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AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR.
UNIT 12, TRIGONOMETRIC FUNCTIONS.

NEW YORK STATE EDUCATION DEPT., ALBANY

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THIS TEACHING GUIDE FOR TRIGONOMETRY IS THE FINAL UNIT OF A SERIES OF 12 UNITS FOR AN EXPERIMENTAL COURSE IN MATHEMATICS FOR GRADE 9. BACKGROUND MATERIAL FOR TEACHERS AS WELL AS QUESTIONS AND ACTIVITIES FOR CLASSROOM PRESENTATIONS ARE PROVIDED. A GLOSSARY OF MATHEMATICAL TERMS FOR THE 12 UNITS CONCLUDES THE REPORT. (RP)

Unit 12

an experimental course in MATHEMATICS FOR THE NINTH YEAR

II 12

THE UNIVERSITY OF THE STATE OF NEW YORK / THE STATE EDUCATION DEPARTMENT
BUREAU OF SECONDARY CURRICULUM DEVELOPMENT / ALBANY

AN EXPERIMENTAL COURSE

IN

M A T H E M A T I C S

FOR THE

NINTH YEAR

*Unit 12. Trigonometric Functions

*Optional

The University of the State of New York
New York State Education Department
Bureau of Secondary Curriculum Development
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AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR

Mathematics 9X

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Mathematics 9X

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FOREWORD

In April 1961, an advisory committee on secondary school mathematics convened at the Department to discuss the direction that secondary mathematics curriculum revision should take. This committee consisted of college and secondary school teachers, supervisors, administrators, and a consultant from one of the national curriculum programs. As a resul of this meeting, the recommendation was made that a revision of the mathematics 7-8-9 program be undertaken immediately.

This publication represents the last of a series containing the units for an experimental course in mathematics for the ninth grade. The other four consist of units 1-4, 5-7, 8-9, and 10-11, respectively. This final publication consists of material for optional unit 12 and a glossary.

The development in unit 12 pertains to an area which some members of the committee believed useful and interesting, but which the committee as a whole did not feel to be an integral part of a fundamental course. Thus, it is again stressed that this material is optional both in content and development, and so may be omitted, or developed in part or fully, along alternate lines. Although unit 12 develops a number of integrated concepts, its omission does not affect the completeness of the presentation in the remaining units.

The materials in the 9X experimental syllabus are based upon the foundations laid in the 7X and 8X experimental syllabuses. Therefore, it is to be understood that the 7X and 8X experimental courses are a prerequisite to the 9X experimental course. As in the 7X and 8X syllabuses, the chief emphasis is placed upon the understanding of basic mathematical concepts as contrasted with the all-too-frequently used program in which the mechanics of mathematics receives the greatest stress. The general approach and content used is that agreed upon by leading mathematical authorities as the most desirable. In the actual teaching of the program major emphasis is placed upon the "discovery process." The principal function of the teacher is to carefully set the stage for learning in an organized fashion such that the pupils will "discover" for themselves the fundamental concepts involved.

The materials in the mathematics 7X, 8X, and 9X experimental syllabuses include much of what today are called the basic ideas and concepts of mathematics. These concepts are those which the pupils will use throughout their study in mathematics. With this material the teacher should be able to aid the pupils to see the beauty of mathematics in terms of the fundamental structure found in mathematical systems. The important unifying concepts included in the new course of study for the ninth grade are:

- Algebraic Expressions and Open Sentences
- Analysis of Algebraic Problems
- The Set of Real Numbers
- Properties of Exponents and Radicals
- Operations with Polynomial Expressions
- Quadratic Equations
- Open Sentences in Two Variables
- Relations and Functions
- Trigonometric Functions

A new mathematical curriculum is not the sole answer to the improvement of mathematics instruction. Most important perhaps is the method of presenting the material. If the teacher develops lesson plans that will allow the pupils to discover concepts for themselves, the teaching and learning of mathematics will become excitingly different and no longer remain the dissemination of rules and tricks.

A special committee was formed to review the 9X syllabus and to make recommendations for the writing of materials. This committee consisted of the following: David Adams, Liverpool High School; Benjamin Bold, Coordinator of Mathematics, High School Division, New York City Board of Education; Mary Challis, Plattsburgh High School; Francis Foran, Garden City Junior High School; Eleanor Maderer, Coordinator of Mathematics, Board of Education, Utica; William Mooar Benjamin Franklin Junior High School, Kenmore; Verna Rhodes, Corning Free Academy; Leonard Simon, Curriculum Center, New York City; Joan Vodek, Chestnut Hill Junior High School, Liverpool; Frank Wohlfort, Coordinator of Mathematics, Junior High School Division, New York City Board of Education.

The original materials for this unit for the 9X syllabus were written by Charles Burdick, coordinator and teacher of mathematics, Oneida Junior High School, Schenectady, and by David Adams. The final draft was developed by Aaron Buchman, associate in mathematics education. The over-all project has been developed under the joint supervision of this Bureau and the office of Frank Hawthorne, Chief, Bureau of Mathematics Education. The final manuscript was edited and prepared for publication by Robert F. Zimmerman, associate in secondary curriculum, and Herbert Bothamley, acting as temporary curriculum associate.

Gordon E. Van Hooft
Chief, Bureau of Secondary Curriculum Development

William E. Young
Director, Curriculum Development Center

UNIT 12: TRIGONOMETRIC FUNCTIONS

Part 1. Background Materials for Teachers

12.1 INTRODUCTION

In the previous unit 11 (Relations and Functions) of the 9X program, the trigonometric functions sine, cosine, and tangent were reviewed as ordered pairs in which the second members were ratios of lengths of sides of right triangles. The definitions used there, by their wording, were necessarily restricted to functions of acute angles. A generalization of these definitions will result in meaningful trigonometric functions of any angle, negative or positive. Thus the work in this unit is an extension of previous concepts which will provide a more meaningful mathematical point of view.

This unit provides an excellent opportunity to combine many of the concepts of coordinate geometry and algebra with those of trigonometry. It is therefore suggested that this unit, although optional, be included in the 9X course because it can serve as a review of many of the concepts learned earlier and as an application of the pupil's previous knowledge to the formation of new concepts.

The unifying concept in the study of trigonometry in this unit is the unit circle. The concept of the unit circle simplifies the study of the trigonometric functions and their relations to one another.

The unit circle is a circle with radius 1 and center at the origin of a rectangular coordinate system. Any point on the circumference of the unit circle can therefore be located on the plane by reference to the coordinate system using an ordered pair of numbers or by reference to an angle formed by an initial ray along the positive side of the x-axis and a terminal ray through the given point. In this way, the relationship between coordinate geometry and trigonometry can be developed. The unit circle is also an effective concept to use in generating the graphs of the trigonometric functions. Using the unit circle as a basis for the graphs of the functions will result in a more meaningful experience than that of copying numbers from a table and plotting them.

The questions and activities used are written in a definite sequence which should be followed, but in a number of instances more questions may be needed to lead the pupils through the discovery process to the desired answer.

12.2 MEANING OF A TRIGONOMETRIC FUNCTION

The following basic concepts are to be developed:

(1) Working definitions of the trigonometric functions

(2) Use of conventions
(3) A trigonometric ratio associated with an angle represents a real number

The purpose of this unit is to develop further the concepts associated with the meaning of trigonometric functions and to develop formulas more useable in form than those used in unit 11 of the 9X course. This will be accomplished by using several conventions and abbreviations. It should be stressed to the pupil that the abbreviated forms still mean the same thing as the original terms. The concept of conventions is then discussed with respect to the lettering of a right triangle. The term convention will be used throughout the unit.

The sine function, for the present, is an association of an angle of a right triangle with the ratio of the length of the leg opposite the given angle to the length of the hypotenuse. Notice that this statement would have no meaning if the reference to the angle were not present. This is why the angle must be present whenever the function is mentioned.

12.3 DEFINITIONS INVOLVING THE GENERAL ANGLE

The following basic concepts are to be developed:

(1) Standard position of an angle
(2) Positive and negative angles
(3) Naming of quadrants
(4) Naming of angles
(5) The significance of rotations of 90°, 180°, 270°, and 360°
(6) Quadrantal angles
(7) Extending the definitions of the trigonometric ratios

The purpose of this section is to make clear several preliminary concepts and ideas that are needed to fully understand the basic trigonometric concepts which will be further developed in this unit. Although a number of the ideas are relatively simple, the pupil's ability in trigonometry is severely limited without some familiarity with these ideas. In question 5 under Standard Position of an Angle, several more examples should be developed by the teacher until the pupils are competent in placing the angles in standard position.

The introduction of the arrow to indicate positive or negative rotation eliminates the necessity of indicating which ray is the initial side and which is the terminal side. Again, in question 3 under Positive and Negative Angles, further examples should be developed by the teacher. The remaining material in the section is self explanatory. The teacher should develop more examples and questions for his group if he feels it is necessary to do so.

The following brief review of some concepts may aid the teacher in his development of the topic with his pupils. The teacher may also wish to review unit 4 of *Mathematics 7X* and unit 5 of *Mathematics 8X*.

An angle is the set of all points contained on two rays that have a common endpoint. This concept is now qualified by adding the idea of directed rotation, one ray being called the initial ray and the other being called the terminal ray. The measure of an angle is the amount of rotation about the common endpoint of a ray coinciding with the initial ray necessary to make this ray coincide with the terminal ray. In this course the unit of measurement of an angle is the degree, which is defined as $\frac{1}{360}$ of a complete rotation. As an additional characteristic of an angle, its measure will be given as a signed number. Rotation in the counterclockwise direction is taken as positive and rotation in the clockwise direction is taken as negative.

The first new concept in this unit is that of an angle in standard position in a coordinate system. Any two rays in a coordinate system that have a common endpoint form an angle. The angle is said to be in standard position if the common endpoint, or vertex, is the point (0,0) and if the initial ray is that part of the x-axis to the right of and including the point (0,0). It is assumed that any angle in a coordinate system is in standard position unless it is stated otherwise.

Given any angle in standard position with any point P with coordinates (x,y) being a point on the terminal ray, the trigonometric ratios of sine, cosine, and tangent of the angle are defined in terms of the abscissa and ordinate of point P and the radius vector of P, which is the distance from the origin to point P along the terminal ray. The abbreviation for the radius vector is r. The definitions of the trigonometric ratios may then be generalized so that they are based only on the angle, and the abscissa, the ordinate, and the radius vector of any point P on the terminal ray of the angle. No triangle is involved in the definitions.

If the angle is designated as θ (spelled theta), the sine of θ is defined as the ratio $\frac{\text{ordinate}}{\text{radius vector}}$. This may be written in abbreviated form $\sin\theta = \frac{y}{r}$.

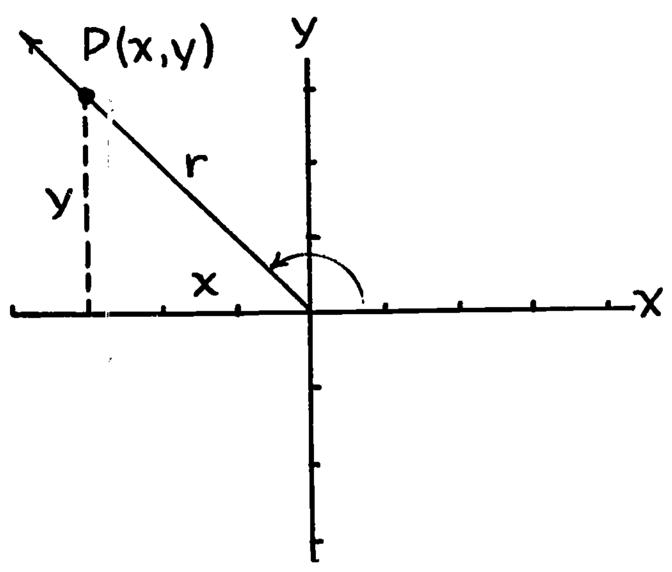
The cosine of θ is defined as the ratio $\frac{\text{abscissa}}{\text{radius vector}}$ and this may be written $\cos\theta = \frac{x}{r}$.

The tangent of θ is defined as the ratio $\frac{\text{ordinate}}{\text{abscissa}}$ and is written $\tan \theta = \frac{y}{x}$. For the teacher's information, three additional ratios are defined. The cosecant, secant, and cotangent of θ are defined as follows: $\csc \theta = \frac{r}{y}$; $\sec \theta = \frac{r}{x}$; $\cot \theta = \frac{x}{y}$.

Notice that cosecant θ, secant θ, and cotangent θ are the respective reciprocals of sine θ, cosine θ, and tangent θ.

It is possible for point P to be any point on the number plane (except for the origin), depending on angle θ. The abscissa and ordinate of P may be positive, negative, or zero, but the radius vector of a point P on the terminal side of angle θ will always be taken as positive. A distance from the origin, other than zero, is negative if it is measured along a ray which is opposite to that containing the terminal ray of an angle θ in standard position.

If any point P on the terminal ray is given, all six trigonometric ratios can be determined. In fact, if any two of the three items, abscissa, ordinate, or radius vector are given, all six trigonometric ratios can be determined. The reason for this is that the relationship among x, y, and r is $x^2 + y^2 = r^2$, which is based on the Pythagorean theorem. This relationship can be demonstrated by selecting any point P whose coordinates are (x,y) on the terminal ray of any given angle and constructing a line segment from P perpendicular to the x-axis. A right triangle is formed whose legs are $|x|$ and $|y|$ and whose hypotenuse is $|r|$; therefore, $x^2 + y^2 = r^2$.



12.4 THE UNIT CIRCLE AND THE LINE $x = 1$

The following basic concepts are to be developed:

(1) Equation of a circle
(2) Signed lengths
(3) The unit circle and the sine and cosine ratios
(4) The line $x = 1$ and the tangent ratio
(5) Relations of the functions of angles in various quadrants

From the work with the equations for the conic sections done previously, the pupils should recognize the equation of a circle.

As an aid to the pupil's memory, if abscissa and ordinate are written alphabetically and then associated with x and y in alphabetical order, it is easy to remember that abscissa is the x value and ordinate is the y value.

The pupils will need protractors, rulers, and compasses to do the work in this section. The work will be facilitated if the ruler is graduated in tenths of an inch. Then an inch can be used as the radius of the unit circle. Another method might be to have the pupils make their own unit and divide it into tenths.

There are several additional concepts which must be clarified in order to understand the trigonometric functions of the general angle. These are not difficult concepts but are new since they involve an understanding of directed line segments and directed rotations.

The radius vector of a given point which is on the terminal ray of an angle is always positive but the abscissa and ordinate of the point may be positive or negative, depending on the quadrant in which the point is located.

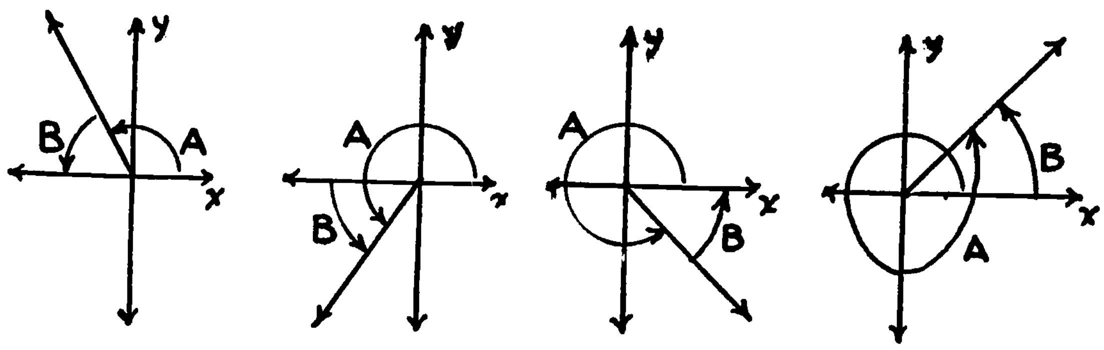
In quadrant I, the abscissa and ordinate of any point P on the terminal ray of an angle are both positive, so all the ratios are positive. In quadrant II, for any Point P on the terminal ray of an angle the abscissa is negative and the ordinate positive, so the sine and cosecant of the angle are positive and all other ratios are negative. In quadrant III, for any point P on the terminal ray of an angle both abscissa and ordinate are negative so only the tangent and cotangent of the angle are positive. All the other ratios are negative. In quadrant IV, for any point P on the terminal ray of an angle the abscissa is positive and the ordinate negative so the cosine and secant of the angle are positive and all other ratios are negative.

The x-axis and y-axis are not contained in any of the quadrants, so special attention should be given to the trigonometric ratios involving angles whose terminal ray form a part of either the x-axis or y-axis. Four such angles are 0°, 90°, 180°, and 270°. Other angles such as 360°, -90°, and so forth are coterminal with some one of the preceding group.

There should be little difficulty in determining the sine and cosine of any of these angles. Any point on the terminal ray may be selected and the ratios determined. For example, to find the cosine of 90°, the point (0,1) may be selected on the terminal ray. Cosine $90^\circ = \frac{1}{1}$ or $\cos 90^\circ = 1$. However, in determining the other trigonometric ratios of some of these four angles the denominator of the ratio is zero. For example, in determining the tangent of 90°, the point (0,2) may be selected on the terminal ray. The tangent of an angle is the ratic of y to x, thus, $\tan 90^\circ = \frac{2}{0}$. Division by zero is undefined and not permitted; therefore, $\tan 90^\circ$ is undefined. Whenever the denominator of the ratio is zero, that ratio is undefined for that angle. As the size of the angles in a sequence of angles increases toward 90°, the tangents of the angles become greater and greater, increasing without bound. Therefore, there is no number which represents $\tan 90^\circ$. Also, the properties of quadrant II angles indicate that as the size of the angles in a sequence of angles decreases toward 90°, the tangents of the angles become more and more negative decreasing without bound. Therefore, if $\tan 90^\circ$ were equal to some number it would have quite contradictory properties. The only correct way to handle ratios such as $\tan 90^\circ$ in which the denominator is zero is to make it clear that that ratio is undefined for that angle. This applies to $\csc 0^\circ$, $\cot 0^\circ$, $\tan 90^\circ$, $\sec 90^\circ$, $\csc 180^\circ$, $\cot 180^\circ$, $\tan 270^\circ$, $\sec 270^\circ$, and all angles coterminal with any of these.

Tables of trigonometric ratios contain ratios only of angles equal to or less than 90°. The reason for this is that any trigonometric ratio of any angle other than those which are undefined, can be determined from such a table. To be able to use such a table, it is necessary to master the concept of reference angles.

For every angle, there is a quadrant I reference angle expressed as a non-negative angle less than 90° whose functions are equal to the absolute values of the corresponding functions of the given angle. In the diagrams on the next page, any trigonometric ratio of angle A is equal to the same trigonometric ratio of its reference angle preceded by the appropriate positive or negative sign. If the given angle is angle A, the measure of the quadrant I reference angle is equal to the measure of the related angle B.



For example, the sine of 315° (which is negative) is equal to the sine of 45° (which is positive) preceded by a negative sign. That is, $\sin 315^{\circ} = - \sin 45^{\circ}$. The cosine of 315° (which is positive) is equal to the cosine of 45° (which is also positive). This may be demonstrated to the pupils by the use of examples in which a point on the terminal ray of an angle greater than 90° is determined and a point with equal radius vector and on the terminal ray of the reference angle is also determined, and then the six trigonometric ratios found for the given angle and its reference angle. The ratios for the reference angle will be the same as the absolute value of the ratios for the given angle. Therefore, any trigonometric ratio, if it exists, of any angle can be determined from a table of trigonometric ratios for angles less than or equal to 90 degrees.

12.5 GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

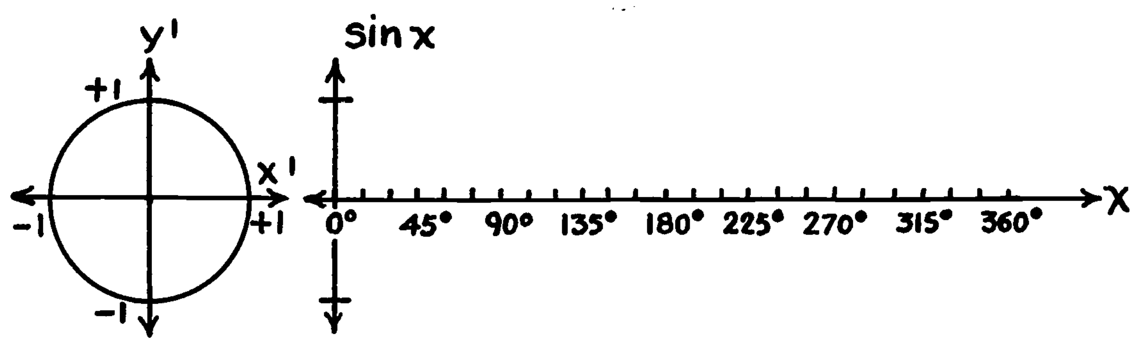
The following basic concepts are to be developed:

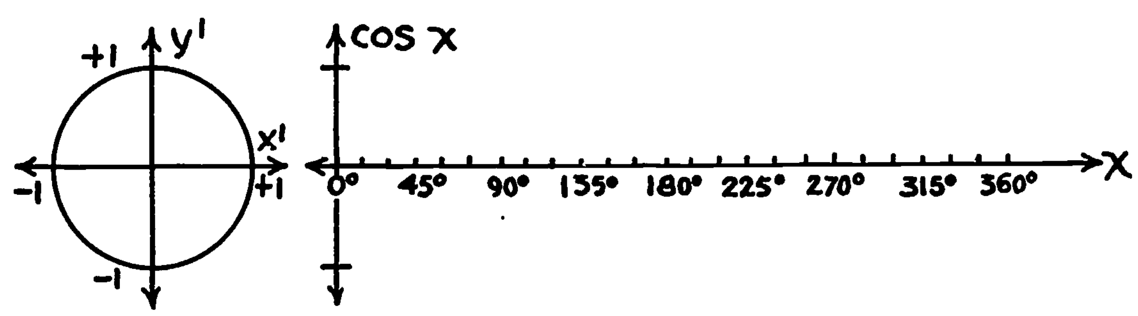
(1) Using measurements to determine the element of a relation
(2) Using measures of angles as abscissas
(3) Graph of the sine function
(4) Graph of the cosine function
(5) Graph of the tangent function

The object of this section is the presentation of the graphs of the functions through the unit circle and the line $x = 1$. The pace of the development is slow at first and then more and more is left to the pupil. If the teacher feels that his group needs more time, he should make the necessary adjustments.

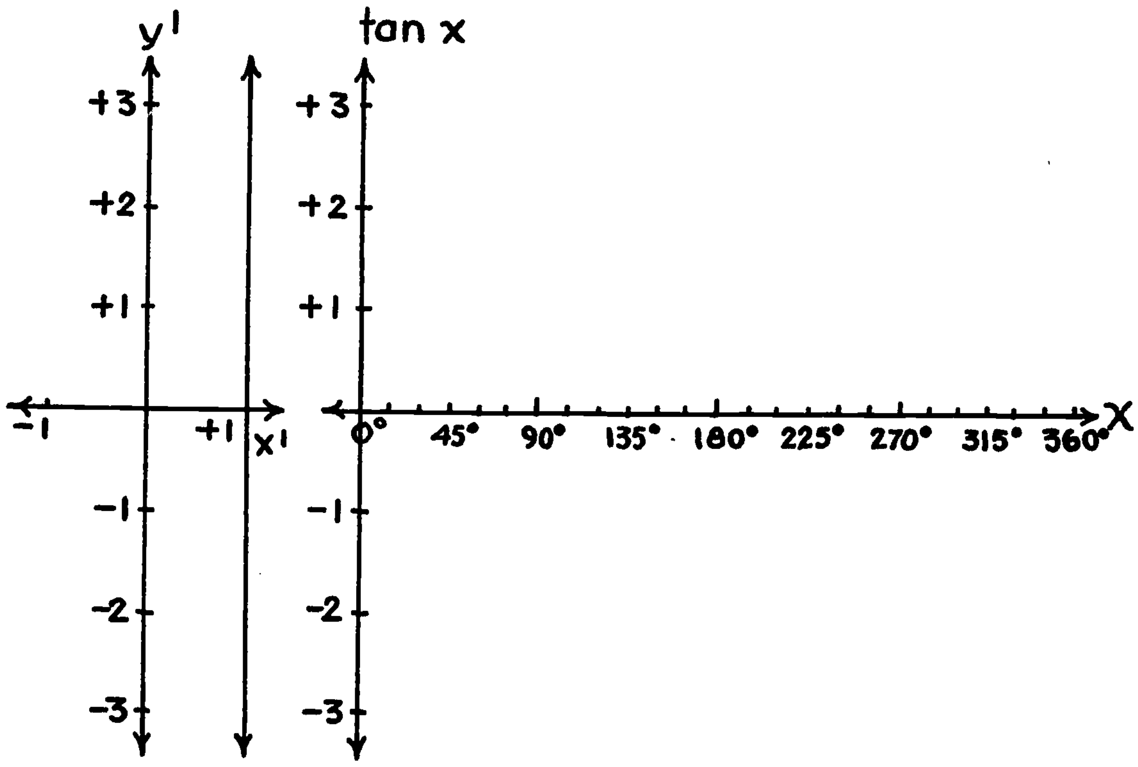
An important concept to develop in this section is that the graphing of the trigonometric functions through the use of the unit circle and the line $x = 1$ involves operating with two related but quite distinct graphs each having its own set of coordinate axes. In the first graph, the angle is in standard position and is measured by an amount of rotation. The ordinate, abscissa, and radius vector of a point which is on both the terminal ray of the angle and on the unit circle or the line $x = 1$, are used to determine the trigonometric ratios associated with the angle. In the second related graph a change is made and the measure of the angle is always the abscissa of a corresponding point. The ordinate of this point has the same measure and sign as some line segment in the first graph. Thus an ordinate, an abscissa, or a radius vector in the first graph may correspond to an ordinate in the second graph, depending upon the function being graphed. This concept is particularly important in graphing the cosine function since ordinates of points in the associated second graph are equal in length and sign to certain corresponding abscissas in the first graph. It is also important in the additional graphs discussed for teacher background - the contangent function, the secant function, and the cosecant function.

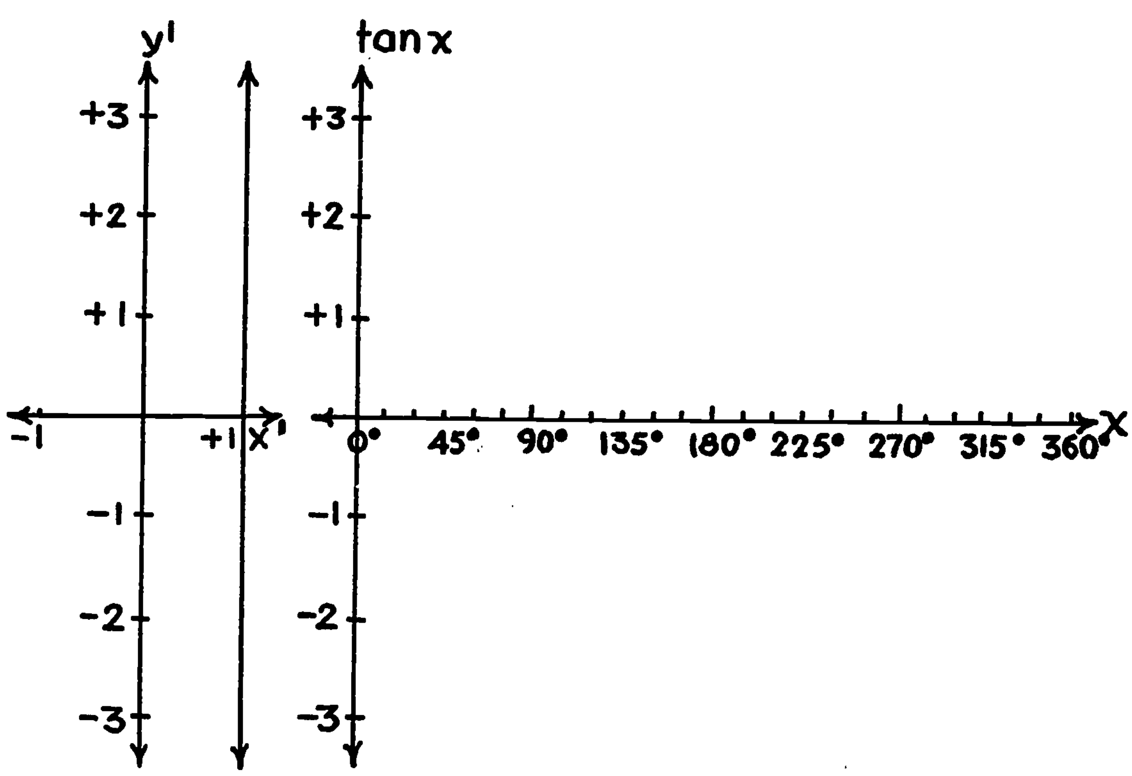
For questions 5 through 24, graphing the sine function and the cosine function, a "ditto" sheet should be prepared as in the following diagram. Each pupil will need one such sheet.





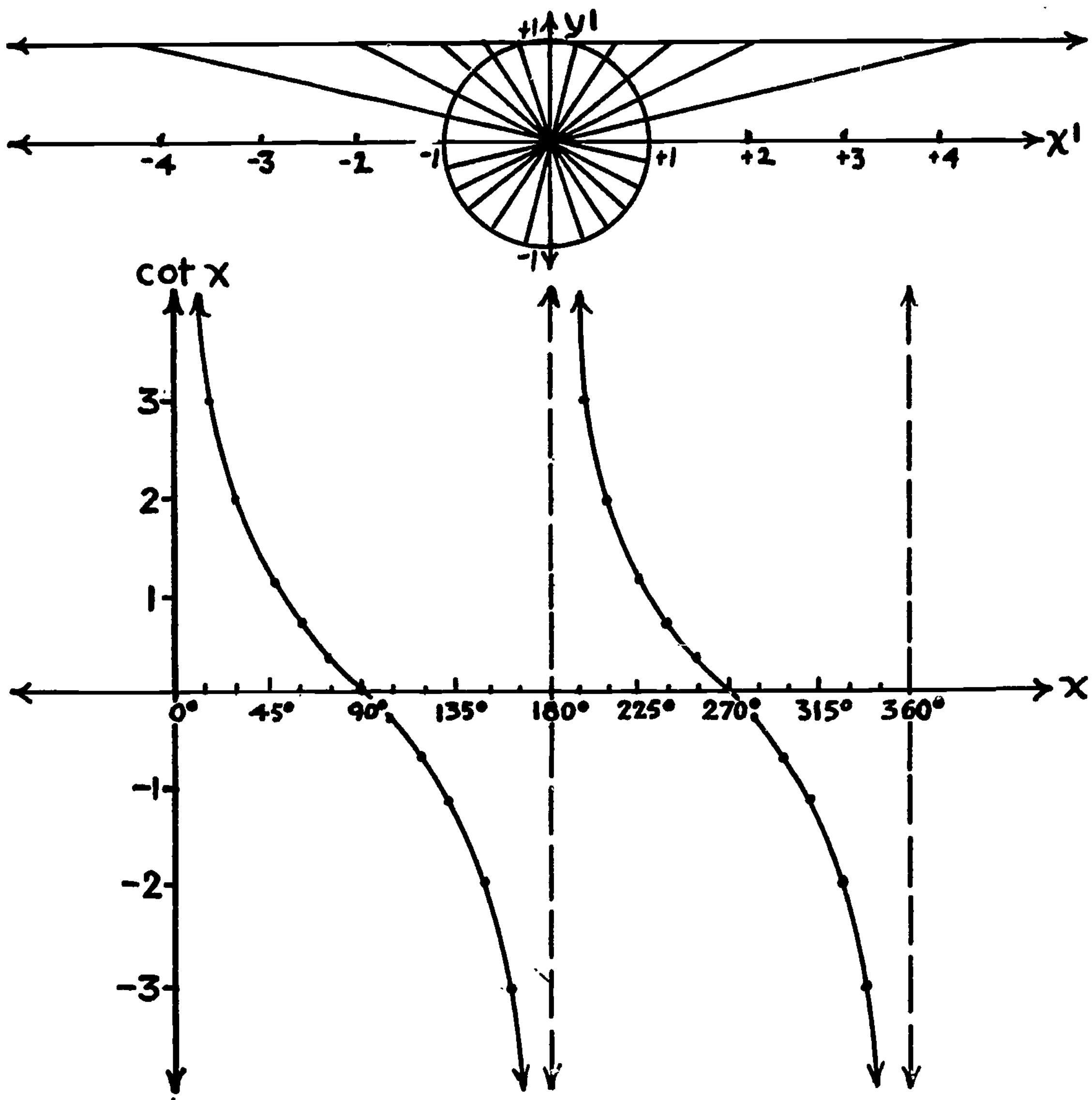
For questions 25 through 30, each pupil will need one half of a ditto sheet prepared as in the following diagram for graphing the tangent function.



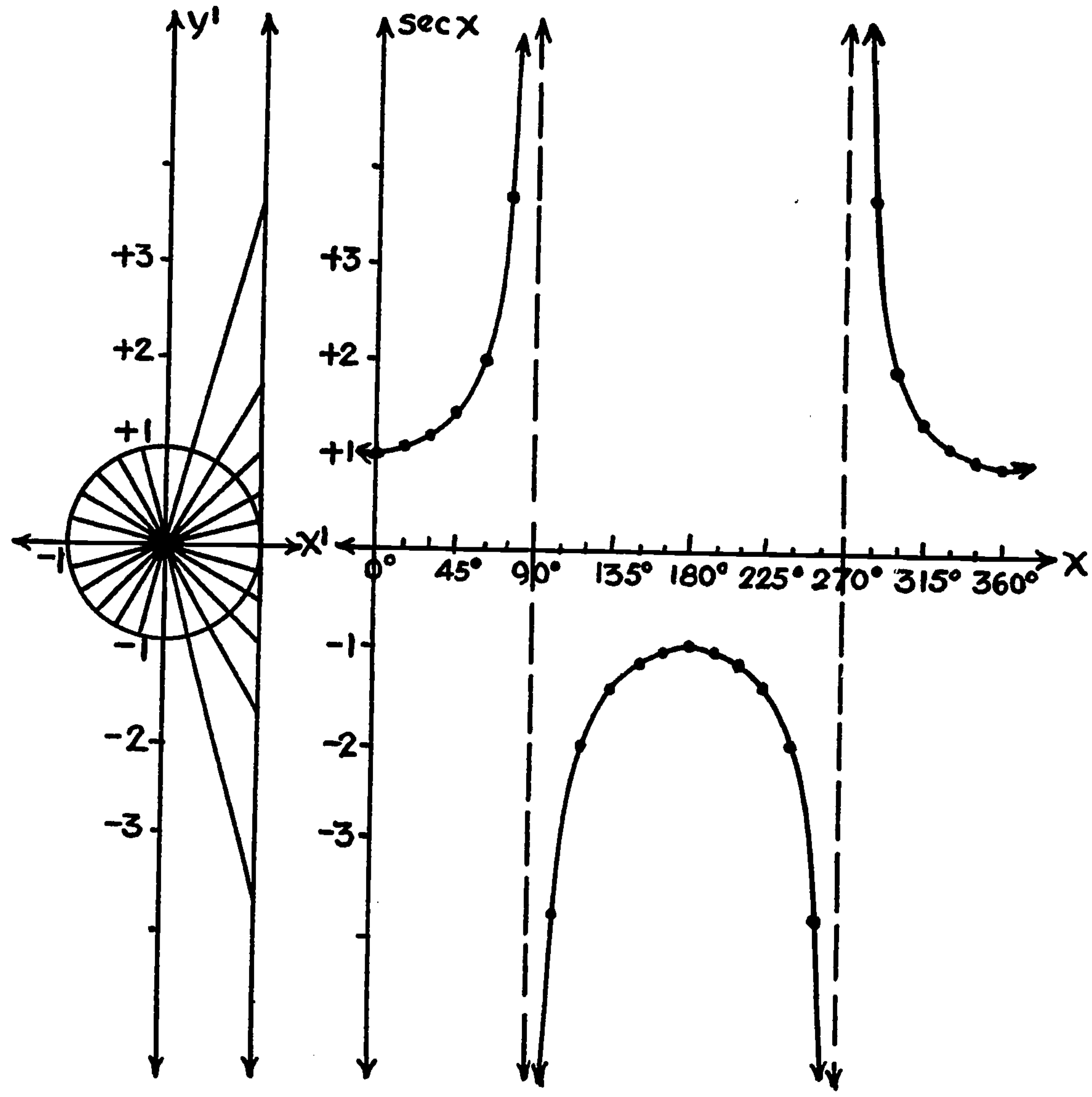


For the teacher's information, the graphs of the cotangent, secant, and cosecant functions are here developed from the unit circle in a manner similar to the development of the other functions. It is not expected that pupils will cover these topics except perhaps an isolated very gifted pupil.

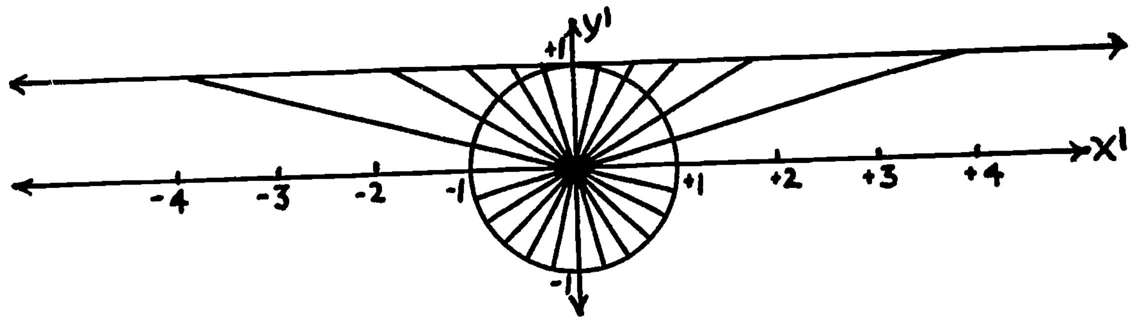
The cotangent is determined by the intersection of the terminal ray or the ray opposite to the terminal ray with a tangent to the unit circle where this tangent intersects the circle at the point (0,1). The ordinates of points in the second graph are equal in length and sign to the abscissas of the corresponding points of intersection determined in the first graph. Again it is important to remember that the measured lengths in the first graph become ordinates in the second graph. Attention must be given to the sign of the segment being copied.

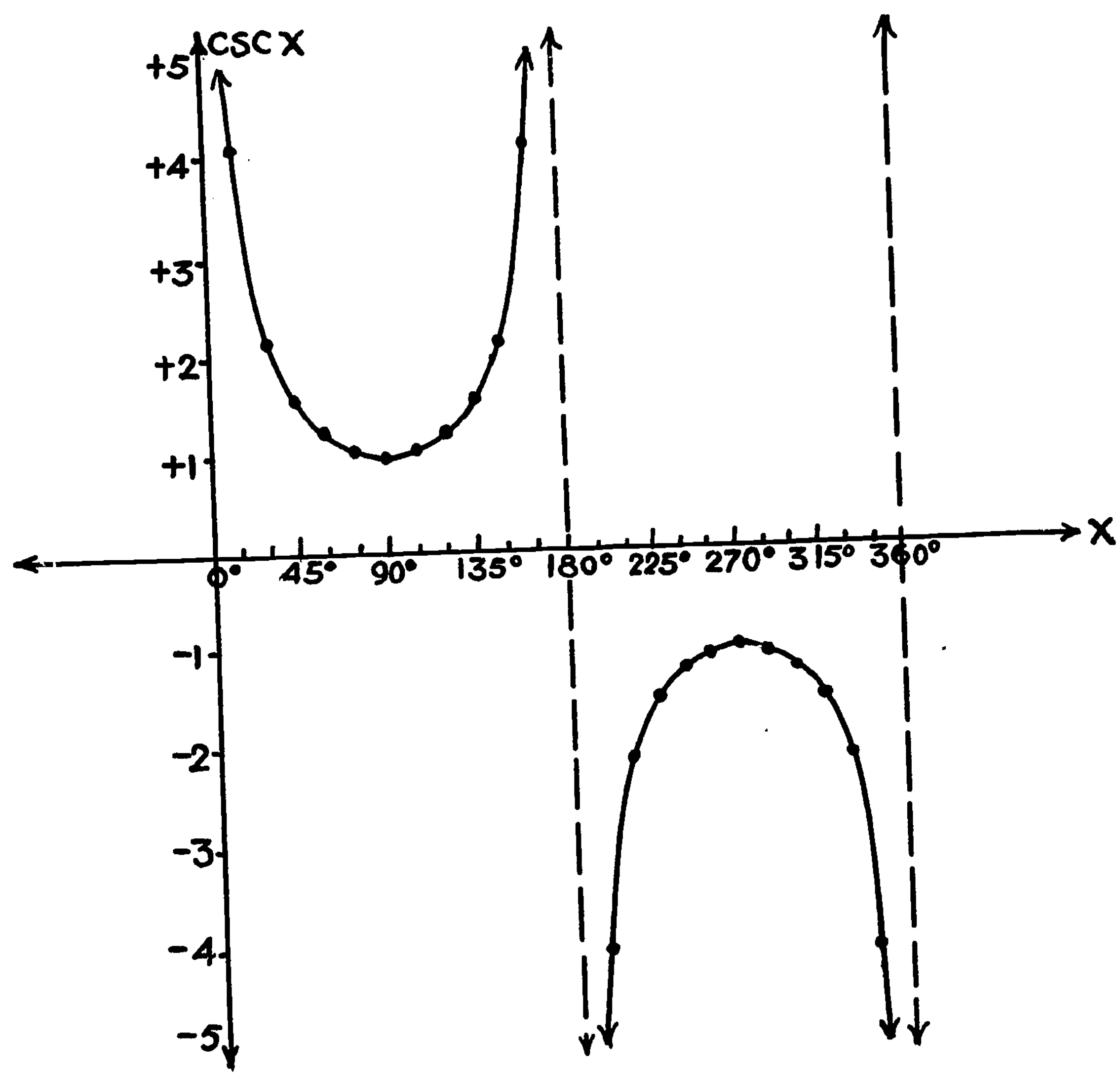


The secant of an angle is the length of the radius vector from the origin to the intersection of the tangent line at (1,0) with the terminal ray of the angle or the ray opposite to the terminal ray. If the point of intersection is on the ray opposite to the terminal ray of the angle, the radius vector of this point is considered to be negative when associated with the angle. The ordinates of points in the second graph are equal in length and sign to the radius vectors of the corresponding points of intersection determined in the first graph.

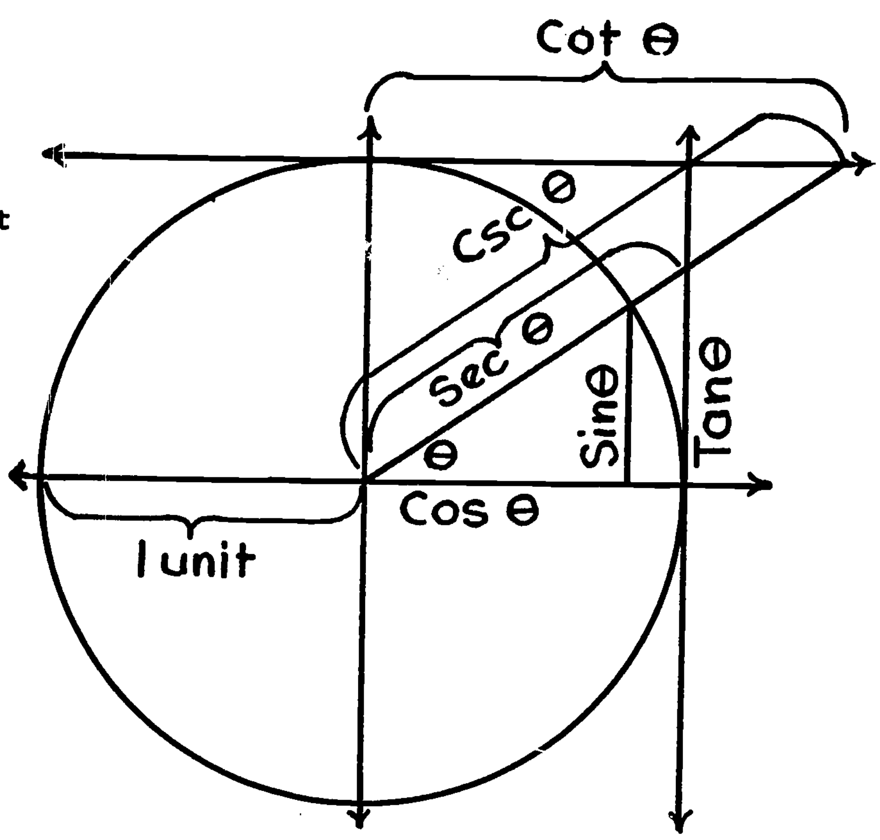


The cosecant of an angle is the length of the radius vector from the origin to the intersection of the tangent line at (0,1) with the terminal ray of the angle or the ray opposite to the terminal ray of the angle. Remarks similar to those made concerning the secant of an angle apply to the graph of cosecant of an angle.



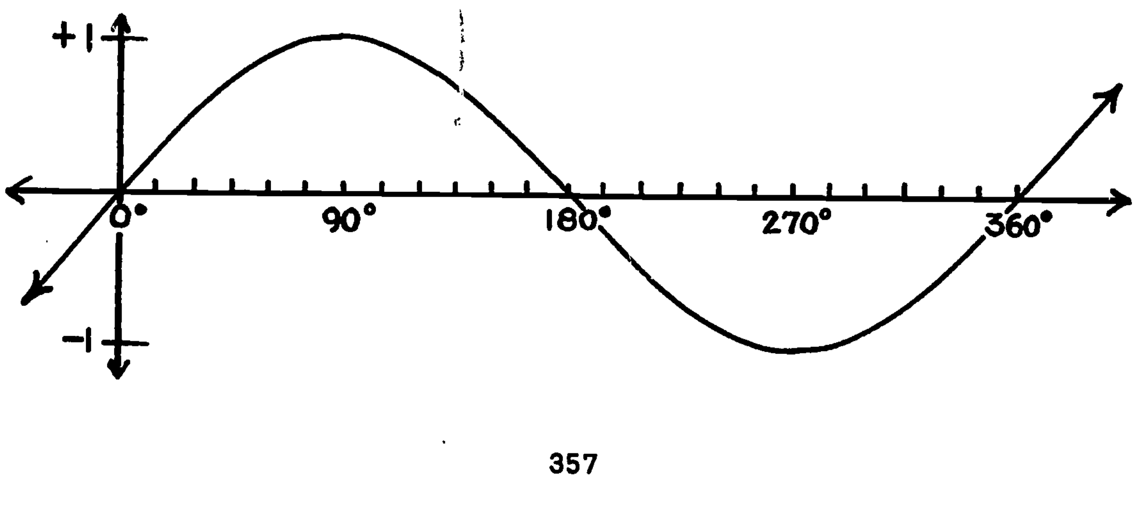


The six line functions are represented on the same unit circle in the diagram at the right.



The following discussion of trigonometric functions and their inverses is intended for teacher background only. Here again, an isolated very gifted pupil may have some questions, but a fuller discussion must be deferred until he studies eleventh year mathematics.

The equation y = sin x defines the relation $\{(x,y) \mid y = \sin x\}$, where y is the ratio defined as the sine of angle x. Here y is a pure number while x is taken as the measure of the angle in degrees. Therefore, to graph this relation in this course, the x-axis will be marked off in degrees. The accompanying graph shows this relation. From this graph it is clear that the vertical line test demonstrates that this relation is a function. No vertical line intersects the graph at more than one point.



However, the horizontal line test shows that the inverse of this function is not a function. There are horizontal lines intersecting the graph which intersect it at more than one point.

The same applies to the equations $y = \cos x$ and $y = \tan x$. These equations define the relations $\{(x,y) \mid y = \cos x\}$ and $\{(x,y) \mid y = \tan x\}$ respectively. Each of these relations is a function as can be seen from the vertical line test of their graphs. The horizontal line test also indicates that the inverse of each of these functions is not a function.

In graphing the tangent function, care must be taken not to have the graph intersect vertical lines at $x = 90^\circ$ and $x = 270^\circ$. The function is undefined at 90° and 270° and no point can be plotted for these angles. However, by plotting the ordered pairs belonging to the function for angles near 90° and 270°, the pupil can obtain some concept of how the second member of the function varies as the angle approaches 90° and 270° from either direction.

<u>Teacher Notes</u>

UNIT 12: TRIGONOMETRIC FUNCTIONS

Part 2. Questions and Activities for Classroom Use

12.1 INTRODUCTION

The questions and activities are designed to introduce the concept of trigonometric functions, applying the concept of function mastered in the previous unit. An important point to emphasize is the difference between a trigonometric ratio and a trigonometric function.

It is intended that the questions and activities serve to show clearly the meaning of trigonometric functions, but there is no intention of developing the topic too deeply, as a thorough study of trigonometric functions is more appropriate for the high school trigonometry and eleventh year mathematics courses. However, the teacher may find that his pupils are capable of pursuing the study of trigonometric functions to a greater depth than the presentation contained in this unit. If this is so, it is suggested that such topics as graphing cosecant, secant, and cotangent functions, and the topic of inverse trigonometric functions be touched upon.

12.2 MEANING OF A TRIGONOMETRIC FUNCTION

Concept: Working definitions of the trigonometric functions.

When last studied, the trigonometric functions were defined in terms of ratios of sides of right triangles.

(1) *What are the names of the trigonometric functions?*

Answer: sine, cosine, and tangent

(2) *What is the definition of the sine function?*

Answer: The sine of an angle is equal to the ratio of the length of the leg opposite the given angle to the length of the hypotenuse.

(3) *What is the definition of the cosine function?*

Answer: The cosine of an angle is equal to the ratio of the length of the leg adjacent to the given angle to the length of the hypotenuse.

(4) *What is the definition of the tangent function?*

Answer: The tangent of an angle is equal to the ratio of the length of the leg opposite the given angle to the length of the leg adjacent to the given angle.

When the trigonometric functions were studied in the previous unit, each function was defined in terms of ratios of the lengths of sides of right triangles. In this unit, more attention will be given to the trigonometric functions as ordered pairs whose second members are ratios more general than the ratios of the lengths of sides of right triangles.

Instead of writing out the definition in sentence form, a shortened form representing the same thing is usually used. This shortened form of the sine ratio follows:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

This is generally read "Sine A equals opposite over hypotenuse." Sin = sine; A is the given angle; opposite corresponds to the length of the leg opposite the given angle A; and the hypotenuse represents the length of the hypotenuse.

The definitions of the cosine and tangent ratios are likewise shortened as follows:

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{and} \quad \tan A = \frac{\text{opposite}}{\text{adjacent}}$$

These definitions are again further shortened by abbreviating in the following manner:

adjacent = adj, opposite = opp, and hypotenuse = hyp

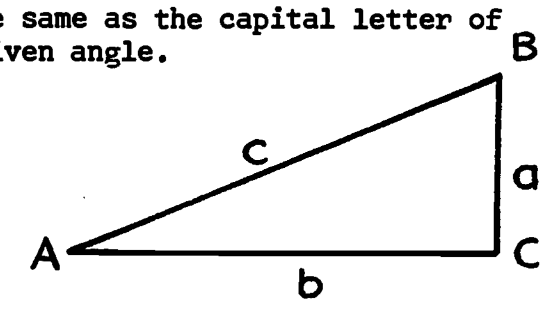
Thus, $\sin A = \frac{\text{opp}}{\text{hyp}}$, $\cos A = \frac{\text{adj}}{\text{hyp}}$, and $\tan A = \frac{\text{opp}}{\text{adj}}$

The capital letter following the name of a function is the letter representing the name of an angle. This angle is the angle to which the definition of the function applies.

Concept: The use of conventions.

In mathematics, when all mathematicians agree to do something in the same way it is called a convention. In lettering triangles and other geometric figures, several conventions are in use.

- All angles are labeled with capital letters.
- The letters are used consecutively.
- In a triangle, the sides are labeled with small letters. The small letters are the same as the capital letter of the angle opposite the given angle.
- In a right triangle, the 90° angle is usually labeled C when using the letters A, B, and C. An example is shown at the right.



Concept: A trigonometric ratio associated with an angle represents a real number.

(5) *Given triangle ABC where a = 1 and c = 2, what is the value of b?*

Answer: $\sqrt{3}$

$c^2 = a^2 + b^2$

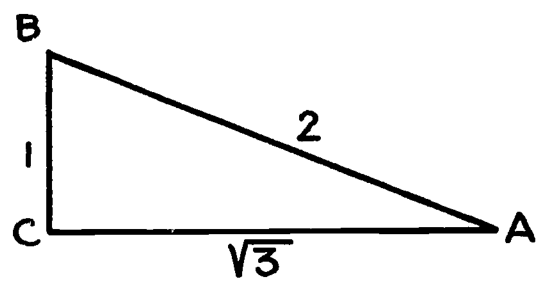
$c^2 - a^2 = b^2$

$\sqrt{c^2 - a^2} = b$

$\sqrt{4 - 1} = b$

$\sqrt{3} = b$

(Pythagorean relation)



Only the positive value of the root is considered because the sides of a triangle are always positive in length.

(6) *What kind of number is b?*

Answer: Irrational

(7) *What is the value of sin A in fractional form?*

Answer: Sin A = $\frac{1}{2}$

(8) *What is the value of sin A in decimal form?*

Answer: Sin A = 0.5000

(9) *What kind of number is 0.5000?*

Answer: Rational

(10) *What is the value of tan A in fractional form?*

Answer: Tan A = $\frac{1}{\sqrt{3}}$

(11) *What kind of number is the numerator of* $\frac{1}{\sqrt{3}}$*?*

Answer: Rational

(12) *What kind of a number is the denominator of* $\frac{1}{\sqrt{3}}$*?*

Answer: Irrational

(13) *If a rational number is divided by an irrational number, is the result rational or irrational?*

Answer: Irrational $\frac{1}{1.7321...} = 0.5773....$

(14) *What two kinds of numbers occur in the trigonometric functions?*

Answer: Rational and irrational

(15) *What number system contains both rational and irrational numbers?*

Answer: Real

(16) *A trigonometric ratio associated with an angle will always be what kind of a number?*

Answer: Real

(17) *Tan A is equal to what?*

Answer: A real number

(18) *Cos A is equal to what?*

Answer: A real number

(19) *Sin A is equal to what?*

Answer: A real number

(20) *What operations can be performed with real numbers?*

Answer: Addition, subtraction, multiplication, division, raising to a power, and taking of a root

(21) *Since tan A, cos A, and sin A are real numbers, what operations can be performed on them?*

Answer: Addition, subtraction, multiplication, division, raising to a power, and taking of roots

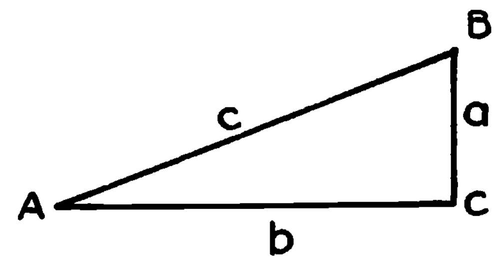
12.3 DEFINITIONS INVOLVING THE GENERAL ANGLE

Up to this point, only trigonometric ratios associated with acute angles of right triangles have been considered. Definitions are limited to angles greater than 0° and less than 90°.

The objective of this section is to extend the definition so as to be able to find the trigonometric ratios associated with any angle.

Concept: Standard position of an angle.

(1) *Given triangle ABC, find the sine of angle A.*



Answer:

$\text{Sin } A = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}$

(2) *Given angle D where D is obtuse, can the previous ideas be used to find the sine of angle D?*

Answer: No

(3) *Is it possible to have an obtuse angle in a right triangle? Explain*

Answer: No. A right triangle contains a 90° angle. If there were also an obtuse angle, that is, an angle greater than 90° and less than 180° in the triangle, there would be more than 180° in the sum of the angles of the right triangle. This is impossible. If the angle being considered is placed in a coordinate system, further properties can be discovered. A review of some angle concepts first will be needed.

(4) *What is a ray?*

Answer: A ray is an infinite portion of a straight line with only one endpoint.

(5) *What is an angle?*

Answer: An angle is the set of all points contained on two rays that have the same endpoint.

(6) *If we include the idea of rotation when thinking about an angle, what is the name given to each of the rays that form an angle?*

Answer: One ray is called the initial ray and the other ray is called the terminal ray.

(7) *What is meant by the measure of an angle?*

Answer: The measure of an angle is the amount of rotation about its common endpoint of a ray coincident with the initial ray necessary to make this ray coincide with the terminal ray. Unless indicated otherwise we shall limit ourselves to rotations not greater than a full rotation.

(8) *What is a unit of measurement of an angle?*

Answer: One unit of measurement of an angle is the degree.

(9) *What is the definition of a degree?*

Answer: A degree is $\frac{1}{360}$ of a complete rotation.

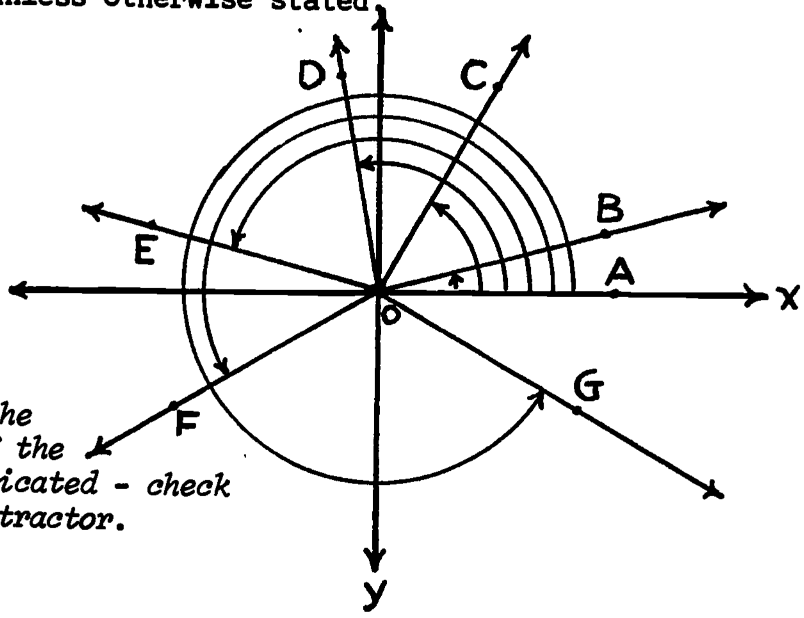
(10) *What is meant by positive and negative angles?*

Answer: If the rotation from the initial ray to the terminal ray is counterclockwise, the rotation is positive and the angle is a positive angle. If the rotation is clockwise, the rotation is negative and the angle is a negative angle.

(11) An angle may be formed in a coordinate system by sets of points that are on two rays which have a common endpoint. If the common endpoint or vertex is the point (0,0) and if the initial ray is that portion of the x-axis to the right of and including the point (0,0), then the angle is said to be in standard position. It shall be assumed that any angle in a coordinate system is in standard position unless otherwise stated.

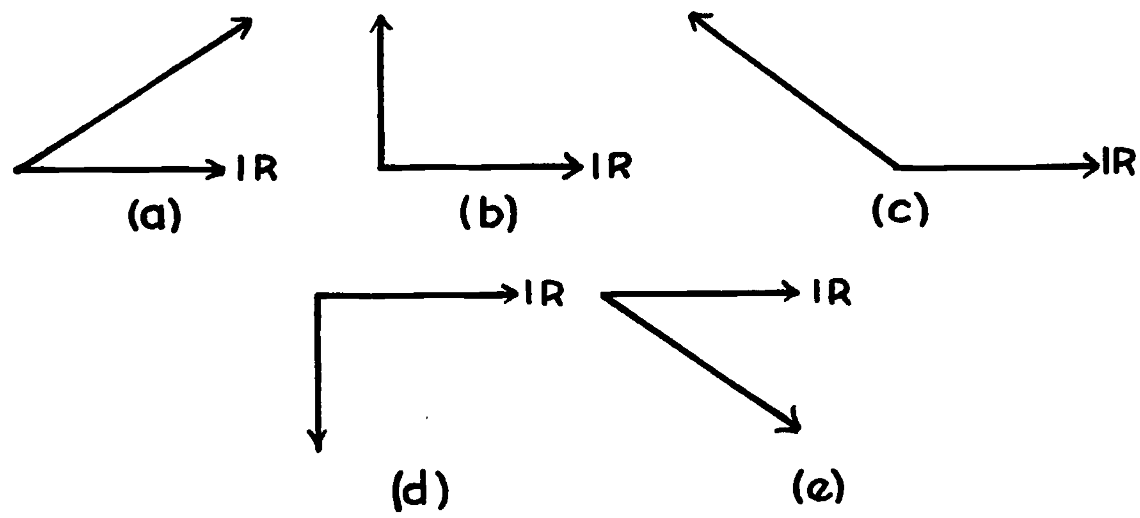
(a) $\angle AOB$
(b) $\angle AOC$
(c) $\angle AOD$
(d) $\angle AOE$
(e) $\angle AOF$
(f) $\angle AOG$

Estimate the measure of the angles indicated - check with a protractor.

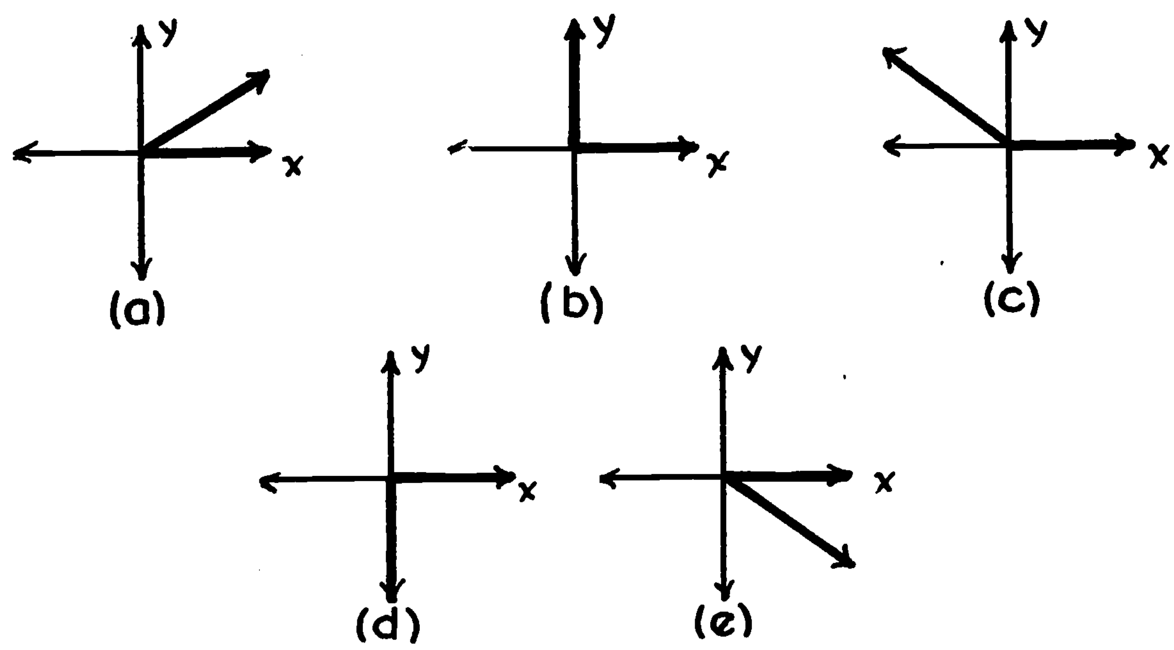


Answers: (a) 15° (d) 165°
(b) 60° (e) 210°
(c) 100° (f) 330°

(12) *Place the following angles in standard position in a coordinate system.* ***NOTE:*** *IR = initial ray.*



Answers:



Concept: Positive and negative angles.

(13) *In how many directions is it possible to rotate in order to start at the initial ray of an angle and end up at the terminal ray?*

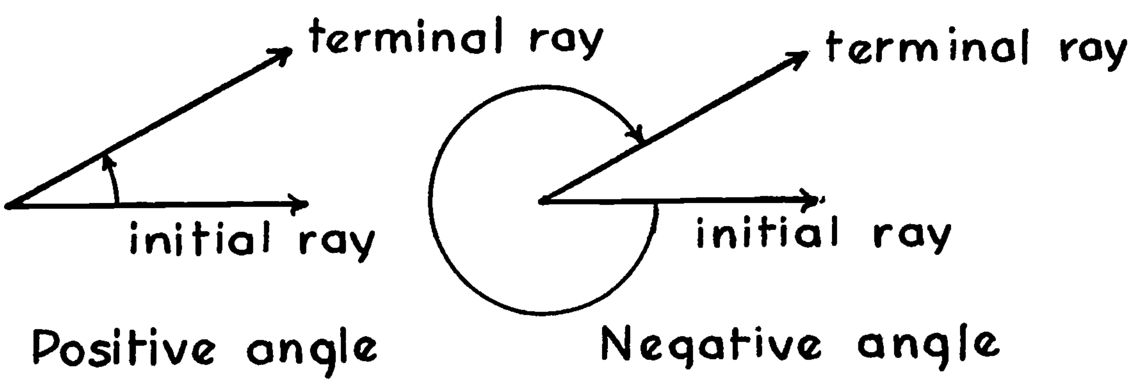
Answer: Two

(14) *What are they?*

Answer: Clockwise and counterclockwise

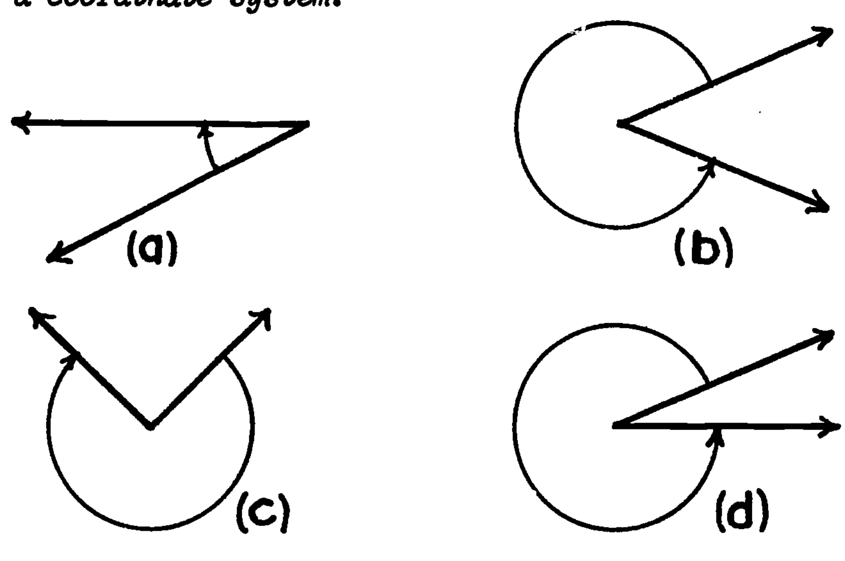
By convention, if an angle is formed by starting at the initial ray and rotating to the terminal ray in a counterclockwise direction, the angle is considered to be positive. If it is formed by rotation in a clockwise direction, the angle is considered to be negative.

Arrows are used to indicate whether the rotation is positive or negative.

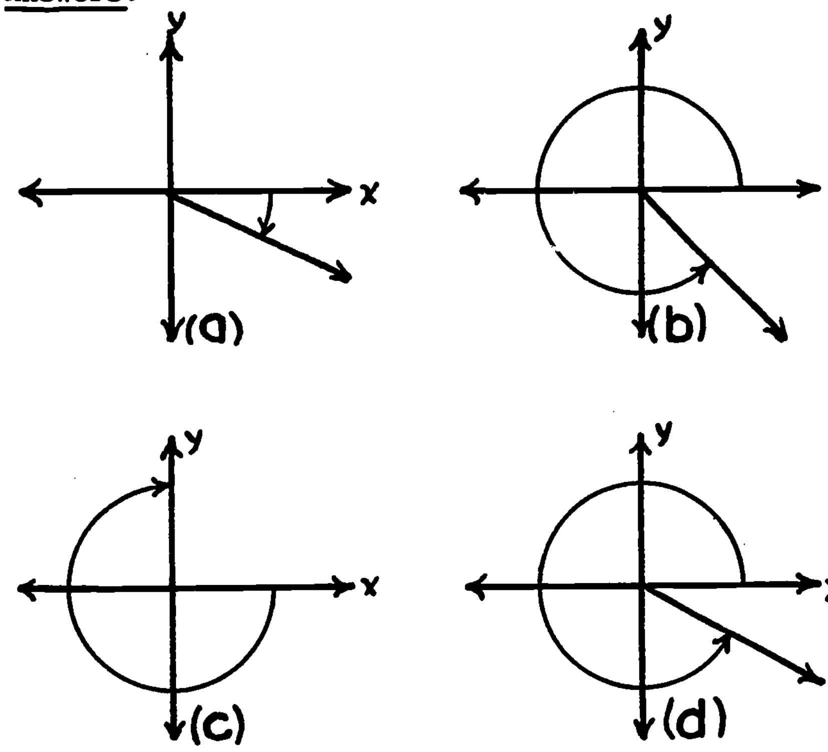


The tail of the arrow starts at the initial ray and is drawn in either a negative or positive direction with the head of the arrow terminating on the terminal ray.

(15) *Place the following angles in standard position on a coordinate system.*



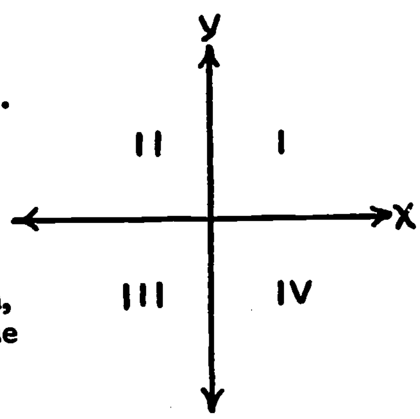
Answers:



Concept: Naming quadrants.

(16) *Into how many sections do the x-and y-axes divide the coordinate plane?*

Answer: Four. These four sections are called quadrants. They are numbered with Roman numerals starting with the quadrant bounded by the positive x-axis and the positive y-axis and going in a counter-clockwise direction, as shown in the diagram at the right.



Concept: Naming angles.

It is possible for the terminal ray of an angle to lie in any quadrant of the coordinate plane or on the boundary between two quadrants. An angle is given the name of the quadrant in which the terminal ray lies. Thus, the terminal ray of a positive acute angle will always lie in the first quadrant. Therefore, positive acute angles are called "first quadrant angles" or "quadrant I angles."

(17) *What kind of angle is an obtuse angle?*

Answer: An obtuse angle is a second quadrant angle.

(18) *What kind of angle is a reflex angle?*

Answer: A third or fourth quadrant angle.

Concept: Measures of fractions of a full rotation.

(19) *If a ray makes one full rotation how many degrees does this represent?*

Answer: 360°

(20) *Starting with the positive x-axis as 0°, through how many degrees does a ray rotate in order to coincide with the positive y-axis?*

Answer: 90°. The positive x-axis is the starting point because the initial ray always lies along it.

(21) *If the positive x-axis is 0°, through how many degrees does a ray rotate in order to coincide with the negative x-axis?*

Answer: 180°

(22) *A similar rotation from the positive x-axis to the negative y-axis represents how many degrees?*

Answer: 270°

(23) *If the rotation is from the positive x-axis around to the positive x-axis again, through how many degrees does the ray rotate?*

Answer: 360°

Concept: Quadrantal angles.

If the terminal ray of any angle falls on one of the axes, it is called a quadrantal angle.

(24) *Name two common quadrantal angles.*

Answer: A right angle and a straight angle

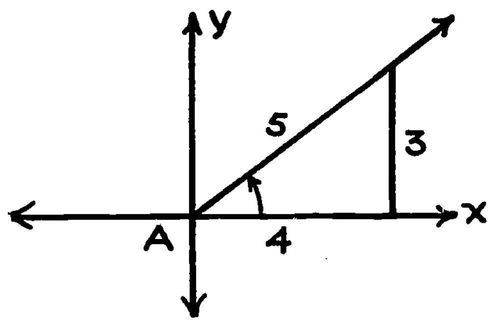
(25) *All quadrantal angles are divisible by what number of degrees?*

Answer: 90°

Concept: **Extending the definitions of the trigonometric ratios.**

(26) *Given a 3-4-5 right triangle, place the triangle so that A, the angle opposite the side labeled 3, is in standard position and the side labeled 4 is along the x-axis.*

Answer:



(27) *Find the value of the sine, cosine, and tangent of A.*

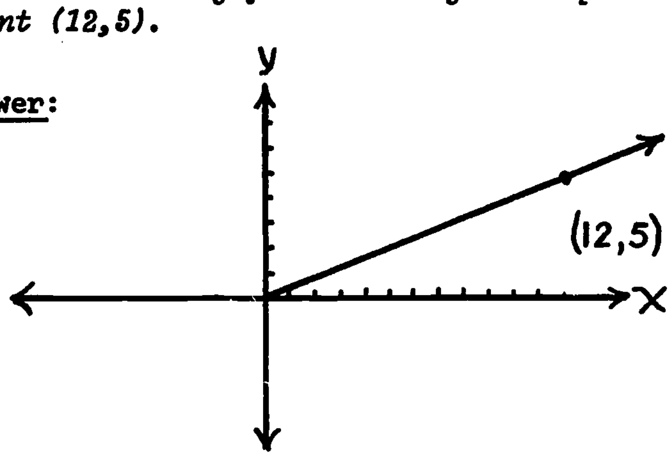
Answer: $\sin A = \frac{3}{5}$ $\cos A = \frac{4}{5}$ $\tan A = \frac{3}{4}$

(28) *If the coordinates of the outer endpoint of the hypotenuse of the triangle were written as an ordered pair, what would the pair be?*

Answer: (4,3)

(29) *Draw the ray from the origin and passing through the point (12,5).*

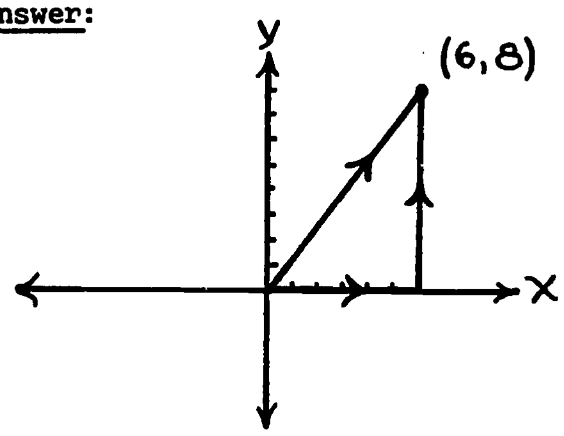
Answer:



In getting from the origin to the point (12,5), there are several paths that can be followed. One path is along the ray from the origin to the point. Another path is to travel along the x-axis to the point on the x-axis equal to the value of the x-coordinate, and after this, travel in a direction parallel to the y-axis a distance equal to the y-coordinate. Both paths end at the same point.

(30) *Using a rectangular coordinate system, draw as in the previous exercise, two different paths from the origin to the point (6,8).*

Answer:



(31) *What kind of a triangle is formed in problem (30)?*

Answer: A right triangle

(32) *Graph any angle in standard position in a coordinate system. Select any point P on the terminal ray of the angle. The distance from the origin to point P along the terminal ray is called the radius vector and is represented by r. Trigonometric ratios may then be defined as ratios of two of the three items: x, the abscissa of point P; y, the ordinate of point P; and r, the radius vector of point P. That is, a trigonometric ratio is a ratio of any two of the three items x, y, and r. How many such ratios are there? Name them.*

Answer: There are six such ratios: $\frac{y}{r}$, $\frac{x}{r}$, $\frac{y}{x}$, $\frac{r}{y}$, $\frac{r}{x}$, and $\frac{x}{y}$.

(33) In the following work, we shall be interested in only three of these ratios. If the angle is represented by θ, the sine of θ is defined as the ratio $\frac{\text{ordinate}}{\text{radius vector}}$ or $\frac{y}{r}$. Cosine θ is the ratio $\frac{\text{abscissa}}{\text{radius vector}}$ or $\frac{x}{r}$ and tangent θ is

$\frac{\text{ordinate}}{\text{abscissa}}$ or $\frac{y}{x}$. *Note:* The symbol θ is a letter in the Greek alphabet and is spelled theta.

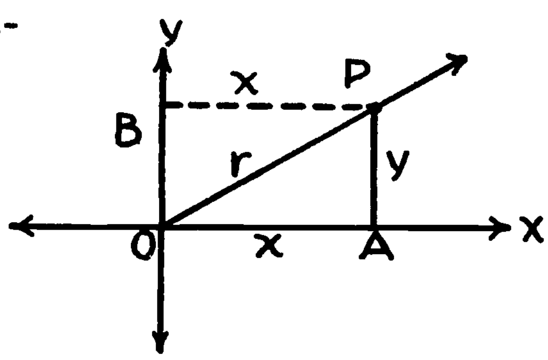
Angle A is an angle in standard position with the terminal ray passing through the point P with coordinates (x,y). The radius vector of P is r. Write the sine, cosine, and tangent of A.

Answer: $\sin A = \frac{y}{r}$, $\cos A = \frac{x}{r}$, $\tan A = \frac{y}{x}$

Note: The pupil must understand that these equations now hold for angle A of any measure whatsoever, positive, negative, and even zero.

(34) *Graph any angle in a standard position in a coordinate system and select any point P on the terminal ray. Draw a line segment from point P perpendicular to the x-axis. What type of triangle is formed? What is the relationship among x, y, and r?*

Answer: A right triangle is formed. The relationship is $x^2 + y^2 = r^2$, based on the Pythagorean theorem.



Note: If PA is perpendicular to the x-axis and PB is perpendicular to the y-axis then the abscissa of point P is BP = x and the ordinate of point P is AP = y. However, OA is equal to BP and is customarily called the abscissa of x and so marked. When x is labeled in this customary manner, then x, y, and r are sides of a right triangle.

(35) *If the abscissa and ordinate of a point are given, can the radius vector be determined? How?*

Answer: If the abscissa and ordinate are given, the radius vector can be determined by the application of the principle that $x^2 + y^2 = r^2$.

(36) *The point (5,12) is on the terminal ray of angle θ. Find the values of the sine, cosine, and tangent of θ.*

Answer: $x^2 + y^2 = r^2$ $\quad 25 + 144 = r^2$ $\quad r = 13$

$\sin\theta = \frac{12}{13}$ $\quad \cos\theta = \frac{5}{13}$ $\quad \tan\theta = \frac{12}{5}$

(37) *The point (-6,8) is on the terminal ray of angle θ. Graph the angle and determine the sine, cosine, and tangent of θ.*

<u>Answer:</u>

$x^2 + y^2 = r^2$

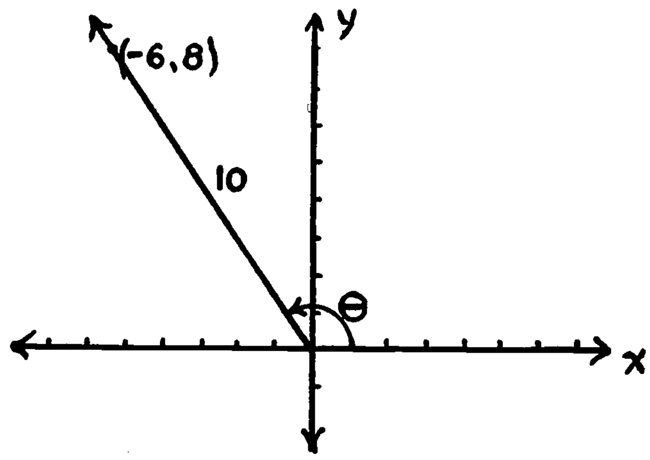
$36 + 48 = r^2$

$r = 10$

$\sin\theta = \frac{8}{10}$ or $\frac{4}{5}$

$\cos\theta = \frac{-6}{10}$ or $-\frac{3}{5}$

$\tan\theta = \frac{8}{-6}$ or $-\frac{4}{3}$



(38) *The point (-1,-1) is on the terminal ray of angle b. Graph the angle and determine the sine, cosine, and tangent of b.*

<u>Answer:</u>

$(-1)^2 + (-1)^2 = r^2$

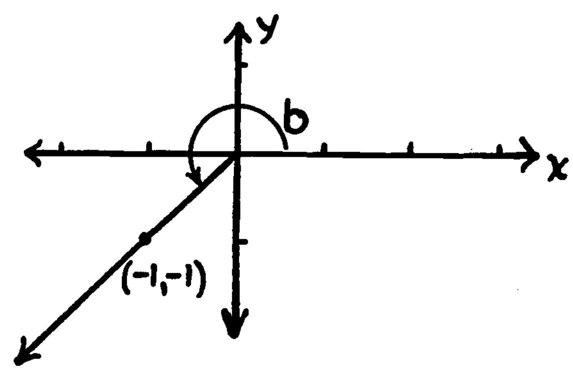
$2 = r^2$

$\sqrt{2} = r$

$\sin b = \frac{-1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$

$\cos b = \frac{-1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$

$\tan b = \frac{-1}{-1}$ or 1



(39) *How can it be determined whether a trigonometric ratio is a positive or a negative number if the abscissa, ordinate, and radius vector are known?*

<u>Answer:</u> If the numerator and denominator that make up the ratio are both positive or both negative, the ratio will be a positive number. If the two items are such that one is positive and the other is negative, the ratio will be a negative number

12.4 THE UNIT CIRCLE AND THE LINE x = 1

Concept: Equation of a circle.

(1) *What equation may be used to find the length of the radius vector when the coordinates (x,y) of a point on the terminal ray are known?*

Answer: Radius vector = $\sqrt{x^2 + y^2}$. Let the radius vector equal r; then, $r = \sqrt{x^2 + y^2}$.

(2) *Square both sides of the equation* $r = \sqrt{x^2 + y^2}$.

Answer: $r^2 = x^2 + y^2$

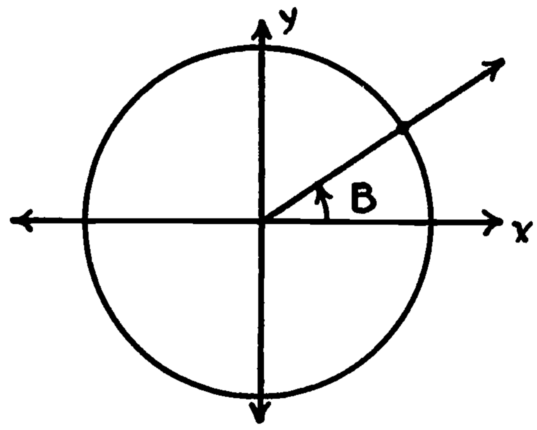
(3) *The expression* $r^2 = x^2 + y^2$ *is the equation for what curve?*

Answer: A circle

(4) *Draw an acute angle B in standard position on a coordinate system and draw a circle with the center at the origin. The terminal ray of angle B intersects the circle at a point on the coordinate plane. Since this point is on the coordinate plane, how can it be identified?*

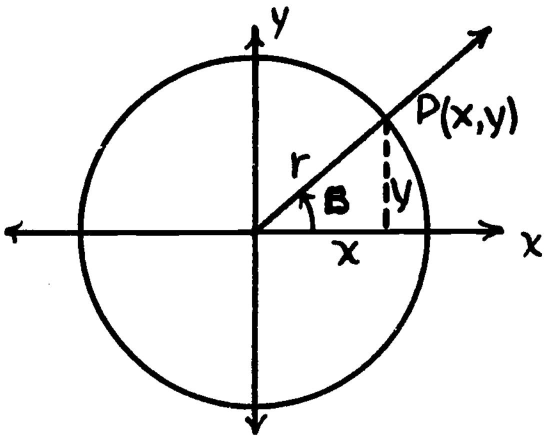
Answer:

The point can be identified by an ordered pair (x,y).



(5) *On the diagram, label the point of intersection of the terminal ray and the circumference of the circle as $P(x,y)$ and label the abscissa, ordinate, and radius vector as x, y, and r respectively. Describe the variation of x, y, and r as the measure of angle B changes.*

Answer: The values of x and y change but the value of r remains constant since point P is on the circle. (See diagram on the following page.)



Concept: Signed lengths.

Later in this unit, a point on a second graph will be located by measuring some distance on a first graph, and transferring this distance to the second graph. However, to locate a point on the coordinate plane we need not only distance, but distance and direction. Thus, we shall talk about the signed length of a line segment. For line segments which correspond to ordinates and abscissas, the sign is determined in the usual manner, to the right and up are positive, to the left and down are negative.

(6) *If a perpendicular is dropped from the point (x,y) to the x-axis, what signed length will it have?*

Answer: y

(7) *What is the signed length on the x-axis from the origin to the base of the perpendicular?*

Answer: x

(8) *What are the values of the trigonometric ratios in terms of x, y and r?*

Answer: $\sin B = \frac{y}{r}$, $\cos B = \frac{x}{r}$, $\tan B = \frac{y}{x}$

Concept: The unit circle and the sine and cosine ratios.

A circle in which the radius equals one is called a *unit circle*. Note that there is no specific size assigned to the unit used.

(9) *If r = 1, what are sin B and cos B equal to?*

Answer: $\sin B = \frac{y}{r} = \frac{y}{1} = y$, $\cos B = \frac{x}{r} = \frac{x}{1} = x$

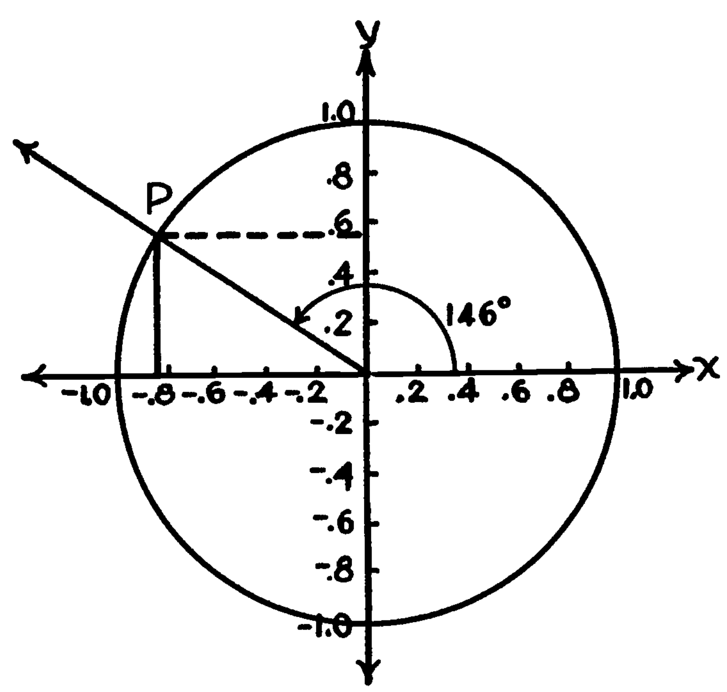
(10) *Compare the values of the sine and cosine ratios with the signed lengths of the sides of the triangle in the unit circle formed by dropping a perpendicular to the x-axis from the point of intersection of the unit circle with the terminal ray of the angle.*

<u>Answer</u>: If the radius of the circle is 1, the value of the sine of the angle is equal to the ordinate and the value of the cosine of the angle is equal to the abscissa of this point of intersection.

Thus, by placing any angle in standard position in a coordinate system drawing a unit circle with center at the origin, and finding the signed lengths of the abscissa and ordinate of the point of intersection of the unit circle and the terminal ray of the angle, the values of the sine and cosine of the angle can be determined. This procedure is the same regardless of the measure of the angle, or the quadrant in which the terminal ray lies.

(11) *Use a protractor to draw an angle of 146° in standard position in a coordinate system. Draw a unit circle with center at the origin, whose radius is 1 decimeter (10 centimeters). Label the point of intersection of the unit circle and the terminal ray of the angle as P. Draw the ordinate of P, measure the ordinate and abscissa of P carefully, and find $\sin 146^{\circ}$ and $\cos 146^{\circ}$.*

<u>Answer</u>: $\sin 146^{\circ} = 0.56$ and $\cos 146^{\circ} = -0.83$



Note: The abscissa of P is shown by the dotted line but is usually not drawn, the equivalent distance on the x-axis being used instead.

Concept: The line x = 1 and the tangent ratio.

(12) *In terms of x, y, and r, what is tan B equal to?*

Answer: $\tan B = \frac{y}{x}$. The radius vector, r, is not involved.

(13) *If (x,y) is a point on the unit circle, is the denominator of the fraction $\frac{y}{x}$ always equal to 1?*

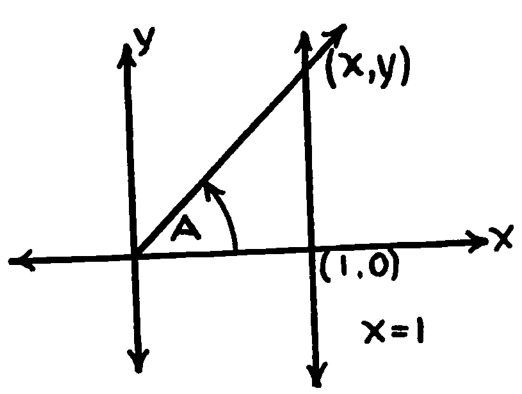
Answer: No, the value of x changes as the point (x,y) moves around the unit circle.

(14) *If the abscissa of a point (x,y) is always to be one, where must the point (x,y) be located?*

Answer: The point (x,y) must be on a line parallel to the y-axis and one unit to the right of the y-axis.

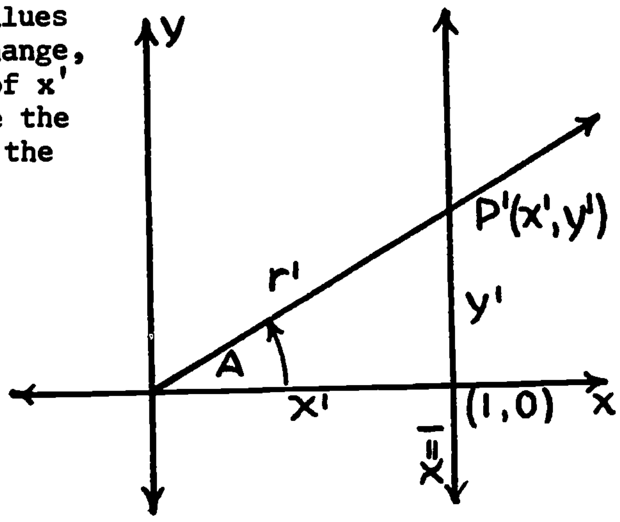
(15) *Draw an acute angle A in standard position in a coordinate system placed on graph paper, choose a scale such that the point (1,0) is between one and three inches from the origin, and draw the line $x = 1$. The terminal ray of angle A intersects the line $x = 1$ at a point on the coordinate plane. Since the point is on the coordinate plane, how can it be identified?*

Answer: The point can be identified by an ordered pair (x,y).



(16) *On the diagram, label the point of intersection of the terminal ray and the line $x = 1$ as $P'(x',y')$ and label the abscissa, ordinate, and radius vector as x', y', and r', respectively. Describe the variation of x', y', and r' as the measure of angle A changes.*

Answer: The values of y' and r' change, but the value of x' remains 1 since the point P' is on the line x = 1



(17) *If $x = 1$, what is tan A equal to?*

Answer: $\tan A = \frac{y'}{x'} = \frac{y'}{1} = y'$

(18) *If the angle is in quadrant I or IV, compare the value of the tangent ratio with the signed lengths of the sides of the triangle formed by part of the x-axis, part of the line $x = 1$, and part of the terminal ray of the angle.*

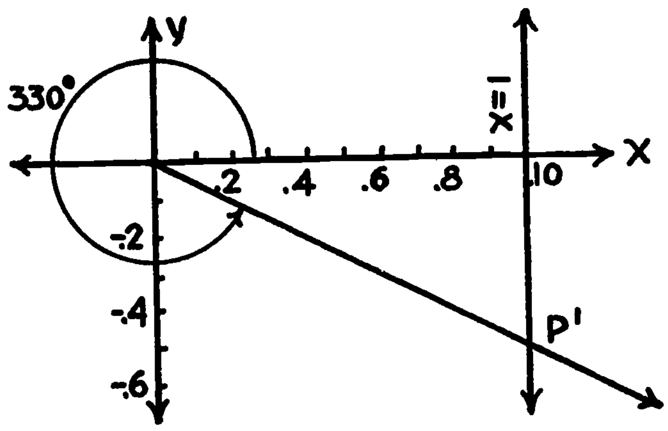
Answer: In this triangle which has one side on the line x = 1, the value of the tangent of the quadrant I or quadrant IV angle is equal to the ordinate of the point of intersection of the line x = 1 and the terminal ray of the angle.

(19) Thus by placing any such angle in standard position in a coordinate system,drawing the line x = 1, and finding the signed length of the ordinate of the point of intersection of the line x = 1 and the terminal ray of the angle, the value of the tangent of the angle can be determined.

Use a protractor to draw an angle of 333° in standard position in a coordinate system. Let the scale be 1 unit equals 1 decimeter (10 centimeters) and draw the line $x = 1$. Label the point of intersection of the line $x = 1$ and the terminal ray of the angle as P'.

Measure the ordinate of P' and find tan 333°.

Answer: $\tan 333^{\circ} = -0.51$ (see diagram on the following page.)



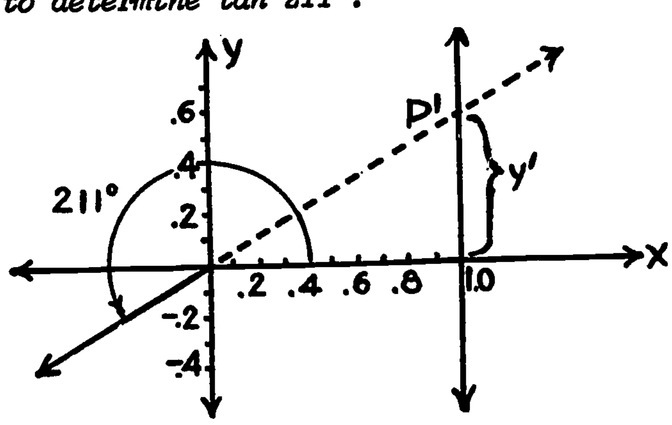
(20) *What difficulties arise in trying to use the procedure exactly as in the last exercise to determine the tangent of any angle?*

Answer: If the angle is a second quadrant angle or a third quadrant angle, or if the measure of the angle is 90°, 180°, or 270°, the terminal ray of the angle and the line $x = 1$ do not intersect.

(21) When the measure of the angle is 90° or 270°, the terminal ray of the angle is parallel to the line $x = 1$ and there is no point of intersection. We shall say that $\tan 90^\circ$ and $\tan 270^\circ$ are undefined. However, if the angle is a second quadrant angle or a third quadrant angle, or if the measure of the angle is 180°, we shall use the ray opposite the terminal ray of the angle to locate a point of intersection with the line $x = 1$. If the angle is θ and the point of intersection is P' with coordinates (x',y'), then for $90^\circ < \theta < 270^\circ$, we again have $\tan \theta = y'$.

Draw a graph to determine tan 211°.

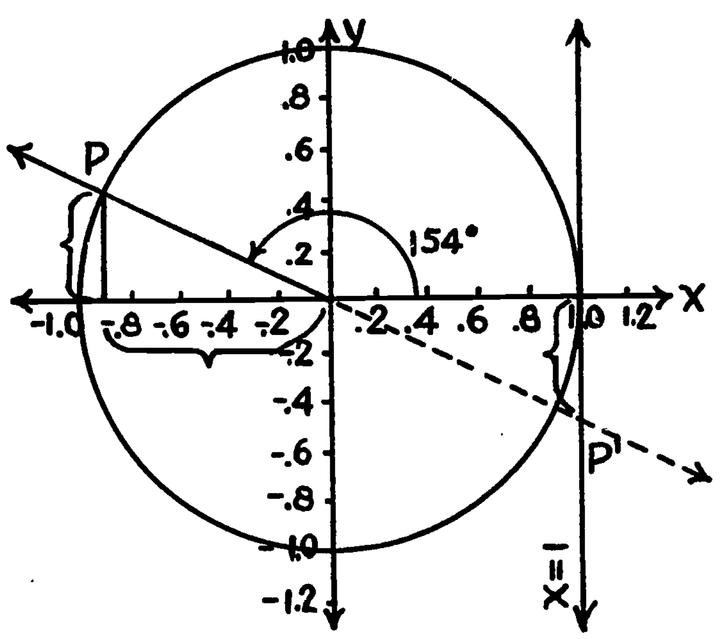
Answer:
$\tan 211^\circ = 0.60$



(22) The line x = 1 and the unit circle may be drawn on the same coordinate system yielding a combined diagram. In this diagram then, the line x = 1 is tangent to the unit circle at the point (1,0), and this figure has some historical significance in the name of the tangent ratio. From a combined diagram, the sine, cosine, and tangent ratios for any angle may be determined by measuring line segments in the figure, except for tan 90° and tan 270° which are undefined.

Draw a graph to determine sin 154°, cos 154°, and tan 154°.

<u>Answer:</u>
sin 154° = 0.44, cos 154° = - 0.90, tan 154° = - 0.49

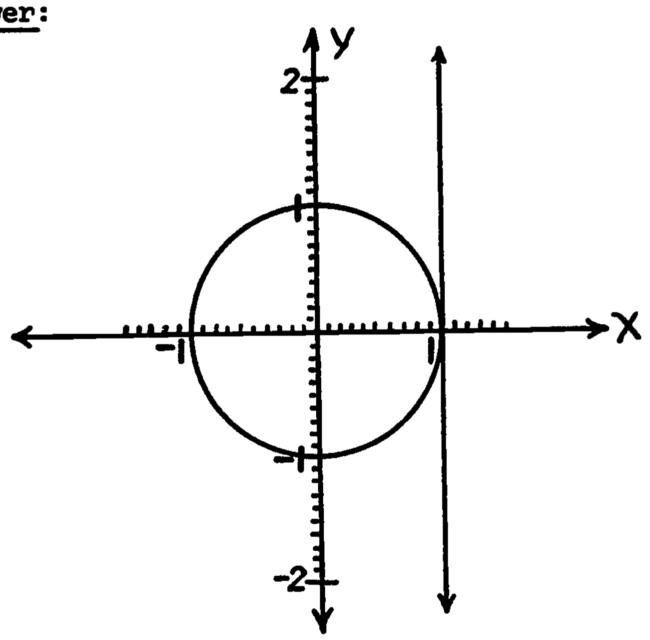


(23) Thus, associated with any angle θ in standard position on a coordinate axes there are two points of intersection which we now find of interest. One we have labeled point P and this is the intersection of the terminal ray of the angle with the unit circle. The other we have called P' and that is the intersection of the terminal ray or the ray opposite to the terminal ray of the angle, with the line x = 1. Note that the line x = 1 is tangent to the unit circle. The signed length of the ordinate of P is sin θ, the signed length of the abscissa of P is cos θ, and the signed length of the ordinate of P' is tan θ. When θ is 90° or 270°, there is no ordinate for P'. In the following exercises, the unit used for the unit circle and the line x = 1, may be one decimeter, one inch, or any length which is designated as one unit. The

length which is taken as one unit should be divided into tenths so that the measured lengths may be expressed to the nearest hundredth of a unit. If graph paper is used, ten divisions or some multiple of ten divisions, may be used as the unit.

On a sheet of graph paper, draw a set of coordinate axes, pick a unit with ten subdivisions, draw a unit circle with center at the origin, and draw the line $x = 1$. Repeat this pattern on four additional sheets of graph paper for use in the following four exercises.

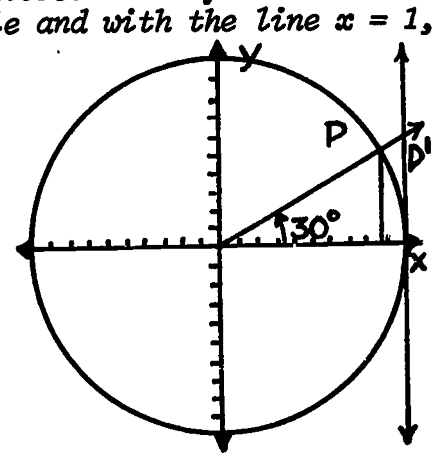
Answer:



Concept: Relations of the functions of angles in the various quadrants.

(24) (a) *Using a graph sheet with the unit circle and the line $x = 1$ drawn on it, draw a 30° angle in standard position on the coordinate system. Label the points of intersection of the terminal ray with the unit circle and with the line $x = 1$, P and P' respectively.*

Answer:



(b) *What is the length of the ordinate of P as a decimal fraction to the nearest hundredth.*

Answer: 0.50

(c) *What is the approximate value of sin 30°?*

Answer: 0.50 (This just happens to be exact.)

(d) *What is the length of the abscissa of P as a decimal fraction to the nearest hundredth.*

Answer: 0.86

(e) *What is the approximate value of cos 30°?*

Answer: 0.86

(f) *What is the length of the ordinate of P' as a decimal fraction to the nearest hundredth.*

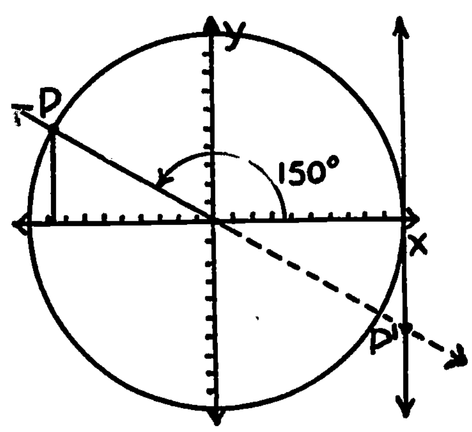
Answer: 0.58

(g) *What is the approximate value of tan 30°?*

Answer: 0.58

(25) (a) *Using a graph sheet as above, draw a 150° angle in standard position and label as above.*

Answer:



Note: It is very important to stress the fact that the values of the functions are generally irrational. The representation here is only to two place accuracy because this is all that can reasonably be expected by working from the diagrams.

(b) *What is the length of the ordinate of P as a decimal fraction to the nearest hundredth?*

Answer: 0.50

(c) *What is the approximate value of sin 150°?*

Answer: 0.50 (Again, this just happens to be exact.)

(d) *What is the length of the abscissa of P as a decimal fraction to the nearest hundredth?*

Answer: -0.86. Note that the direction on the x-axis is negative.

(e) *What is the approximate value of cos 150°?*

Answer: -0.86

(f) *What is the length of the ordinate of P' as a decimal fraction to the nearest hundredth?*

Answer: -0.58. Note that the ray opposite to the terminal ray of the angle is used and also the direction parallel to the y-axis is negative.

(g) *What is the approximate value of tan 150°?*

Answer: -0.58

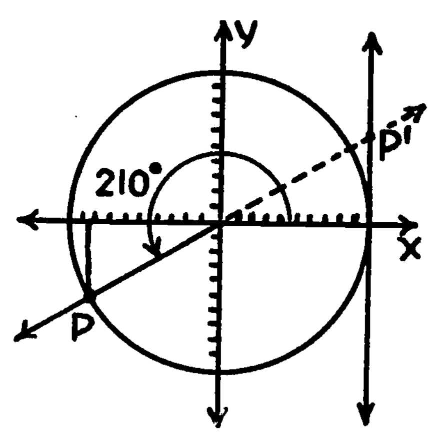
(26) *Using the same method, find the sin 210°, cos 210°, and tan 210°.*

Answer:

sin 210° = -0.50

cos 210° = -0.86

tan 210° = 0.58. Note that the ray opposite to the terminal ray of the angle is used and also the direction parallel to the y-axis is positive.



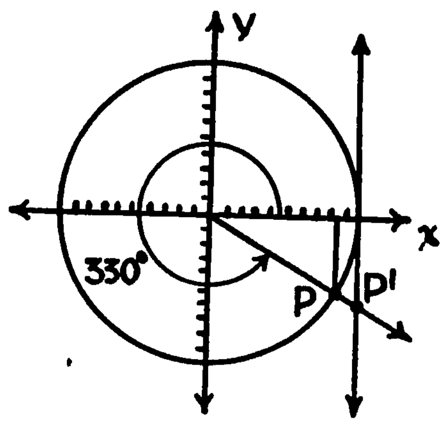
(27) *Using the last sheet of prepared graph paper, find sin 330°, cos 330°, and tan 330°.*

Answer:

sin 330° = -0.50

cos 330° = 0.86

tan 330° = -0.58



(28) *Make a chart of the sine, cosine, and tangent of 30°, 150°, 210°, and 330°.*

Answer:

	30°	150°	210°	330°
Sin	+0.50	+0.50	-0.50	-0.50
Cos	+0.86	-0.86	-0.86	+0.86
Tan	+0.58	-0.58	+0.58	-0.58

(29) *Write the absolute value of each item in the chart in the previous exercise.*

Answer:

(a) $|\text{Sin } 30^\circ| = +0.50$

(b) $|\text{Sin } 150^\circ| = +0.50$

(c) $|\text{Sin } 210^\circ| = +0.50$

(d) $|\text{Sin } 330^\circ| = +0.50$

(e) $|\text{Cos } 30^\circ| = +0.86$

(f) $|\text{Cos } 150^\circ| = +0.86$

(g) $|\text{Cos } 210^\circ| = +0.86$

(h) $|\text{Cos } 330^\circ| = +0.86$

(i) $|\text{Tan } 30^\circ| = +0.58$

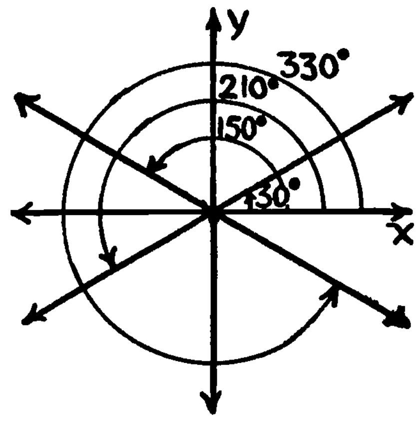
(j) $|\text{Tan } 150^\circ| = +0.58$

(k) $|\text{Tan } 210^\circ| = +0.58$

(l) $|\text{Tan } 330^\circ| = +0.58$

(30) *Using only one set of axes, draw the angles whose measures are 30°, 150°, 210°, and 330° respectively, in standard position.*

Answer: In the diagram on the next page note that the other three rays are reflections of the terminal ray of the 30° angle with respect to the x-axis, the y-axis, and the origin.



(31) *Describe how the positions of the terminal rays of the 150° angle, the 210° angle, and the 330° angle are related to the position of the terminal ray of the 30° angle.*

Answer: The terminal ray of the 150° angle and the terminal ray of the 30° angle are symmetric with respect to the y-axis. The terminal ray of the 210° angle and the terminal ray of the 30° angle are symmetric with respect to the origin. The terminal ray of the 330° angle and the terminal ray of the 30° are symmetric with respect to the x-axis.

(32) *An angle in the 2nd, 3rd, or 4th quadrants which is related to a first quadrant angle in one of the ways described in the previous exercise, is said to have that first quadrant angle as a reference angle. What reference angle is associated with each of the following: 140°, 200°, 310°.*

Answer: The reference angle of 140° is 40°, the reference angle of 200° is 20°, and the reference angle of 310° is 50°.

(33) The concept of reference angle may be used in conjunction with a table of the trigonometric functions to check the accuracy of the measurements used to determine the trigonometric ratios from a graph. Thus, since $|\tan 145^\circ| = |\tan 35^\circ|$ and from a table $\tan 35^\circ \approx 0.7002$, then, to the nearest hundredth $|\tan 145^\circ| \approx 0.70$. Although this does not tell us whether $\tan 140^\circ$ is positive or negative, it is a useful check.

Use a table of the trigonometric functions to find each of the following to the nearest hundredth:
$|\cos 321^{\circ}|$, $|\tan 255^{\circ}|$, $|\sin 148^{\circ}|$.

<u>Answers</u>: $|\cos 321^{\circ}| = |\cos 39^{\circ}| \approx 0.7771 \approx 0.78$
$|\tan 225^{\circ}| = |\tan 75^{\circ}| \approx 3.7321 \approx 3.73$
$|\sin 148^{\circ}| = |\sin 32^{\circ}| \approx 0.5299 \approx 0.53$

12.5 GRAPHS OF TRIGONOMETRIC FUNCTIONS

In unit 11 of this course, portions of the sine function, the cosine function, and the tangent function were graphed by selecting entries in a table of trigonometric functions, and using these entries to compose the ordered number pairs which determined points on the graph. In this section, the abscissas of the desired points will still be measures of angles, but the ordinates of the desired points will be determined by measurements in a second coordinate system containing the unit circle and the line $x = 1$.

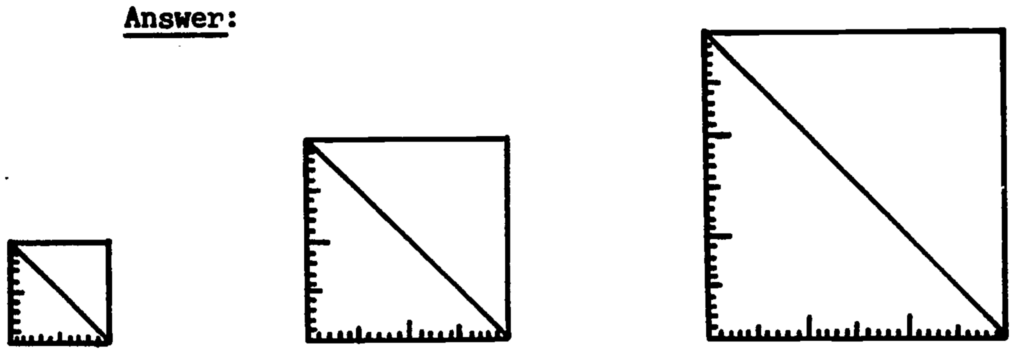
Concept: Using measurements to determine the elements of a relation.

(1) *What is meant by a relation?*

<u>Answer</u>: A relation is a set of ordered number pairs.

(2) There are several ways in which to obtain the members of the ordered number pairs which are the elements of a relation. Thus the members of the pairs may be paired entries in a table or they may be paired numbers associated in some formula or equation involving two variables. It is also possible to obtain ordered number pairs by associating two corresponding measurements, which may or may not be of the same kind. Thus two linear measurements may be associated such as the lengths of the sides and diagonals in a series of squares.
Using graph paper or a unit divided into tenths, draw three squares with sides one unit, two units, and three units respectively. Measure the diagonal of each square to the nearest hundredth of a unit and list the set of ordered pairs which associate a side with a corresponding diagonal. In each ordered pair listed, write the side of the square as the first member and the corresponding diagonal as the second member.

Answer:

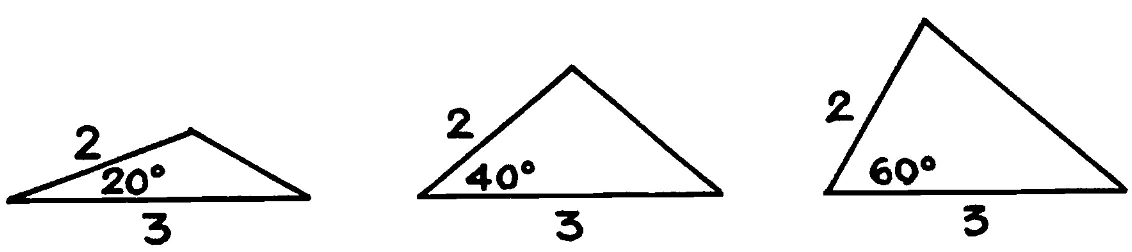


{(1,1.41), 2,2.83), (3,4.24)}

(3) Measurements of different kinds may also be associated. Thus the measure of an angle may be associated with the measure of a side.

Draw three triangles with the following measurements: the base of each is 3 units long, a second side is 2 units long, and the included angles are 20°, 40°, and 60° respectively. Measure the third side of each triangle to the nearest hundredth of a unit and list the set of ordered pairs which associate the measure of an angle with the measure of the opposite side.

Answer:



{(20,1.31), (40,1.95), (60,2.65)}

Concept: Using measures of angles as abscissas.

(4) It is most important to see that if the set {(20,1.31), (40,1.95), (60,2.65)} obtained in the previous exercise, is to be plotted on a coordinate system, then the numbers 20, 40, and 60, which are the first members of the ordered number pairs, must be the abscissas of the points plotted, and that these distances are measured along the x-axis. That is, although 20, 40, and 60 originally referred to certain rotations which are the measures of certain angles, now that these numbers are represented by abscissas of points, these abscissas are lengths measured along the x-axis.

If the distance from the origin to the point on the x-axis which represents 20 degrees is 1 unit, determine the distance from the origin to the point on the x-axis which represents (a) 40° (b) 45° (c) 50° (d) 60°.

Answers:

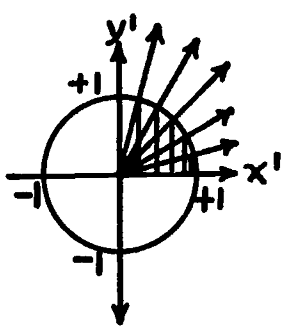
(a) 2 units (b) $2\frac{1}{4}$ units (c) $2\frac{1}{2}$ units (d) 3 units

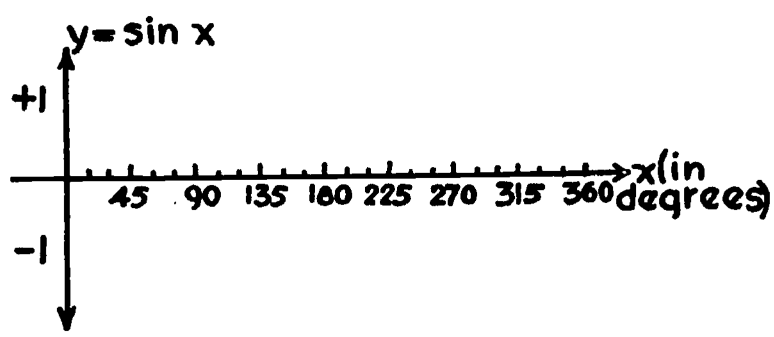
Concept: Graph of the sine function.

(5) For the following exercises, the pupil will use either the dittoed sheets prepared by the teacher (see section 12.5 of *Background Material for Teachers*) or he will draw pairs of coordinate systems as illustrated in that section. The pupil may use graph paper if he does not use the dittoed materials. If graph paper is not used, a ruler or pair of compasses will be needed to "transfer" the lengths of line segments.

On the first coordinate system containing the unit circle, draw in standard position angles of measure 0°, 15°, 30°, 45°, 60°, 75°, and 90°. For each angle draw the ordinate of the point of intersection of the terminal ray of the angle and the unit circle.

Answer:





(6) *The signed length of the ordinate of the point of intersection of the terminal ray of the angle and the unit circle determines the value of which trigonometric ratio associated with that angle?*

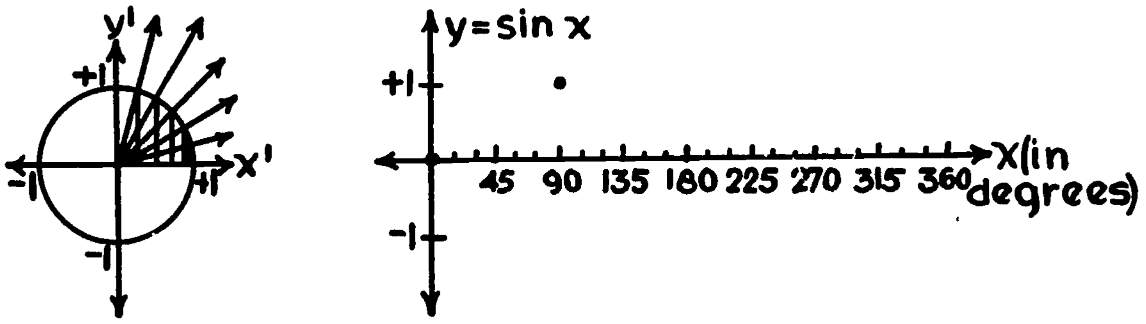
Answer: The sine ratio

(7) *What is the length of the ordinate of the point of intersection of the terminal ray and the unit circle when the measure of the angle is (a) 0° (b) 90°?*

Answers: (a) 0 (b) 1

(8) *On the second coordinate system, plot the points (0,0) and (90,1).*

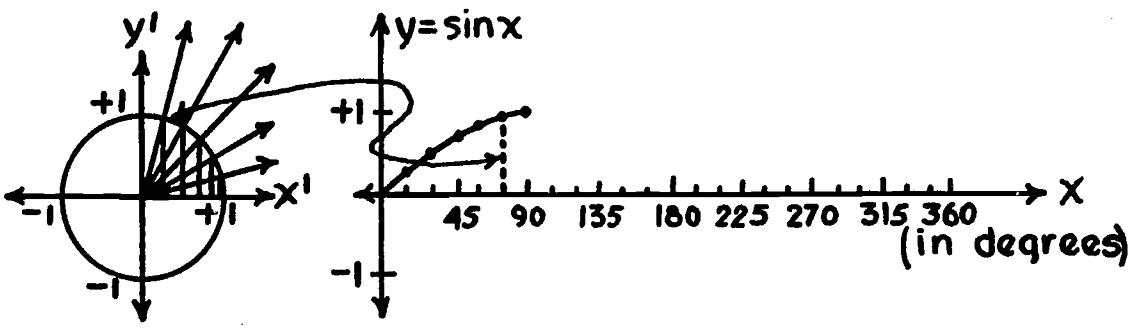
Answer:



(9) If graph paper is not used, use compasses or rulers to "transfer" the lengths of line segments from the first coordinate system to the corresponding measures on the second coordinate system. These lengths, which in this graph happen to be ordinates in the first coordinate system, will, in the second coordinate system, be ordinates of points whose abscissas are measures of angles.

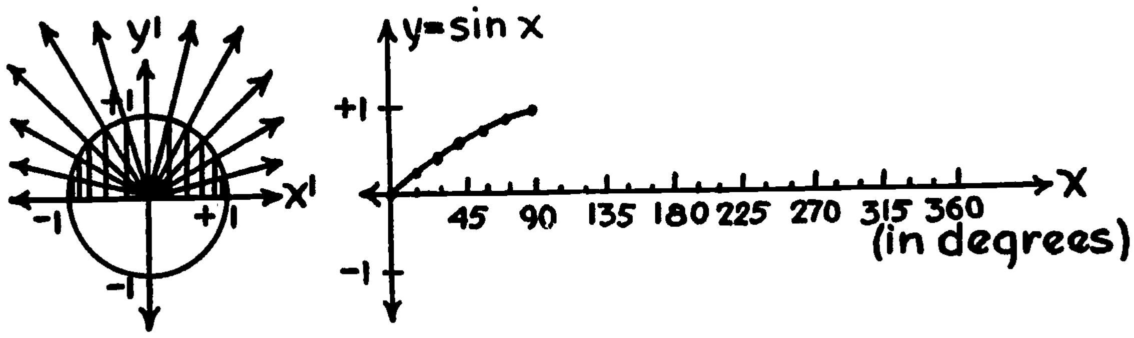
Plot the points whose abscissas are 15, 30, 45, 60, and 75 respectively, and whose ordinates are the respective sines of the angles as determined by the segments in the first coordinate system. Connect the seven points which are now plotted, with a smooth curve.

Answer:



(10) *In the first coordinate system containing the unit circle, continue drawing further members of the set of angles by now drawing in standard position angles whose measures are 105°, 120°, 135°, 150°, 165°, and 180° respectively. As before, for each angle draw the ordinate of the point of intersection of the terminal ray and the unit circle.*

Answer:

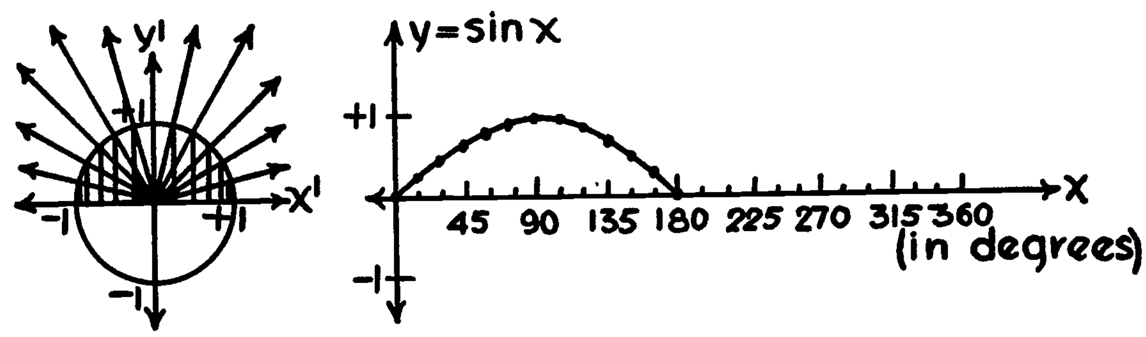


(11) *Are the ordinates drawn in exercise (10) positive or negative quantities?*

Answer: The ordinate of each of the points except the one on the x-axis, represents a positive quantity. The ordinate of the point on the terminal ray of the angle with measure 180° is 0 since the point is on the x-axis.

(12) *In the second coordinate system, plot the points whose abscissas are 105, 120, 135, 150, 165, 180 and whose ordinates are the respective sines of the angles as determined by segments in the first coordinate system. Continue the smooth curve through the additional points now plotted.*

Answer:

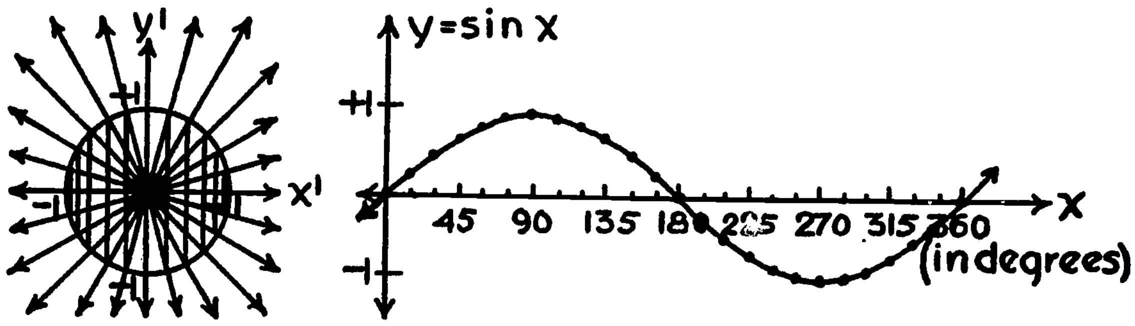


(13) *Do protractors have to be used to draw the remaining angles in the set of angles 0°, 15°, ..., 330°, 360°?*

Answer: No. The terminal rays of the angles in this set, whose measures are greater than 180° but less than or equal to 360°, are symmetric with respect to the origin with the terminal rays of the angles already drawn, so that respective pairs of rays form a straight line.

(14) *Draw in standard position in the first coordinate system, the angles whose measures are respectively 195°, 210°, ..., 360°. Using the same procedure as previously, continue the graph of the sine relation, $y = \sin x$, until the point whose abscissa is $x = 360$. In future, more advanced work, you will find that this graph continues in both directions, that is, x may be a negative quantity or a number greater than 360. Therefore, continue the graph a short distance beyond the first and last points and use arrow heads to indicate that the graph goes on in both directions.*

Answers:



(15) *In the graph of the previous exercise, what statement can be made about the ordinates of the points whose abscissas are 195, 210, ..., 330, 345?*

Answer: They represent negative quantities. They are second coordinates of points below the x-axis.

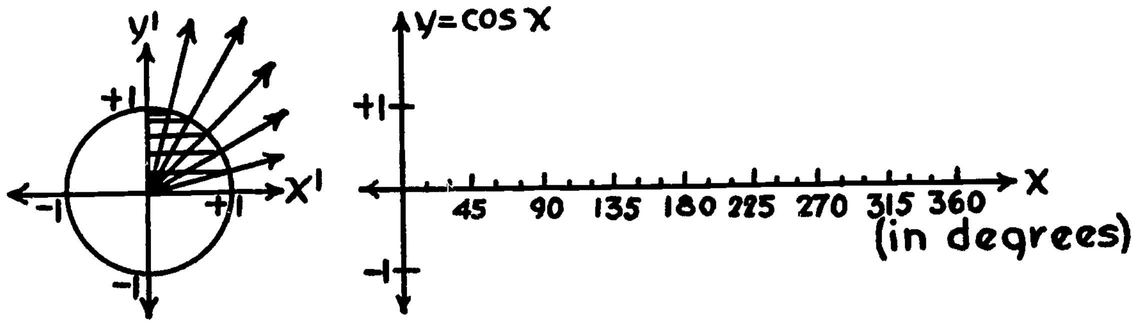
(16) *Use the vertical line test to determine whether the relation, $y = \sin x$, is a function.*

Answer: Any vertical line intersects the graph of the relation, y = sin x, at not more than one point. Therefore, the relation, y = sin x, is a function.

Concept: Graph of the cosine function.

(17) *Use the second dittoed sheet or the second pair of coordinate systems drawn on graph paper for the following work. On the first coordinate system of the pair, the one containing the unit circle, draw in standard position angles of measure 0°, 15°, 30°, 45°, 60°, 75°, and 90°. For each angle draw the abscissa of the point of intersection of the terminal ray of the angle and the unit circle. It is more convenient to use the actual abscissa rather than the equivalent distance on the x-axis in the following work.*

Answer:



(18) *The signed length of the abscissa of the point of intersection of the terminal ray of the angle and the unit circle determines the value of which trigonometric ratio associated with that angle?*

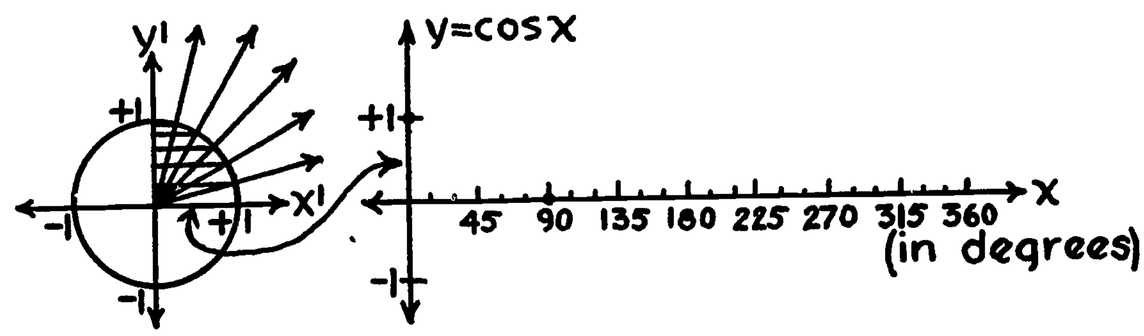
Answer: The cosine ratio

(19) *What is the length of the abscissa of the point of intersection of the terminal ray and the unit circle when the measure of the angle is (a) 0° (b) 90°?*

Answers: (a) 1 (b) 0

(20) *On the second coordinate system, plot the points (0,1) and (90,0).*

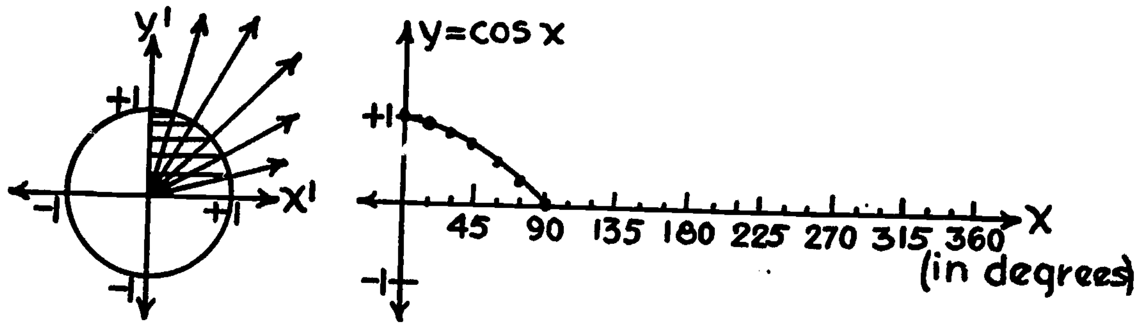
Answer:



(21) It is most important to see clearly that the numerical value of each cosine ratio is represented by an abscissa of a point in the first graph but by an ordinate of a point in the second graph. The abscissa of each point in the second graph is still the measure of an angle. Whether a certain quantity is shown by an ordinate or an abscissa depends only upon its place in the ordered number pair.

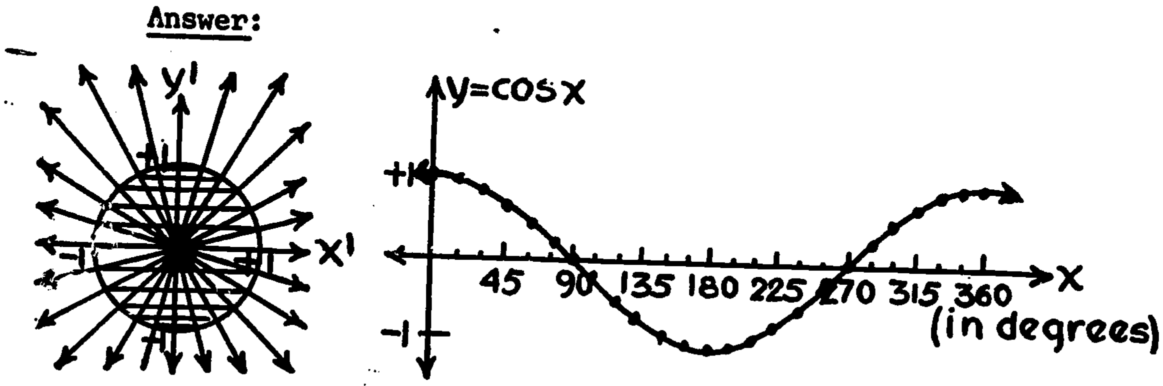
On the second coordinate system plot the points whose abscissas are 15, 30, 45, 60, and 75 respectively, and whose ordinates are the respective cosines of the *angles as determined by the segments in the first coordinate system. Connect the points which are now plotted, with a smooth curve.*

Answer:



(22) *Continuing in a manner similar to that used to plot the sine function, plot points of the cosine relation, y = cos x, until the point whose abscissa is x = 360. Since abscissas in the first coordinate system are being made equal to ordinates in the second coordinate system, and these have different directions, check carefully that negative quantities are being made equal to negative quantities and positive quantities to positive quantities. Connect the plotted points with a smooth curve, and as before, continue the graph a short distance beyond the first and last points, using arrow heads to indicate that the graph goes on in both directions. Curve the extensions downward to indicate that the maximum ordinate of the graph is 1 unit.*

Answer:



(23) *Use the vertical line test to determine whether the relation, y = cos x, is a function.*

Answer: Any vertical line intersects the graph of the relation, y = cos x, at not more than one point. Therefore, the relation, y = cos x, is a function.

(24) *Examine the graphs of the functions, $y = \sin x$ and $y = \cos x$, and remembering that each pattern is repeated over and over again in both directions, find a relation between the two graphs. Test your conclusion.*

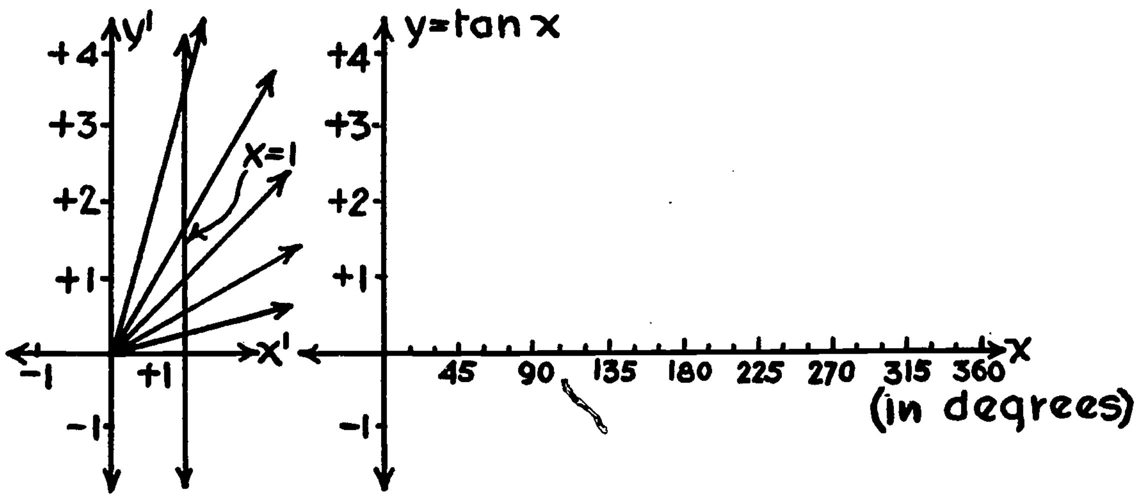
Answer: The graphs of y = sin x and y = cos x are the same curves except that x = 0 is at a different place on the curve for y = sin x than for y = cos x. If the y-axis of the graph of y = cos x is placed at 90 on the graph of y = sin x, and the two x-axes are on one another, the curves also match and fall one upon the other. This can be seen by holding the two graphs up to the light. *Note:* To have the curves match completely, the patterns of the graphs that were drawn must be repeated without end in both directions.

Concept: Graph of the tangent function.

(25) In this work, the pupil will use either the dittoed sheet prepared by the teacher (see section 12.5 of *Background Material for Teachers*) or he will draw a pair of coordinate systems as illustrated in that section. The pupil may use graph paper if he does not use the dittoed materials. As before, if graph paper is not used, a ruler or pair of compasses will be needed to "transfer" the lengths of line segments.

On the first coordinate system containing the line $x = 1$, draw in standard position angles of measure, 0°, 15°, 30°, 45°, 60°, 75°, and 90°.

Answer:



(26) *Does the terminal ray of each angle drawn intersect the line $x = 1$?*

Answer: The terminal ray of the angle of measure 90° does not intersect the line x = 1 because the ray and the line are parallel.

(27) *If an angle of 89° were drawn in standard position on the first coordinate system, would the terminal ray of the angle intersect the line $x = 1$?*

Answer: They would intersect because the ray and the line would not be parallel.

(28) *The value of which trigonometric ratio associated with an angle is determined by the signed length of the ordinate of the point of intersection of the terminal ray of the angle and the line $x = 1$?*

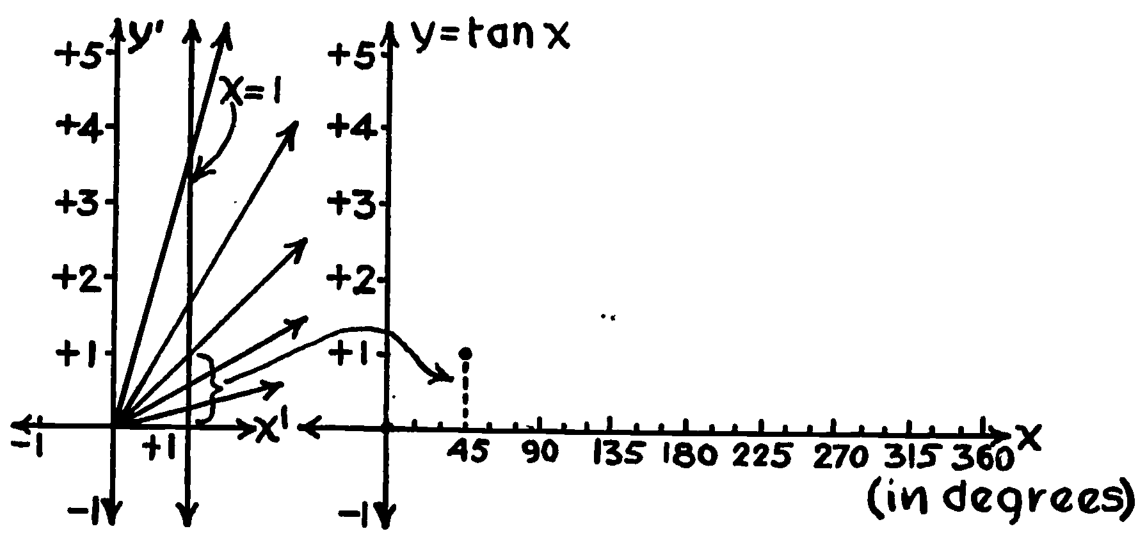
Answer: The tangent ratio

(29) *What is the length of the ordinate of the point of intersection of the terminal ray and the line $x = 1$ when the measure of the angle is (a) 0° (b) 45°?*

Answers: (a) 0 (b) 1

(30) *On the second coordinate system, plot the points (0,0) and (45,1).*

Answer:

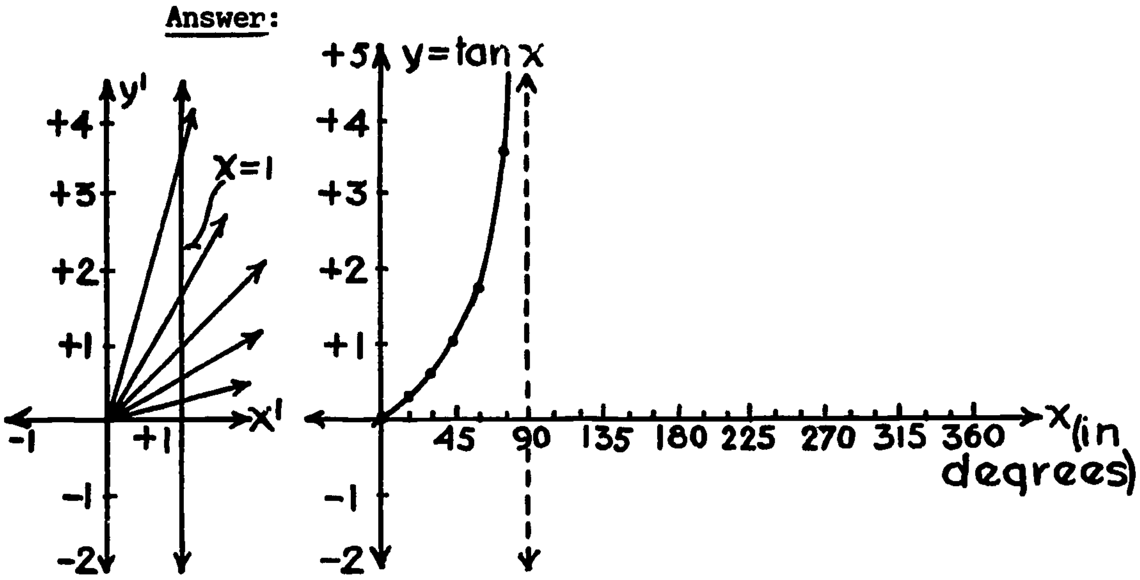


(31) It is clear that when the measure of the angle is 90°, there is no segment that can be transferred from the first coordinate system to the second. From previous work it is known that tan 90° is undefined. Also it can be seen that if we were to draw angles whose measure were

less than 90° but close to 90°, the points of intersection of the terminal rays and the line $x = 1$ would have ordinates of great length.

On the second coordinate system plot the points whose abscissas are 15, 30, 60, and 75 respectively, and whose ordinates are the respective tangents of the angles as determined by the lengths of segments in the first coordinate system. Connect the six points which are now plotted with a smooth curve and continue the curve past the last point plotted by making it slope sharply upward. However, make sure that no point on the graph of $y = \tan x$ has an abscissa of 90, that is, the graph must not intersect the line $x = 90$.

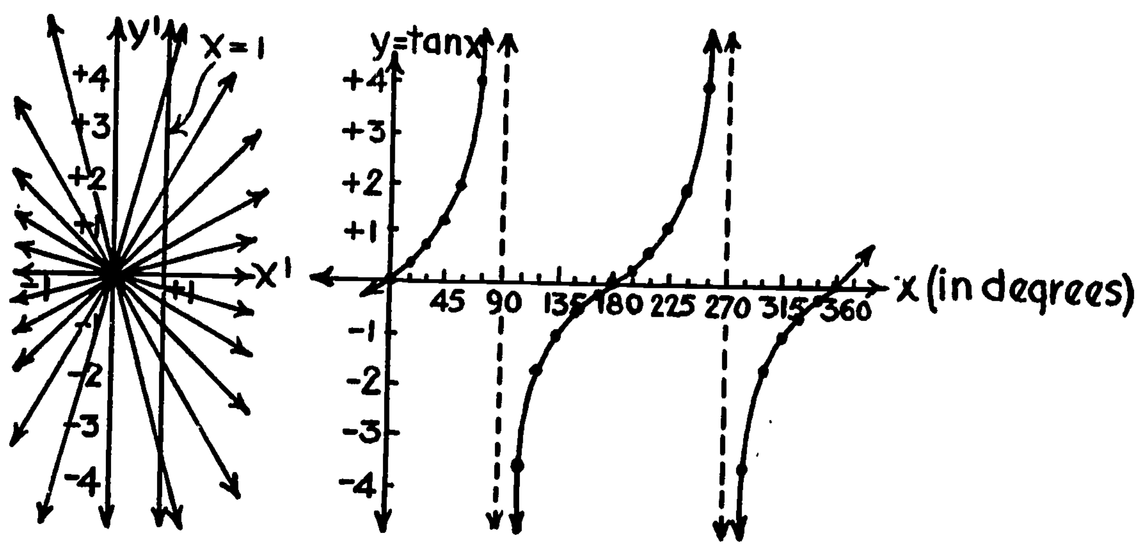
Answer:



(32) If the graph of $y = \tan x$ is continued by "transferring" segments from the first coordinate system to the second, there are several properties that must be recalled and kept clearly in mind. First, in considering angles whose measure varies from 0° to 360°, remember that $\tan 90^\circ$ and $\tan 270^\circ$ are both undefined. Secondly, remember that the tangents of angles in the second and fourth quadrants are negative, so it must be distinguished very carefully between positive and negative ordinates to be "transferred." Lastly, remember that for each angle whose measure is greater than 90° but less than 270°, we find the point of intersection of the line $x = 1$ with the ray opposite the terminal ray of the angle.

Continuing in a manner similar to that used to plot the sine function and the cosine function, plot points of the tangent relation, $y = \tan x$, until the point whose abscissa is $x = 360$. We find that we cannot connect all the points with a single smooth curve because of the *interruptions at 90 and 270. Points on the curve with abscissas close to 90 but less than 90 have ordinates which are large positive quantities, while those which have abscissas close to 90 but greater than 90 have ordinates which are negative quantities of large absolute value. Thus the two portions of the curve continue in opposite directions in the vicinity of the line $x = 90$, and this should be indicated by the arrowheads. A similar situation occurs in the vicinity of the line $x = 270$, and here too the arrowheads should indicate this. Connect the successive plotted points, other than at these two critical places, by smooth curves, and as before, continue the graph a short distance beyond the first and last points, using arrowheads to indicate that the graph goes on in both directions.*

Answer:



(33) *Use the vertical line test to determine whether the relation, $y = \tan x$, is a function.*

Answer: Any vertical line which intersects the graph of the relation, y = tan x, does so at not more than one point. Therefore, the relation, y = tan x, is a function.

GLOSSARY OF TERMS

Abscissa The first number of an ordered number pair. The horizontal coordinate in a rectangular coordinate system.

Absolute value If x is positive, its absolute value is x; if x is negative, its absolute value is -x; and if x is zero, its absolute value is zero.

Angle The set of all points in the plane contained in two rays that have the same endpoint.

Angle (measure of) The measure of an angle is the measure of the amount of rotation of the initial side of the angle about its vertex necessary to make the initial side coincide with the position of the terminal side. If the unit of measurement is taken as the degree, this unit is $\frac{1}{360}$ of a complete rotation. Counterclockwise rotation is taken as positive and clockwise rotation is taken as negative.

Associative property for addition In any sum of several terms, any pair of successive terms may be added first, any other pair of successive terms may be added next, and so on. $a + (b + c) = (a + b) + c$

Associative property for multiplication In any product of several factors any pair of successive factors may be multiplied first, any other pair of successive factors may be multiplied next, and so on. $(ab)c = a(bc)$

Binomial A polynomial of two terms.

Cancellation law for rationals A rational expression may be reduced to a simpler rational expression by applying the principle of equality of rational expressions: $\frac{ab}{ac} = \frac{b}{c}$.

Cartesian plane The set of all points defined by the set U x U of ordered pairs, where U is the set of all real numbers.

Cartesian set The set U x U of ordered pairs, where U is any given set of numbers.

Closure A set of numbers has the property of closure under a given operation if, when any two elements of the set are combined under the given operation, the result is always a unique element of the set.

Coefficient Each factor of a product is the coefficient of the product of the other factors.

Coefficient numerical If an algebraic term consists of the product of a constant and a variable or variables, the constant is called the numerical coefficient of the term.

Commutative property for addition The result of combining two elements under addition does not depend on the order in which the addends are taken. a + b = b + a

Commutative property for multiplication The result of combining two elements under multiplication does not depend on the order in which the factors are taken. ab = ba

Complex fraction A fraction which has a fraction for the numerator or the denominator or both.

Coordinates The numbers in an ordered number pair which identify a point in the Cartesian plane.

Cosecant of an angle The ratio of the radius vector of point *P* to the ordinate of point *P*, where point *P* is any point on the terminal ray of the given angle, the angle being in standard position on a coordinate system.

Cosine of an angle The ratio of the abscissa of point *P* to the radius vector of point *P*, where point *P* is any point on the terminal ray of the given angle, the angle being in standard position on a coordinate system.

Cosine of an acute angle in a right triangle The ratio of the length of the leg adjacent to the given angle to the length of the hypotenuse.

Cotangent of an angle The ratio of the abscissa to the ordinate of point *P*, where point *P* is any point on the terminal ray of the given angle, the angle being in standard position on a coordinate system.

Degree of a polynomial The degree of the highest degree term of the polynomial. The degree of a term of one variable is the exponent of that variable. A term in several variables has its degree equal to the sum of the exponents of its variables.

Dependent system of equations A system of equations which consists of equivalent equations. A system of equations the graphs of which are coincident lines.

Discriminant The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $D = b^2 \quad 4ac$.

Distributive principle The product of a monomial by a polynomial is equal to the sum of the products of the monomial by each term of the polynomial. $a(b + c) = ab + ac$

Domain The set of all abscissas of a relation.

Element of a set A member of a set. Any object or concept contained in the set.

Equality of rational expressions $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

Equation A statement of equality between two quantities.

Equation (conditional) An equation whose solution set is a proper subset of the replacement set of the variable. An equation whose solution set consists of at least one but not all the elements of the replacement set of the variable.

Equation (internally, inconsistent) An equation whose solution set is the null set.

Equation (linear) An equation whose highest degree term in the variable (or variables) is first degree.

Equation (quadratic) An equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

Equivalent Possessing equal value but not necessarily equal form.

Euler circle A circle used as a graphical representation of a set.

Exponent (fractional) A fraction at the right and above an expression, the denominator denoting the principal root of the expression and the numerator the power of this root.

Exponent (negative) A negative number to the right and above an expression indicating that in addition to the operations indicated by the absolute value of the exponent, the reciprocal of the resulting expression is to be taken.

$$x^{-3} = \frac{1}{x^3}$$

Exponent (positive integral) A positive integer above and to the right of an expression indicating how many times the expression is to be taken as a factor.

$y^5 = (y)(y)(y)(y)(y)$

Exponent (zero) A zero written above and to the right of an expression, indicating the result of subtracting exponents when dividing an expression by itself, $\frac{x^3}{x^3} = x^0$.

The value of any nonzero expression with zero exponent is defined to be 1.

Factor One of two or more quantities multiplied together to give a product.

Factoring To resolve into factors.

Function A relation in which no element in the domain has more than one element in the range associated with it.

Function (linear) A function defined by a first degree equation. The graph of $\{(x,y) \mid y = mx + b\}$ is a straight line.

Function (quadratic) A function defined by a quadratic equation.

Hypotenuse The side of a right triangle opposite the right angle.

Identity An equation that is always true. An equation whose solution set is identical with the replacement set of the variable or variables. An equation that is true for all values of the variables contained in the equation.

Identity element for addition in a given set If x is any element in the given set, the identity element for addition in that set is the element y such that $x + y = x$.

Identity element for multiplication in a given set If x is any element in the given set, the identity element for multiplication in that set is the element y such that $xy = x$.

Inconsistent system of equations A system of equations whose solution set is the null set; a system consisting of equations whose solution sets have no intersection.

Independent system of two linear equations Two linear equations in two variables having exactly one number pair which satisfies both equations.

Inequality A statement that one quantity is less than or greater than another. a > b is read "a is greater than b" or "b is less than a."

Inequality (absolute) An inequality whose solution set is identical to the replacement set of the variable. An inequality that is true for all possible values of the variable. $x + 5 > x$

Inequality (conditional) An inequality whose solution set is a proper subset of the replacement set of the variable; at least one but not all the elements in the replacement set are elements in the solution set. $x + 5 > 7$

Inequality (internally inconsistent) An inequality whose solution set is the null set. No element in the replacement set of the variable is an element in the solution set. $x + 5 < x$

Integers (set of) The set $I = \{0,+1,-1,+2,-2,\ldots\}$.

Integral expression An expression in which no variable, when written with a positive exponent, appears in a denominator.

Inverse (additive) The additive inverse of the number x is the number y such that $x + y = 0$. The additive inverse of x is written $-x$.

Inverse of a function The function, if it exists, obtained by interchanging the domain and range of the original function.

Inverse (multiplicative) The multiplicative inverse, or reciprocal, of the number x is the number y such that the product xy is equal to the identity element for multiplication: $xy = 1$. The multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$, $a \neq 0$.

Inverse operations If $a \theta b = c$ and $c \Omega b = a$, then θ and Ω are inverse operations.

Irrational number A real number that is not rational; a real number that is not expressible as an integer or quotient of integers.

Leg of a right triangle Either of the sides of the right triangle adjacent to the right angle.

Like terms Two or more terms having the same identical variable or variables and the exponents of corresponding variables the same, terms which are identical except for the numerical coefficient.

Lines (concurrent) Lines which have a point in common.

Lines (coincident) Two lines such that all points of either one lie on the other.

Natural numbers The set of counting numbers, the set $N = \{1,2,3,\ldots\}$.

Null set The empty set. The set which contains no element, represented as $\{\}$ or $\emptyset$.

Number line (real) The straight line in which every point represents a real number and every real number is represented by a point.

Monomial An expression of one term.

One-to-one correspondence The relationship between two sets such that for every element in the first set there is one distinct corresponding element in the second set, and for every element in the second set there is one distinct corresponding element in the first set.

Open phrase An expression or polynomial containing one or more variables.

Open sentence An equation or inequality containing at least one variable, the truthfulness of the sentence depending on the element which is substituted for the variable.

Ordered number pair A pair of numbers associated in a particular order. The numbers may represent the abscissa and ordinate, in that order, of a point in a coordinate plane.

Ordinate The coordinate of a point in a plane which is the distance from the x-axis measured along a line parallel to the y-axis.

Pi The name of the Greek letter which corresponds to the Roman P. The symbol π denotes the ratio of the circumference to the diameter of any circle.

Polynomial A rational integral algebraic expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$, where n is any non-negative integer.

Prime number An integer which has no integral factors except itself and one. The number 1 is usually excluded.

Prime factors All the prime numbers or prime polynomials that will exactly divide the given number or polynomial.

Prime polynomial A polynomial which has no polynomial factors except itself and constants in a particular field.

Product The result of multiplication.

Pythagorean theorem The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

Quadratic formula $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ This indicates the roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

Quotient The result of division.

Radical The indicated root of an expression.

Radicand The expression under a radical sign.

Radius vector of a point The distance from the point to the origin. This is the length of the straight line segment whose endpoints are the given point and the origin, respectively.

Range of a relation The set of ordinates of a relation.

Rational expression An expression in which no variable is an irreducible radical or is under a fractional exponent.

Rational number Any number that can be expressed as an integer or as a quotient of two integers.

Real numbers Rational and irrational numbers that do not contain an even root of a negative number.

Reciprocal The multiplicative inverse. If $a \neq 0$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. If $a = 0$, $\frac{a}{b}$ has no reciprocal.

Reflexive property of equality For any expression a, $a = a$.

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Relation Any set of ordered number pairs. A relation may be defined by the solution set of an equation, by a graph, by a table of values, or by a listing of the members of a set of ordered pairs.

Replacement set of a variable The set of numbers whose elements may be used as successive replacements for the variable.

Root of an equation A number which when substituted for the variable in the equation reduces it to an identity. The n^{th} root of a number is the number which, when taken as a factor n times or raised to the n^{th} power, results in the given number.

Root (extraneous) An apparent root, derived in the process of solving an equation, which is not a root and therefore is not an element in the solution set of the equation. It is generally introduced by squaring each side of the given equation or by multiplying each side of an equation by a variable.

Root (principal) The principal root of a real number is the positive real root in the case of roots of positive numbers, the negative real root in the case of odd roots of negative numbers.

Secant of an angle in a coordinate plane The reciprocal of the cosine of the angle. The ratio of the radius vector of point *P* to the abscissa of point *P*, where point *P* is any point on the terminal ray of the angle, the angle being in standard position on a coordinate system.

Set A collection or group of objects or concepts.

Set (Cartesian) The Cartesian set of U is the set of all ordered pairs of numbers such that the first and second members of each pair are each elements of the given set of numbers. The same element may be used for the first and second members of a pair.

Set (complement of) The complement of set *A* is the set of all elements in the universal set that are not in set *A*.

Sets (disjoint) Sets whose intersection is the null set. Sets that contain no common elements.

Sets (equivalent) Sets which can be put into one-to-one correspondence. Sets containing the same number of elements.

Sets (intersection of) The intersection of sets A and B is the set of all elements that are in both sets; to be in the intersection, an element must be in set A and also in set B.

Sets (matching) Sets which have a one-to-one correspondence between them. Equivalent sets.

Sets (union of) The union of set *A* and set *B* is the set of all elements that are either in set *A* or in set *B* or in both set *A* and set *B*.

Solution set of an open sentence The set of all elements which when substituted for the variable in the open sentence results in a true statement.

Simultaneous equations Equations whose solution set contains at least one element but not all elements of the replacement set.

Sine of an angle in a coordinate plane The ratio of the ordinate of point *P* to the radius vector of point *P*, where point *P* is any point on the terminal ray of the angle, the angle being in standard position on a coordinate system.

Sine of an acute angle in a right triangle The ratio of the length of the leg opposite the angle to the length of the hypotenuse.

Slope of the graph of a linear equation The ratio $\frac{y_1 - y_2}{x_1 - x_2}$ where (x_2,y_2) and (x_1,y_1) are any two points on the graph of the linear equation.

Subset Set *A* is a subset of set *B* if every element in set *A* is also an element in set *B*. Every set is a subset of itself.

Subset (proper) Set *A* is a proper subset of set *B* if set *A* is a subset of set *B* and set *A* does not contain all the elements of set *B*.

Standard position of an angle in a coordinate plane An angle in a coordinate plane is said to be standard position if the initial ray of the angle is the part of the x-axis to the right of (0,0) and the vertex of the angle is the point (0,0).

Symmetric property of equality For every two expressions a and b, if a = b, then b = a.

Tangent of an angle in a coordinate plane The ratio of the ordinate to the abscissa of point *P*, where point *P* is any point on the terminal ray of the angle, the angle being in standard position on a coordinate system.

Tangent of an acute angle in a right triangle The ratio of the length of the leg opposite the given angle to the length of the leg adjacent to the given angle.

Term An algebraic expression written as a product or quotient of numerals or variables or both.

Transitive property of equality For any expressions a, b, and c, if a = b and b = c, then a = c.

Trinomial A polynomial of three terms.

Variable A symbol, usually a letter in an open phrase or sentence replaceable by a member of a defined set of numbers called the replacement set of the variable.

Venn diagram A diagram consisting of a rectangle to represent the universal set and three mutually intersecting Euler circles, each representing a set. The three circles intersect in all possible combinations.

Whole numbers The set of natural numbers and zero.

X-intercept The abscissa of an ordered number pair whose ordinate is zero.

Y-intercept The ordinate of an ordered number pair whose abscissa is zero.