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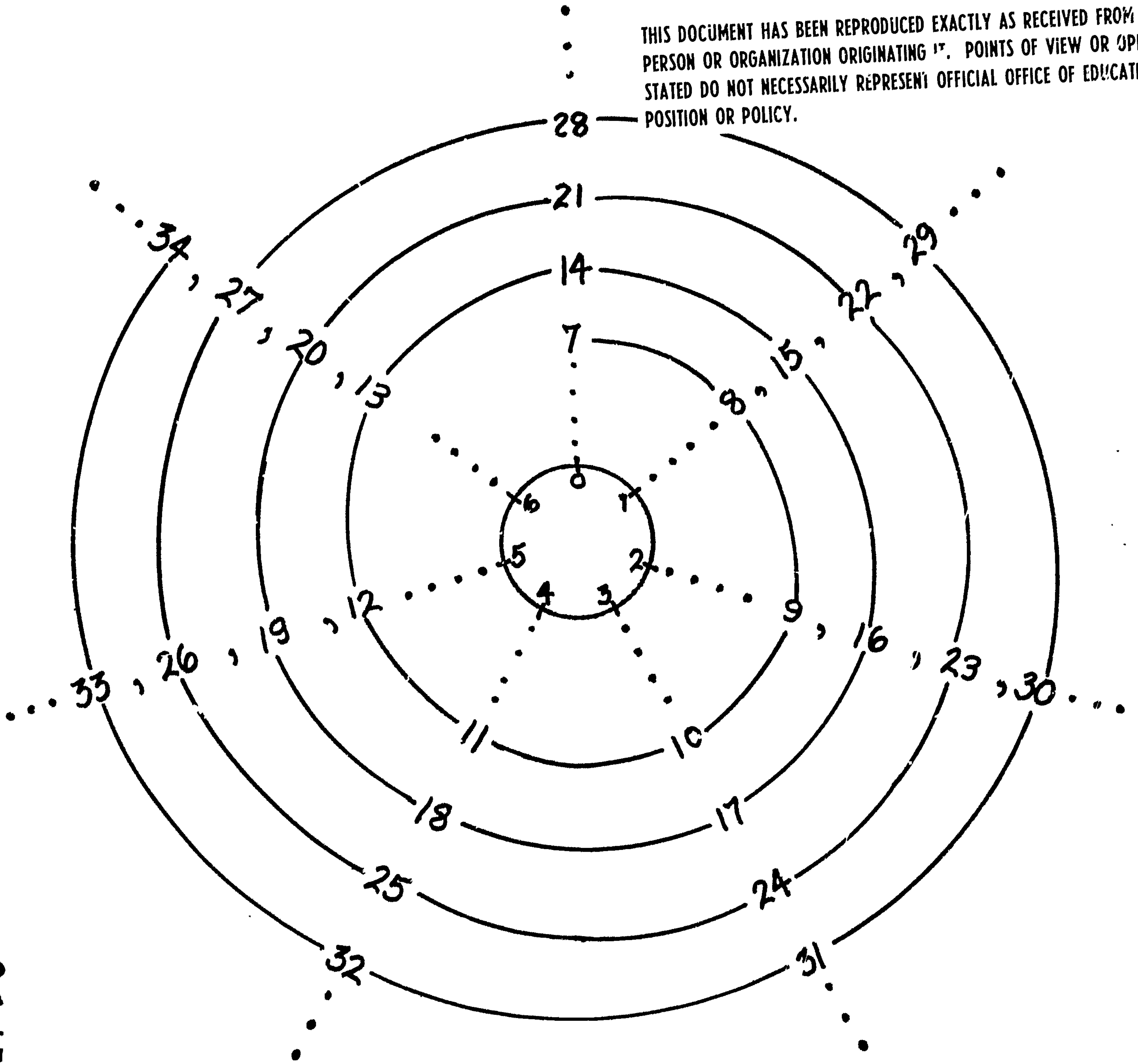
This booklet, one of a series, has been developed for the project, A Program for Mathematically Underdeveloped Pupils. A project team, including inservice teachers, is being used to write and develop the materials for this program. The materials developed in this booklet include (1) addition, subtraction, and multiplication in modulo seven (related to the days of the week), (2) congruency and equivalence classes, and (3) some basic properties of whole numbers for the indicated operations in modulo seven. (RP)



MODULO SEVEN

ESEA Title III

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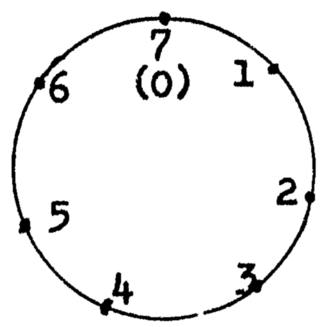
MODULO ARITHMETIC

The set of whole numbers continues on and on and we never reach a "largest whole number." We can always add one to a number and reach a larger number. In modulo arithmetic we will work with a largest number. It is called the modulo. In our regular work with numbers, we can represent numbers on a straight line that goes on and on. In modulo arithmetic, numbers can be represented on a circle. Let's choose something you already know about to work in our new mathematical system. The days of the week would be a good choice as we use them over and over. Let's give each day of the week a number.

Sunday	-	1st day of the week.
Monday	-	2nd day of the week.
Tuesday	-	3rd day of the week.
Wednesday	-	4th day of the week.
Thursday	-	5th day of the week.
Friday	-	6th day of the week.
Saturday	-	7th day of the week.

After the 7th day, we go back to the 1st and begin all over again. This is an example of a cycle. Now put these numbers, representing the days of the week, on a circle.

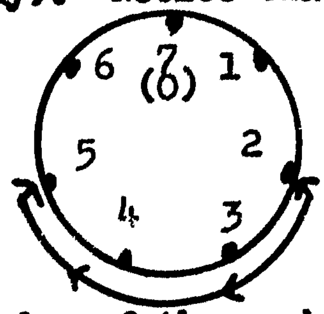


Addition

Now suppose we try some problems related to the days of the week. Suppose it is the 2nd day of the week (Monday). What day of the week will it be in 3 days?

2nd day
+ 3 days
5th day

This was easy because we began with the 2 on our circle and counted 3 days. We end up at the 5th day (Thursday). Notice that we add by counting clockwise.



Let the first number be the day of the week and the second number be the number of days we are adding on. Show these on a circle.

Day of the week -	2nd day	3rd day	4th day
Days to add -	+ 4 days	+ 4 days	+ 3 days
	6th day	7th day	7th day

Would we arrive at the same answer if we let the second number be the day of the week and the first number be the number of days we are adding on?

Day of the week -	4th day	4th day	3rd day
Days to add -	+ 2 days	+ 3 days	+ 4 days
	6th day	7th day	7th day

As you can see, in each case, the answer is the same.

Suppose we start at some number on our circle and add 7 days. Would the answer be the same as the number you started with? For example, will 7 days from Thursday still be Thursday? Will 7 days from Friday still be Friday?

1st day (Sunday)	3rd day (Tuesday)
+ 7 days	+ 7 days
1st day (Sunday)	3rd day (Tuesday)

As you see, to add 7 in this system is much like adding <u>zero</u> in our regular system. Try adding <u>any</u> multiple of 7, such as 14, 21, 28, to any number on the circle. If you add by counting in a clockwise direction, then you end up at your starting point.

2nd	2nd	2nd
+ 14	+ 21	+ 28
2nd	2nd	2nd

Could numbers, other than whole numbers, arise in our system? Could it be part Friday and part Saturday, or must it be either one or the other? You are right, we won't have fractions. Do you also notice that our answer must be on the circle and the <u>largest</u> number here is a 7? When we get to <u>7</u>, counting, we start over again. Try some porblems that normally add to a number greater than 7.

	3rd + 6	4th + 8	6th + 9	7th + 15
Regular addition answer	9	12	15	22
Modulo answer	2nd	5th	1st	1st

Our <u>modulo answer</u> is the remainder, when 7 is divided into each <u>regular addition answer</u>.

$$\begin{array}{r} 1 \\ 7\overline{)9} \\ \underline{7} \\ 2 \end{array} \qquad \begin{array}{r} 1 \\ 7\overline{)12} \\ \underline{7} \\ 5 \end{array} \qquad \begin{array}{r} 2 \\ 7\overline{)15} \\ \underline{14} \\ 1 \end{array} \qquad \begin{array}{r} 3 \\ 7\overline{)22} \\ \underline{21} \\ 1 \end{array}$$

Can you explain why these remainders would be our answers?

Think about the problems where we added a number greater than 7. Could we have divided the number by 7 and just added the remainder rather than the dividemd? Look at the case where we added 8 days to the 4th. Let's write it in this way.

4th + 8 = 4th + (7 + 1)

To add 7 and then 1 would be the same as adding 8.

4th + 7 + 1 =
4th + 1 = 5th

As you know, to add 7 would merely carry us once around the circle, so we add 1 to 4 rather than 8. Remember that adding 7 or a multiple of 7 is like adding 0 in our regular system or a multiple of 0 which is 0.

The idea of adding to numbers on our circle can be much easier if we use the idea of divisors and remainders. In our case, the divisor will be 7. Now if you divide 7 into any whole number what are the possible remainders?

Remainder	Set of numbers that give this remainder	
0	{7, 14, 21, 28, 35, ...}	= A
1	{1, 8, 15, 22, 29, ...}	= B
2	{2, 9, 16, 23, 30, ...}	= C
3	{3, 10, 17, 24, 31, ...}	= D
4	{4, 11, 18, 25, 32, ...}	= E
5	{5, 12, 19, 26, 33, ...}	= F
6	{6, 13, 20, 27, 34, ...}	= G

The capital letters on the right of each set are used to refer to the sets. For example, every number in set B has a remainder of 1 when divided by 7. The remainder is shown on the left.

Now suppose we have to add one of the numbers in set C to a number on our circle. It does not make any difference which one you choose from this set. The answer will be the same. They all have a remainder of 2 when all cycles of 7 are divided out. If this is the case, we will use the smallest number in place of a larger number when adding.

Number on our circle	5	5	5	5	5
Number added	2	9	16	30	37
	7	7	7	7	7

Count around the circle using these examples and see if you don't end up at 7 in each case.

Now select the example where we added 30 to 5. Choose the closest multiple of 7 that is closest to 30. It is 28. Now write:

$$
\begin{aligned}
5 + 30 &= \\
5 + (28 + 2) &= \\
(5 + 28) + 2 &= \\
5 + 2 &= 7
\end{aligned}
$$

As you know, if you add a multiple of 7 to 5, you will end back at 5. If you add 28, it merely sends you completely around the circle 4 times.

Could you make an addition table for your circle? Below is a table that is not complete. Can you complete this table by counting around the circle? Since adding 7 is the same as adding <u>zero</u>, we will use a zero in place of 7.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4							
5							
6	6						5

Activities for Addition

What day of the week will it be:

1. 5 days after Saturday?
2. 30 days after Monday?
3. 65 days after Friday?
4. 105 days after Wednesday?
5. 210 days after Wednesday?

Solve these open sentences (counting on the circle).

6. $5 + 3 = \square$
7. $6 + 4 = \square$
8. $5 + 6 = \square$
9. $5 + 7 = \square$
10. $6 + 7 = \square$
11. $3 + 7 = \square$
12. $5 + \square = 5$
13. $6 + \square = 6$
14. $2 + \square = 2$

15. Solve the following open sentence by putting the same number (modulo 7) in each box.

$$\square + \square = \square$$

Subtraction

How would subtraction work in this system? You could count back, counterclockwise rather than clockwise. Try a few examples.

1. Begin at 4 and count back 3. $\begin{array}{r} 4 \\ \underline{-3} \\ 1 \end{array}$

2. Begin at 3 and count back 5. $\begin{array}{r} 3 \\ \underline{-5} \\ 5 \end{array}$

3. Begin at 2 and count back 6. $\begin{array}{r} 2 \\ \underline{-6} \end{array}$

Could you get these answers without counting back? In the case where we subtracted 5 from 3, it appears the 5 is greater than 3. What if you add 7 to the 3 and make the problem 10 minus 5 rather than 3 minus 5? The answer is the same.

$$3 - 5 = 5$$
and
$$10 - 5 = 5$$

Will it work for example 3?

$$2 - 3 = 6$$
and
$$9 - 3 = 6$$

Can you explain why this gives the same answer? Sure, to add 7 or any multiple of 7 is the same as adding zero, because it puts you right back where you started.

As we think a little more carefully about the numbers on our circle, we find that each number on the circle represents all numbers that give a particular remainder when divided by 7. Some illustrations will aid in clearing up this idea. If we subtract 23 from a number on our circle, say 3, we have:

$$3 - 23 = 1$$
or
$$3 - 2 = 1$$

The number 23 has a remainder of 2 when divided by 7. Then you would get the same answer if you subtracted any number that had a remainder of 2 when you divided it by 7. Then rather than subtracting 23, we could subtract the smallest whole number that gives a remainder of 2. This is 2, since 2 divided by 7 gives a remainder of 2. As you see, we always work with the smallest number and this will always be one of the numbers on our circle.

Two ideas have been used in our subtraction work:

1. Adding 7 or a multiple of 7 to the minuend as in the case of 3 - 23 = 1. We added 21 (7 x 3) to 3 and subtracted 23. 24 - 23 = 1.
2. Dividing the subtrahend by 7 and using the remainder as the new subtrahend. As in the case of 3 - 23 = 1, we divided 23 by 7 and used the remainder, 2, as the new subtrahend. 3 - 2 = 1.

Activities for Subtraction

What day of the week was it:

1. 6 days before Monday?
2. 35 days before Thursday?
3. 44 days before Sunday?
4. 102 days before Monday?
5. 56 days before Tuesday?

Solve these open sentences.

6. $6 - \square = 2$
7. $3 - \square = 4$
8. $4 - \square = 5$
9. $1 - \square = 1$
10. $3 - \square = 3$
11. $5 - \square = 5$
12. $7 - \square = 2$
13. $1 - \square = 7$
14. $6 - \square = 7$
15. $3 - \square = 7$

16. Solve the following open sentence by putting the same number in each box.

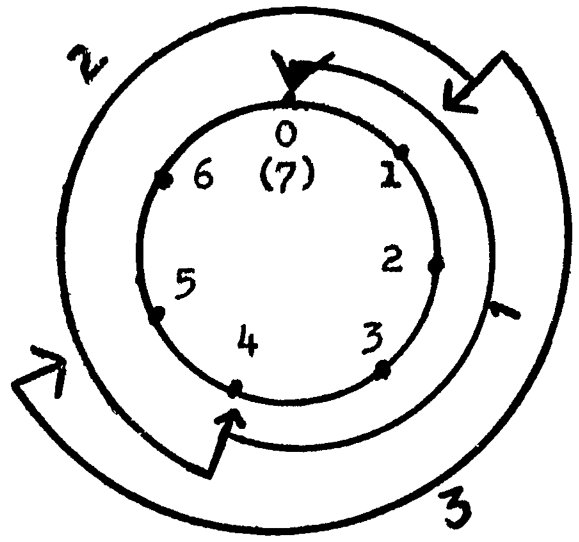
$$\square - \square = \square$$

Multiplication

The numbers on our circle 0, 1, 2, 3, 4, 5, 6, can be used to make a multiplication table. We will get our product in our regular way and then divide 7 into our product. The remainder will be the answer in our modulo system.

For example: Regular $3 \times 4 = 12$.
$12 \div 7 = 1$ with remainder 5.
Modulo $3 \times 4 = 5$.

To illustrate this answer, starting at zero, show 3 moves with 4 days in each move.



or $3 \times 4 = (4 + 4) + 4$
$= \quad 1 + 4$
$= 5$

It may be a little clearer if you show this as repeated addition. We could also list our sets of numbers that give the same remainder when divided by 7. The least number to give that remainder will be our answer, since these are the only numbers on our circle. It represents all the other numbers of the set and appears on our circle. The least number in each set is circled. Let a capital letter represent each set.

{A = 0, 7, 14, 21, 28, 35, 42, 49, ...}
{B = 1, 8, 15, 22, 29, 36, 43, 50, ...}
{C = 2, 9, 16, 23, 30, 37, 44, 51, ...}
{D = 3, 10, 17, 24, 31, 38, 45, 52, ...}}
{E = 4, 11, 18, 25, 32, 39, 46, 53, ...}
{F = 5, 12, 19, 26, 33, 40, 47, 54, ...}
{G = 6, 13, 20, 27, 34, 41, 48, 55, ...}

This may give a clear idea of our multiplication. Now multiply: 6 x 5 = 30. Thirty is in set C and has a remainder of 2 when divided by 7. The smallest number in set C with this same property is 2. Then 6 x 5 = 2.

The multiplication table is incomplete. Can you complete this table?

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4					
5	0	5					
6	0	6	5	4	3	2	1

Check some of these answers by doing regular multiplication and finding the product in one of the sets above. Then your modulo answer is the smallest number in that set. In other words it is a number on our modulo circle.

5 x 4 = 20; 20 is in set G
then 5 x 4 = 6

As you know, we could simply divide 20 by 7 and the remainder is our answer.

$$20 \div 7 = 2 \text{ remainder } 6$$

Congruency

Select any two numbers that are in the same set. Subtract one of these numbers from the other. Is 7 an exact divisor of the difference? Try two more. Suppose we select 41 and 13 from set G and then subtract 13 from 41. We get: $41 - 13 = 28$ and 7 is a divisor of 28.

In our system we are using 7 and it is called our modulus. If 7 divides the difference of any two numbers, we say that one of the numbers "is congruent to" the other number - modulo 7. The symbol "$\equiv$", using three straight line segments means "is congruent to." As an example: $41 \equiv 13$ (modulo 7), since 7 divides the difference, $41 - 13$.

Two numbers must come from the same set if they are congruent. This is to say that they must have the same remainder when divided by 7.

Test this idea by selecting a number from two different sets. Take their difference. Does 7 divide the difference? Say we select 28 from set A and 11 from set E.

$28 - 11 = 17$ and 7 is not a divisor of 17
Then 28 "is not congruent to" 11 (modulo 7)

EXERCISES

1. Construct a modulo 4 addition and multiplication table using the seasons. For example:

Spring - 1st
Summer - 2nd
Fall - 3rd
Winter - 4th (or 0)

2. Use a clock to make a modulo 12 addition table.

3. Use your modulo 7 addition and multiplication table to solve the following number sentences.

a) $\square + 3 = 1$	b) $\square + 6 = 2$
c) $6 + 4 = \square$	d) $5 + \square = 3$

4. Use your modulo 7 multiplication table to solve these number sentences.

a) $4 \times \square = 3$ (Note: Another form of writing this is: $3 \div 4 = \square$)

b) $\square \times 5 = 4$

c) $3 \times 6 = \square$

d) $6 \times 6 = \square$

5. Could the months of the year be used to illustrate a modulo 12 system?

6. The following is a summary of some basic properties of whole numbers for our regular operation. Illustrate each using modulo 7. Is each property true for modulo 7?

Let a, b, and c be any whole numbers.

A. The <u>Commutative Property</u> holds for addition and multiplication.
($a + b = b + a$ and $a \times b = b \times a$)

B. The <u>Associative Property</u> holds for addition and multiplication.
$(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$

C. The <u>Distributive Property</u> holds for multiplication <u>over</u> addition.
$a\ (b + c) = (a \times b) + (a \times c)$

D. The <u>Identity Element</u> for addition is "zero" and the <u>Identity Element</u> for multiplication is 1.
$a + 0 = a$ and $a \times 1 = a$

MODULO 7

Illustration of Terms

<u>Associative property</u> - the result of a process is not changed if the <u>grouping</u> is changed.

1) Addition <u>is</u> associative.

$$(2 + 5) + 3 = 2 + (5 + 3)$$
$$7 + 3 = 2 + 8$$
$$10 = 10$$

2) Multiplication is associative.

$$(2 \times 5) \times 3 = 2 \times (5 \times 3)$$
$$10 \times 3 = 2 \times 15$$
$$30 = 30$$

3) Division <u>is</u> <u>not</u> associative.

$$(80 \div 4) \div 2 \neq 80 \div (4 \div 2)$$
$$20 \div 2 \neq 10 \div 2$$
$$10 \neq 5$$

<u>Clockwise</u> - the direction in which the hands of a clock move. The arrows show a clockwise direction.

<u>Construct</u> - to make by putting together parts; to make a thing in such a way that it will meet the requirements of a certain set of rules.

<u>Commutative</u> <u>property</u> - the result of a process is not changed if the process is done in a different <u>order</u>.

1) Addition <u>is</u> commutative.

$$2 + 4 = 4 + 2$$
$$6 = 6$$

2) Multiplication <u>is</u> commutative.

$$4 \times 3 = 3 \times 4$$
$$12 = 12$$

3) Subtraction <u>is</u> <u>not</u> commutative.

$$10 - 4 \neq 4 - 10$$
$$6 \neq -6$$

4) Division <u>is</u> <u>not</u> commutative.

$$4 \div 2 \neq 2 \div 4$$

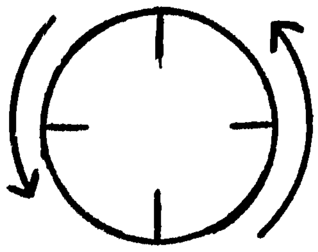
Congruency - a relation between two numbers which means that they both have the same remainder when they are divided by a fixed number. The fixed number is called the modulo.

1) A congruency relation modulo 7 exists among the numbers 4, 11, 32, 60, because they each have a remainder of 4 when they are divided by 7.

2) There is a modulo 2 congruency relation among the numbers 0, 2, 4, 10, 102, because all of these numbers have 0 remainder when they are divided by 2. In fact, there is a modulo 2 congruency relation among all even numbers because all even numbers have a remainder of 0 when divided by two.

Counterclockwise - the opposite of the direction in which the hands of a clock move.

The arrows show a counterclockwise direction.



Cycle - a group of things which keep repeating in the same order.

1) The months of the year form a cycle since they keep repeating, and their order does not change.

2) The batting order of the members of a baseball team is a cycle.

Distributive property - the result of a process is not changed when one part is spread out over another part.

1) Multiplication is distributive over addition.

$$
\begin{aligned}
2 \times (5 + 3) &= (2 \times 5) + (2 \times 3) \\
2 \times (8) &= 10 + 6 \\
16 &= 16
\end{aligned}
$$

2) Addition is not distributive over multiplication.

$$
\begin{aligned}
5 + (6 \times 2) &\neq (5 + 6) \times (5 + 2) \\
5 + 12 &\neq 11 \times 7 \\
17 &\neq 77
\end{aligned}
$$

3) Multiplication is distributive over subtraction.

$$
\begin{aligned}
5 \times (6 - 4) &= (5 \times 6) - (5 \times 4) \\
5 \times 2 &= 30 - 20 \\
10 &= 10
\end{aligned}
$$

4) Multiplication is not distributive over division.

$$
\begin{aligned}
7 \times (24 \div 3) &\neq (7 \times 24) \div (7 \times 3) \\
7 \times 8 &\neq 168 \div 21 \\
56 &\neq 8
\end{aligned}
$$

Dividend - a number that is to be divided by another number.

```
      6 quotient
2 ⟌ 13 dividend
     12
      1 remainder
```

Divides evenly - a division which has a zero remainder.

7 divides 56 evenly.

```
      8
7 ⟌ 56
     56
      0 remainder
```

Divisor - a number which is divided into another number.

Identity element - an element which when applied to another element under a rule (such as addition, subtraction, multiplication or division) leaves the element unchanged.

1) In addition and subtraction zero is the identity element.
   $6 + 0 = 6$; $19 - 0 = 19$; $100 + 0 = 100$

2) In multiplication and division, one is the identity element.
   $1 \times 14 = 14$; $21 \div 1 = 21$

3) If we are working with a congruency relation modulo 2, then any number which is a multiple of 2 is an identity element for addition and subtraction.

Minuend - a number from which another number is subtracted.

```
  17 minuend
-  4 subtrahend
  13
```

Multiple of a number - the result of multiplying that number by a whole number.

1) all even numbers are multiples of two.

2) some multiples of 3 are: (0, 3, 6, 9, 12, 15, 18, 21, 24)

3) some multiples of 6 are: (0, 6, 12, 18, 24, 36, 42, 48)

Product - the result of multiplying two numbers.

```
 12
x 5
 60 product
```

Remainder - the part of the dividend which is "left over" or not evenly divisible by the divisor.

Subtrahend - a number which is subtracted from another number.

```
  28 minuend
 - 9 subtrahend
  17
```