are movable and static respectively, it was decided to incorporate both types of items in the group test. The items used would have to require the quantitative comparisons children are known to make. Another requirement was that the children should manipulate any movable objects rather than watch a demonstration and make judgments relative to that demonstration. This requirement was based on a central theme in Piaget's theory that overt actions are gradually transformed into mental operations [25, pp. 120f]. Moreover, it was thought that active involvement by the child would be a safeguard against wandering minds. Another consideration was that the children should be at individual stations (unable to view others at work) arranged so that the examiner could view each child. The ground rules are summarized below:

(1) A group test was to be constructed.
(2) The items were to be based on the types of quantitative comparison which children are known to make.
(3) The test was to contain items in which the objects were movable in nature as well as static.
(4) The children should manipulate the movable objects.
(5) Each child should be at an individual station.

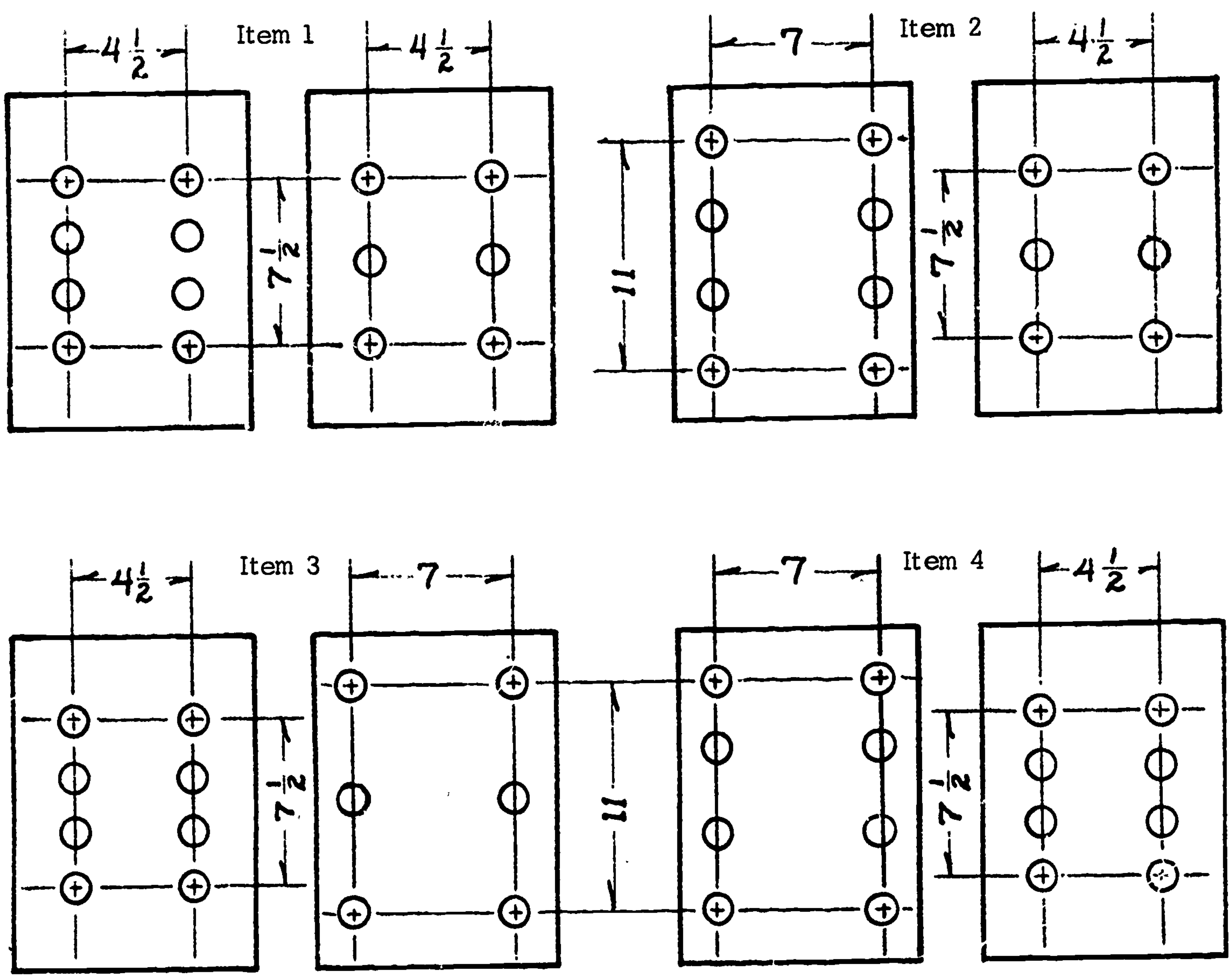
An individual test of conservation of numerousness already existed which served as a point of departure [29, pp. 12f]. The items of this test involved only static objects. Diagrammatic pictures of the items of the test are shown in Figures 1-3.

Five of the twelve items were retained and modified into pictorial form. They were Item 4 of Part 1, Items 1, 3, and 4 of Part 2, and Item 4 of Part 3. The correlations between all possible pairs of these items were significant, which warranted their selection [29, p. 26].

The test of conservation of numerousness used in this study is shown in Figure 4 approximately 14.5% of original size. The directions for the test, which follow, are self-explanatory. The discs referred to were shotgun wads painted black. The first four items are warmup items to teach the children how to respond.

W-1 Look at the squares on both pages. Are there the same number of squares on both pages? Or does one page have more than the other? Show me by pointing. Don't talk out loud. If you think both pages have the same number of squares, put a finger on both pages. (Make sure the children are using both hands.) If you think one page has more squares on it, put your finger on that page. Don't take it away until I tell you. Turn your book to the pages with the bee at the top.

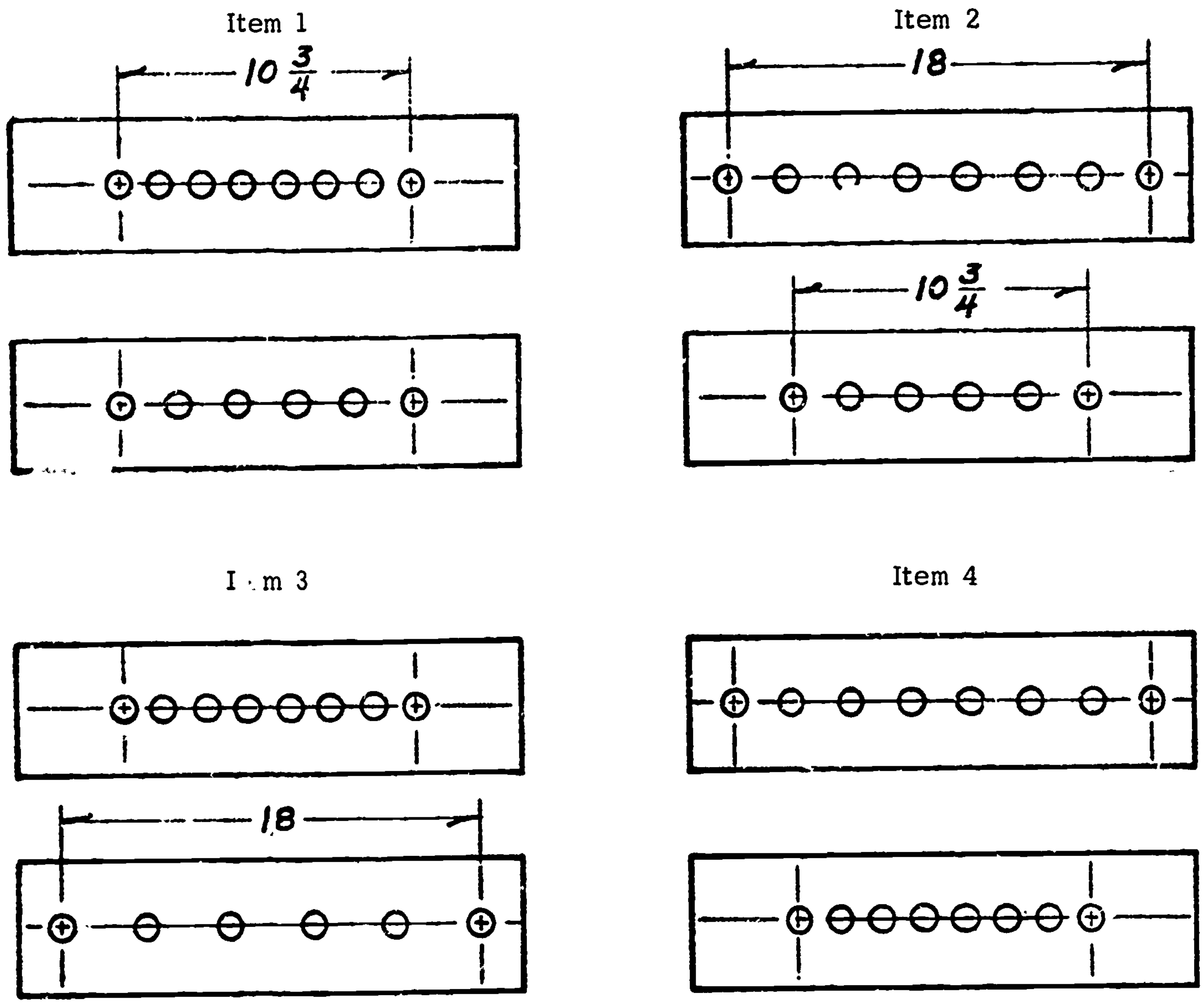
W-2 Look at the squares on both pages. Remember what you are supposed to do with



Note: All dimensions . . inches.

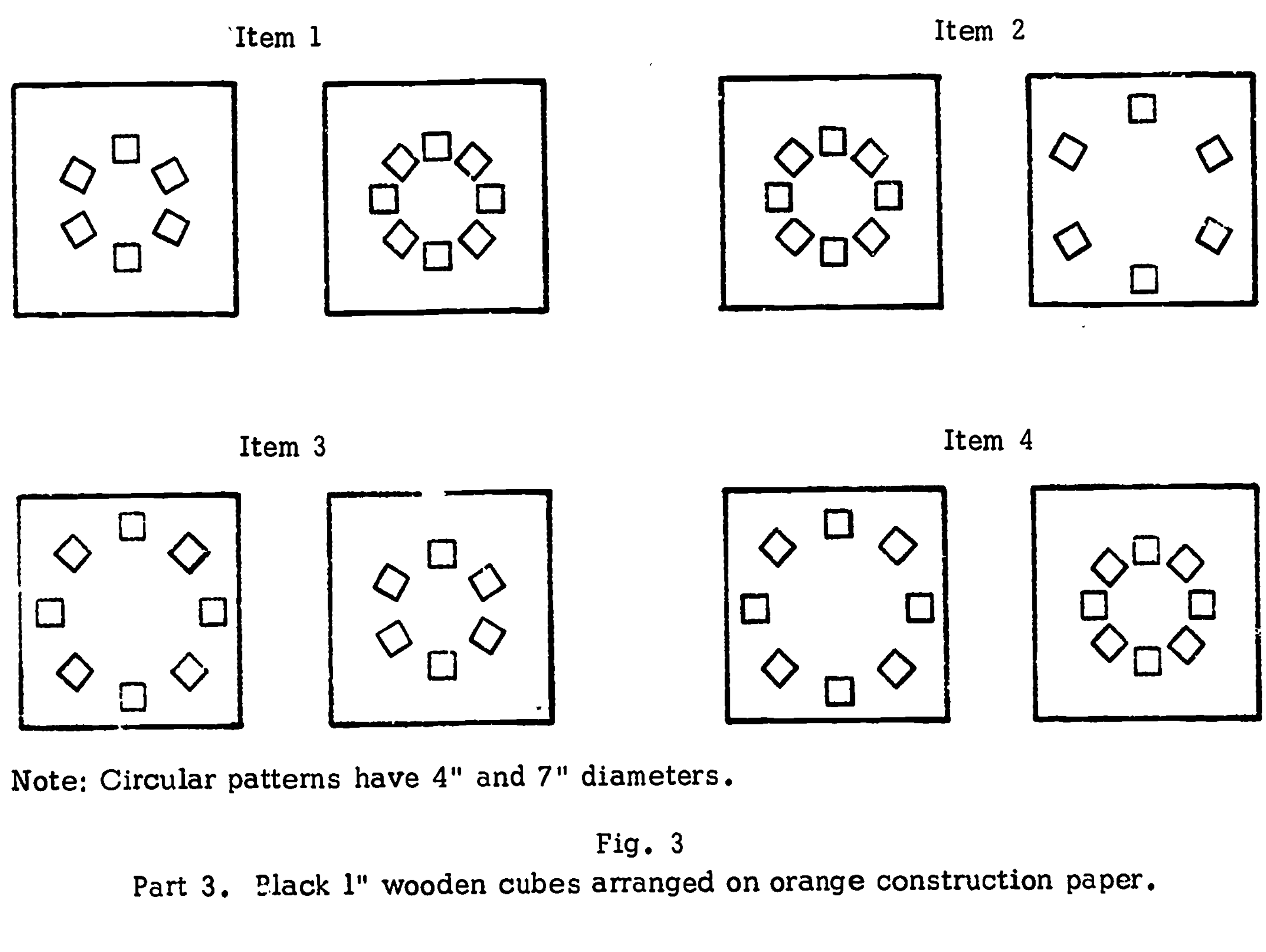
Fig. 1

Part 1. White styrofoam balls about 1 1/2" in diameter arranged on orange construction paper.



Note: All dimensions in inches.

Fig. 2
Part 2. Black checkers of 1 1/4 " diameter arranged on orange construction paper.



Note: Circular patterns have 4" and 7" diameters.

Fig. 3
Part 3. Black 1" wooden cubes arranged on orange construction paper.

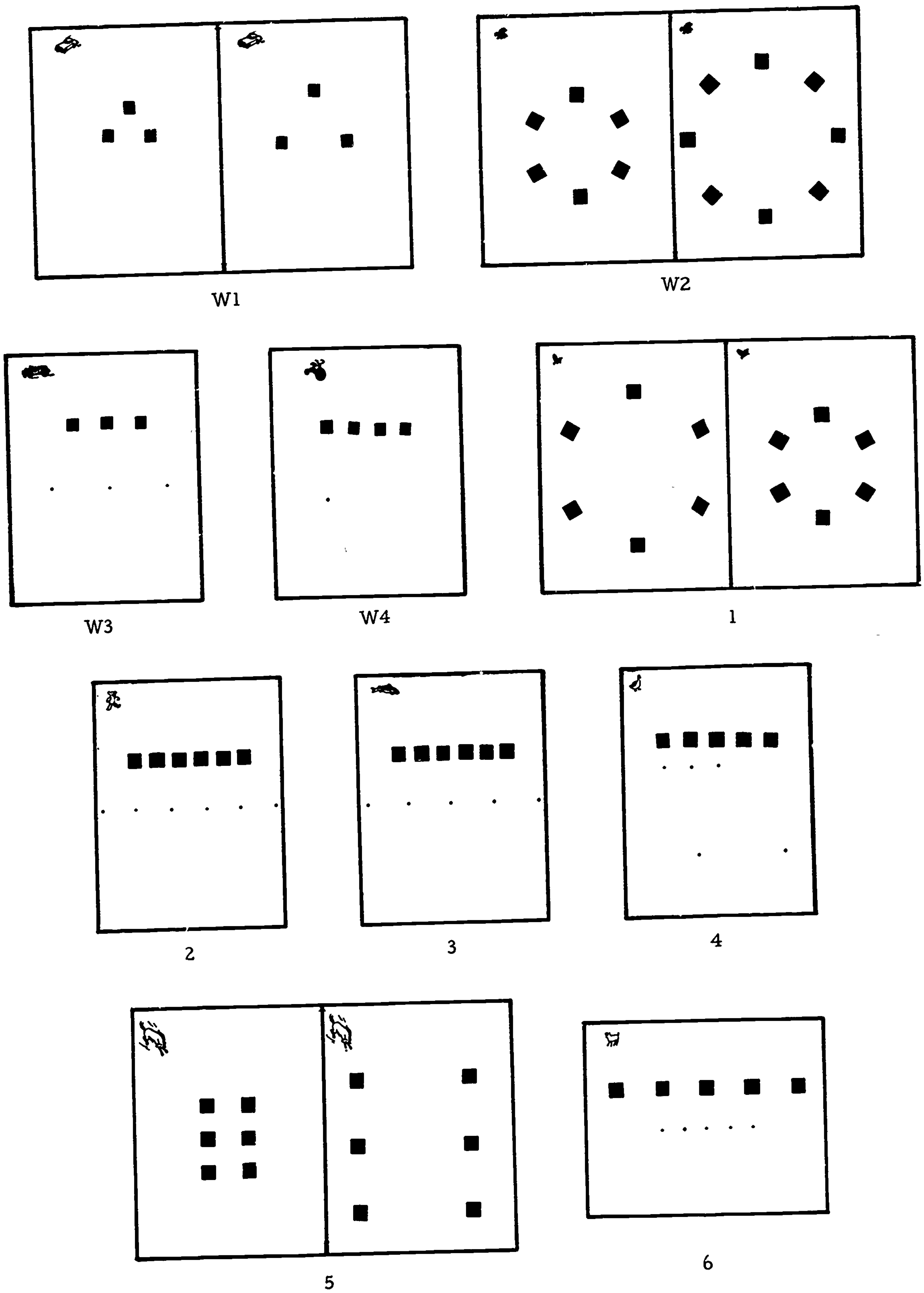


Figure 4. Test of Conservation of Numerousness

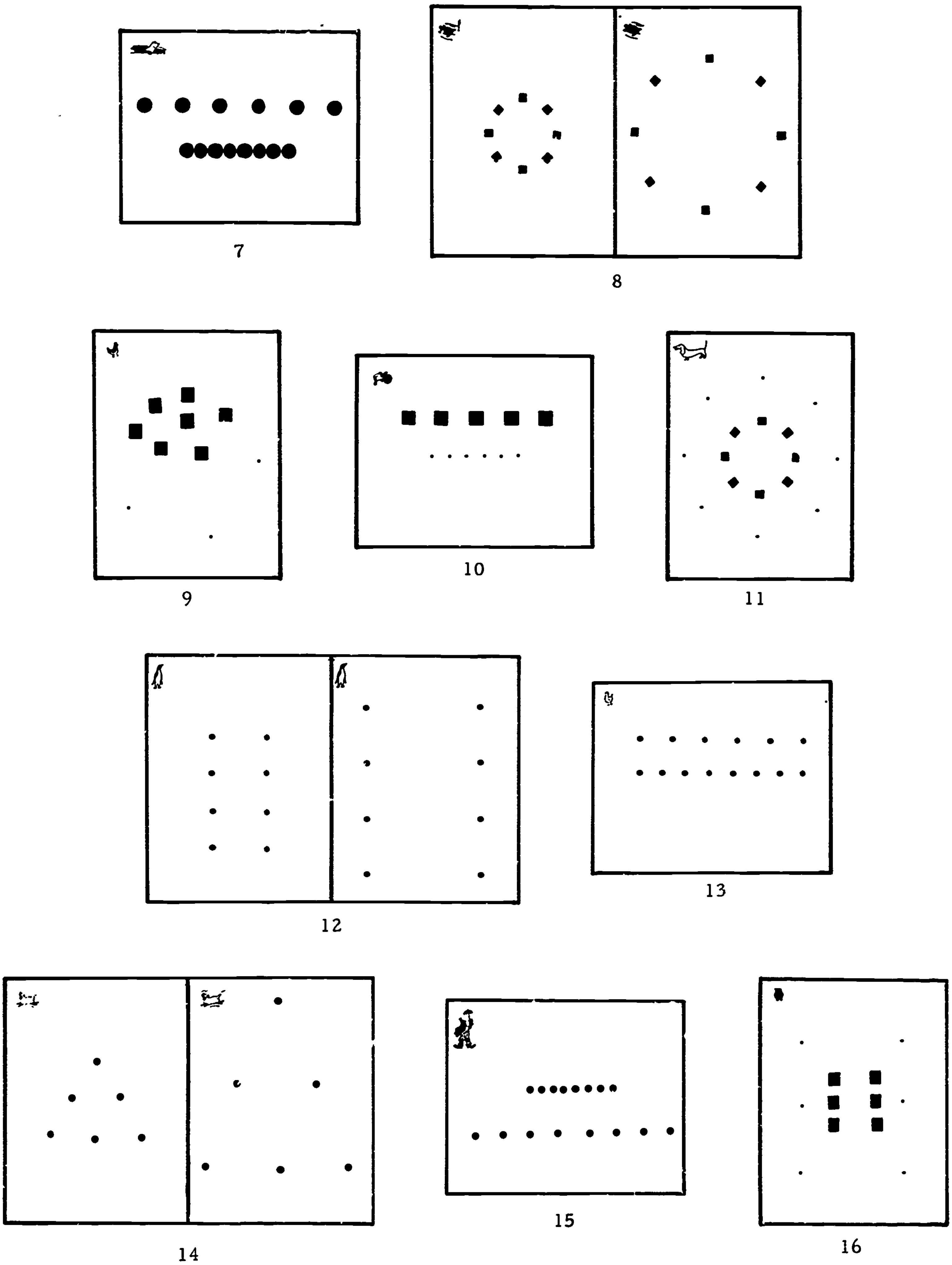


Figure 4 (continued)

your hands. Listen carefully. Are there the same number of squares on both pages? Or does one page have more than the other? Show me by pointing. Turn to the page with the car at the top.

W-3 Three discs. Put the discs on the squares. Notice there are the same number of discs as squares. Now move the discs to cover the dots. Are there the same number of discs as squares? Or are there more of one than the other? Show me by pointing. (Make sure they point with both hands or one depending on whether they think they are the same, etc.). Turn to the page with the tricycle at the top.

W-4 One disc. Place the disc on the dot. Are there the same number of squares as discs? Or are there more of one than the other? Show me. Turn to the page with the butterfly at the top.

1. Look at the squares on both pages. Are there the same number of squares on both pages? Or does one page have more than the other? Show me. Turn to the page with the teddy bear at the top.

2. Six discs. Put the discs on the squares. Now cover the dots with the discs. Are there the same number of discs as squares? Or are there more of one than the other? Show me. Turn to the page with the fish at the top.

3. Five discs. Put the discs on the squares. Now cover the dots with the discs. Are there the same number of squares as discs? Or are there more of one than the other? Show me. Turn to the page with the duck at the top.

4. Five discs. Put the discs on the squares. Move the discs to cover the dots. Are there the same number of discs as squares? Or are there more of one than the other? Show me. Turn to the page with the horse at the top.

5. Look at the squares on both pages. Are there the same number of squares on both pages? Or does one page have more than the other? Show me. Turn to the page with the sheep at the top.

6. Five discs. Put the discs on the squares. Move the discs to cover the dots. Are there the same number of discs as squares? Or are there more of one than the other? Show me. Turn to the page with the bear at the top.

7. Look at the dots in both rows. Are there the same number of dots in both rows? Or does one row have more than the other? Show me. Turn to the page with the turtle at the top.

8. Look at the squares on both pages. Are there the same number on both pages? Or does one page have more than the other? Show me. Turn to the page with the chicken at the top.

9. Seven discs. Put the discs on the squares. Move some of the discs to cover the dots. Are there the same number of discs as squares? Or are there more of one than the other? Show me. Turn to the page with the tractor at the top.

10. Six discs. Cover each dot with a disc. Are there the same number of squares as discs? Or are there more of one than the other? Show me. Turn to the page with the dog at the top.

11. Eight discs. Put the discs on the squares. Move the discs to cover the dots. Are there the same number of discs as squares? Or are there more of one than the other? Show me. Turn to the page with the penguin at the top.

12. Look at the dots on both pages. Are there the same number of dots on both pages? Or does one page have more than the other? Show me. Turn to the page with the chicken at the top.

13. Look at the dots in both rows. Are there the same number of dots in both rows? Or does one row have more than the other? Show me. Turn to the page with the sheep at the top.

14. Look at the dots on both pages. Are there the same number of dots on both pages? Or does one page have more than the other? Show me. Turn to the page with the clown at the top.

15. Look at the dots in both rows. Are there the same number of dots in both rows? Or does one row have more than the other? Show me. Turn to the page with the owl at the top.

16. Six discs. Put the discs on the squares. Move the discs to cover the dots. Are there the same number of squares as discs? Or are there more of one than the other? Show me.

Since they were training items, the warmup items involved gross quantitative comparisons or involved such a small number of objects that children would have a very easy time answering them.

Eight of the test items involved objects that were static and eight involved movable objects. Of the eight items involving static objects, six involved comparison of two equal sets, three of six objects per set and three of eight objects per set. The geometrical configuration varied across these six items with configurations of (1) circles, (2) rectangles, (3) lines, and (4) triangles because comparisons of two equal sets of objects are easier in a rectangular configuration than in a circular or a linear configuration [29, p. 24]. Two of the eight items under consideration involved comparisons of two sets of objects arranged in lines—one of six objects and one of eight objects. These items were included to provide some floor in the test. It has been noted earlier that if two rows of objects have equal length but greater density, an intensive quantitative judgment will suffice for a correct comparison of the numbers of objects in the two sets. One of the two items was exactly of this nature. The other item had the row of eight objects shorter than the row of six objects. Actually, an intensive comparison should be necessary for a correct response, but children who are only capable of gross comparisons will answer the item correctly if they focus on density, which seems to be the most likely focusing. The six items with the same number of objects per set definitely require an extensive quantitative comparison for a correct solution. Since it is the extensive quantitative comparison that makes the numerical correspondence possible, if a child makes a correct comparison by using one-to-one correspondence then he will be said to have made a comparison of extensive quantity and the correspondence will be said to have quantified the two sets for the child. If a child makes a correct comparison by counting, then, because the three stages in coordination of cardinal and ordinal numbers correspond to the three stages in seriation and to the three stages in cardinal correspondence, the child will be said to have made a comparison of extensive quantity.

The remaining eight items of the test involved objects which the child moves. Five of these items had situations in which the child had to compare two sets of objects with the same number in each set. These items varied in many ways from the corresponding six in the first eight discussed above. One of the most striking differences was that in the items with movable objects, the one-to-one correspondence was actually established by the children before they were asked to compare the two sets in their final state. Many children counted the two sets in their final state to make a correct comparison and did not rely on the initial correspondence. Whether the correspondence was numerical for these children is questionable. However, the fact that they did count may indicate that these children rely on a counting process to quantify the two sets in question, rather than on the correspondence, as a result of the classroom training they have received. Since the purpose is not to study modes of correct responses to the items relative to conservation of numerousness, the point of view will be taken that any correct response will be acceptable.

THE PILOT STUDIES

Two pilot studies of the test were actually conducted. The first pilot study was conducted using kindergarten, first- and second-grade children. After the results of this pilot study were obtained and analyzed, it was ascertained that the directions to the children were too complex, especially for the kindergarten children. A type of the directions used follows: "If you think there are more squares on one page, mark that page. If you think there are the same number of squares on both pages, mark both pages." The kindergarten children would often mark a certain page, i.e., left or right, and would then continue to mark the same page throughout the remainder of the test. As a result of this pilot study, the directions were revised and the warmup items were added.

A second pilot study was then conducted using kindergarten children from Cottage Grove School, Monona Grove, Wisconsin. The investigators felt that it was only necessary to use kindergartners since, if they could follow the directions, the first graders would also be able to follow them.

Table 1 gives the frequency distribution of total scores from the second pilot study. It is easy to see that the distribution of total scores departed grossly from the normal distribution. It is interesting to consider the following theoretical frequency distribution based on guessing (see Table 2). The probability p for a correct response was taken to be 1/2 as well as the probability q for an incorrect response; it was assumed that the children selected one of the two alternatives given to them. The gross differences in the two distributions seem to indicate that these kindergartners were basing their responses not on guessing but on other judgments.

A Hoyt-Reliability coefficient of .91 was computed along with an item analysis [4]. The item analysis is described on page 25 and is summarized in Table 3. Item 9 was particularly difficult for these children. It was retained in the final version of the test to increase ceiling. The internal-consistency reliability of .91 indicates the item was not detrimental to the total test. The difficulty levels of the items, excluding number 9, ranged from .27 to .51, which indicates the test was not easy for the kindergarten children. The X_{50} points were all fairly close to the mean (ranging from 1.08 to -.19) with good betas, when they exist. The betas indicate that the items were functioning well.

In order to be sure that the instrument developed was measuring the same thing as an individual test, similar items using physical

Table 1

Distribution of Total Scores; Pilot Study
N = 37

Total Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Mean	Std. Dev.
Frequency	4	3	6	2	3	1	1	0	2	2	1	2	6	1	1	2	0	6.46	4.99

Table 2

Theoretical Frequency Distribution Based on Guessing

Total Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	0	0	.07	.32	1.03	2.5	4.5	6.46	7.27	6.46	4.5	2.5	1.03	.32	.07	0	0

Table 3

Item Analysis of Test of Conservation of Numerousness

Item	Frequency Correct	Difficulty	R	X_{50}	Beta
1 s	19	.51	.64	-.05	.84
2	18	.49	.83	.04	1.53
3	10	.27	.56	1.08	.68
4	15	.41	-	.21	-
5 s	19	.51	.99	-.03	10.36
6	16	.43	-	.15	-
7 s	20	.54	.52	-.19	.60
8 s	13	.35	.99	.38	8.09
9	1	.02	.48	4.00	.54
10	18	.48	.77	.04	1.21
11	16	.43	-	.15	-
12 s	16	.43	-	.16	-
13 s	17	.46	.37	.27	.40
14 s	11	.30	.70	.75	.97
15 s	14	.38	-	.29	-
16	16	.43	-	.17	-

*s denotes those items in which no movement was involved.

Table 4

Correlation Between Corresponding Items:
Group Test and Individual Test

	Item															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Correlation	.35	.39	.54	.77	.80	.61	.45	.61	-.06	.32	.49	.54	.19	.31	.36	.78

objects were individually administered by the same experimenter to the same 37 children three weeks after the group test. The internal consistency reliability of the individual test was .88, and the correlation of total scores between the two tests was .84. Table 4 gives the correlation by items for the two tests. The ninth item continued to operate strangely, with essentially a zero correlation, which can be attributed to the difficulty of the item. The large range (.19 to .80) of the remaining item correlations points out the many variables which may affect the responses to the items and indicates a high degree of instability in a conservation test with a small number of items. The correlation of .84 between the total scores only indicates good stability between the total scores of the group test and the individual test for these kindergarten children.

DEVELOPMENT OF INSTRUCTIONAL PROGRAM

A pilot study on teaching conservation of numerousness was carried out during the school year 1965-66.[3] Twenty-six fifteen-minute lessons were taught once each week by the classroom teachers in the cities of Sheboygan and Glendale, Wisconsin. Knowledge gained from the study and teacher-pupil reactions to the lessons were very useful in designing the present study. Several observations made by the teachers in the pilot study reveal some of the reasons for the direction of the present study: not enough time to develop concepts; too much time on one activity without a change of routine; pictures on worksheets were too small, illegible, and/or too closely clustered for small children to work with; transitivity lessons too difficult for primary children; kindergartners could not always perform the paper and pencil exercises; and some lessons were boring due to too much repetition.

The lessons developed for the present study were designed to eliminate the criticisms of the pilot study. Of primary concern in the development of the new lessons, was a sound progression of learning activities for a given concept and also for the individual lessons. The lessons were prepared so that activities would progress from physical action to concrete manipulation to semi-concrete illustrations. A final step of numerical expressions was never included because the investigators did not want to introduce the variable of relating conservation to numerical symbols. The elimination of this step also did away with any need for paper and pencil type exercises.

The activities of the various lessons proceeded in the fashion indicated above. Whenever a new concept was introduced, the activities required the children to be involved physically. This meant that they either walked, ran, played games or became involved in a similar group activity. The second step was to have them demonstrate the concept by manipulating physical objects. Only those things that would be readily available in any school system were used. The third level was the semi-concrete phase in which the children observed the teacher or another child demonstrate the concept on the flannel board or peg board.

The number of lessons used was cut from 26 in the pilot study to 12 for the present study. All of the lessons involving transitivity were eliminated because of the difficulty teachers encountered in presenting them to the children. The new lessons utilized the teaching sequence listed above and incorporated those activities from the pilot study considered profitable. There was no particular reason for using only twelve lessons or for choosing a thirty-minute period instead of fifteen, except for the criticisms already listed. It was felt that as much, or more, could be accomplished in the present experiment by presenting the concepts over a shorter period of time.

The new lessons included the following activities: one-to-one correspondence, as many as, perceptual rearrangement, more than, fewer than, additions, and subtractions. The lessons began with many tasks involving one-to-one correspondence and the term "as many as" was used extensively. Piaget contends,

> ... learning is possible in the case of these logical-mathematical structures, but on one condition—that is, that the structure which you want to teach to the subjects can be supported by simpler, more elementary, logical-mathematical structures..... an example .... the conservation of number in the case of one-to-one correspondence [26, p. 16].

The inclusion of additive and subtractive activities is supported by Smedslund's studies on the conservation of substance where he employed plasticine to teach this idea [27].

Much more emphasis is placed on the active participation of the children in the new lessons than in those used in the pilot study. In addition, the lessons include those activities Piaget has described as most profitable.

[3] This project was supervised by Barbara Lamphere Boe who was a doctoral candidate at the University of Wisconsin.

> Good pedagogy must involve presenting the child with situations in which he himself experiments, in the broadest sense of that term—trying things out to see what happens, manipulating things, manipulating symbols, posing questions and seeking his own answers, reconciling what he finds one time with what he finds at another, comparing his findings with those of other children [26, p. 2].

The activities are varied and a definite change is effected at least three times in each lesson. This is done to alleviate any problems that might arise because of short attention spans and to capitalize on the use of concrete experiences. We know from other experiments that these are most useful [9].

Much of the progression of difficulty in the twelve lessons was based on the investigators' experience in teaching primary children and supervising student teachers of primary children. The progression followed this order: (1) One-to-one correspondence (as many as); (2) transpositions of equivalent sets, i.e., changing the perceptual arrangement of the two sets but not the number involved; (3) more than (one set possessing fewer objects than the other); (5) transpositions of sets involving more than and fewer than; (6) addition of elements to one set to make it have as many as the other set; (7) subtraction of elements from one set to make it have as many as the other set; (8) and finally combining two sets to make a set of specified size.

Before the lessons were tried on a sample population, many kindergarten teachers expressed concern that the kindergarten children could not remain attentive throughout a thirty-minute lesson. To check this and other concerns, the investigators asked Miss Dawn Kloften of Franklin School, Racine, Wisconsin to select the most immature and the slowest children from her kindergarten, first-grade, and second-grade classes to use for a trial run of the lessons during the fall semester, 1966. On the whole, the lessons proved very teachable. The children enjoyed the experiences and could follow the directions, and attention span proved to be no problem. Miss Kloften made several suggestions for minor changes or variations which were incorporated into the lessons used in the experiment. The text of those lessons is in the Appendix.

III

THE STUDY

As stated earlier, this study was designed to test the feasibility of increasing the ability of kindergarten and first grade children to conserve numerousness through a carefully developed set of lessons which are to be taught one each week for thirty minutes over a twelve-week period.

SUBJECTS

The subjects for this study were randomly selected from the kindergarten and first-grade classes of the Oconomowoc, Wisconsin, school district. One rural three-room school was eliminated from the study because it did not have a kindergarten class. Table 5 contains the mean IQ's of the groups involved in the study. The IQ's ranged from 76 to 142. The ages ranged from 5 years 2 months to 6-4 for the kindergarten children and from 6-1 to 8-1 for the first grade children. The mean age for both the experimental and the control group of kindergarten children was 5-8. In the first grade, the mean age was 6-8 for the experimental group and 6-8 1/2 for the control group.

Oconomowoc is a small city with a population of about 8,000 from a wide range of socioeconomic levels. The city is a bedroom community since many of the residents commute to work in Milwaukee—a distance of some 30 miles. The city has several small industries. Most of the working class is gainfully employed in service occupations, retailing, industry, or farming.

A demographic study conducted by the school system in 1960 characterized the community as having a higher than average number of parents who felt that a college education was essential for their children.

The school district covers an area of 115 square miles in three counties and 13 municipalities. The population of this area is approximately 20,000. Six of the seven elementary schools in the district were used for this study. They include: Ashipun, Greenland, Ixonia, Okauchee, Park Lawn, and Summit schools. Ashipun is a good sized rural consolidated elementary school of grades K-6 with 271 pupils. Greenland elementary is a moderately large metropolitan school containing 553 children. Ixonia is a small elementary school (122 students) in a town five miles west of Oconomowoc. Okauchee elementary school has 321 children. It is in a small lake front community about two miles east of Oconomowoc. Park Lawn is another moderately

Table 5

Mean IQ Scores

Group	IQ
Kindergarten	
Experimental	104.742
Control	106.097
All Kindergarten	105.419
First Grade	
Experimental	110.475
Control	109.272
All First Grade	109.873
All Experimental	107.608
All Control	107.685
All Groups	107.646

Table 6

Number of Classes and Students by School

	No. of Classes		No. of Students		
School	K	1	K	1	Total
Ashipun	2	1	41	41	82
Greenland	6	3	144	69	213
Ixonia	1	1	34	16	50
Okauchee	2	2	42	54	96
Park Lawn	4	3	84	94	178
Summit	3	3	72	63	135
Grand Totals	18	13	417	337	754

large elementary school (580 pupils) in the city of Oconomowoc. Summit elementary is a large consolidated elementary school two miles south of the city. It has 513 students—most of whom are bused to the school. Table 6 below indicates the number of kindergarten and first grade classes in each school. It also summarizes the size of the classes.

METHOD

The method of stratified random sampling was used to assign the children to the treatment groups. Wherever possible, twelve children for the experimental group and twelve for the control group were selected from each of the classes involved with an equal number of boys and girls in each of the treatment groups. Table 7 below indicates the number of children from each school assigned to each treatment group at the beginning of the study.

The first grade class at Ixonia has only sixteen pupils so each of these children was randomly assigned to the experimental and control groups. Ashipun has only ten in both treatment groups at the kindergarten level because the morning section had only ten girls and twelve boys. There was an outbreak of chicken pox when the kindergarten class at Ixonia was being tested, so it was possible to test only ten children in each of the treatment groups. Four children dropped from the study when they moved out of the district: two in the kindergarten experimental and one in the first-grade experimental group at Ixonia, one in the kindergarten control group at Summit School.

The pretest of conservation of numerousness was administered to all of the subjects in groups of five students each during the period January 16-30, 1967. The children were separated by 20" X 30" posterboard screens so that they could not observe each other's work. The posttest was administered in the same manner at the end of the twelve-week experimental period. The lessons ended on May 4; testing began on May 8, 1967 and was completed on May 18, 1967. These tests were administered by Dr. Harold Harper.

The Lorge-Thorndike Intelligence Test, Level 1, Form A, was used to determine the IQ levels of the children. The nonverbal test was administered to the first-grade classes and some of the kindergarten students by Dr. Harper. Many of the kindergarten children were tested by the teacher hired to conduct the experimental lessons.

The intelligence test was administered to class units at the first-grade level in two sittings of 15-20 minutes each. Part 1, the vocabulary test, was given in the first sitting. Parts 2 and 3 were administered at the second sitting. The short attention spans of the children necessitated the shorter testing periods. The kindergarten children were given the IQ tests in groups of six. They were screened so that they could not see each other's work but seated so that the examiner could easily view every child. These tests were administered during the months of March and April.

The children who were randomly assigned to the control groups, Treatment 2, received the same arithmetic instruction they normally would during the twelve-week period of experi-

Table 7

Number of Subjects in Treatment and Control Groups by School

School	Grade	$Treatment_1$ (Experimental)	$Treatment_2$ (Control)
Ashipun	K a.m.	10	10
	1	12	12
Greenland	K p.m.	12	12
	1	12	12
Ixonia	K a.m.	10	10
	1	8	8
Okauchee	K p.m.	6	6
	1	12	12
Park Lawn	K a.m.	12	12
	1	12	12
Summit	K p.m.	12	12
	1	12	12

mentation. As the school system uses the Scott-Foresman series, there is no formal arithmetic instruction at the kindergarten level; the teachers used many different, informal approaches to teaching number concepts in the various kindergarten classrooms.

Children assigned to Treatment 1, the experimental treatment, received the twelve half-hour weekly lessons in conservation of numerousness. These lessons were taught by the same teacher each week in order to eliminate the teacher variable. The teacher (Mrs. Jane Edwards) is a substitute teacher for the Oconomowoc school system who has had full-time teaching experience as a first-grade teacher and has substituted in grades K-6 in Oconomowoc and Chicago.

Dr. Harper demonstrated the lessons for Mrs. Edwards by teaching them to a group of randomly chosen kindergarten children from a class not included in the experiment. Each lesson was demonstrated during the week preceding its use as a model for the routine to be followed in teaching the experimental groups. Both teachers agreed on changes, deletions, and additions before the experimental groups were actually exposed to the concepts.

To eliminate the variable of weekly time for participating in the experiment, a schedule of rotated lesson times was established. The assignments to this schedule were arranged on the basis of the kindergarten group that was being used in a given school. Selection of the kindergarten groups was random except for Ixonia school which only has a morning kindergarten class. This schedule of times was constructed so that no one group would have the advantage, or disadvantage, of having Mrs. Edwards for the first lesson, last lesson, etc. during each week. The schedule is shown in Figure 5 and instructions to the classroom teachers which appeared below the schedule when it was distributed in the district are included at the bottom of the schedule.

The experimenter met with all of the kindergarten and first-grade teachers prior to the execution of the project. The teachers were familiarized with the general purpose of the experiment but were not informed of the nature of the experimental lessons.

RELIABILITY AND CORRELATIONAL STUDIES

Since two grades were involved, separate internal-consistency reliability coefficients [28, p. 156] were computed for each grade on the pretest of conservation of numerousness by the use of an appropriate computer program [6]. An item analysis, also computed, involved the following item statistics: (1) X_{50}, (2) beta, (3) r_b, and (4) a difficulty index [4]. Underlying these four statistics is the concept of an item characteristic curve, which "is a smooth curve fitted to the proportion of persons at each criterion score level who made the particular responses being studied [4, p. 24f]."

In order to utilize the parameters of the normal curve, the assumption that the item characteristic curve has the form of the integrated normal function (normal ogive) must be made [4, p. 25]. Once this assumption is made, the definition of X_{50} and β may then be given.

> The parameters of the item characteristic curve which specify the normal ogive fitted to the item response data are the following:
>
> X_{50}, the criterion score at which the probability of correct response is .5.
>
> β, a measure of the steepness of the item characteristic curve which specifies the capability of the item to discriminate between individuals possessing various amounts of the criterion ability [5, p. 11f].

The difficulty of an item "corresponds to the area under the item characteristic curve [5, p. 29]." r_b is the point biserial correlation of an item with the total test.

A correlation study was conducted on IQ and the pretests of conservation of numerousness.

EXPERIMENTAL DESIGN

The experimental design was a $6 \times 2 \times 2$ factorial design with two covariates [35, p. 578]. The factors were Schools (S), Sex (X), and Experimental Versus Control group (T). The dependent measure was the posttest of conservation of numerousness. The two covariates were the scores on the pretest of conservation of numerousness and the Lorge-Thorndike nonverbal intelligence test. All of the factors were considered to be fixed factors. A diagram of the design is given in Figure 6. As Factor S is at six levels, the scheme would be repeated six times. Y_{ijkn} represents the n^{th} observation on the dependent measure in cell ijk. X_{ijkn} and Z_{ijkn} represent the n^{th} observation in cell ijk for the two covariates. n_{ijk} represents the number of observations in cell ijk. The analysis of covariance is shown in Figure 7.

In the ANCOVA table, $M = \Sigma\, n_{ijk}$. The covariance analysis was performed for the scores within each grade—i.e., two separate analyses

Figure 5

Schedule of One-half Hour Teaching Lessons for Conservation of Numerousness Experiment

School		Weeks											
		1	2	3	4	5	6	7	8	9	10	11	12
Park Lawn	K	Ma1[a]	Ta1	Wa1	Ma2	Ta2	Wa2	Ma1	Ta1	Wa1	Ma2	Ta2	Wa2
	1	Ma2	Ta2	Wa2	Ma1	Ta1	Wa1	Ma2	Ta2	Wa2	Ma1	Ta1	Wa1
Greenland	K	Mp1	Tp1	Wp1	Mp2	Tp2	Wp2	Mp1	Tp1	Wp1	Mp2	Tp2	Wp2
	1	Mp2	Tp2	Wp2	Mp1	Tp1	Wp1	Mp2	Tp2	Wp2	Mp1	Tp1	Wp1
Ashipun	K	Ta1	Wa1	Ma1	Ta2	Wa2	Ma2	Ta1	Wa1	Ma1	Ta2	Wa2	Ma2
	1	Ta2	Wa2	Ma2	Ta1	Wa1	Ma1	Ta2	Wa2	Ma2	Ta1	Wa1	Ma1
Summit	K	Tp1	Wp1	Mp1	Tp2	Wp2	Mp2	Tp1	Wp1	Mp1	Tp2	Wp2	Mp2
	1	Tp2	Wp2	Mp2	Tp1	Wp1	Mp1	Tp2	Wp2	Mp2	Tp1	Wp1	Mp1
Ixonia	K	Wa1	Ma1	Ta1	Wa2	Ma2	Ta2	Wa1	Ma1	Ta1	Wa2	Ma2	Ta2
	1	Wa2	Ma2	Ta2	Wa1	Ma1	Ta1	Wa2	Ma2	Ta2	Wa1	Ma1	Ta1
Okauchee	K	Wp1	Mp1	Tp1	Wp2	Mp2	Tp2	Wp1	Mp1	Tp1	Wp2	Mp2	Tp2
	1	Wp2	Mp2	Tp2	Wp1	Mp1	Tp1	Wp2	Mp2	Tp2	Wp1	Mp1	Tp1

The experimental teaching lessons will begin the week of February 6 and continue through May 5, 1967. Mrs. Edwards will be the teacher for the small group sessions. In order to protect the material used in the lessons, the principal investigators have requested that the lessons <u>not</u> be observed by other teachers or parents. At the end of the experiment the entire text of the experimental lessons will be made available to all interested persons. The results of the study will be published and distributed to the Oconomowoc school administrators.

[a] Read as "Monday, a.m., first period."

Figure 6

Diagram of Design

S	X	T			C		
1	b	Y_{1111}	X_{1111}	Z_{1111}	Y_{1121}	X_{1121}	Z_{1121}
		Y_{1112}	X_{1112}	Z_{1112}	Y_{1122}	X_{1122}	Z_{1122}
		⋮	⋮	⋮	⋮	⋮	⋮
		$Y_{111n_{111}}$	$X_{111n_{11}}$	$Z_{111n_{11}}$	$Y_{112n_{112}}$	$X_{112n_{112}}$	$Z_{112n_{112}}$
	g	Y_{1211}	X_{1211}	Z_{1211}	Y_{1221}	X_{1221}	Z_{1221}
		Y_{1212}	X_{1212}	Z_{1212}	Y_{1222}	X_{1222}	Z_{1222}
		⋮	⋮	⋮	⋮	⋮	⋮
		$Y_{121n_{121}}$	$X_{121n_{121}}$	$Z_{121n_{121}}$	$Y_{122n_{122}}$	$X_{122n_{122}}$	$Z_{122n_{122}}$
⋮		⋮	⋮	⋮	⋮	⋮	⋮

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Figure 7

ANCOVA Analysis

Source of Variation	df_l	MS^a	F
S	s-1	MS'_s	Each F ratio
X	x-1	MS'_x	is obtained by
T	t-1	MS'_t	dividing each
SX	(s-1)(x-1)	MS'_{sx}	mean square by
ST	(s-1)(t-1)	MS'_{st}	MS'_e
XT	(x-1)(t-1)	MS'_{xt}	
SXT	(s-1)(x-1)(t-1)	MS'_{sxt}	
Error	M - sxt - 2	MS'_e	

[a]The primes denote adjusted mean squares.

were conducted—to obtain the maximal within class homogeneity of the regression coefficients resulting from the linear regression of Y on X and Y on Z. Moreover, due to the randomization procedure followed, these regression coefficients should be homogeneous, which is a basic assumption in the analysis of covariance.

HYPOTHESES

The null hypothesis of particular interest to the investigators (for each grade) are stated below. The test of conservation of numerousness was used in the test of these hypotheses.

1. There is no difference in the adjusted mean scores observed between the treatment and control groups.
2. There is no difference in the adjusted mean scores observed between boys and girls.
3. There is no difference in the adjusted mean scores observed among schools.
4. There is no difference in the adjusted mean scores observed in treatment and control groups across schools.
5. There is no difference in the adjusted mean scores observed between treatment and control groups across sex.

Other hypotheses of secondary importance were tested as dictated by the statistical design.

IV

RESULTS OF THE STUDY

RELIABILITY AND CORRELATIONAL STUDIES

Extensive internal-consistency reliability studies were conducted on the pre- and post-test of conservation of numerousness. Table 8 contains reliability coefficients from the pretest. Both of these reliabilities were substantial, especially the reliability for the kindergarten, which agreed favorably with the reliability of .91 obtained in the pilot study.

Table 8

Internal Consistency Reliabilities of the Pretest of Conservation of Numerousness

Grade	No. of Students	Reliability
K	124	.87
1	136	.75

The frequency distribution of total scores for the pretest is given by grade in Table 9. There was a great difference between the two distributions: At the kindergarten level, children had total scores at each point of the scale except the highest; at the first-grade level, however, children scored at only 12 of the 16 points with no scores at the three lowest points. This was not consistent with data reported from a previous individual test of conservation of numerousness described earlier [29, p. 23] on which 128 of 341 first-grade children tested in the spring of 1966 scored at a low level. The reason for such a large discrepancy is not at this time clear. Whether it is a function of the two different tests, the two different populations, or both is yet to be determined. An informal analysis of the effects of the three dimensions on which the items differed, based on the item analysis reported in Table 10, may contribute to an understanding of the observed discrepancy in the distributions of the two tests. Item 9 has not been considered in the discussion because of its difficulty.

The first dimension to be considered is the arrangement of objects in static configurations or with manipulations by the children. Items 1, 5, 7, 8, 12, 13, 14, and 15 were of the first type, and Items 2, 3, 4, 6, 10, 11, and 16 were of the second type. For the kindergarten the ranges of the difficulties were .33 - .58 and .36 - .56 and the mean difficulties were .47 and .46 respectively.

The second dimension to be considered is number of objects in the two sets to be compared—the same number in each set or a different number in each set. Items 1, 2, 4, 5, 6, 8, 11, 12, 14, 15, and 16 were of the first type and items 3, 7, 10 and 13 were of the second type. The ranges of the difficulties were .33 - .58 and .36 - .54 and the mean difficulties were .46 and .47 respectively for the kindergarten.

The last dimension to be considered is the geometrical configurations of rectangles, circles, lines and triangles. Items that fall in the various categories are as follows: rectangles—5, 12, and 16; circles—1, 8, and 11;

Table 9

Frequency Distribution of Total Scores: Pretest of Conservation of Numerousness

Grade	Total Score																	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Mean
Kindergarten	6	12	12	9	6	4	6	7	10	6	8	12	12	9	3	2	0	7.02
First	0	0	0	3	1	1	0	5	4	7	10	11	14	27	28	22	3	12.2

Table 10

Item Analyses: Kindergarten and First Grades: Pretest

Item No.	Difficulty		r		X_{50}		β	
	K	1	K	1	K	1	K	1
1	.43	.76	.78	.68	.23	-1.02	1.23	.93
2	.42	.82	.68	.63	.30	-1.43	.93	.80
3	.36	.47	.35	.47	.99	.16	.38	.53
4	.43	.84	.91	.61	.19	-1.62	2.21	.77
5	.58	.90	.98	1.10	-.20	-1.14	5.19	-
6	.52	.93	1.01	.99	-.06	-1.46	-	6.72
7	.54	.88	.34	.48	-.29	-2.39	.36	.55
8	.35	.71	.81	.82	.45	- .66	1.40	1.44
9	.08	.13	.26	.29	5.30	3.81	.27	.31
10	.52	.82	.48	.50	-.12	-1.86	.55	.57
11	.47	.88	.92	.80	.09	-1.39	2.29	1.33
12	.49	.89	.93	1.06	.02	-1.16	2.56	-
13	.44	.64	.27	.63	.52	- .57	.28	.81
14	.49	.89	.88	.93	.02	-1.32	1.87	2.50
15	.33	.73	.82	.67	.53	- .90	1.44	.91
16	.56	.93	1.08	1.03	.13	-1.46	-	-

lines—2, 3, 4, 6, 7, 10, 13 and 15; and triangles—14. The ranges of difficulties are .49 - .58; .35 - .47; .33 - .52; and .49; the mean difficulties are .54, .42, .46, and .49 respectively, for the kindergarten.

A surprising observation is that no differences in the difficulty of the two categories related to number of objects in the sets was apparent for the kindergarten or for the first grade. The observation is surprising since it was assumed that for those items, an extensive quantitative comparison would be necessary. For the first grade, one of two factors could have been operating; i.e., either these children were able to work in situations demanding extensive quantitative comparisons or the items were of such a nature that they did not introduce enough perceptual conflict for the children to be forced to make a judgment while ignoring perception. Because of the previous data collected on the individual test [29, p. 23], it would seem that the latter would be a more logical factor affecting the students' responses. The kindergarten children who scored at or below seven on the pretest made extensive quantitative comparisons less frequently than gross quantitative comparisons, since the items which involved a comparison of two sets with the same number in each set was 7.5 and the mean frequency for the items which involved a comparison of two sets with different numbers in each set was 15.8.

The internal consistency reliability studies for the posttest of conservation were conducted relative to four populations, (1) the experimental kindergarten group, (2) the experimental first-grade group, (3) the control kindergarten group, and (4) the control first-grade group. Table 11 contains the reliabilities for each of these four groups. The reliabilities continued to be substantial in the case of the kindergartners but not in the case of the first graders. Since the first graders found the posttest quite easy, one would expect low reliabilities. Table 12 contains the frequency distribution for each of the four groups mentioned above.

Table 11

Internal-Consistency Reliabilities of Posttest of Conservation of Numerousness

Grade	Treatment	
	Experimental	Control
K	.86	.90
1	.70	.46

All of the groups had a higher mean score than was reported in Table 9 for their respective grades. At least three factors could account for this gain: 1) maturation, 2) familiarity with the test, and 3) experiences with number concepts. The first is not subject to experimental control. It could be only assessed in the case of a total lack of familiarity with the test and no experience involving number

Table 12

Frequency Distribution of Total Scores: Posttest of Conservation of Numerousness

Grade	Group	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Mean
		Total Score																	
Kindergarten	E	1	1	3	1	0	0	3	1	1	2	1	7	13	13	6	5	1	10.98
	C	2	2	4	5	0	4	2	2	2	1	8	4	7	7	5	6	1	9.2
First Grade	E	0	0	0	0	0	0	0	1	0	0	3	4	9	18	23	12	1	13.3
	C	0	0	0	0	0	0	1	0	1	1	2	1	3	12	17	25	3	13.7

Table 13

Correlations Between Tests of Conservation and Lorge-Thorndike Intelligence Test

Grade	Pretest	Posttest
K	.33**	.39**
1	.20*	.16

*p < .05
**p < .01

concepts. The second is subject to experimental control. The authors feel that the separation of twelve weeks was enough to reduce carry-over effects to a minimum; however, some retest familiarity could have been present. The third is also subject to experimental control and was controlled in all respects except one, the day-by-day classroom experiences the teachers provided. These experiences included much informal emphasis on counting.

Table 13 contains the correlations between the tests of conservation and the Lorge-Thorndike Intelligence Test, nonverbal, for the kindergartners and the first graders. Three of the correlations are significantly different from a zero correlation, but all are small.

RESULTS OF THE TEACHING EXPERIMENT

The analysis of covariance was chosen for this study to increase the precision of the randomized experiment. Two things were of major concern to the investigators, (1) the innate capacity of the children and (2) the effects of maturation. For these reasons the pretest of conservation of numerousness scores and the Lorge-Thorndike intelligenc quotients were used as the covariates in the final analysis of the data. Tables 14 and 15 summarize this analysis.

Table 14

Analysis of Covariance for Kindergarten Conservation of Numerousness Test Scores

Source of Variation	df	MS[a]	F
S (Schools)	5	14.48	1.09
X (Sex)	1	14.81	1.12
T (Treatment)	1	74.68	5.62*
SX	5	7.65	.58
ST	5	19.07	1.44
XT	1	.48	.04
SXT	5	18.16	1.37
Error	95	13.28	

[a]Adjusted Mean Squares

*p < .03

Table 15

Analysis of Covariance for First-Grade Conservation of Numerousness Test Scores

Source of Variation	df	MS[a]	F
S (Schools)	5	4.45	2.02
X (Sex)	1	2.22	1.01
T (Treatment)	1	1.79	.81
SX	5	2.92	1.33
ST	5	1.28	.58
XT	1	.002	.001
SXT	5	.32	.15
Error	109	2.20	

[a]Adjusted Mean Squares

The adjusted mean for the experimental group was 10.97 and for the control group, 9.29.

On the basis of these data, only one of the null hypotheses stated earlier was rejected—and that only for kindergarten groups. It is Hypothesis 1,, "There is no difference in the adjusted mean scores observed between the treatment and control groups." The mean square for treatments was 74.68 with 1 and 95 degrees of freedom, which yields an _F_ ratio of 5.62 which is significant at the .03 level of confidence. This significance was in favor of the experimental group which implies that the twelve lessons used in this study were effective in enhancing the kindergarten children's acquisition of conservation of numerousness. None of the hypotheses were rejected for the first-grade children—the lessons had no apparent effect on their learning.

Further secondary conclusions were drawn on the basis of the data obtained. Many of the first graders in the study were apparently conserving when the pretest was administered. The experimental teacher indicated that there was a consistent high level of interest on the part of the kindergartners in the twelve experimental lessons but she also reported some apathy and disinterest on the part of many first graders—due no doubt to the fact that they already understood the material. The concern about the length of the lessons for the kindergartners proved to be of no consequence. Activities were varied enough to maintain interest and to compensate for the short attention spans.

It was also concluded that it is possible to improve conservation abilities among children whose mean age at the beginning of instruction is 5 years and 8 months. This is a younger age than was successfully taught in any of the other studies reviewed earlier.

Children in the control groups were subjected to many and varied informal number experiences by individual kindergarten teachers. Some used only their own materials, games, and original activities—mostly related to counting. Others employed their own materials plus such things as suggested in the SRA GCMP teachers manual for kindergarten, the AAAS Science—A Process Approach, the Follet Kindergarten Readiness Book, and the Continental Press worksheets.

If any one of these teacher's approaches standing alone were sufficient to produce an outcome comparable to that produced in conjunction with the experimental treatment, an interaction of schools by treatments would have occurred. In that no interaction occurred, it was concluded that the twelve lessons supplementing the child's normal activities were the most effective method of teaching conservation of numerousness in this experiment. Also, the emphasis placed on the progressive development of concepts (i.e., physical activities, use of concrete objects and semi-concrete illustrations before abstraction) seems to have been fruitful.

CONCLUSION

The purpose of this study was two-fold: first, to produce an instrument capable of testing children's strengths and weaknesses in the basic number concept of conservation of numerousness; and second, to produce a set of lessons which would enhance children's acquisition of this basic number concept. The investigators feel that both aspects of the purpose were accomplished.

The testing instrument may have some weaknesses for first-grade children but seems to be fairly reliable in measuring kindergartners' abilities in conservation of numerousness.

The experimental lessons were successful in that they produced significant differences in acquisition of conservation of numerousness among kindergarten children in favor of the experimental groups. The lessons had no apparent effect on first-grade children's performance on the test of conservation of numerousness and would thus seem inappropriate for use at that level.

IMPLICATIONS

1. Further experimentation is needed to perfect the testing instrument. Though the present form of the test functioned well in this study, especially with kindergarten children, the investigators feel it should be further refined to give a higher ceiling. This would make it more useful with first-grade children.

The investigators feel that further refinement and variation of the physical materials may make it possible to test even larger groups of children at one sitting.

The investigators are of the opinion that the test in its present form would be more useful in predicting success in first-grade mathematics than current readiness tests.

2. The twelve 30-minute lessons might well become part of the mathematics curriculum for kindergartners. Because these lessons produced significant results in favor of the experimental kindergarten groups, the investigators feel they would be most inappropriate for the basic number experiences of kindergarten children. The Oconomowoc Public

Schools are making them available to all of the kindergarten teachers during the 1967-68 school year.

3. Further experimentation with the lessons should be conducted in schools made up of socially deprived children. As the study progressed in Oconomowoc, the investigators became aware of the fact that this quiet little Midwest community was not as typical as was desired in that the children came from homes that are in the middle and upper socio-economic class. For this reason it seems necessary to replicate the study in several communities and particularly those containing large segments of socially deprived children.

4. Further experimentation with the progressive development of concepts should be conducted to determine the most effective technique for teaching basic number concepts. The teaching progression used in this study was to engage children first in physical action, second in manipulation of concrete materials, and third in the use of semi-concrete illustrations at the flannel board. Does the sequence make a difference? Are there more appropriate and productive sequences that might be employed?

The investigators feel that the sequences used here were very successful and we would encourage teachers to utilize these steps in teaching basic number concepts.

5. The kindergarten children involved in this study should be observed in succeeding years to see if these lessons have a long range effect on their achievement in the elementary mathematics program. The study conducted by Steffe [2,] in 1966 indicated that first-grade children who possessed conservation of numerousness were more successful in problem solving than those who had not reached this stage. If possession of this skill enhances a student's performance in mathematics, an advantage might well appear in Grades 1, 2, and 3. The investigators plan to observe the achievement of the subjects involved in this study for the next three years to see whether the experimental children show any significant difference in achievement in mathematics over the control children.

APPENDIX

LESSON I

Objectives:

1. To introduce the concept of one-to-one correspondence.
2. To impress the children with the fact that two sets ("groups") can be put in one-to-one correspondence.

Duration:

30 minutes

Materials needed:

1. Record (march music) and record player
2. Paper hats (two colors)

Background:

One-to-one correspondence of two sets is an important pre-number concept. A set is a collection of objects such as buttons, boys, glasses. (The teacher may wish to discuss sets with the children prior to some of the experiences included in these units; however, it is the feeling of the investigators that children will understand the term as it is used in connection with the prescribed activities.) Given two sets, say a set of four children wearing red hats and a set of four children wearing green hats, for each red hat there is a green hat and vice versa. That is, the members of the two sets can be matched in a one-to-one fashion and the two sets are said to be in one-to-one correspondence. Two of these matchings are:

$g_1 \longleftrightarrow r_1$		$g_1 \longleftrightarrow r_3$
$g_2 \longleftrightarrow r_2$	or	$g_2 \longleftrightarrow r_2$
$g_3 \longleftrightarrow r_3$		$g_3 \longleftrightarrow r_1$
$g_4 \longleftrightarrow r_4$		$g_4 \longleftrightarrow r_4$

There are 24 possible matchings for these two sets. It is important for the children to realize that no matter which way they are matched, there is a green hat for every red hat and there is a red hat for every green hat.

The activities which follow are designed to help the children begin to understand this concept. A partner will be one child of a pair of children.

Activities:

1. Split your group into two equal sets of children. If you have an odd number of children, you may need to act as a partner for one of the children.
2. Give one set the green paper hats and the other set the red paper hats.
3. Have the children line up in two lines to form pairs.

Red hats	0	0	0	0
	|	|	|	|
Green hats	X	X	X	X

4. Ask, "Does everyone have a partner?"
   "Does each one with a red hat have a partner with a green hat?"
   "Does each one with a green hat have a partner with a red hat?"
5. Have the children with green hats change places. Then ask, "Does each one with a green hat now have a partner with a red hat?"
   "Does each child in this group (pointing to red) have a partner?"
6. Change again and again asking similar questions.
7. Have the two groups exchange hats and repeat the preceding activities.
8. Remove the hats and have the children choose different partners to form two new equal sets.
9. Line them up and ask questions similar to those posed in the preceding exercise, then form the children into two circles (one inside of the other). (It is helpful to have the circles marked on the floor with masking tape or chalk.) Have them hold hands so they know that the person opposite each child is his partner.

```
          0
          |
          X
  0—X           X—0
          X
          |
          0
```

10. Prepare record and record player.
11. Explain to the children that when the music starts, the outside circle will march while the inside circle stands at attention. When the music stops, the children in the inside circle will turn and face their new partners. Then ask questions like, "Who is your new partner?" "Does everyone have a partner?" "Do you see your old partner?"
12. Now have the inside circle march while the outside circle stands at attention and repeat the activities and questions used in number 11.
13. Explain to the children that this time when the music starts the inside circle will move in one direction and the outside circle will move in the other direction. They are to march in a circular fashion until the music stops. When it stops, they should take the hand of the child who is opposite to determine each person's new partner. (It may be necessary to draw arrows on the floor so that the children in each circle will know which way they are to march.)
14. Each time the music stops repeat questions similar to those posed earlier to be sure that each child recognizes that he will always have a partner even though it is not the same partner each time.
15. Have the inside circle change places with the outside circle and repeat some of the activities and questions.

    Other questions could be, "Do you have your original partner?" "Does each one still have a partner?" "Can you find your original partner?" "Can you find your last partner?" etc.
16. Repeat these activities until the thirty minute period is consumed or until interest wanes. DO NOT USE MORE THAN 30 MINUTES!

LESSON II

Objectives:

To introduce the idea of "as many as" based on one-to-one correspondence.

Duration:

30 minutes

Materials needed:

1. Counters for the children (two different kinds)(macaroni and lima beans)
2. Flannel board and cutouts for the teacher.

Background:

In Lesson I, we introduced the children to the notion that two sets of children could be placed in one-to-one correspondence with each other, and that two sets have one-to-one correspondence if for each member of one set there is a matching member of the other set.

In the activities for today's lesson, there are two important concepts to keep in mind, i.e.: (1) the children do not need to know how to count to successfully complete the activities involved (in fact, counting should not be mentioned); and (2) members of sets must be paired in two ways. For every element in one set there must be a matching element in the other set and _vice versa_.

To see that the pairing or matching must be taken two ways, consider the following example:

Set A:	Button 1	Button 2	Button 3	Button 4
Set B:	Needle 1	Needle 2	Needle 3	

In this instance, each needle can be matched with a button but each button cannot be paired with a needle. Hence, we cannot say that there are as many needles as buttons.

In the activities in today's lesson we want the children to become familiar with the use of the term "as many as" and they _must not count_. One way to avoid the counting is to have the children work with larger sets of objects—say twelve to fifteen objects in each set.

Activities:

1. "Does anyone remember what we did last week?" "Who can tell us?"
2. Conduct a brief review by lining up the children's chairs in a straight row and ask, "Is there a chair for each child?" "Sit down on the chairs." "Are there any left over?" "Does each child have a chair?" etc.
3. Have the children place their macaroni and lima beans in two rows across their desks so they can put the two sets in one-to-one correspondence with each other. (It helps to have them line up one kind first then the other.) Then say: "Is there a macaroni for each lima bean?" "Is there a lima bean for each macaroni?" "Each macaroni is paired with a lima bean, what can we say about these two sets?" The desired response is to get the children to use the term "as many as" to indicate that the sets are of the same size. Encourage them to use the term "as many as" in talking about the two sets.
4. Take two sets of felt figures. Place one set on the felt board in a straight row. Ask a child to put the other set in a row

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right under the one you put up. (Use ducks, rabbits, fish, cats, dogs, or the like.) Then ask questions similar to those in activity number 3 above, leading the children to the use of the term "as many as." Then summarize this step by saying, "Each duck is paired with a rabbit and each rabbit is paired with a duck. We have as many ducks as we have rabbits. We have matched the set of rabbits with the set of ducks. What can we say about the two sets?"

5. The teacher places two different sets on the flannel board only this time use squares, discs, stars, or the like. Instead of making horizontal rows, make vertical columns pairing the members of the two sets. Now pose questions again similar to those used in the preceding activities.
6. Place two different sets on the flannel board and have one more object in one set than in the other set. Then ask, "do the sets match?" "Are there as many apples as there are pears?" "Are there as many pears as there are apples?"

LESSON III

Objective:

To develop the children's ability to find out if one set has "as many as" another set.

Duration:

30 minutes per week

Materials needed:

1. Felt board and cut outs
2. Counters for the children (lima beans or crayons)

Background:

In Lessons I and II, we introduced the children to the notion that two sets are matched if the members can be paired in one-to-one correspondence, i.e., if for every element in one set there is an element in the other set with which it can be paired or matched. This week's lesson will be a continuation of this idea.

Activities:

1. The teacher places ten apples and ten pears on the flannel board in two clusters (not in rows or columns). Then ask, "How can we find out if the set of apples matches the set of pears?" If any child says "pair them," let him come to the flannel board and do the pairing. If no one suggests it, you make the suggestion and then have a child arrange the sets pairing the elements. Next ask, "What can we say about these two sets?" Encourage use of "as many as" by posing questions similar to those used in Lessons I and II.
2. Have the children take out ten of their lima beans or a box of crayons. Pair the children off. Then explain the rules for a simple game on matching. The children can keep score if they want to.

The game is played like this. One child takes a number of counters without his partner seeing how many he took. The other player takes some of his counters to see if he can match the set of his opponent. They take turns matching each opponent's set. The person who is able to match his partner's set gets a point. If he is not able to match it his opponent gets the point.

Continue this game until time is consumed or interest decreases. Have the children change partners if time permits.

LESSON IV

Objective:

To introduce the concept that the elements of a set may be moved about without changing the numerousness of the set.

Duration:

30 minutes per week

Materials needed:

1. A flannel board and cutouts (including at least ten "macaroni" and ten "lima beans.")
2. Children will need ten macaroni and ten lima beans.

Background:

In the first three enrichment lessons, the children were introduced to the concepts of one-to-one correspondence and "as many as." In this week's activities, we wish to extend the idea of "as many as" by transforming one of the sets of objects, i.e., bunching them together or spreading them apart. Once again, we will employ one-to-one correspondence to "test" the concept of "as many as," both before and after the transformation. It is important that the child has his own objects with which he may work.

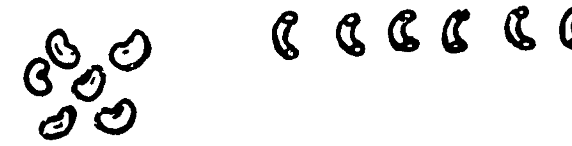
The first activity is a review of the previous programs and will be done simultaneously by the teacher and pupils. The major portion of the time should be devoted to the development of the constancy of numerousness of a set of objects no matter how they are moved about. In the succeeding activities, the teacher will present two sets of objects in one-to-one correspondence. Here the emphasis is upon the constancy of numerousness of a set of objects regardless of how these objects are arranged.

There is no particular reason for the teacher and the children to use the specific materials that are listed in these lessons; however, it is important for the children's concrete devices to be similar in appearance to those that the teacher uses on the flannel board.

1. Have two groups of children come to the front of the room (five in each group). Arrange them in linear matched (or paired) groups. Then ask, "Are there as many in this group as in this group?" "Does each child have a partner?"
2. Then leave one group where it was lined up and ask the other group to form a circle apart from the other group and ask, "Are there as many children in the circle as there are in the line?" "Are there as many in the line as in the circle?" "How can we find out?" Encourage them to see that we could pair them off again if necessary.
3. The teacher will ask the children to put a lima bean on their desks for every one she puts on the flannel board. She then places six lima beans on the board—slowly so the children can follow her actions and reproduce the same objects on their desks.
4. The teacher then asks them to arrange the same number of macaroni on their desks so that the physical arrangement looks something like that on the flannel board.

Then ask:
"Are there as many macaroni as lima beans?"
"Do the sets match?"
"Is there one lima bean for each macaroni?"

5. "Rearrange your sets on the flannel board so that one set is still in a row and the other set is grouped in a cluster to one side like this":



Then ask:
"Are there as many lima beans as macaroni?"
"Are there as many macaroni as lima beans?"
"How can we find out if there are as many lima beans as macaroni?"
If a child says, "pair them," let him do it at the flannel board. If no one suggests this, you should do so and ask some child to make the arrangement. After they are matched ask, "Do the two sets match?" "Are there as many macaroni as lima beans?" "Are there as many lima beans as macaroni?"

6. Arrange two new sets on the flannel board using figures like animals, birds, fruit, or etc. Try different arrangements and repeat questions similar to those used in activity #3 above.
7. Have different children perform similar arrangements on the flannel board.

LESSON V

Objective:

To continue the development of the concept of constancy of numerousness in exercises involving "as many as" and movement of sets.

Duration:

30 minutes

Materials needed:

1. Felt cutouts—sets of five and six in different categories
2. Books and chairs
3. Counters for children

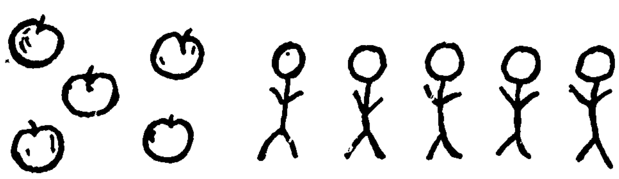
Background:

The lesson for this week is similar to the one presented to the children last week. This concept is rather difficult for children to grasp. When we change the configuration of a set, children have a tendency to base their judgment of the numerousness of the set on the space over which the members are distributed instead of relating the elements involved to their first observation of the original arrangement of the objects. There will be more emphasis in thi lesson on the semi-concrete development this concept.

Activities:

1. Arrange a row of ten chairs at the front of the classroom. Ask ten children to sit in the chairs and then ask, "Are there as many chairs as children?" "Are there any chairs left over?" Then ask the children to return to their seats. (If this is done with a group larger than ten, then have the ten children stand together in a position apart from the chairs). Now ask: "Are there as many children as chairs?" "Are there as many chairs as children?" "How could we be sure?"
2. Have a child place a reading book on each of the ten chairs at the front of the room. Ask, "Are there as many books as chairs?" "Are there as many chairs as books?" "Are there any books without chairs?" "Are there any chairs without books?"

   Take the books from the chairs and stack them on a table or the teacher's desk at the front of the room. Ask, "Are there as many books as chairs?" "Are there as many chairs as books?" "How could we find out?"
3. Use five apple cutouts (or other objects) and five children cutouts and place them in a pattern like, that indicated below.



   Then ask, "Is there an apple for each child?" How could we find out?" If a child suggests matching or pairing the cutouts, let him do so as the group watches.
4. Place six baseball mitts on the flannel board in any arrangement as long as they appear to be grouped. Then put six baseballs on the flannel board in a different location. Ask, "Are there as many baseballs as mitts?" "How could we find out?" Regroup them and ask questions similar to those used above.
5. Have the children take out some counters and form two sets that match. Then have them pull the two sets to opposite sides of their desks and ask, "Are there as many in this set as this set?" (If they are not certain have them pair the two sets off again.)

   Pull them to opposite sides again. Have them add one counter to each set and repeat the questions asked above.

   Have them repeat this adding to and also taking away equal numbers of counters from each set and pose appropriate questions.

LESSON VI

Objective:

To extend and refine the idea that the elements of a set may be moved about without changing the numerousness of the set.

Duration:

30 minutes

Materials:

1. An 18" X 3" (or longer) piece of poster board
2. Spring clothespins (two different colors)
3. A piece of pegboard
4. Golf tees
5. Coffee stirring sticks or tongue depressors
6. Paper hats from lesson 1

Background:

This lesson is an extension of the concepts stressed during the two preceding weeks. The previous grouping of objects was related to bunching the objects in different locations. The lesson this week will stress also the idea that numerousness does not change when the objects are spread apart. As before, the one-to-one correspondence will be used as a "test" of the concept "as many as," both before and after transformation of the sets. The major emphasis is on the constancy of numerousness of a set in spite of various transformations that are performed.

Activities:

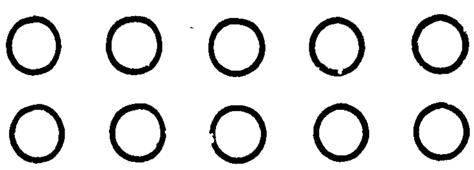
1. Have two sets of five children come to the front of the room. Give one group five red hats and the other five green hats. Have them pair off at arms length in two straight rows. Ask questions to establish the one-to-one correspondence of the two sets.

   Have one row remain in position while you have the other row move closely together (still in a row, however). Then ask, "Are there as many in the red hat row as the green hat row?" "Are there as many in the green hat row as in the red hat row?" "How could we find out?" Have them pair off again to verify their answers.

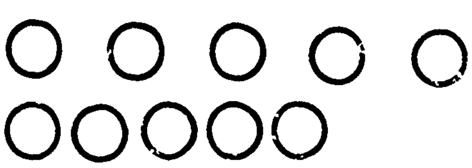
   Next, have one row remain in position while you have the other one spread farther apart. Repeat questions similar to those posed before.

2. The teacher and the pupils will establish a one-to-one correspondence between two sets of objects. This can be done with a flannel board on a pegboard. In either case the children ought to have counters similar to those the teacher uses. For example, if the teacher uses the pegboard and golf tees, having the children put a lima bean on their desks for every golf tee (white) the teacher puts in the board will serve as a similar visual pattern. (Be sure to spread the tees far enough apart so that each child can easily distinguish the separation of the elements in the set.)

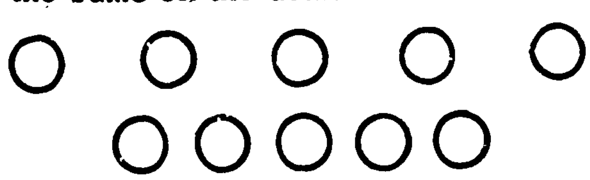
   Arrange a pattern with two sets of five elements in matched rows. Have the children do the same at their seats.



   Now have them move the bottom row tightly together while the top row remains as it was. (You do the same on the demonstration board.) Then ask questions similar to those used in activity #1.

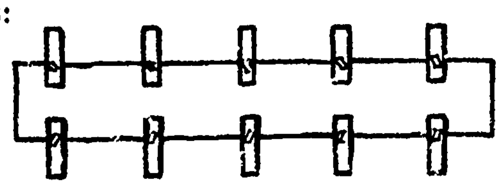


   After they check by putting them back in one-to-one correspondence, have them spread out the top row while you do the same on the demonstration board.



   Ask questions similar to those in the last transformation.

3. Arrange beans in one row and a cluster at the bottom of their desks. Ask similar questions.
4. Take the strip of poster board and the spring clothespins and arrange them like this:



   Ask, "Are there as many green clothespins as red ones?" "Are there as many red clothespins as green ones?"

   Then move one set closer together and ask, "Are there as many green pins as red pins?" "Are there as many red pins as green pins?" "How can we tell?" Let a child pair them off again to check the one-to-one correspondence.

   Repeat these questions with one row spread out over the entire length of the poster board strip.
5. Another variation might be to have one row of clothespins at one end on top of the poster board and the other row at the opposite end on the bottom of the poster board. Then repeat questions similar to those posed in other activities in this lesson.
6. Continue with related activities until time is consumed.

LESSON VII

Objective:

To introduce the concepts of "more than" and "fewer than."

Duration:

30 minutes

Materials:

1. Counters for the children
2. Flannel board and cutouts
3. Books
4. Crayons
5. Record and record player

Background:

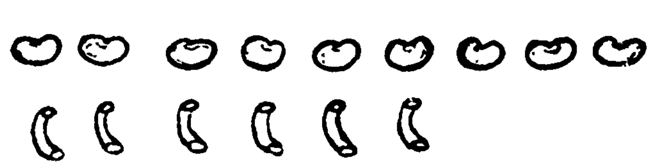
Throughout the previous programs, both sets being considered have contained the same number of elements. The emphasis has been on such facts as (1) for every element in one of the sets, there is a corresponding element in the other and *vice versa*; and (2) no matter how a set of elements is regrouped, the numerousness of the set remains the same (this is called the *conservation of number* or the *conservation of numerousness*). This latter idea could be shown to be true by re-establishing the one-to-one correspondence between the transformed set and the set which remained unchanged (the latter set is often called a *model set*).

In this lesson, one of the sets will continue to be a model set while the other set will be changed with respect to the ideas of (1) more objects than those in the model set or (2) fewer objects than those in the model set. When discussing objects, the terms "more than" and "fewer than" are used exclusively. Do not use "larger than" or "smaller than." Do not have the children count at any time. To determine if more than or fewer than exists, attempt to establish a one-to-one correspondence. If one set does not have a partner in the other, then one set (1) does not have as many elements as the other, (2) has "fewer than" the other. The set that has no partner for some of its elements is the set containing more elements than the other set, or that set that has "more than" the other.

Activities:

1. Have twelve books on the teacher's desk. Have each of the (ten or fewer) children take a book, and ask, "Does everyone have a book?" "Does every book belong to someone?" "Do we have more books than children?" If there appears to be some hesitation to the children's responses, carefully show that some of the books do not belong to anyone; that more children are needed in order to have each book belong to a child.

   "We have more books than children, because there are some books that do not belong to anyone."
2. Put seven crayons on a table at the front of the room. Ask each child to go pick up a crayon. When the crayons are expended ask, "Does everyone have a crayon?" "When there are not enough crayons so each child can have one, we say there are fewer crayons than children." We have fewer crayons because there are some children who do not have a crayon."
3. Give five books to one child and three to another child. Ask, "Who has more books, Bob or Susan?" "Does Bob have more books than Susan?" "How can we tell?" "Are there more books in this pile than this pile?"
4. Do the same as in activity #3 but this time stress the term "fewer than."
5. Have the children take out their macaroni and lima beans. Use your felt cutouts of the same objects and ask the children to duplicate on their desks what they see you put on the flannel board. Put up nine lima beans and six macaroni in rows like this:



   Check the children to be sure they have one set larger than the other. (It is not necessary that they have exactly nine and six.) Have the children make up some questions about these sets. Ask the following questions if the children do not pose them: "Is there a lima bean for each macaroni?" "Does every macaroni have a lima bean for a partner?" "If there are some lima beans without macaroni partners, what can we say about the lima beans?" "What can we say about the macaroni—are there more macaroni than lima beans or fewer macaroni than lima beans?" "Which set has more?" "Which set has fewer?"
6. Repeat this activity with seven lima beans and six macaroni.
7. If time permits you could play a short game of musical chairs. Put out enough chairs so that each child has a seat. Then have them stand and take away a chair. Ask, "Are there more children than chairs?" "How can we find out?" Other pertinent questions could be posed occasionally during the game to emphasize "more than" and "fewer than."

LESSON VIII

Objective:

To extend the ideas of "more than" and "fewer than" and one-to-one correspondence.

Duration:

30 minutes

Materials:

1. Counters for children (lima beans and macaroni)
2. Felt board cutouts for teacher

Background:

The lesson this week is an extension of the concepts of "more than" and "fewer than" which was introduced in the last lesson. If there are any activities which the teacher was unable to cover in last week's lesson, these should be used as an introduction to this lesson. The emphasis this week will be more abstract than it was last week. The activities will be centered more on visual observation than on physical manipulation.

Activities:

1. Have the children take out about eight lima beans and eight macaroni. (Do not stress the number but be sure they have two equivalent sets.) "How can we be sure we have as many lima beans as we have macaroni?" We hope they suggest matching the two sets by putting them in one-to-one correspondence. Check to be sure each child does this. You will then

have a double-check to insure that each child has two equivalent sets.

2. Ask them to bunch the two sets in different places on their desk. Then ask, "Are there as many macaroni as lima beans?" (and vice versa). Make certain they all think there are the same number in each set. (If there are still some who are uncertain, have them verify it by matching the sets again.)

   Now have them take one lima bean away and put it back in thei[illegible] box. Ask, "Are there as ma[illegible] beans as macaroni?" "How c[illegible] we tell?" (Let them match them if they need to.) "What can we say about these two sets?" Urge them to use the terms "more than" and "fewer than." (It may be wise at this point to do the same thing with the children at the front of the room.)

3. Use your flannel board and some colorful sets of objects. Start by putting up three rabbits and four apples in grouped clus[illegible] Have the children make a simil[illegible]ngement at the[illegible] Ask, "Which set has more than the other set?" "Which set has fewer than the other set?" "How do you know?" If they say the set of apples looks larger, ask them to watch carefully as you put one more object (alternately) in each set until you have a set of seven and a set of eight. Then repeat your questions.

4. Now put up another rabbit and see if they can determine that there are "as many" in one set as in the other. Have them answer more questions and verify their answers by matching the sets.

5. Next, take one apple away and "bunch" the two sets in two different locations on the flannel board. Repeat the questions posed in activity #3.

6. Let some of the children demonstrate sets of "more than" and "fewer than" on the flannel board. If you want to make a game of this, you can split the class in two teams and let them keep score with "Xs" on the chalkboard. One member of one team would put up some sets and ask a member of the other team if there are "more than," "fewer than," or "as many as" in each set. A correct response merits an "X" on the board for the team replying. After the game ends, you could have them decide which team had "more than" and/or "less than" the other in terms of the number of "Xs" on the chalkboard.

LESSON IX

Objectives:

To move from the concepts of more than and fewer than to construction of equivalent sets having "as many as."

Duration:

30 minutes

Materials:

The children will need twenty counters (lima beans)
Flannel board animals
Plastic cups (6)

Background:

The lessons to this point have stressed as many as, more than, and fewer than. The lesson this week will be a play on these concepts which will lead eventually to the additive and subtractive idea of number.

Activities:

1. Have the ten children form two lines in different places at the front of the room. Put six in one line and four in the other. Ask, "Which line has more than the other?" "Which line has fewer than the other?" "How can we tell?" (Have the two lines pair off to confirm their statements.)
2. With the two lines next to each other, ask, "How could we make this short line have as many as the other line?" (The children will no doubt suggest adding two more children to the short line. Tell them that we are not allowed to do this ... we must use only the ten children we have.) Lead them to see that by taking one from the long line and putting that person in the short line we make both lines the same length—or one will then have as many as the other.
3. Now put eight children in one line and two in the other and repeat the activities and questions of #2 above.
4. Have the children go to their desks and take out two sets of lima beans of twelve and eight. (You will have to help the kindergartners with this. Two even-numbered sets of most any size will work.) Have them line up the pile they think is more than the other first. Ask, "Are these sets the same size?" "How can you tell?" (If they suggest pairing let them do so—otherwise you should encourage it.)

"Using just these counters, how can we change the sets so they will both have the same number?" (They should observe that they can take one from the larger set and put it in the smaller set until they are of the same size.)

5. Repeat the activities in #4 above changing the size of the sets. You could use 14 and 6, or 11 and 9, etc. This stage of manipulation should receive much attention.
6. Display a model set of six rabbits on the flannel board. "How would we pick up enough to feed the rabbits?" (Have some plastic bowls such as cottage cheese containers that could be placed on the table in front of the rabbits; or, use flannel pictures of bowls. Also, use felt discs to represent food.) Extend questioning by using 12 "pieces of food" placing one piece in each bowl and then asking appropriate questions.

LESSON X

Objective:

To continue transformations where we make equivalent sets from two sets of different size.

Duration:

30 minutes

Materials:

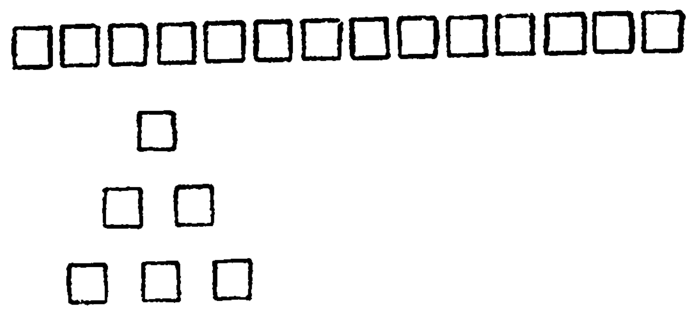
Counters for the children
Flannel board and felt cutouts of squares or circles

Background:

Last week's lesson was an introduction to the idea that two sets of unequal size could be transformed to make two sets of equal size. Most of the activities were at a concrete level. This week we will continue our attention to this concept at the concrete and semi-concrete levels. This concept is an important forerunner to solving problems involving addition and subtraction. This lesson will also serve as a review of terminology.

Activities:

1. Start by having the children take out two sets of lima beans—any number they happen to pick up will do. Then ask, "How can we make one set have as many as the other set?" (They should do this by pairing the two sets. If after they pair them they are still not equivalent, have them put the odd counter back in their counter box.)
2. Have them take four away from one set and put them with the other set. Then say, "Point to the set that has more than the other set." "Point to the set that has fewer than the other set." "How can we make them both have the same number?"
3. Repeat activity #2 using the children and paper hats.
4. Place fourteen squares in a row on the flannel board and six squares in a corner as a cluster, like this:



Ask, "Who can point to the set that has more than the other?" "Who can point to the set that has fewer than the other?" "How can we check this?" "Who will pair them off for us?" "How can we make it so one set has as many as the other?" (Let the children take turns forming different sized sets. You might encourage them to ask their own questions using ` ` terms "as many as" "more than" a than.") Also ask questions -e there more ...than or fewer ... than when we first saw them on the flannel board?"

5. They could use other cutouts to do things like putting up six ducks and three apples and then have them tell how many more apples are needed so each duck will have an apple. They should complete the visual arrangement by supplying the apples needed to form the two equivalent sets.

LESSON XI

Objective:

To continue emphasis on constancy of number in operations involving transformations related to addition and subtraction.

Duration:

30 minutes

Materials:

Counters for the children—eight of two different kinds
Flannel board house and felt figures
Paper hats from Lesson 1

Background:

The last few lessons have treated the idea of transformations in constructing equivalent sets. This lesson will employ the model set as a check on equivalence and also as an introduction to the idea of the constancy of number in an addition operation.

Activities:

1. Ask five children to come to the front of the room and put on the green hats. (Do not mention the number of children in the group.) They should form a row at arm's length.

   Have five others put on the red hats and stand in a group somewhere apart from the green hats.

   Ask, "Are there as many with green hats as red hats?" "How can we tell?" Have them pair off.

   Then have two of the reds move to one side of the room and three move to the other side away from the greens. Ask, "Are there as many red hats as green hats?"

   If they are uncertain have them pair off again.

   Repeat this type of activity making all of the combinations possible which sum to five or six, i.e., 3 + 2, 2 + 3, 4 + 1, 1 + 4, etc. Ask similar questions each time emphasizing the fact that the total does not change even though our red hat subsets change locations and size.

2. Have the children take out ten lima beans and ten macaroni. Tell them to put their lima beans in a straight row across their desks and group the macaroni in two sets apart from the lima beans. (You should do the same at the flannel board. Do not be concerned that everyone has his macaroni grouped the same as yours or as the other children.)

   Ask, "Are there as many lima beans as macaroni?" "Check to be sure."

   After they have paired them off ask them to pull three macaroni away to one position and the others to a different position from the lima beans. Then ask appropriate questions.

   Ask, "Are there as many macaroni as lima beans on your desk now?" If they seem uncertain have them check their sets. Repeat this activity and similar questions placing different size sets of macaroni in various places on their desk tops. You or a child should repeat the same combination at the flannel board.

3. Another variation which has much interest for the children is to use a felt house and cutouts of children and adults. One child would come to the flannel board and name the members of his family as he placed felt figures in a row. (Adults together and children together) Keep that child at the board while you ask another to put a row of discs under the felt figures. The row of discs will then be used for a model set.

   Ask the first child to show what his family does at different times during the day. Example—say, "Show me what your family is doing after dinner in the evening." The child would then place the various individuals in the rooms and talk briefly about their activities. Then ask the class, "Are there as many people in the house as there are discs in our model set?" "How do we know?" "Are there more people in the living room than there are in the kitchen?" "Are there more people in the living room than in the model set?" etc.

LESSON XII

Objective:

To strengthen concepts of "as many as," "more than," and "fewer than" through replication of model sets.

Duration:

30 minutes

Materials:

Counters for children (pop bottle caps)
Flannel board
Worksheets

Background:

This is the final lesson on conservation of numerousness. The lesson will review concepts that have been stressed throughout the past twelve weeks but do it in a manner different from most of the previous lessons. In this lesson, we shall combine concrete and abstract tasks to both review and assess.

Activities:

1. Give each child six pop bottle caps. Give the children the mimeographed worksheets. Ask them to take the two with the horse at the top and place them side by side on their desks. Then have them place a counter on each dot on the two sheets.

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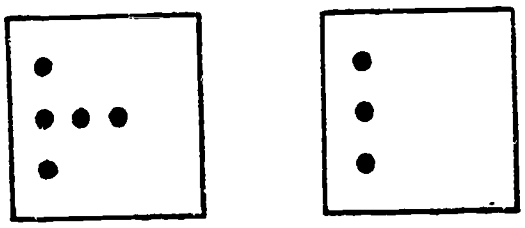
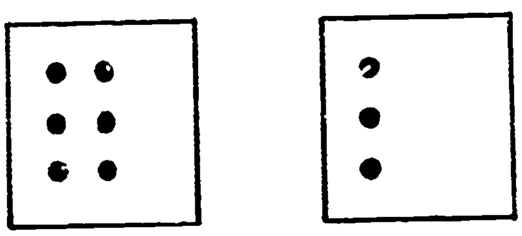
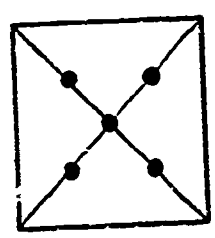


After they have placed their counters, say, "Point to the sheet that has more than the other." "What can we say about the other sheet?" "Make the sheet with 'fewer than' have the same number as the sheet with 'more than.'" "Now what can we say about the two sheets?"

2. "Take the two sheets with the tractor at the top and place them side by side on your desk. Place your counters on the dots on both sheets. Point to the sheet that has fewer than the other one. What can we say about the other sheet? Make the sheet with fewer than have the same number as the sheet that has more than. Now what can we say about the two sheets?"



3. "Take the sheet with the lamb at the top. Put your counters on the sheet so it looks like this ...



(Use the flannel board to show this pattern.) "Leave the middle counter where it is and move the others toward the corners on the lines. Are there more counters now, or were there more before we moved them, or are there the same number now as before?"

Have them move them back where they were at first and then remove the center counter. Ask questions similar to those posed in the last exercise.

4. Have them put the five counters back where they were and then move the top two counters out on the lines to the corners. Ask questions similar to those used earlier to emphasize the idea of "as many as." Different separations may be used.

5. Use the sheets with the square, circle, triangle, and rectangle. Each worksheet has a set of boxes at the top of the page. Have the children put a counter in each box and determine that they accept the fact that, "There are as many counters as boxes"; that, "The set of boxes can tell us if we still have as many counters as before."

There are three pages for this exercise. For each sheet the children will take their counters from the boxes and place them on the geometric figure.

At each step ask if there "are as many counters as there were at the beginning." That is, are there still as many counters as boxes at the top of the first page? Then ask, "What do you notice about the counters as you go from one page to the next?" "Each time you change the shape in which you place the counters, but you still have as many counters as before. We can take any given set of counters and place them in any shape and there will always be as many counters as those with which we started."

On the triangle sheet take one away from each child and ask questions.

On the rectangle sheet give one more to each child and ask questions.

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